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Local triviality for continuous field C^*-algebras

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Let X be a finite dimensional compact metrizable space. Let A be a separable continuous field C*-algebra over X with all fibers isomorphic to the same stable Kirchberg algebra $\mathcal{D}$. We show that if $\mathcal{D}$ has finitely generated K-theory and it satisfies the Universal Coefficient Theorem in KK-theory, then there exists a dense open subset U of X such that the ideal $A(U)$ is locally trivial. The assumptions that the space X is finite dimensional and that the K-theory of the fiber is finitely generated are both necessary.

1 Introduction

Continuous field C*-algebras play the role of bundles of C*-algebras (in the sense of topology) as explained in [1]. Continuous field C*-algebras appear naturally since any separable C*-algebra A with Hausdorff primitive spectrum X has a canonical continuous field structure over X with fibers the primitive quotients of A [5]. The bundle structure that underlies a continuous field C*-algebra is typically not locally trivial even if all fibers are mutually isomorphic, see [4, Example 8.4].

A point $x \in X$ is called singular for A if $A(U)$ is nontrivial (i.e. $A(U)$ is not isomorphic to $\mathrm{C}_0(U) \otimes \mathcal{D}$ for some C*-algebra $\mathcal{D}$) for any open set U that contains x. The singular points of A form a closed subspace of X. If all points of X are singular for A we say that A is nowhere locally trivial.

An example of a unital continuous field C*-algebra A over the unit interval with mutually isomorphic fibers and such that $A \otimes \mathcal{K}$ is nowhere locally trivial was constructed in [4]. In that example, all the fibers $A(x)$ of A are isomorphic to the same Kirchberg algebra $\mathcal{D}$ with $K_0(\mathcal{D}) \cong \mathbb{Z}^\infty$ and $K_1(\mathcal{D}) = 0$. We will argue below that the complexity of the continuous field A ultimately reflects the property of the K-theory of the fiber of not being finitely generated. On the other extreme, even if the K-theory of the fiber vanishes, a field can be nowhere locally trivial if the base space is infinite dimensional. Indeed, a unital separable

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continuous field C*-algebra A over the Hilbert cube was constructed in [3] with the property that all fibers are isomorphic to the Cuntz algebra $\mathcal{O}_2$, but nevertheless $A \otimes \mathcal{K}$ is nowhere locally trivial. The structure of continuous field C*-algebras with Kirchberg algebras as fibers over a finite dimensional space was studied by the second named author in [2] and [3]. In the present paper we use results from those articles to prove the following result on local triviality.

Theorem 1.1. Let X be a finite dimensional metrizable compact space, and let $\mathcal{D}$ be a stable Kirchberg algebra that satisfies the UCT and such that $K_*(\mathcal{D})$ is finitely generated. Let A be a separable continuous field C*-algebra over X such that $A(x) \cong \mathcal{D}$ for all $x \in X$. Then there exists a dense open subset U of X such that $A(U)$ is locally trivial. □

Recall that a C*-algebra satisfies the Universal Coefficient Theorem in KK-theory (abbreviated UCT) if and only if it is KK-equivalent to a commutative C*-algebra [11]. The two examples that we reviewed earlier show that both assumptions, that the space X is finite dimensional and that the K-theory of the fiber is finitely generated, are necessary.

We show in Example 3.7 that the conclusion of Theorem 1.1 cannot be strengthen. Indeed, given a nowhere dense closed subset F of a compact metrizable space X and any stable Kirchberg C*-algebra $\mathcal{D}$ with nontrivial K-theory, we construct a continuous field C*-algebra A over X with all fibers isomorphic to $\mathcal{D}$, and such that the set of singular points of A coincides with F.

2 Preliminaries

In this section, we recall a number of concepts and results that we use in the proof of the main theorem.

2.1 $\mathrm{C}_0(X)$-algebras and Continuous Fields

Let X be a locally compact Hausdorff space. A $\mathrm{C}_0(X)$-*algebra* is a C*-algebra A endowed with a $*$-homomorphism θ from $\mathrm{C}_0(X)$ to the center $Z(M(A))$ of the multiplier algebra $M(A)$ of A such that $\mathrm{C}_0(X)A$ is dense in A; see [6]. If $U \subset X$ is an open set, $A(U) = \mathrm{C}_0(U)A$ is a closed ideal of A. If $Y \subseteq X$ is a closed set, the restriction of $A = A(X)$ to Y, denoted $A(Y)$, is the quotient of A by the ideal $A(X \setminus Y)$. The quotient map is denoted by $\pi_Y : A(X) \to A(Y)$. If Y reduces to a point x, we write $A(x)$ for $A(\{x\})$ and π_x for $\pi_{\{x\}}$. The C*-algebra $A(x)$ is called the fiber of A at x. The image $\pi_x(a) \in A(x)$ of $a \in A$ is denoted by $a(x)$. The following lemma collects some basic properties of $\mathrm{C}_0(X)$-algebras, see [2].

Lemma 2.1. Let A be a $\mathrm{C}_0(X)$-algebra and let $B \subset A$ be a $\mathrm{C}_0(X)$-subalgebra. Let $a \in A$ and let Y be a closed subset of X. Then:

(1) The map $x \mapsto \|a(x)\|$ is upper semicontinuous.

(2) $\|\pi_Y(a)\| = \sup\{\|\pi_x(a)\| \mid x \in Y\}$.

(3) If $a(x) \in \pi_x(B)$ for all $x \in X$, then $a \in B$.

(4) The restriction of $\pi_x : A \to A(x)$ to B induces an isomorphism $B(x) \cong \pi_x(B)$ for all $x \in X$.

$\square$

A $\mathrm{C}_0(X)$-algebra such that the map $x \mapsto \|a(x)\|$ is continuous for all $a \in A$ is called a *continuous* $\mathrm{C}_0(X)$-algebra or a *continuous field* C^*-algebra. A C^*-algebra A is a continuous $\mathrm{C}_0(X)$-algebra if and only if A is the C^*-algebra of continuous sections of a continuous fields of C^*-algebras over X in the sense of [5].

2.2 **Semiprojectivity**

A separable C^*-algebra $\mathcal{D}$ is *semiprojective* if for any C^*-algebra A and any increasing sequence of two-sided closed ideals (J_n) of A with $J = \overline{\bigcup_n J_n}$, the natural map $\varinjlim Hom(D, A/J_n) \to Hom(D, A/J)$ induced by $\pi_n : A/J_n \to A/J$ is surjective. If we weaken this condition and require only that the above map has dense range, where $Hom(D, A/J)$ is given the point-norm topology, then $\mathcal{D}$ is called *weakly semiprojective*.

The following is a generalization of a result of Loring [8, Thm. 19.1.3]; it is proved along the same general lines.

Proposition 2.2. Let D be a separable weakly semiprojective C^*-algebra. For any finite set $\mathcal{F} \subset D$ and any $\epsilon > 0$ there exists a finite set $\mathcal{G} \subset D$ and $\delta > 0$ such that for any inclusion of C^*-algebras $B \subset A$ and any $*$-homomorphism $\varphi : D \to A$ with $\varphi(\mathcal{G}) \subset_\delta B$, there is a $*$-homomorphism $\psi : D \to B$ such that $\|\varphi(c) - \psi(c)\| < \epsilon$ for all $c \in \mathcal{F}$. If in addition $K_*(D)$ is finitely generated, then we can choose $\mathcal{G}$ and δ such that we also have $K_*(\psi) = K_*(\varphi)$. $\square$

Remarks 2.3. By work of Neubüser [9], H. Lin [7] and Spielberg [13], a Kirchberg algebra $\mathcal{D}$ satisfying the UCT and having finitely generated K-theory groups is weakly semiprojective. It is shown in [2, Prop. 3.11] that if a Kirchberg algebra D is weakly semiprojective, then so is its stabilization $\mathcal{D} = D \otimes \mathcal{K}$. $\square$

The following result gives necessary and sufficient K-theory conditions for triviality of continuous fields whose fibers are Kirchberg algebras. See also [3] for other generalizations.

Theorem 2.4 ([2])**.** Let X be a finite dimensional compact metrizable space. Let A be a separable continuous field over X whose fibers are stable Kirchberg algebras satisfying the UCT. Let $\mathcal{D}$ be a stable Kirchberg algebra that satisfies the UCT and such that $K_*(\mathcal{D})$ is finitely generated. Then A is isomorphic to $\mathrm{C}_0(X) \otimes \mathcal{D}$ if and only if there is $\sigma : K_*(\mathcal{D}) \to K_*(A)$ such that $\sigma_x : K_*(\mathcal{D}) \to K_*(A(x))$ is bijective for all $x \in X$. $\square$

2.3 **Approximation of Continuous Fields**

In this subsection, we state a corollary of a result on the structure of continuous fields proved in [2, Thm. 4.6]. In this, the property of weak semiprojectivity is used to approximate a continuous field A by continuous fields A_k given by n-pullbacks of trivial continuous fields. We shall use this construction several times in the sequel.

First, we recall the notion of pullback for C*-algebras. The pullback of a diagram

$$A \xrightarrow{\pi} C \xleftarrow{\gamma} B$$

is the C*-algebra

$$E = \{(a,b) \in A \oplus B \mid \pi(a) = \gamma(b)\}.$$

We are going to use pullbacks in the context of continuous field C*-algebras.

Definition 2.5. Let X be a metrizable compact space, and let $\mathcal{D}$ be a C*-algebra. Suppose that $X = Z_0 \cup Z_1 \cup \ldots \cup Z_n$, where $\{Z_j\}_{j=0}^n$ are closed subsets, and write $Y_i = Z_0 \cup Z_1 \cup \ldots \cup Z_i$. The notion of an n*-pullback of trivial continuous fields* with fiber $\mathcal{D}$ over X is defined inductively by the following data. We are given continuous fields E_i over Y_i with fibers isomorphic to $\mathcal{D}$ and fiberwise injective morphisms of fields $\gamma_{i+1} : \mathrm{C}(Y_i \cap Z_{i+1}) \otimes \mathcal{D} \to E_i(Y_i \cap Z_{i+1})$, $i \in \{1, \ldots, n-1\}$, with the following properties:

(i) $E_0 = \mathrm{C}(Y_0) \otimes \mathcal{D} = \mathrm{C}(Z_0) \otimes \mathcal{D}$.

(ii) E_1 is the field over $Y_1 = Y_0 \cup Z_1$ defined by the pullback of the diagram (where $\pi = \pi_{Y_0 \cap Z_1}$)

$$E_0(Y_0) \xrightarrow{\pi} E_0(Y_0 \cap Z_1) \xleftarrow{\gamma_1 \circ \pi} \mathrm{C}(Z_1) \otimes \mathcal{D}.$$

(iii) In general, E_{i+1} is the field over $Y_{i+1} = Y_i \cup Z_{i+1}$ defined as the pullback of the diagram

$$E_i(Y_i) \xrightarrow{\pi} E_i(Y_i \cap Z_{i+1}) \xleftarrow{\gamma_{i+1} \circ \pi} \mathrm{C}(Z_{i+1}) \otimes \mathcal{D}$$

where $\pi = \pi_{Y_i \cap Z_{i+1}}$. We call the continuous field $E = E_n(Y_n) = E_n(X)$ an n-pullback (of trivial fields). Observe that all its fibers are isomorphic to $\mathcal{D}$. □

Remark 2.6. (a) If E is an n-pullback of trivial continuous fields with fiber $\mathcal{D}$ over X, then E_i is an i-pullback and $E_i(Z_i) \cong \mathrm{C}(Z_i) \otimes \mathcal{D}$ for all $i = 0, 1, ..., n$.

(b) If $V \subset X$ is a closed set such that $V \cap (Z_{i+1} \cup \ldots \cup Z_n) = \emptyset$, then $E(V) \cong E_i(V)$. If in addition $V \subset Z_i$, then it follows that $E(V) \cong E_i(V) \cong \mathrm{C}(V) \otimes \mathcal{D}$.

□

Notation. We denote by $\mathcal{D}_n(X)$ the class of continuous fields with fibers isomorphic to $\mathcal{D}$ which are n-pullbacks of trivial fields in the sense of Definition 2.5 and which have the additional property that the spaces Z_i that appear in their representation as n-pullbacks are finite unions of closed subsets of X of the form $\overline{U(x,r)}$, where $U(x,r) = \{y \in X : d(y,x) < r\}$ is the open ball of center x and radius r for a fixed metric d for the topology of X. □

Definition 2.7. Let A be a C*-algebra. We say that a sequence of C*-subalgebras $\{A_n\}$ is exhaustive if for any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$ and any n_0, there exists $n \geq n_0$ such that $\mathcal{F} \subset_\epsilon A_n$. In the case of continuous fields we will require that A_n are C$_0(X)$-subalgebras of A. □

The condition $\mathcal{F} \subset_\epsilon B$ means that for each $a \in \mathcal{F}$ there is $b \in B$ such that $\|a - b\| < \epsilon$. The following is a corollary of a result from [2]. We shall use it to approximate a continuous field A by exhaustive sequences consisting of n-pullbacks of trivial continuous fields.

Theorem 2.8. ([2, Thm. 4.6]) Let $\mathcal{D}$ be a stable Kirchberg algebra that satisfies the UCT. Suppose that $K_*(\mathcal{D})$ is finitely generated. Let X a be finite dimensional compact metrizable space and let A be a separable continuous field over X such that all its fibers are isomorphic to $\mathcal{D}$. For any finite set $\mathcal{F} \subset A$ and any $\epsilon > 0$, there exists $B \in \mathcal{D}_n(X)$ with $n \leq dim(X)$ and a fiberwise injective C(X)-linear $*$-homomorphism $\eta : B \to A$ such that $\mathcal{F} \subset_\epsilon \eta(B)$. □

Theorem 4.6 of [2] does not state that the sets Z_i that give the n-pullback structure of B are finite unions of closures of open balls. However, this additional condition will be satisfied if in the proof of [2, Thm. 4.6] on page 1866 one chooses the closed sets U_{i_k} to be finite unions of sets of the form $\overline{U(x,r)}$.

2.4 A simple algebraic lemma

The following elementary lemma collects some useful properties of finitely generated abelian groups. It is singled out in this subsection because it will be used repeatedly in the sequel, sometimes without further reference. A proof is included for completeness.

Lemma 2.9. Let G be a finitely generated abelian group.

(i) If G is finite, then a map $\alpha : G \to G$ is bijective if and only if α is injective if and only if α is surjective.

(ii) Any surjective homomorphism $\eta : G \to G$ is bijective.

(iii) In a commutative diagram of group homomorphisms

$$\begin{tikzcd} G \arrow[rr, "\gamma"] \arrow[dr, "\alpha"'] & & G \\ & G, \arrow[ur, "\beta"'] & \end{tikzcd}$$

if α is not bijective, then γ is not bijective.

□

Proof. (i) This is obvious. (ii) Since G is a finitely generated abelian group, $G \cong \mathbb{Z}^k \oplus T$ where $k \geq 0$ and T is a finite torsion group. To prove the statement, consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, "\eta"] & \mathbb{Z}^k \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & T \arrow[r] & G \arrow[r] & \mathbb{Z}^k \arrow[r] & 0. \end{tikzcd}$$

One can represent η as $\begin{pmatrix} \alpha & 0 \\ \gamma & \beta \end{pmatrix}$, where $\gamma : \mathbb{Z}^k \to T$. Note that α is surjective (and hence bijective) since η is surjective. By the five lemma, β is surjective and so it must be bijective by (i). Applying the five lemma again, we see that η is bijective.

(iii) If γ were bijective, β would be surjective and hence bijective by (ii). Since α is not bijective, this is a contradiction. ∎

3 Local triviality

In this section we prove our main result, Theorem 1.1. The main technical result of the paper is the following.

Theorem 3.1. Let X be a finite dimensional metrizable compact space, and let $\mathcal{D}$ be a stable Kirchberg algebra that satisfies the UCT and such that $K_*(\mathcal{D})$ is finitely generated. Let A be a separable continuous field C*-algebra over X such that $A(x) \cong \mathcal{D}$ for all $x \in X$. Then there exists a closed subset V of X with non-empty interior such that $A(V) \cong \mathrm{C}(V) \otimes \mathcal{D}$. □

To prove this, we need several lemmas.

Lemma 3.2. Let $\phi : A \to B$ be a $*$-homomorphism of trivial fields over a compact metrizable space X with all the fibers isomorphic to $\mathcal{D}$. Suppose that there is $x \in X$ such that $K_*(\phi_x) : K_*(A(x)) \to K_*(B(x))$ is not bijective. If $K_*(\mathcal{D})$ is finitely generated, then there exists a neighborhood V of x such that $K_*(\phi_v) : K_*(A(v)) \to K_*(B(v))$ is not bijective for any $v \in V$. □

Proof. We can view ϕ as being given by a continuous map $\tilde{\phi} : X \to Hom(\mathcal{D}, \mathcal{D})$ defined by $\tilde{\phi}(x) = (\phi_x : d \mapsto \phi(1 \otimes d)(x) = \phi_x(d))$, where $Hom(\mathcal{D}, \mathcal{D})$ is equipped with the point-norm topology. Since $K_*(\mathcal{D})$ is finitely generated, the map $Hom(\mathcal{D}, \mathcal{D}) \to Hom(K_*(\mathcal{D}), K_*(\mathcal{D}))$ is locally constant. This concludes the proof. ∎

Lemma 3.3. Let X be a metrizable compact space and let A be a continuous field in the class $\mathcal{D}_n(X)$. Then for any open set $U \subset X$ and $x \in U$, there is an open set V such that $x \in \overline{V} \subset U$ and $A(\overline{V}) \cong \mathrm{C}(\overline{V}) \otimes \mathcal{D}$. □

Proof. We use the notation from Definition 2.5 with A in place of E. Let $i \in \{0, 1, ..., n\}$ be the largest number with the property that $x \in Z_i$. Set $X_i = \bigcup_{i=i+1}^{n} Z_i$ if $i < n$ and $X_n = \emptyset$. Then $dist(x, X_i) > 0$ since X_i is a closed set. Let W be an open ball centered at x such that $\overline{W} \subset U$ and $\overline{W} \cap X_i = \emptyset$. By the definition of $\mathcal{D}_n(X)$, there exist $z \in Z_i$ and $r > 0$ such that $x \in \overline{U(z,r)} \subset Z_i$. Since $x \in W \cap \overline{U(z,r)}$ and W is open, there must be a sequence $z_n \in W \cap U(z,r)$ which converges to x. Setting $V = W \cap U(z,r)$, we have that $x \in \overline{V} \subset \overline{W} \subset U$ and $\overline{V} \cap X_i = \emptyset$. Because $\overline{V} \subset \overline{U(z,r)} \subset Z_i$, it follows that $A(\overline{V})$ is trivial by Remark 2.6(b). ∎

Lemma 3.4. Let $\mathcal{D}$ be a stable Kirchberg algebra such that $K_*(\mathcal{D})$ is finitely generated. Let $\{\mathcal{D}_n\}$ be an exhaustive sequence for $\mathcal{D}$ with inclusion maps $\phi_n : \mathcal{D}_n \hookrightarrow \mathcal{D}$. Suppose that $K_*(\mathcal{D}_n) \cong K_*(\mathcal{D})$ for all $n \geq 1$. Then, there exists $n_1 < n_2 < \ldots < n_k < \ldots$ such that $K_*(\phi_{n_k})$ is bijective for all k. □

Proof. For the sake of simplicity, we give the proof only for K_0. Let $K_0(\mathcal{D})$ be generated by classes of projections $e_i \in \mathcal{D}$, $i = 1, \ldots, r$. Since $\{\mathcal{D}_n\}$ is exhaustive, there exist $n_1 < n_2 < \ldots < n_k < \ldots$ such that $dist(e_i, \mathcal{D}_{n_k}) < 1$ for $i = 1, \ldots, r$ and $k \geq 1$. By functional calculus, it follows immediately that the maps $K_0(\phi_{n_k}) : K_0(\mathcal{D}_{n_k}) \to K_0(\mathcal{D})$ are surjective. Then they must be bijective by Lemma 2.9. ∎

Let us recall that a continuous field A over X is nowhere trivial if there is no open subset $V \neq \emptyset$ of X such that $A(V) \cong \mathrm{C}_0(V) \otimes \mathcal{D}$ for some C*-algebra $\mathcal{D}$.

Lemma 3.5. Let X be a finite dimensional metrizable compact space, and let $\mathcal{D}$ be a stable Kirchberg algebra that satisfies the UCT and such that $K_*(\mathcal{D})$ is finitely generated. Let A be a separable continuous field C*-algebra over X with all fibers isomorphic to $\mathcal{D}$. Let $B \in \mathcal{D}_n(X)$ be such that there exists a $\mathrm{C}(X)$-linear and fiberwise injective $*$-homomorphism $\phi : B \to A$. If A is nowhere trivial, then for any nonempty set $U \subset X$ there exists an open nonempty set W such that $\overline{W} \subset U$, $B(\overline{W})$ is trivial and for all $v \in \overline{W}$, $K_*(\phi_v)$ is not bijective. □

Proof. By Lemma 3.3 there is an open set $V \neq \emptyset$ such that $\overline{V} \subset U$ and $B(\overline{V}) \cong \mathrm{C}(\overline{V}) \otimes \mathcal{D}$. After replacing X by $\overline{V}$ and U by V and restricting both B and A to $\overline{V}$ we may assume without any loss of generality that $B = \mathrm{C}(X) \otimes \mathcal{D}$. By Theorem 2.8, there is an exhaustive sequence $\{\phi_k : A_k \to A\}$ such that $A_k \in \mathcal{D}_{l_k}(X)$ with $l_k \leq dim(X)$. Let us regard $\mathcal{D}$ as the subalgebra of constant functions of $B = \mathrm{C}(X) \otimes \mathcal{D}$ and denote by j the corresponding inclusion map. Applying Proposition 2.2 for the weakly semiprojective C*-algebra $\mathcal{D}$ and the map $\phi \circ j$, after passing to a subsequence of $(A_k)_k$, if necessary, we construct a sequence of $*$-homomorphisms $\psi_k^0 : \mathcal{D} \to A_k$ such that their canonical $\mathrm{C}(X)$-linear extension $\psi_k : B \to A_k$ form a sequence of diagrams

$$\begin{tikzcd} B \arrow[rr, "\phi"] \arrow[dr, "\psi_k"'] & & A \\ & A_k \arrow[ur, "\phi_k"'] & \end{tikzcd},$$

satisfying that $\|\phi_k\psi_k(b) - \phi(b)\| \to 0$ for all $b \in B$, as $k \to \infty$. Since $K_*(\mathcal{D})$ is finitely generated and satisfies the UCT, we can moreover arrange that $[\phi_k \circ \psi_k \circ j] = [\phi \circ j] \in KK(\mathcal{D}, A)$ for all $k \geq 1$. This follows from [2, Prop. 3.14 and Thm. 3.12]. In addittion, since there is an isomorphism $KK_X(\mathrm{C}(X, \mathcal{D}), A) \cong KK(\mathcal{D}, A)$, it follows that $[\phi_k \circ \psi_k] = [\phi] \in KK_X(B, A)$ and hence

$$[(\phi_k)_v \circ (\psi_k)_v] = [\phi_v] \in KK(B(v), A(v)) \tag{1}$$

for all points $v \in X$ and $k \geq 1$, see [10].

Since A is nowhere trivial, it follows from Theorem 2.4 that there exists $x \in U$ such that $K_*(\phi_x)$ is not bijective. By applying Lemma 3.4 to the exhaustive sequence $\{(\phi_k)_x : A_k(x) \to A(x)\}$ we find a k, which we now fix, such that $K_*((\phi_k)_x)$ is bijective. It follows that $K_*((\psi_k)_x)$ is not bijective for this fixed k.

Let V be the open set given by Lemma 3.3 applied to A_k, U and x. Then $x \in \overline{V} \subset U$ and $A_k(\overline{V}) \cong \mathrm{C}(\overline{V}) \otimes \mathcal{D}$. Restricting the diagram above to $\overline{V}$, we obtain a diagram

$$\begin{tikzcd} B(\overline{V}) \arrow[rr, "\phi"] \arrow[dr, "\psi_k"'] & & A(\overline{V}) \\ & A_k(\overline{V}) \arrow[ur, "\phi_k"'] & \end{tikzcd}$$

where both $B(\overline{V})$ and $A_k(\overline{V})$ are trivial and $K_*((\phi_k)_v) \circ K_*((\psi_k)_v) = K_*(\phi_v)$ for all $v \in \overline{V}$ as a consequence of (1). Since $K_*((\psi_k)_x)$ is not bijective, by Lemma 3.2 there is $r > 0$ such that $K_*((\psi_k)_v)$ is not bijective for all $v \in \overline{V} \cap U(X, r)$. Let W be an open ball whose closure is contained in $V \cap U(x, r) \subset U$. It follows by Lemma 2.9 (iii) that $K_*(\phi_v)$ is not bijective for any $v \in \overline{W}$. ∎

Proof of Theorem 3.1. By Theorem 2.8, there is an exhaustive sequence $\{A_k\}_k$, such that $A_k \in \mathcal{D}_{l_k}(X)$, $l_k \leq dim(X)$ and that the maps $\phi_k : A_k \to A$ are $\mathrm{C}(X)$-linear and fiberwise injective $*$-homomorphisms for all k. Seeking a contradiction suppose for each open set $V \neq \emptyset$, $A(\overline{V}) \ncong \mathrm{C}(\overline{V}) \otimes \mathcal{D}$.

Apply Lemma 3.5 to $\phi_1 : A_1 \to A$ to find an open set $V_1 \neq \emptyset$ such that $K_*((\phi_1)_v)$ is not bijective for all $v \in \overline{V}_1$. Next, apply Lemma 3.5 again for $\phi_2 : A_2(\overline{V}_1) \to A(\overline{V}_1)$ and V_1 to find a nonempty open set V_2 such that $\overline{V}_2 \subset V_1$ and $K_*((\phi_2)_v)$ is not bijective for all $v \in \overline{V}_2$. Using the same procedure inductively, one finds a sequence of open sets $\{V_k\}_k$ with $V_k \supset \overline{V}_{k+1}$, such that $K_*((\phi_k)_v)$ is not bijective for all $v \in \overline{V}_k$ and $k \geq 1$.

Choose $x \in \bigcap_{k=1}^{\infty} \overline{V}_k$ and note that $\{A_k(x)\}_k$ is an exhaustive sequence for $A(x)$ such that none of the maps $K_*((\phi_k)_x) : K_*(A_k(x)) \to K_*(A(x))$ are bijective. By Lemma 3.4 this implies that $K_*(A(x)) \ncong K_*(\mathcal{D})$, and this is a contradiction. ∎

Proof of Theorem 1.1. Let $\mathcal{U}$ be the family of all open subsets U of X such that $A(U)$ is trivial. Since X is compact metrizable, we can find a sequence $\{U_n\}_n$ in $\mathcal{U}$ whose union is equal to the union of all elements of $\mathcal{U}$. If we set $U_\infty = \bigcup_n U_n$, then U_∞ is dense in X by Theorem 3.1. Since $A(U_\infty) = \varinjlim_n \{A(U_1 \cup \cdots \cup U_n)\} = \varinjlim_n \{A(U_1) + \cdots + A(U_n)\}$, we see immediately that $A(U_\infty)$ is locally trivial. Indeed the ideal $A(U_\infty)(U_n)$ of $A(U_\infty)$ determined by the open set U_n is equal to $A(U_n) \cong \mathrm{C}_0(U_n) \otimes \mathcal{D}$.

∎

Corollary 3.6. Fix $n \in \mathbb{N} \cup \{\infty\}$. Let X be a finite dimensional compact metrizable space and A be a continuous field over X such that $A(x) \cong \mathcal{O}_n \otimes \mathcal{K}$ for all $x \in X$. Then there exists a closed subset V of X with nonempty interior such that $A(V) \cong \mathrm{C}(V) \otimes \mathcal{O}_n \otimes \mathcal{K}$. $\square$

The following example shows that the conclusion of Theorem 1.1 is in a certain sense optimal. Indeed, given a nowhere dense closed subset F of any compact metrizable space X and a stable Kirchberg C*-algebra $\mathcal{D}$ with nontrivial K-theory, we construct a continuous field C*-algebra A over X with all fibers isomorphic to $\mathcal{D}$, and such that the set of singular points of A coincides with F. It is worth mentioning that the Lebesgue measure of a nowhere dense closed subset of $[0,1]^n$ can be any nonnegative number < 1.

Example 3.7. Let F be a closed nowhere dense subset of X and set $U = X \setminus F$. Let $\mathcal{D}$ be a stable Kirchberg algebra with $K_*(\mathcal{D}) \neq 0$. Fix an injective $*$-homomorphism $\gamma : \mathcal{D} \to \mathcal{D}$ which factors through $\mathcal{O}_2$. It follows that $K_*(\gamma) = 0$. Define a continuous field C*-algebra over X by

$$A = \{f \in C(X) \otimes \mathcal{D} \mid f(x) \in \gamma(\mathcal{D}), \quad \forall x \in F\}.$$

It is clear that $A(U) \cong \mathrm{C}_0(U) \otimes \mathcal{D}$. We show that F is the set of singular points of A. Let $z \in F$ and let W be an open neighborhood of z. We will show that $A(W) \ncong C_0(W) \otimes \mathcal{D}$. To this purpose we verify that if V is a compact neighborhood of z such that $V \subset W$ then $A(V)$ is not trivial.

Let us observe that

$$A(V) = \{f \in \mathrm{C}(V) \otimes \mathcal{D} \mid f(x) \in \gamma(\mathcal{D}), \quad \forall x \in V \cap F\}.$$

We will assume that $K_0(\mathcal{D}) \neq 0$. The case $K_1(\mathcal{D}) \neq 0$ is similar. By Lemma 2.1(4) applied to the inclusion $A \subset C(X) \otimes \mathcal{D}$, the projection maps onto fibers $\pi_x : A \to A(x)$ can be identified with the evaluation maps $f \mapsto f(x)$. Thus $\pi_x(A) = \mathcal{D}$ if $x \in U$ and $\pi_x(A) = \gamma(\mathcal{D})$ if $x \in F$. Seeking a contradiction, let us assume that $A(V)$ is trivial. Then we can find a $*$-homomorphism $\phi : \mathcal{D} \to A(V)$ such that $\pi_x \circ \phi : \mathcal{D} \to A(x)$ is a $*$-isomorphism for all x in V. In particular $\pi_z \circ \phi : \mathcal{D} \xrightarrow{\cong} \gamma(\mathcal{D}) = A(z)$. Let $e \in \mathcal{D}$ be a projection such that $[e] \neq 0$ in $K_0(\mathcal{D})$. Then there is a projection $p_0 \in \mathcal{D}$ such that $\pi_z(\phi(e)) = \gamma(p_0)$ and so $[\gamma(p_0)] \neq 0$ in $K_0(\gamma(\mathcal{D})) = K_0(A(z))$. Let $p \in A(V)$ be the constant projection equal to $\gamma(p_0)$. By Lemma 2.1(1) there is a closed neighborhood V_0 of z that is contained in V such that $\|\pi_x(\phi(e)) - \pi_x(p)\| < 1$ for all $x \in V_0$. Since F has empty interior, there is a point $y \in V_0 \setminus F$. Then $\|\pi_y(\phi(e)) - \pi_y(p)\| < 1$ and so $\|\pi_y(\phi(e)) - \gamma(p_0)\| < 1$. This implies that $[\pi_y(\phi(e))] = [\gamma(p_0)] = 0$ in $K_0(\mathcal{D}) = K_0(A(y))$. On the other hand since $\pi_y \circ \phi$ is a $*$-isomorphism, we must have $K_0(\pi_y \circ \phi)([e]) \neq 0$ since $[e] \neq 0$. We reached a contradiction and this completes the proof.

$\square$

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