# GROUP QUASI-REPRESENTATIONS AND ALMOST FLAT BUNDLES

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Abstract. We study the existence of quasi-representations of discrete groups $G$ into unitary groups $U(n)$ that induce prescribed partial maps $K_0(C^*(G)) \to \mathbb{Z}$ on the K-theory of the group C*-algebra of $G$. We give conditions for a discrete group $G$ under which the K-theory group of the classifying space $BG$ consists entirely of almost flat classes.

## 1. Introduction

The notions of almost flat bundle and group quasi-representation were introduced by Connes, Moscovici and Gromov [4] as tools for proving the Novikov conjecture for large classes of groups. The first example of a topologically nontrivial quasi-representation is due to Voiculescu for $G = \mathbb{Z}^2$, [27]. In this paper we use known results on the Novikov and the Baum-Connes conjectures to derive the existence of topologically nontrivial quasi-representations of certain discrete group $G$, as well as the existence of nontrivial almost flat bundles on the classifying space $BG$, by employing the concept of quasidiagonality.

A discrete completely positive asymptotic representation of a C*-algebra $A$ consists of a sequence $\{\pi_n : A \to M_{k(n)}(\mathbb{C})\}_n$ of unital completely positive maps such that $\lim_{n\to\infty} \|\pi_n(aa') - \pi_n(a)\pi_n(a')\| = 0$ for all $a, a' \in A$. The sequence $\{\pi_n\}_n$ induces a unital $*$-homomorphism

$$A \to \prod_n M_{k(n)}(\mathbb{C}) / \sum_n M_{k(n)}(\mathbb{C})$$

and hence a group homomorphism $K_0(A) \to \prod_n \mathbb{Z}/\sum_n \mathbb{Z}$. This gives a canonical way to push forward an element $x \in K_0(A)$ to a sequence of integers $(\pi_{n\,\sharp}(x))$, which is well-defined up to tail equivalence; two sequences are tail equivalent, $(y_n) \equiv (z_n)$, if there is $m$ such that $x_n = y_n$ for all $n \geq m$.

In the first part of the paper we study the existence of discrete asymptotic representations of group C*-algebras that interpolate on K-theory a given

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group homomorphism $h : K_0(C^*(G)) \to \mathbb{Z}$. We rely heavily on results of Kasparov, Higson, Yu, Skandalis and Tu [15], [12], [30], [24], [16], [26]. For illustration, we have the following:

**Theorem 1.1.** *Let $G$ be a countable, discrete, torsion free group with the Haagerup property. Suppose that $C^*(G)$ is residually finite dimensional. Then, for any group homomorphism $h : K_0(C^*(G)) \to \mathbb{Z}$ there is a discrete completely positive asymptotic representation $\{\pi_n : C^*(G) \to M_{k(n)}(\mathbb{C})\}_n$ such that $\pi_{n\,\sharp}(x) \equiv h(x)$ for all $x \in K_0(I(G))$.*

Here $I(G)$ is the kernel of the trivial representation $\iota : C^*(G) \to \mathbb{C}$. By contrast, any finite dimensional unitary representation of $G$ induces the zero map on $K_0(I(G))$. The groups with the Haagerup property are characterized by the requirement that there exists a sequence of normalized continuous positive-definite functions which vanish at infinity on G and converge to 1 uniformly on finite subsets of G. The conclusion of Theorem 1.1 also holds if $G$ is an increasing union of residually finite amenable groups, see Theorem 3.4. The class of groups considered in Theorem 1.1 contains all countable, torsion free, amenable, residually finite groups (also the maximally periodic groups) and the surface groups [17]. Moreover, this class is closed under direct products and free products (see [10], [3]). If we impose a weaker condition, namely that $C^*(G)$ is quasidiagonal, then in general we need two asymptotic representations in order to interpolate $h$, see Theorem 3.3. Theorem 1.1 remains true if we replace the assumption that $G$ has the Haagerup property by the requirements that $G$ is uniformly embeddable in a Hilbert space and that the assembly map $\mu : RK_0(BG) \to K_0(C^*(G))$ is surjective. Let us recall that Hilbert space uniform embeddability of $G$ implies that $\mu$ is split injective, as proven by Yu [30] if the classifying space $BG$ is finite and by Skandalis, Yu and Tu [24] in the general case. We will also use a strengthening of this result by Tu [26] who showed that $G$ has a gamma element. In conjunction with a theorem of Kasparov [15] this guarantees the surjectivity of the dual assembly map $\nu : K^0(C^*(G)) \to RK^0(BG)$ for countable, discrete, torsion free groups which are uniformly embeddable in a Hilbert space.

The notion of almost flat K-theory class was introduced in [4] as a tool for proving the Novikov conjecture. In the second part of the paper we pursue a reverse direction. Namely, we use known results on the Baum-Connes and the Novikov conjectures to derive the existence of almost flat K-theory classes by employing the concept of quasidiagonality.

**Theorem 1.2.** *Let $G$ be a countable, discrete, torsion free group which is uniformly embeddable in a Hilbert space. Suppose that the classifying space*

*$BG$ is a finite simplicial complex and that the full group C*-algebra $C^*(G)$ is quasidiagonal. Then all the elements of $K^0(BG)$ are almost flat.*

The class of groups considered in Theorem 1.2 is closed under free products, by [1] and [2]. If $G$ can be written as a union of amenable residually finite groups (as is the case if $G$ is a linear amenable group), then $C^*(G)$ is quasidiagonal. It is an outstanding open question if all discrete amenable groups have quasidiagonal C*-algebras [29].

Voiculescu has asked in [29] if there are invariants of a topological nature which can be used to describe the obstruction that a C*-algebra be quasidiagonal. One can view Theorem 1.2 as further evidence towards a topological nature of quasidiagonality, since it shows that the existence of non-almost flat classes in $K^0(BG)$ represents an obstruction for the quasidiagonality of $C^*(G)$.

The fundamental connection between deformations of C*-algebras and K-theory was discovered by Connes and Higson [5]. They introduced the concept of asymptotic homomorphism of C*-algebras which formalizes the intuitive idea of deformations of C*-algebras. An asymptotic homomorphism is a family of maps $\varphi_t : A \to B$, $t \in [0, \infty)$ such that for each $a \in A$ the map $t \to \varphi_t(a)$ is continuous and bounded and the family $(\varphi_t)_{t\in[0,\infty)}$ satisfies asymptotically the axioms of $*$-homomorphisms. There is a natural notion of homotopy for asymptotic homomorphisms. E-theory is defined as homotopy classes of asymptotic homomorphisms from the suspension of $A$ to the stable suspension of $B$, $E(A, B) = [[C_0(\mathbb{R}) \otimes A, C_0(\mathbb{R}) \otimes B \otimes \mathcal{K}]]$. The introduction of the suspension and of the compact operators $\mathcal{K}$ yields an abelian group structure on $E(A, B)$. Connes and Higson showed that $E$-theory defines the universal half-exact C*-stable homotopy functor on separable C*-algebras. In particular the KK-theory of Kasparov factors through $E$-theory. A similar construction based on completely positive asymptotic homomorphisms gives a realization of KK-theory itself as shown by Larsen and Thomsen [13].

While E-theory gives in general maps of suspensions of C*-algebras it is often desirable to have interesting deformations of unsuspended C*-algebras. In joint work with Loring [8], [6], we proved a suspension theorem for commutative C*-algebras $A = C_0(X \setminus x_0)$, where $X$ is a compact connected space and $x_0 \in X$ is a base point. Specifically, we showed that the reduced K-homology group $\widetilde{K}_0(X) = K_0(X, x_0)$ is isomorphic to the homotopy classes of asymptotic homomorphisms $[[C_0(X \setminus x_0), \mathcal{K}]]$. One can replace the compact operators $\mathcal{K}$ by $\cup_{n=1}^{\infty} M_n(\mathbb{C})$ and conclude that the reduced K-homology of $X$ classifies the deformations of $C_0(X)$ into matrices. The case of $X = \mathbb{T}^2$

played an important role in the history of the subject. Indeed, Voiculescu [28] exhibited pairs of almost commuting unitaries $u, v \in U(n)$ whose properties reflect the non-triviality of $H^2(\mathbb{T}^2, \mathbb{Z})$. One can view such a pair as associated to a quasi-representation of $C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$. If the commutator $\|uv - vu\|$ is sufficiently small, then there is an induced pushforward of the Bott class that represents the obstruction for perturbing $u, v$ to a pair of commuting unitaries, [28], [9]. It is therefore quite natural to investigate deformations of C*-algebras associated to non-commutative groups. In view of Theorem 1.1 we propose the following:

*Conjecture. If $G$ is a discrete, countable, torsion free, amenable group, then the natural map $[[I(G), \mathcal{K}]] \to KK(I(G), \mathcal{K}) \cong K^0(I(G))$ is an isomorphism of groups.*

This is verified if $G$ is commutative. Indeed, $I(G) \cong C_0(\widehat{G} \setminus x_0)$ and $\widehat{G}$ is connected since $G$ is torsion free, so that we can apply the suspension result of [6].

Manuilov, Mishchenko and their co-authors have studied various aspects and applications of quasi-representations and asymptotic representations of discrete groups. The paper [18] is a very interesting survey of their contributions. The notion of quasi-representation of a group is used in the literature in several non-equivalent contexts, to mean several different things, see [22].

## 2. Quasi-representations and K-theory

**Definition 2.1.** *Let $A$ and $B$ be unital $C^*$-algebras. Let $F \subset A$ be a finite set and let $\varepsilon > 0$. A unital completely positive map $\varphi : A \to B$ is called an $(F, \varepsilon)$-homomorphism if $\|\varphi(aa') - \varphi(a)\varphi(a')\| < \varepsilon$ for all $a, a' \in F$. If $B$ is the $C^*$-algebra of bounded linear operators on a Hilbert space, then we say that $\varphi$ is an $(F, \varepsilon)$-representation of $A$. We will use the term quasi-representation to refer to an $(F, \varepsilon)$-representation where $F$ and $\varepsilon$ are not necessarily specified.*

An important method for turning K-theoretical invariants of $A$ into numerical invariants is to use quasi-representations to pushforward projections in matrices over $A$ to scalar projections. Consider a finite set of projections $\mathcal{P} \subset M_m(A)$. We say that $(\mathcal{P}, F, \varepsilon)$ is a $K_0$-triple if for any $(F, \varepsilon)$-homomorphism $\varphi : A \to B$ and $p \in \mathcal{P}$, the element $b = (id_m \otimes \varphi)(p)$ satisfies $\|b^2 - b\| < 1/4$ and hence the spectrum of $b$ is contained in $[0, 1/2) \cup (1/2, 1]$. We denote by $q$ the projection $\chi(b)$, where $\chi$ is the characteristic function of the interval $(1/2, 1]$. It is not hard to show that for any finite set of projections $\mathcal{P}$ there exist a finite set $F \subset A$ and $\varepsilon > 0$ such that $(\mathcal{P}, F, \varepsilon)$ is a $K_0$-triple. If $(\mathcal{P}, F, \varepsilon)$ is a $K_0$-triple, then any $(F, \varepsilon)$-homomorphism

$\varphi : A \to B$ induces a map $\varphi_\sharp : \mathcal{P} \to K_0(B)$ defined by $\varphi_\sharp(p) = [q]$. Let $\mathrm{Proj}(A)$ denote the set of all projections in matrices over $A$. It is convenient to extend $\varphi_\sharp$ to $\mathrm{Proj}(A)$ by setting $\varphi_\sharp(p) = 0$ if $b = (id_m \otimes \varphi)(p)$ does not satisfy $\|b^2 - b\| < 1/4$. If $\varphi$ were a $*$-homomorphism, then $\varphi$ would induce a map $\varphi_* : K_0(A) \to K_0(B)$. Intuitively, one may think of $\varphi_\sharp$ as a substitute for $\varphi_*$.

Two sequences $(a_n)$ and $(b_n)$ are called tail-equivalent if there is $n_0$ such that $a_n = b_n$ for $n \geq n_0$. Tail-equivalence is denoted by $(a_n) \equiv (b_n)$ or even $a_n \equiv b_n$, abusing the notation.

We will also work with discrete completely positive asymptotic morphisms $(\varphi_n)_n$. They consists of a sequence of contractive completely positive maps $\varphi_n : A \to B_n$ with $\lim_{n\to\infty} \|\varphi_n(aa') - \varphi_n(a)\varphi_n(a')\| = 0$ for all $a, a' \in A$. If in addition each $B_n$ is a matricial algebra $B_n = M_{k(n)}(\mathbb{C})$, then we call $(\varphi_n)_n$ a discrete asymptotic representation of $A$. A discrete completely positive asymptotic morphism $(\varphi_n)_n$ induces a sequence of maps $\varphi_{n\,\sharp} : \mathrm{Proj}(A) \to K_0(B_n)$. Note that if $p, q \in \mathrm{Proj}(A)$ have the same class in $K_0(A)$, then $\varphi_{n\,\sharp}(p) \equiv \varphi_{n\,\sharp}(q)$.

For any $x \in K_0(A)$, we fix projections $p, q \in \mathrm{Proj}(A)$ such that $x = [p]-[q]$ and set $\varphi_{n\,\sharp}(x) = \varphi_{n\,\sharp}(p) - \varphi_{n\,\sharp}(q) \in K_0(B_n)$. The sequence $(\varphi_{n\,\sharp}(x))$ depends on the particular projections that we use to represent $x$ but only up to tail-equivalence. While in general the maps $\varphi_{n\,\sharp} : K_0(A) \to K_0(B_n)$ are not group homomorphisms, the sequence $(\varphi_{n\,\sharp}(x))$ does satisfy $(\varphi_{n\,\sharp}(x + y)) \equiv (\varphi_{n\,\sharp}(x) + \varphi_{n\,\sharp}(y))$ for all $x, y \in K_0(A)$.

A subset $B \subset L(H)$ is called quasidiagonal if there is an increasing sequence $(p_n)$ of finite rank projections in $L(H)$ which converges strongly to $1_H$ and such that $\lim_{n\to\infty} \|[b, p_n]\| = 0$ for all $b \in B$. $B$ is block-diagonal, if there is a sequence $(p_n)$ as above such that $[b, p_n] = 0$ for all $b \in B$ and $n \geq 1$. Let $A$ be a separable C*-algebra. Let us recall that the elements of $KK(A, \mathbb{C})$ can be represented by Cuntz pairs, i.e. by pair of $*$-representations $\varphi, \psi : A \to L(H)$, such that $\varphi(a) - \psi(a) \in K(H)$, for all $a \in A$.

**Definition 2.2.** *Let $A$ be a separable C*-algebra. An element $\alpha \in KK(A, \mathbb{C})$ is called quasidiagonal if it can be represented by a Cuntz pair $(\varphi, \psi) : A \to L(H)$ with the property that the set $\psi(A) \subset L(H)$ is quasidiagonal. In this case let us note that the set $\varphi(A) \subset L(H)$ must be also quasidiagonal. Similarly, we say that $\alpha$ is residually finite dimensional if it can be represented by a Cuntz pair with the property that the set $\psi(A)$ is block-diagonal. We denote by $KK_{qd}(A, \mathbb{C})$ the subset of $KK(A, \mathbb{C})$ consisting of quasidiagonal* *classes and by $KK_{rfd}(A,\mathbb{C})$ the subset of $KK(A,\mathbb{C})$ consisting of residually finite dimensional classes. It is clear that $KK_{rfd}(A,\mathbb{C})\subset KK_{qd}(A,\mathbb{C})$, that $KK_{qd}(A,\mathbb{C})$ is a subgroup of $KK(A,\mathbb{C})$ and that $KK_{rfd}(A,\mathbb{C})$ is a subsemigroup.*

*We say that $A$ is K-quasidiagonal if $KK_{qd}(A,\mathbb{C}) = KK(A,\mathbb{C})$ and that $A$ is K-residually finite dimensional if $KK_{rfd}(A,\mathbb{C}) = KK(A,\mathbb{C})$.*

**Remark 2.3.** *Let $A$ be a separable $C^*$-algebra. It was pointed out by Skandalis [23] that for any given faithful $*$-representation $\pi : A \to L(H)$ such that $\pi(A)\cap K(H) = \{0\}$, one can represent all the elements of $KK(A,\mathbb{C})$ by Cuntz pairs where the second map is fixed and equal to $\pi$. It follows that a separable quasidiagonal $C^*$-algebra is K-quasidiagonal and a separable residually finite dimensional $C^*$-algebra is K-residually finite dimensional. More generally, if $A$ is homotopically dominated by $B$ and $B$ is K-quasidiagonal or K-residually finite dimensional then so is $A$. Let us note that the Cuntz algebra $O_2$ is K-residually finite dimensional while it is not quasidiagonal.*

The following lemma and proposition are borrowed from [7]. For the sake of completeness, we review briefly some of the arguments from their proofs. Let $B$ be a unital C*-algebra and let $E$ be a right Hilbert $B$-module. If $e, f \in L_B(E)$ are projections such that $e - f \in K_B(E)$, we denote by $[e, f]$ the corresponding element of $KK(\mathbb{C}, B) \cong K_0(B)$.

**Lemma 2.4.** *Let $B$ be a unital $C^*$-algebra and let $E$ be a right Hilbert $B$-module. Let $e, f \in L_B(E)$ and $h \in K_B(E)$ be projections such that $e - f \in K_B(E)$, and $\|eh-he\| \le 1/9$, $\|fh-hf\| \le 1/9$, $\|(1-h)(e-f)(1-h)\| \le 1/9$. Then*

$$Sp(heh) \cup Sp(hfh) \subset [0, 1/2) \cup (1/2, 1],$$
$$[e, f] = [\chi(heh), \chi(hfh)] \in KK(\mathbb{C}, B) \cong K_0(B).$$

*Proof.* One shows that if $e', f' \in L_B(E)$ are projections such that $e' - f' \in K_B(E)$ and $\|e-e'\| < 1/2$, $\|f-f'\| < 1/2$, then $[e, f] = [e', f']$. This is proved using the homotopy $(\chi(e_t), \chi(f_t))$ where $e_t = (1-t)e+te'$, $f_t = (1-t)f+tf'$, $0 \le t \le 1$. Then one applies this observation to conclude that

$$[e, f] = [\chi(x)+\chi(x'), \chi(y)+\chi(y')] = [\chi(x)+\chi(x'), \chi(y)+\chi(x')] = [\chi(x), \chi(y)],$$

where $x = heh,\ x' = (1-h)e(1-h),\ y = hfh,\ y' = (1-h)f(1-h)$. $\square$

Let $A, B$ be separable C*-algebras. An element $\alpha \in KK(A,\mathbb{C})$ induces a group homomorphism $\alpha_* : K_0(A \otimes B) \to K_0(B)$ via the cup product

$$KK(\mathbb{C}, A \otimes B) \times KK(A,\mathbb{C}) \to KK(\mathbb{C}, B), \quad (x, \alpha) \mapsto x \circ (\alpha \otimes 1_B).$$

Here we work with the maximal tensor product.

**Proposition 2.5.** *Let $A$ be a separable unital $C^*$-algebra and $\alpha \in KK_{qd}(A,\mathbb{C})$. There exist two discrete asymptotic representations $(\varphi_n)_n$ and $(\psi_n)_n$ consisting of unital completely positive maps $\varphi_n : A \to M_{k(n)}(\mathbb{C})$ and $\psi_n : A \to M_{r(n)}(\mathbb{C})$ such that for any separable unital $C^*$-algebra $B$, the map $\alpha_* : K_0(A\otimes B) \to K_0(B)$ has the property that*

$$\alpha_*(x) \equiv (\varphi_n \otimes id_B)_\sharp(x) - (\psi_n \otimes id_B)_\sharp(x),$$

*for all $x \in K_0(A \otimes B)$. If $\alpha \in KK_{rfd}(A,\mathbb{C})$ then all $\psi_n$ can be chosen to be $*$-representations.*

*Proof.* Represent $\alpha$ by a Cuntz pair $\varphi, \psi : A \to L(H)$ with $\varphi(a) - \psi(a) \in K(H)$, for all $a \in A$, and such that the set $\psi(A)$ is quasidiagonal. Therefore there is an increasing approximate unit $(p_n)_n$ of $K(H)$ consisting of projections such that $(p_n)_n$ commutes asymptotically with both $\varphi(A)$ and $\psi(A)$. Let us define contractive completely positive maps $\varphi_n, \psi_n : A \to L(p_nH)$ by $\varphi_n(a) := p_n\varphi(a)p_n$ and $\psi_n(a) := p_n\psi(a)p_n$. Without any loss of generality we may assume that $x$ is the class of a projection $e \in A \otimes B$. It follows from the definition of the Kasparov product that

$$\alpha_*(x) = [(\varphi \otimes id_B)(e), (\psi \otimes id_B)(e)] \in KK(\mathbb{C}, B).$$

On the other hand, the sequence of projections $p_n \otimes 1_B \in K(H) \otimes B$ commutes asymptotically with both projections $(\varphi \otimes id_B)(e)$ and $(\psi \otimes id_B)(e)$ and moreover

$$\lim_{n\to\infty} \|p_n \otimes 1_B \left((\varphi \otimes id_B)(e) - (\psi \otimes id_B)(e)\right) p_n \otimes 1_B\| = 0,$$

since the sequence $(p_n \otimes 1_B)_n$ forms an approximative unit of $K(H) \otimes B$. It follows now from Lemma 2.4 that for all sufficiently large $n$

$$[(\varphi \otimes id_B)(e), (\psi \otimes id_B)(e)] = (\varphi_n \otimes id_B)_\sharp(e) - (\psi_n \otimes id_B)_\sharp(e).$$

It is standard to perturb $\varphi_n$ and $\psi_n$ to completely positive maps such that $\varphi_n(1)$ and $\psi_n(1)$ are projections. Finally, let us note that $\psi_n$ is a $*$-homomorphism if $p_n$ commutes with $\psi$. □

## 3. Asymptotic representations of group C*-algebras

We use the following notation for the Kasparov product:

$$KK(A,B) \times KK(B,C) \to KK(A,C), \quad (y,x) \mapsto y \circ x.$$

In the case of the pairing $K_i(B) \times K^i(B) \to \mathbb{Z}$ we will also write $\langle y, x\rangle$ for $y \circ x$. We are mostly interested in the map

$$K^i(C^*(G)) \to \mathrm{Hom}(K_i(C^*(G)), \mathbb{Z}), \tag{1}$$

induced by the pairing above for $B = C^*(G)$. If $G$ has the Haagerup property, then it was shown in [25] that $C^*(G)$ is KK-equivalent with a commutative C*-algebra and hence the map (1) is surjective. Assuming that $G$ is a countable, discrete, torsion free group that is uniformly embeddable in a Hilbert space, we are going to verify that the map (1) is split surjective whenever the assembly map $\mu : RK_i(BG) \to K_i(C^*(G))$ is surjective.

Following Kasparov [15], for a locally compact, $\sigma$-compact, Hausdorff space $X$ and $C_0(X)$-algebras $A$ and $B$ we consider the representable K-homology groups $RK_i(X)$, the representable K-theory groups $RK^i(X)$ and the bivariant theory $\mathcal{R}KK_i(X; A, B)$. If $Y$ is compact, then $RK_i(Y) = KK_i(C(Y), \mathbb{C})$ and $RK^i(Y) = KK_i(\mathbb{C}, C(Y))$. Suppose now that $X$ is locally compact, $\sigma$-compact and Hausdorff. Then

$$RK_i(X) \cong \varinjlim_{Y \subset X} RK_i(Y) = \varinjlim_{Y \subset X} KK_i(C(Y), \mathbb{C})$$

where $Y$ runs over the compact subsets of $X$. Kasparov [15, Prop. 2.20] has shown that

$$RK^i(X) \cong \mathcal{R}KK_i(X; C_0(X), C_0(X)).$$

Moreover, if $Y \subset X$ is a compact set, then the restriction map $RK^i(X) \to RK^i(Y)$ corresponds to the map

$$\mathcal{R}KK_i(X; C_0(X), C_0(X)) \to \mathcal{R}KK_i(Y; C(Y), C(Y)) \cong KK_i(\mathbb{C}, C(Y)).$$

It is useful to introduce the group

$$LK^i(X) = \varprojlim_{Y \subset X} RK^i(Y),$$

where $Y$ runs over the compact subsets of $X$. If $X$ is written as the union of an increasing sequence $(Y_n)_n$ of compact subspaces, then as explained in the proof of Lemma 3.4 from [16], there is a Milnor $\varprojlim^1$ exact sequence:

$$0 \to \varprojlim{}^1 RK^{i+1}(Y_n) \to RK^i(X) \to \varprojlim RK^i(Y_n) \to 0.$$

The morphism $RK^i(X) \to \mathrm{Hom}(RK_i(X), \mathbb{Z})$ induced by the pairing $RK_i(X) \times RK^i(X) \to \mathbb{Z}$ factors through the morphism

$$\begin{aligned} \varprojlim RK^i(Y_n) =& LK^i(X) \to \mathrm{Hom}(RK_i(X), \mathbb{Z}) = \mathrm{Hom}(\varinjlim RK_i(Y_n), \mathbb{Z}) \\ &\cong \varprojlim \mathrm{Hom}(RK_i(Y_n), \mathbb{Z}) \end{aligned}$$

given by the projective limit of the morphisms $RK^i(Y_n) \to \mathrm{Hom}(RK_i(Y_n), \mathbb{Z})$.

If $X$ is a locally finite separable CW-complex then there is a Universal Coefficient Theorem [16, Lemma 3.4]:

$$0 \to Ext(RK_{i+1}(X), \mathbb{Z}) \to RK^i(X) \to \mathrm{Hom}(RK_i(X), \mathbb{Z}) \to 0. \tag{2}$$

In particular, it follows that the map $LK^i(X) \to \mathrm{Hom}(RK_i(X), \mathbb{Z})$ is surjective.

Let us recall the construction of the assembly map $\mu : RK_i(BG) \to K_i(C^*(G))$ and of the dual map $\nu : K^i(C^*(G)) \to RK^i(BG)$ as given in [15]. Kasparov considers a natural element

$$\beta_G \in \mathcal{R}KK(BG; C_0(BG), C_0(BG) \otimes C^*(G))$$

(which we denote here by $\ell$ as it corresponds to Mischenko's "line bundle" on $BG$). If $G$ is a discrete countable group then it is known [15, §6] that $EG$ and $BG$ can be realized as locally finite separable CW-complexes. Write $BG$ as the union of an increasing sequence $(Y_n)_n$ of finite CW-subcomplexes. Let $\ell_n$ be the image of $\ell$ in

$$\mathcal{R}KK(Y_n; C(Y_n), C(Y_n) \otimes C^*(G)) \cong KK(\mathbb{C}, C(Y_n) \otimes C^*(G))$$

under the restriction map induced by the inclusion $Y_n \subset BG$.

The map $\mu_n : RK_i(Y_n) \to K_i(C^*(G))$ is defined as the cap product by $\ell_n$:

$$KK(\mathbb{C}, C(Y_n) \otimes C^*(G)) \times KK_i(C(Y_n), \mathbb{C}) \to KK_i(\mathbb{C}, C^*(G))$$

$$(\ell_n, z) \mapsto \mu_n(z) = \ell_n \circ (z \otimes 1).$$

The assembly map $\mu : RK_i(BG) \to K_i(C^*(G))$ is the inductive limit homomorphism $\mu := \varinjlim \mu_n$. The homomorphism $\nu : K^i(C^*(G)) \to RK^i(BG)$ is defined as the cap product by $\ell$:

$$\begin{aligned} \mathcal{R}KK(BG; C_0(BG), C_0(BG) \otimes C^*(G)) \times KK_i(C^*(G), \mathbb{C}) \\ \longrightarrow \mathcal{R}KK_i(BG; C_0(BG), C_0(BG)) \end{aligned}$$

$$(\ell, x) \mapsto \nu(x) = \ell \circ (1 \otimes x).$$

Let $\nu_n : K^i(C^*(G)) \to RK^i(Y_n)$ be obtained by composing $\nu$ with the restriction map $RK^i(BG) \to RK^i(Y_n)$. Noting that $\nu_n$ is also given by the cap product by $\ell_n$, Kasparov has shown that

$$\nu_n(x) \circ z = \mu_n(z) \circ x$$

for all $x \in K^i(C^*(G))$ and $z \in RK_i(Y_n)$, [15, Lemma 6.2]. The assembly map induces a homomorphism $\mu^* : \mathrm{Hom}(K_i(C^*(G)), \mathbb{Z}) \to \mathrm{Hom}(RK_i(BG), \mathbb{Z})$. Since

$$\mathrm{Hom}(RK_i(BG), \mathbb{Z}) \cong \mathrm{Hom}(\varinjlim RK_i(Y_n), \mathbb{Z}) \cong \varprojlim \mathrm{Hom}(RK_i(Y_n), \mathbb{Z})$$

and since the equalities $\nu_n(x) \circ z = x \circ \mu_n(z)$ are compatible with the maps induced by the inclusions $Y_n \subset Y_{n+1}$, we obtain that the following diagram is commutative

$$\begin{tikzcd} K^i(C^*(G)) \arrow[r] \arrow[d, "\nu"'] & \mathrm{Hom}(K_i(C^*(G)),\mathbb{Z}) \arrow[d, "\mu^*"] \\ RK^i(BG) \arrow[r, two heads] & \mathrm{Hom}(RK_i(BG),\mathbb{Z}) \end{tikzcd} \tag{3}$$

where the horizontal arrows correspond to natural pairings of K-theory with K-homology. The map $RK^i(BG) \to \mathrm{Hom}(RK_i(BG),\mathbb{Z})$ is surjective by (2).

In view of the previous discussion, by combining results of Kasparov [15] and Tu [26], one derives the following.

**Theorem 3.1.** *Let $G$ be a countable, discrete, torsion free group. Suppose that $G$ is uniformly embeddable in a Hilbert space. Then for any group homomorphism $h : K_i(C^*(G)) \to \mathbb{Z}$ there is $x \in K^i(C^*(G))$ such that $h(\mu(z)) = \langle \mu(z), x\rangle$ for all $z \in RK_i(BG)$.*

*Proof.* For a discrete group $G$ which admits a uniform embedding into a Hilbert space it was shown in [26, Thm. 3.3] that $G$ has a $\gamma$-element. Since $G$ is torsion free, we can take $\underline{B}G = BG$. If $G$ has a $\gamma$-element, it follows by Theorem 6.5 and Lemma. 6.2 of [15] that the dual map $\nu : KK_i(C^*(G),\mathbb{C}) \to RK^i(BG)$ is split surjective. Therefore, in the diagram above, the composite map $K^i(C^*(G)) \to \mathrm{Hom}(RK_i(BG),\mathbb{Z})$, $x \mapsto \langle \nu(x), \cdot\rangle$ is surjective. This shows that if $h : K_i(C^*(G)) \to \mathbb{Z}$ is a group homomorphism, then $\mu^*(h) = h \circ \mu = \langle \nu(x), \cdot\rangle$ for some $x \in K^i(C^*(G))$. Since the diagram above is commutative, we obtain that $h \circ \mu = \langle \nu(x), \cdot\rangle = \langle \mu(\cdot), x\rangle$. $\square$

The following proposition is more or less known; for example, it is implicitly contained in [11]. Let $\iota$ be the trivial representation of $G$, $\iota(s) = 1$ for all $s \in G$.

**Proposition 3.2.** *Let $\mu : RK_0(BG) \to K_0(C^*(G))$ be the assembly map. Then for any unital finite dimensional representation $\pi : C^*(G) \to M_m(\mathbb{C})$, $\pi_* \circ \mu = m \cdot \iota_* \circ \mu$.*

*Proof.* Write $BG$ as the union of an increasing sequence $(Y_n)_n$ of finite CW-subcomplexes. Let $z \in RK_0(Y_n)$ for some $n \geq 1$ and let $x = [\pi] \in K^0(C^*(G))$. The equality $\nu_n(x) \circ z = \mu_n(z) \circ x$ becomes $\langle \nu_n(x), z\rangle = \pi_*(\mu_n(z))$. The Chern character makes the following commutative:

$$\begin{tikzcd} RK^0(Y_n) \times RK_0(Y_n) \arrow[r] \arrow[d, "ch^* \times ch_*"'] & \mathbb{Z} \arrow[d, hook] \\ H^{even}(Y_n,\mathbb{Q}) \times H_{even}(Y_n,\mathbb{Q}) \arrow[r] & \mathbb{Q} \end{tikzcd}$$

Thus $\langle ch^*(\nu_n(x)), ch_*(z)\rangle = \pi_*(\mu_n(z))$. Since $x$ is the class of a unital finite dimensional representation $\pi : C^*(G) \to M_n(\mathbb{C})$, it follows that $\nu_n(x)$ is simply the class of the flat complex vector bundle $[V] = \pi_*(\ell_n)$ over $Y_n$. On the other hand, if $V$ is a flat vector bundle, then $ch^*(V) = rank(V) = m = \dim(\pi)$ by [14]. Therefore, for any unital $m$-dimensional representation $\pi$, $\pi_*(\mu_n(z)) = m \cdot \langle 1, ch_*(z)\rangle$. By applying the same formula for the trivial representation $\iota : C^*(G) \to \mathbb{C}$, we get $\iota_*(\mu_n(z)) = \langle 1, ch_*(z)\rangle$. It follows that $\pi_*(\mu_n(z)) = m \cdot \iota_*(\mu_n(z))$. $\square$

Recall that we denote by $I(G)$ the kernel of the trivial representation $\iota : C^*(G) \to \mathbb{C}$. Since the extension $0 \to I(G) \to C^*(G) \to \mathbb{C} \to 0$ is split, $K_0(C^*(G)) \cong K_0(I(G)) \oplus \mathbb{Z}$.

**Theorem 3.3.** *Let $G$ be a countable, discrete, torsion free group that is uniformly embeddable in a Hilbert space. Let $h : K_0(C^*(G)) \to \mathbb{Z}$ be a group homomorphism.*

*(i) If $C^*(G)$ is K-quasidiagonal, then there exist two discrete completely positive asymptotic representations $\{\pi_n : C^*(G) \to M_{k(n)}(\mathbb{C})\}_n$ and $\{\gamma_n : C^*(G) \to M_{r(n)}(\mathbb{C})\}_n$ such that $\pi_{n\,\sharp}(x) - \gamma_{n\,\sharp}(x) \equiv h(x)$ for all $x \in \mu(RK_0(BG))$.*

*(ii) If $C^*(G)$ is K-residually finite dimensional, then there is a discrete completely positive asymptotic representation $\{\pi_n : C^*(G) \to M_{k(n)}(\mathbb{C})\}_n$ such that $\pi_{n\,\sharp}(x) \equiv h(x)$ for all $x \in K_0(I(G)) \cap \mu(K_0(BG))$.*

*Proof.* Part (i) follows from Theorem 3.1 and Proposition 2.5 for $A = C^*(G)$ and $B = \mathbb{C}$. For part (ii) we observe that if $\gamma_n$ is a $*$-representation, then $\gamma_* = 0$ on $K_0(I(G))$ by Proposition 3.2. $\square$

**Theorem 3.4.** *Let $G$ be a countable, discrete, torsion free group. Suppose that $G$ satisfies either one of the conditions (a) or (b) below.*

*(a) $G$ has the Haagerup property and $C^*(G)$ is K-residually finite dimensional.*

*(b) $G$ is an increasing union of residually finite, amenable groups.*

*Then, for any group homomorphism $h : K_0(C^*(G)) \to \mathbb{Z}$ there is a discrete completely positive asymptotic representation $\{\pi_n : C^*(G) \to M_{k(n)}(\mathbb{C})\}_n$ such that $\pi_{n\,\sharp}(x) \equiv h(x)$ for all $x \in K_0(I(G))$.*

*Proof.* Recall that the assembly map is an isomorphism for groups with the Haagerup property by a result of Higson and Kasparov [12], and that these groups are also embeddable in a Hilbert space. Thus, if $G$ satisfies (a), then the conclusion follows from Theorem 3.3(ii). Suppose now that $G$ satisfies (b). Thus $G = \bigcup_i G_i$ where $G_i$ are residually finite, amenable groups and $G_i \subset G_{i+1}$. Then $C^*(G) = \overline{\bigcup_i C^*(G_i)}$ and $K_0(C^*(G)) \cong \varinjlim K_0(C^*(G_i))$. Similarly, $I(G) = \overline{\bigcup_i I(G_i)}$ and $K_0(I(G)) = \varinjlim K_0(I(G_i))$.

Let $\theta_i : K_0(C^*(G_i)) \to K_0(C^*(G))$ be the map induced by the inclusion $C^*(G_i) \subset C^*(G)$. Let $h$ be given as in the statement of the theorem. By the first part of the theorem, for each $i$, there is a discrete completely positive asymptotic representation $(\pi_n^{(i)})_n$ of $C^*(G_i)$ such that $\pi_{n\,\sharp}^{(i)}(x) \equiv h(\theta_i(x))$ for all $x \in K_0(I(G_i))$. By Arveson's extension theorem, each $\pi_n^{(i)}$ extends to a unital completely positive map $\bar{\pi}_n^{(i)}$ on $C^*(G)$. Since $C^*(G)$ is separable, $K_0(I(G))$ is countable and $K_0(I(G)) = \varinjlim K_0(I(G_i))$, it follows that there is a sequence of natural numbers $r(1) < r(2) < \dots$ such that $(\bar{\pi}_{r(i)}^{(i)})_i$ is a discrete completely positive asymptotic representation of $C^*(G)$ such that $\bar{\pi}_{r(i),\,\sharp}^{(i)}(x) \equiv h(x)$ for all $x \in K_0(I(G))$. □

## 4. Almost flat K-theory classes

In this section we use the dual assembly to derive the existence of almost flat K-theory classes on the classifying space $BG$ if the group C*-algebra of $G$ is quasidiagonal. It is convenient to work with an adaptation of the notion of almost flatness to simplicial complexes, see [19].

**Definition 4.1.** *Let $Y$ be a compact Hausdorff space and let $(U_i)_{i\in I}$ be a fixed finite open cover of $Y$. A complex vector bundle $E \in \mathrm{Vect}_m(Y)$ is called $\varepsilon$-flat if is represented by a cocycle $v_{ij} : U_i \cap U_j \to U(m)$ such that $\|v_{ij}(y) - v_{ij}(y')\| < \varepsilon$ for all $y, y' \in U_i \cap U_j$ and all $i, j \in I$. A K-theory class $\alpha \in K^0(Y)$ is called almost flat if for any $\varepsilon > 0$ there are $\varepsilon$-flat vector bundles $E, F$ such that $\alpha = [E] - [F]$. This property does not depend on the cover $(U_i)_{i\in I}$.*

**Remark 4.2.** *The set of all almost flat elements of $K^0(Y)$ form a subring denoted by $K^0_{af}(Y)$. If $f : Z \to Y$ is a continuous map, then $f^*(K^0_{af}(Y)) \subset K^0_{af}(Z)$.*

The following proposition gives a method for producing $\varepsilon$-flat vector bundles. Let $Y$ be a finite simplicial complex with universal cover $\widetilde{Y}$ and fundamental group $G$. Consider the flat line-bundle $\ell$ with fiber $C^*(G)$, $\widetilde{Y} \times_G C^*(G) \to Y$, where $G \subset C^*(G)$ acts diagonally, and let $P$ be the corresponding projection in $M_m(\mathbb{C}) \otimes C(Y) \otimes C^*(G)$. Consider a discrete asymptotic representation $\{\varphi_n : C^*(G) \to M_{k(n)}(\mathbb{C})\}_n$ and set $F_n = (\mathrm{id}_m \otimes id_{C(Y)} \otimes \varphi_n)(P)$. Since $\|F_n^2 - F_n\| \to 0$ as $n \to \infty$, $E_n := \chi(F_n)$ is a projection in $M_{mk(n)}(C(Y))$ such that $\|E_n - F_n\| \to 0$ as $n \to \infty$.

**Proposition 4.3.** *For any $\varepsilon > 0$ there is $n_0 > 0$ such that for any $n \geq n_0$ there is an $\varepsilon$-flat vector bundle on $Y$ which is isomorphic to the vector bundle given by the idempotent $E_n$.*

*Proof.* We rely on a construction and results of Phillips and Stone from [20, 21], see also [18]. A simplicial complex is locally ordered by giving a partial ordering $\mathbf{o}$ of its vertices in which the vertices of each simplex are totally ordered. The first barycentric subdivision of any simplicial complex has a natural local ordering [21, §1.4]. Thus we may assume that $Y$ is endowed with a fixed local ordering $\mathbf{o}$. Let $Y$ have vertices $I = \{1, 2, ..., m\}$. We denote by $Y^k$ the set of $k$-simplices of $Y$. Given $r \geq 1$, a $U(r)$-valued lattice gauge field $\mathbf{u}$ on the simplicial complex $Y$ is a function that assigns to each 1-simplex $\langle i, j\rangle$ of $Y$ an element $u_{ij} \in U(r)$ subject to the condition that $u_{ji} = u_{ij}^{-1}$, see [21, Def. 3.2]. Consider the cover of $Y$ by dual cells $(V_i)_{i\in I}$ [21, A.1].

Phillips and Stone show that for a fixed locally ordered finite simplicial complex $Y$ as above there is a function $h : [0, +\infty) \to [0, 1]$ with $\lim_{t\to\infty} h(t) = 0$ and which has the following property. Let $\mathbf{u}$ be a $U(r)$-valued lattice gauge field on $Y$ for some $r \geq 1$. Suppose that

$$\|u_{ij}u_{jk} - u_{ik}\| \leq \delta \tag{4}$$

for all 2-simplices $\langle i, j, k\rangle$ (with vertices so $\mathbf{o}$-ordered). Then there is a cocycle $v_{ij} : V_i \cap V_j \to U(r)$, $\langle i, j\rangle \in Y^1$, such that

$$\sup_{x\in V_i\cap V_j} \|v_{ij}(x) - u_{ij}\| < h(\delta).$$

The functions $v_{ij}(x)$ are constructed by an iterative process, based on the skeleton of $Y$. At each stage of the construction one takes affine combinations of functions defined at a previous stage, starting with the constant matrices $u_{ij}$. It follows that for each $i \in I$, there exists a fixed small open tubular neighborhood $U_i$ of $V_i$ which is affinely homotopic to $V_i$, such that the cover $(U_i)_{i\in I}$ has the following property. For any $U(r)$-valued lattice gauge field $\mathbf{u}$ on $Y$ that satisfies (4), there is a cocycle $v_{ij} : U_i \cap U_j \to U(r)$, $\langle i, j\rangle \in Y^1$ such that

$$\sup_{x\in U_i\cap U_j} \|v_{ij}(x) - u_{ij}\| < 2h(\delta).$$

We are going to use the asymptotic representation $(\varphi_n)_n$ as follows. Using trivializations of $\ell$ to $U_i$ one obtains group elements $s_{ij} \in G$ for $\langle i, j\rangle \in Y^1$ giving a constant cocycle on $U_i \cap U_j$ that represents $\ell$, so that $s_{ij}^{-1} = s_{ji}$ and $s_{ij} \cdot s_{jk} = s_{ik}$ whenever $\langle i, j, k\rangle \in Y^2$.

If $(\chi_i)_{i\in I}$ are positive continuous functions with $\chi_i$ supported in $U_i$ and such that $\sum_{i\in I} \chi_i^2 = 1$, then $\ell$ is represented by an idempotent

$$P = \sum_{i,j\in I} e_{ij} \otimes \chi_i\chi_j \otimes s_{ij} \in M_m(\mathbb{C}) \otimes C(Y) \otimes C^*(G).$$

Here $m = |I|$ and $(e_{ij})$ is the canonical matrix unit of $M_m(\mathbb{C})$. It follows that for all $n$ sufficiently large, $(id_m \otimes id_{C(Y)} \otimes \varphi_n)_\sharp(P)$ is given by the class of a projection $E_n$ with $\|E_n - F_n\| < 1/2$, where $F_n = (id_m \otimes id_{C(Y)} \otimes \varphi_n)(P)$.

$$F_n = \sum_{i,j\in I} e_{ij} \otimes \chi_i\chi_j \otimes \varphi_n(s_{ij}) \in M_m(\mathbb{C}) \otimes C(Y) \otimes M_{k(n)}(\mathbb{C}).$$

For $v \in GL_k(\mathbb{C})$ we denote by $w(v)$ the unitary $v(v^*v)^{-1/2}$. Fix $n$ sufficiently large so that $\varphi_n(s_{ij}) \in GL_{k(n)}(\mathbb{C})$. For each ordered edge $\langle i, j\rangle \in Y^1$ we set $u_{ij} = w(\varphi_n(s_{ij}))$ and $u_{ji} = u_{ij}^{-1}$. This will define a $U(k(n))$-valued lattice gauge field on the ordered simplicial complex $Y$. Fix $\varepsilon > 0$ such that $4m^2\varepsilon < 1/2$ and choose $\delta > 0$ such that $h(\delta) < \varepsilon/2$. Since $(\varphi_n)_n$ is an asymptotic representation, there is $n_0 > 0$ such that if $n \geq n_0$ then

$$\|\varphi_n(s_{ij}) - u_{ij}\| < \varepsilon/2 \tag{5}$$

for all $\langle i, j\rangle \in Y^1$ and $\|u_{ij}u_{jk} - u_{ik}\| \leq \delta$ for all 2-simplices $\langle i, j, k\rangle$. By the result of Phillips and Stone quoted above, there exists a cocycle $v_{ij} : U_i \cap U_j \to U(k(n))$ such that

$$\|v_{ij}(x) - u_{ij}\| < h(\delta) < \varepsilon/2 \tag{6}$$

for all $x \in U_i\cap U_j$. It follows that $\|v_{ij}(x)-v_{ij}(x')\| < \varepsilon$ for all $x, x' \in U_i\cap U_j$ and all $i, j \in I$ and hence the idempotent

$$e_n(x) = \sum_{i,j\in I} e_{ij} \otimes \chi_i(x)\chi_j(x)v_{ij}(x), \quad x \in Y$$

gives an $\varepsilon$-flat vector bundle on $Y$. From (5) and (6) we have

$$\|v_{ij}(x) - \varphi_n(s_{ij})\| < \varepsilon \tag{7}$$

for all $x \in U_i\cap U_j$ and $\langle i, j\rangle \in Y^1$ . Using (7) we see that $\|e_n-F_n\| \leq 2m^2\varepsilon < 1/2$ and hence $\|e_n - E_n\| \leq \|e_n - F_n\| + \|E_n - F_n\| < 1$. It follows that $E_n = we_nw^{-1}$ for some invertible element $w$. This shows that the isomorphism class of the vector bundle given the idempotent $E_n$ is represented by an $\varepsilon$-flat vector bundle since we have seen that $e_n$ has that property. □

Let $Y$ be a finite simplicial complex with universal cover $\widetilde{Y}$ and fundamental group $G$ and let $\ell$ be the corresponding flat line-bundle with fiber $C^*(G)$. Recall that the Kasparov product $K_0(C(Y) \otimes C^*(G)) \times KK(C^*(G), \mathbb{C}) \to K^0(Y)$ induces a map $\nu : KK(C^*(G), \mathbb{C}) \to K^0(Y)$, $\nu(\alpha) = [\ell] \circ (\alpha \otimes 1)$.

**Corollary 4.4.** $\nu(KK_{qd}(C^*(G), \mathbb{C})) \subset K^0_{af}(Y)$.

*Proof.* This follows from Propositions 2.5 and 4.3. □

**Theorem 4.5.** *Let $G$ be a countable, discrete, torsion free group which is uniformly embeddable in a Hilbert space. Suppose that the classifying space $BG$ is a finite simplicial complex and that the full group $C^*$-algebra $C^*(G)$ is K-quasidiagonal. Then all the elements of $K^0(BG)$ are almost flat.*

*Proof.* We have seen in the proof of Theorem 3.1 that under the present assumptions on $G$, the dual assembly map $\nu : KK(C^*(G), \mathbb{C}) \to K^0(BG)$ is surjective. Since $C^*(G)$ is K-quasidiagonal by hypothesis (this holds for instance if $C^*(G)$ is quasidiagonal as observed in Remark 2.3), we have that $KK(C^*(G), \mathbb{C}) = KK_{qd}(C^*(G), \mathbb{C})$. The result follows now from Corollary 4.4. $\square$

From Theorem 4.5 one can derive potential obstructions to quasidiagonality of group C*-algebras.

**Remark 4.6.** *Let $G$ be a countable, discrete, torsion free group which is uniformly embeddable in a Hilbert space and such that the classifying space $BG$ is a finite simplicial complex. If not all elements of $K^0(BG)$ are almost flat, then $C^*(G)$ is not quasidiagonal.*

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