Functional Analysis, Spring 2020
Homework # 1
due on Wednesday January 22 in class

1. Let (X, d) be a metric space. Let (x_n) and (y_n) be two convergent sequences in X. Namely, suppose that $x_n \to x$ and $y_n \to y$. Show that $d(x_n, y_n) \to d(x, y)$.

2. Show that a normed linear space X is a Banach space if and only if for every sequence (x_n) of vectors in X, the condition $\sum_{n=1}^{\infty} \|x_n\| < \infty$ implies the convergence of $\sum_{n=1}^{\infty} x_n$ in X.

3. Let T be a compact Hausdorff space and let $C(T)$ denote the linear space of all continuous functions from T to $\mathbb{C}$, endowed with the norm

$$\|f\|_\infty = \sup\{|f(t)| : t \in T\}.$$

Show that $C(T)$ is a Banach space.

(It is fine to give a solution for the case when T is a closed and bounded set in R^n endowed with the usual euclidean distance.)

(One possible approach is to show that $C(T)$ is a closed normed linear subspace of $\ell^\infty(T)$.)

4. Show that the space $C[0, 1] = \{f : [0, 1] \to \mathbb{C},\ f \text{ continuous}\}$ with the norm $\|f\|_1 = \int_0^1 |f(t)|\, dt$ is not a Banach space.