Functional Analysis, Spring 2020

Homework #8

This assignment is due on Wednesday, March 11

1) Let A be a unital Banach algebra. Prove that the map $\varphi : GL(A) \to GL(A)$, $\varphi(a) = a^{-1}$ is continuous.

2) Let X, Y be Banach spaces and let $T \in L(X, Y)$. Show that $T(X)$ is a closed subspace of Y if and only if there is $C > 0$ such that

$$\|Tx\| \geq C\, dist(x, \ker(T)), \quad \forall x \in X.$$

Notation: if $\emptyset \neq N \subset X$, $dist(x, N) = \inf\{\|x - n\|; n \in N\}$.

3) Define $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$S(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...).$$

(a) Find $\|S\|$ and $\|S^*\|$.

(b) Show that for any $\lambda \in \mathbb{C}$, $\ker(S - \lambda I) = \{0\}$.

(c) Show that for any $|\lambda| < 1$, $\ker(S^* - \lambda I) \neq \{0\}$. What is the dimension of $\ker(S^* - \lambda I)$?

(d) Determine $\sigma(S)$ and $\sigma(S^*)$.

Hint: Consider the relationship between $\sigma(T^*)$ *and* $\sigma(T)$ *for* $T \in L(H)$*, where* H *is a Hilbert space.*