Functional Analysis, Spring 2020
Homework #10
This assignment is due on Wednesday, April 8

1. Let H be a Hilbert space and assume that $T \in L(H)$ is a normal operator. Suppose that $\sigma(T) = \{a + bi \, : \, 1 \le a \le 2, \, 1 \le b \le 2\}$.

(a) Compute $\|T^{-1}\|$.

(b) Compute $\|T^* + T^{-1}\|$.

2. Let H be a Hilbert space and let $T \in L(H)$.

(a) Suppose that $\langle Tx, x \rangle \ge 0$ for all $x \in H$. Show that $T = T^*$ and that $\sigma(T) \subset [0, \infty)$.

(b) Suppose that there is $c > 0$ such that $\langle Tx, x \rangle \ge c\langle x, x \rangle$ for all $x \in H$. Show that T is invertible.

3. Let H be a Hilbert space and assume that $T \in L(H)$ is a normal operator. Suppose that $\lambda \in \sigma(T)$ is an accumulation point in the sense that for any $\epsilon > 0$ the set $(\lambda - \epsilon, \lambda + \epsilon) \cap \sigma(T)$ is infinite.

Prove that there is an orthonormal sequence $(x_n)_{n=1}^{\infty}$ in H such that $\lim_{n \to \infty} \|(T - \lambda)x_n\| = 0$.