Functional Analysis, Spring 2020
Homework #7
This assignment is due on Wednesday, March 4

1) Let H be the Hilbert space $L^2([0,1], dt)$. Let $\varphi : [0,1] \to \mathbb{C}$ be a continuous function. Define $T \in L(H)$ by $(Tx)(t) = \varphi(t)x(t)$ for all $t \in [0,1]$ and $x \in H$. T is called the operator of multiplication by φ and is often denoted by M_φ.

Show that the spectrum of T, $\sigma_{L(H)}(T) = \{\varphi(t) : t \in [0,1]\}$.

Hint: If an operator T is invertible, then T is bounded below. This means that there is $C > 0$ such that $\|Tx\| \geq C\|x\|$ for all $x \in H$.

2). Let $(H_n)_{n=1}^{\infty}$ be a family of Hilbert spaces and let $H = \bigoplus_{n=1}^{\infty} H_n$ be the Hilbert space which is the direct sum of the family $(H_n)_{n=1}^{\infty}$ as defined in HW5, question 3. If $T_n \in L(H_n)$ and $\sup_n \|T_n\| < \infty$ for all $n \geq 1$, define $T \in L(H)$ by

$$T(x_1, x_2, ..., x_n, ...) = (T_1x_1, T_2x_2, ..., T_nx_n, ...).$$

(a) Show that T is invertible in $L(H)$ if and only if each T_n is invertible in $L(H_n)$ and $\sup_n \|T_n^{-1}\| < \infty$.

(b) Show that

$$\overline{\cup_{n=1}^{\infty} \sigma(T_n)} \subset \sigma(T).$$

3) Given a nonempty compact subset K of $\mathbb{C}$, find a bounded linear operator $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ whose spectrum is equal to K, i.e. $\sigma_{L(\ell^2(\mathbb{N}))}(T) = K$.