Functional Analysis, Spring 2020
Homework #5
This assignment is due on Wednesday, February 19 in class

1. Prove that (a) holds in a real Hilbert space and (b) in a complex Hilbert space:

$$(a) \quad \langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2).$$

$$(b) \quad \langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 + i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2).$$

2. Let $(H_n)_{n=1}^{\infty}$ be a sequence of Hilbert spaces. By definition

$$H = \bigoplus_{n=1}^{\infty} H_n$$

consists of all sequences $x = (x_n)_{n=1}^{\infty}$ with the property that $\|x\|^2 := \sum_{n=1}^{\infty} \|x_n\|^2 < \infty$. Show that H is a Hilbert space with respect the scalar product

$$\langle x, y\rangle = \sum_{n=1}^{\infty} \langle x_n, y_n\rangle.$$

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Note that one needs to verify that the scalar product is well defined (i.e. the corresponding series is convergent) and to verify completeness of H.

3. Let $(H_n)_{n=1}^{\infty}$ be a a sequence of Hilbert spaces and let $T_n \in L(H_n)$. Define $T : \bigoplus_{n=1}^{\infty} H_n \to \bigoplus_{n=1}^{\infty} H_n$ by

$$T(x_1, x_2, ..., x_n, ...) = (T_1x_1, T_2x_2, ..., T_nx_n, ...).$$

Show that $\|T\| = \sup_n \|T_n\|$.