GROUP QUASI-REPRESENTATIONS AND INDEX THEORY

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ABSTRACT. Let M be a closed connected manifold and let D be an elliptic operator on M. Let G be a discrete countable group and let $\widetilde{M} \to M$ be a principal G-bundle. Connes and Moscovici showed that this data defines an analytic index $ind_{\ell^1(G)}(D) \in K_0(\ell^1(G))$. If B is a unital tracial C*-algebra, we give a formula for the trace of the image of $ind_{\ell^1(G)}(D)$ in $K_0(B)$ under the map induced by a quasi-representation of G in B. As an application, we reprove and generalize a formula of Exel and Loring to surface groups.

1. INTRODUCTION

Let M be a closed connected Riemannian manifold with fundamental group G and let D be an elliptic operator on M. Connes, Gromov and Moscovici [5] introduced the notion of almost flat bundle and proved an index theorem showing that the pushforward of the equivariant index of D under a quasi-representation of G coming from parallel transport in an almost flat bundle E on M is equal to the index of D twisted by E. They used this result to show that the signature with coefficients in an almost flat bundle is a homotopy invariant.

In this paper we take a dual approach where the input data involves a group quasi-representation rather than an almost flat bundle. In conjunction with a finite dimensional approximation technique based on quasi-diagonality, this approach enables us to use the classic Atiyah-Singer index theorem [1] and respectively the Mishchenko-Fomenko index theorem [20] to give a new proof and a generalization of the index theorem of Connes, Gromov and Moscovici that allows for almost flat bundles E with fibers projective Hilbert modules over a tracial C*-algebra, see Theorem 3.6. In the last section, we reprove and generalize the Exel-Loring formula [9] to surface groups, by reinterpreting it as an index theorem, see Theorem 4.2.

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Let G be a discrete countable group, let $\mathcal{G} \subset G$ be a finite set and let $\omega > 0$. Let B be a unital C*-algebra. A map $\pi : G \to B$ is called a $(\mathcal{G}, \omega)$-unitary representation if it satisfies the following conditions:

(i) $\pi(1) = 1$, $\|\pi(s)\| \leq 1$ for all $s \in G$.

(ii) $\|\pi(s^{-1}) - \pi(s)^*\| < \omega$ for all $s \in \mathcal{G}$,

(iii) $\|\pi(st) - \pi(s)\pi(t)\| < \omega$ for all $s, t \in \mathcal{G}$.

Let A be a unital involutive Banach algebra, let $\mathcal{G} \subset A$ be a finite set and let $\omega > 0$. A unital linear contraction $\pi : A \to B$ such that $\|\pi(a^*) - \pi(a)^*\| < \omega$ and $\|\pi(aa') - \pi(a)\pi(a')\| < \omega$ for all $a, a' \in \mathcal{G}$ is called a $(\mathcal{G}, \omega)$-$*$-representation of A in B. We will use the term quasi-representation to refer to a $(\mathcal{G}, \omega)$-unitary representation or to a $(\mathcal{G}, \omega)$-$*$-representation where $\mathcal{G}$ and ω are not necessarily specified. Let us note that a quasi-representation of G induces a quasi-representation of the involutive Banach algebra $\ell^1(G)$ by the formula: $\widehat{\pi}(\sum_{s \in G} \lambda_s \cdot s) = \sum_{s \in G} \lambda_s \pi(s)$. Moreover, given a finite set $\widehat{\mathcal{G}} \subset \ell^1(G)$ and $\widehat{\omega} > 0$ there exist a finite set $\mathcal{G} \subset G$ and $\omega > 0$ such that if $\pi : G \to B$ is a $(\mathcal{G}, \omega)$-unitary representation, then $\widehat{\pi} : \ell^1(G) \to B$ is a $(\widehat{\mathcal{G}}, \widehat{\omega})$-$*$-representation. An important method for turning K-theoretical invariants of G into numerical invariants is to use quasi-representations $\pi : G \to M_k(\mathbb{C})$ to pushforward idempotents in matrices over $\ell^1(G)$ or $C^*(G)$ to scalar idempotents [5]. It is then very natural to seek quasi-representations that act non-trivially on K-theory. This question is addressed in [7] where we study the existence of sequences of quasi-representations $\pi_n : C^*(G) \to M_{k(n)}(\mathbb{C})$ that induce prescribed maps $K_0(C^*(G)) \to \mathbb{Z}$ and we give conditions for G under which all the K-theory classes in $K^0(BG)$ are almost flat. The first example of a topologically nontrivial quasi-representation is due to Voiculescu [27] for $G = \mathbb{Z}^2$.

Suppose that M is a closed smooth connected manifold. Let G be a discrete countable group and let $\widetilde{M} \to M$ be a principal G-bundle. The diagonal action of G on $\widetilde{M} \times \ell^1(G)$ defines a flat $\ell^1(G)$-bundle $\widetilde{M} \times_G \ell^1(G) \to M$ which gives an element $\ell \in K_0(C(M) \otimes \ell^1(G))$ (Mishchenko's line bundle). Using Lafforgue's cap product (see [16]) $KK^{ban}(\mathbb{C}, C(M) \otimes \ell^1(G)) \times KK^{ban}(C(M), \mathbb{C}) \to KK^{ban}(\mathbb{C}, \ell^1(G))$, $(y, z) \mapsto y \circ (z \otimes 1)$, one defines a map $\mu : K_0(M) \to K_0(\ell^1(G))$, by $\mu(z) = \ell \circ (z \otimes 1)$. For Banach algebras A, B we denote by $A \otimes B$ the completion of the algebraic tensor product with respect to the maximal Banach norm such that $\|a \otimes b\| \leq \|a\|\|b\|$.

The goal of the paper is to provide a formula for the push-forward of $\mu(z)$, $z \in K_0(M)$, under a quasi-representation π of $\ell^1(G)$. To make this precise, we write $\mu(z) = [e_0] - [e_1]$ where e_0, e_1 are idempotents in $M_m(\mathbb{C}) \otimes \ell^1(G)$. If $\pi : \ell^1(G) \to M_k(\mathbb{C})$ is sufficiently multiplicative (this depends on the

choice of the representatives e_0, e_1), then $(id_m \otimes \pi)(e_i)$ can be perturbed canonically to idempotents f_i in $M_{mk}(\mathbb{C})$, $i = 0, 1$. We are interested in computing the integer $\pi_\sharp(\mu(z)) := [f_0] - [f_1] \in K_0(\mathbb{C}) = \mathbb{Z}$. More generally, if $\pi : \ell^1(G) \to B$ is a sufficiently multiplicative quasi-representation in a unital C*-algebra B and τ is a tracial state of B, we would like to compute $\tau(\pi_\sharp(\mu(z)))$ where $\pi_\sharp(\mu(z))$ is defined similarly. We reserve the notation $\mu_{\ell^1} : K_0(M) \to K_0(\ell^1(G))$ for a related map defined in [16]. Its definition involves a projection $p \in \ell^1(G, C_0(\widetilde{M}))$ (constructed from a cutoff function) and the descent map $j_G : KK_G^{ban}(C_0(\widetilde{M}), \mathbb{C}) \to KK^{ban}(\ell^1(G, C_0(\widetilde{M})), \ell^1(G))$. The author does not know if $\mu_{\ell^1} = \mu$ in this general context. If $\widetilde{M} \to M$ is the universal cover of M, then μ_{ℓ^1} is the Lafforgue's ℓ^1-version of the assembly map and it is known that the images of $\mu_{\ell^1}(z)$ and $\mu(z)$ coincide under the map $K_0(\ell^1(G)) \to K_0(C^*(G))$, see [16]. We will verify that if $\mathcal{G}$ is sufficiently large and ω is sufficiently small, then for any $(\mathcal{G}, \omega)$-unitary representation $\pi : G \to B$, $\pi_\sharp(\mu_{\ell^1}(z)) = \pi_\sharp(\mu(z))$, see Corollary 3.5. The computation $\pi_\sharp(\mu_{\ell^1}(z))$ is given by Theorem 3.6. This is obtained by combining a technical result, Theorem 3.2 (see also Corollary 3.5), with the Mishchenko-Fomenko index theorem [20], [24]. The case of a matricial C*-algebra $B = M_k(\mathbb{C})$ yields a result that is closely related to the index theorem of Connes, Gromov and Moscovici [5]. Our approach, which is very different from [5], is based on descriptions of the isomorphism $KK^{ban}(\mathbb{C}, A) \cong K_0(A)$ and of the cap product $KK^{ban}(\mathbb{C}, A \otimes B) \times KK^{ban}(A, \mathbb{C}) \to KK^{ban}(\mathbb{C}, B)$ that involve compressions by quasi-central idempotents, see Corollary 2.4 and Proposition 2.5. In the last part of the paper we bring a new perspective and generalize the Exel-Loring formula [9] to surface groups by realizing it as an index formula.

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2. Idempotents in Banach algebras and KK^{ban}-theory

In this section we study various perturbation properties of idempotents in a Banach algebra. Proposition 2.5 is a key tool for the rest of the paper, for it allows us to approach index computations via compressions by quasi-central idempotents. We work with Lafforgue's KK^{ban}-theory [16] in a simplified version which does require the use of dual pairs. This equivalent approach is used in the survey [15]. A written account of why the two approaches are equivalent is given in [22].

We begin by a short review of the setup from [15]. Let A be a unital Banach algebra. A right Banach A-module is a Banach space E endowed with a right action of A such that $x \cdot 1 = x$ and $\|xa\| \le \|x\|\|a\|$ for all $x \in E$ and $a \in A$. A left Banach A-module F is defined similarly. We denote by $E \otimes_A F$ the completion of $E \otimes^{alg} F$ under the largest seminorm such that $\|x \otimes ay - xa \otimes y\| = 0$ and $\|x \otimes y\| \le \|x\|\|y\|$, for all $x \in E$, $y \in F$, $a \in A$.

Let E and F be right Banach A-modules. The space of $\mathbb{C}$-linear continuous maps $T : E \to F$ such that $T(xa) = T(x)a$ for $x \in E$ and $a \in A$ is denoted by $L_A(E, F)$ or $L_A(E)$ if $E = F$. The ideal $K_A(E)$ of A-compact operators is defined as the closure of the linear span of the A-rank one operators, i.e. those of the form $S \circ T$ where $S \in L_A(A, E)$ and $T \in L_A(E, A)$. A Fredholm module over A consists of a $\mathbb{Z}/2$-graded right Banach A-module E and an odd operator $T \in L_A(E)$ such that $T^2 - id_E \in K_A(E)$. If (E, T) is a Fredholm module over A and $\theta : A \to B$ is a unital morphism then $(E \otimes_A B, T \otimes 1)$ is a Fredholm module over B. Let $A[0, 1]$ be the Banach algebra of continuous functions from $[0, 1]$ to A endowed with norm $\|f\| = \sup_{t \in [0,1]} \|f(t)\|_A$. Let $\theta_0, \theta_1 : A[0, 1] \to A$ be the evaluations at 0 and 1. Two Fredholm modules over A are homotopic if they are the images under θ_0 and θ_1 of a Fredholm module over $A[0, 1]$. Let us recall that for any unital Banach algebra A there is a functorial bijection between $K_0(A)$ and the set of homotopy classes of Fredholm modules over A, see [16, Thm. 1.2.8]. $E^{ban}(A, B)$ is the set of isomorphism classes of triples (E, π, T), where E is a $\mathbb{Z}/2$-graded Banach right B-module, $\pi : A \to L_B(E)$ is a morphism of Banach algebras that takes values in even operators, and $T \in L_B(E)$ is an odd operator that satisfies $\pi(a)(T^2 - id_E) \in K_B(E)$ and $\pi(a)T - T\pi(a) \in K_B(E)$ for any $a \in A$. One defines $KK^{ban}(A, B)$ as the set of homotopy classes in $E^{ban}(A, B)$, where the homotopy relation is defined using $E^{ban}(A, B[0, 1])$. If we are in a situation when $\alpha \in KK^{ban}(A, B)$ is given by the class of a triplet

$$(1) \qquad \left(H \oplus H, \pi = \begin{pmatrix} \pi_0 & 0 \\ 0 & \pi_1 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),$$

where H is a right Banach B-module, then we will also write $\alpha = [\pi_0, \pi_1]$. Here $\pi_i : A \to L_B(H)$ have the property that $\pi_0(a) - \pi_1(a) \in K_B(H)$, for all $a \in A$. In the case $A = \mathbb{C}$, we will also write $\alpha = [p_0, p_1]$ where p_i is the idempotent $p_i = \pi_i(1)$ and $p_0 - p_1 \in K_B(H)$. While a general intersection product is not available in KK^{ban}, there are cap products

$$(2) \qquad K_0(A) \times KK^{ban}(A, B) \to K_0(B), \quad (x, z) \mapsto x \circ z,$$

(3) $\qquad K_0(A \otimes D) \times KK^{ban}(A, B) \to K_0(B \otimes D) \quad (x, z) \mapsto x \circ (z \otimes 1).$

The product (3) is defined by combining the product (2) with the morphism $KK^{ban}(A, B) \to KK^{ban}(A \otimes D, B \otimes D)$. Recall that $E^{ban}(\mathbb{C}, B)$ is the set of isomorphism classes of Fredholm modules over B and $KK^{ban}(\mathbb{C}, B) \cong K_0(B)$. With this in mind, assuming that A is unital, one can describe the product (2) as follows (see [15]). If p is an idempotent in A, and $(E, \pi, T) \in E^{ban}(A, B)$, then the product of $[p] \in K_0(A)$ with $\alpha = [E, \pi, T] \in KK^{ban}(A, B)$ is the class of the Fredholm module over B equal to

(4) $\qquad\qquad\qquad (Im\,\pi(p), \pi(p)T\pi(p)).$

When p is an idempotent in $M_k(A)$, we use the image of p by $id_k \otimes \pi : M_k(A) \to L_B(E^k)$. If α is of the form (1), then the class of (4) is equal to

(5) $\qquad\qquad\qquad [\pi_0(p), \pi_1(p)],$

since both $\pi(p)(\pi(p)T\pi(p) - T)$ and $(\pi(p)T\pi(p) - T)\pi(p)$ are in $K_B(H \oplus H)$. Given $0 < \varepsilon \le 1/2$ we consider the following neighborhood of $\{0, 1\}$:

$$V(\varepsilon) = \{\lambda \in \mathbb{C} : |\lambda| < \varepsilon\} \cup \{\lambda \in \mathbb{C} : |\lambda - 1| < \varepsilon\}.$$

Let B be a unital Banach algebra. Let $x \in B$ and suppose that its spectrum $Sp(x)$ is contained in $V(\varepsilon)$. Then we denote by $\chi(x)$ the idempotent

$$\chi(x) = \frac{1}{2\pi i} \int_C (\lambda - x)^{-1} dz$$

where $C = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1/2\}$. Note that $\chi(x)$ is well-defined whenever $\|x^2 - x\| < 1/4$. Indeed, suppose that $\|x^2 - x\| < \varepsilon^2$. If $\lambda \in Sp(x)$, then $\lambda^2 - \lambda \in Sp(x^2 - x)$ and so $|\lambda^2 - \lambda| \le \|x^2 - x\| < \varepsilon^2$. It follows that $\lambda \in V(\varepsilon)$.

Definition 2.1. *Let $\pi : A \to B$ be a unital linear contraction of Banach algebras. Let $p \in M_m(\mathbb{C}) \otimes A$ be an idempotent and let $x = (id_m \otimes \pi)(p) \in M_m(\mathbb{C}) \otimes B$. We define $\pi_\sharp(p) = [\chi(x)] \in K_0(B)$ whenever $\|x^2 - x\| < 1/4$ and set $\pi_\sharp(p) = 0$ otherwise.*

One can fix for each K-theory element $\alpha \in K_0(A)$, two idempotents e_α, f_α such that $\alpha = [e_\alpha] - [f_\alpha]$ and define $\pi_\sharp : K_0(A) \to K_0(B)$ by $\pi_\sharp(\alpha) = \pi_\sharp(e_\alpha) - \pi_\sharp(f_\alpha)$. While this definition of $\pi_\sharp$ on $K_0(A)$ depends on the choices we make for e_α, f_α, the ambiguity can be eliminated by passing to maps that are more and more multiplicative. Specifically, if $\alpha = [e'_\alpha] - [f'_\alpha]$ for another set of idempotents e'_α, f'_α, then $\pi_\sharp(e_\alpha) - \pi_\sharp(f_\alpha) = \pi_\sharp(e'_\alpha) - \pi_\sharp(f'_\alpha)$ if $\pi : A \to B$ is a unital linear contraction such that $\|\pi(aa') - \pi(a)\pi(a')\| < \omega$ for all a, a' in a sufficiently large finite set $\mathcal{G}$ and sufficiently small ω.

Lemma 2.2. *Let B be a unital Banach algebra and let E be a Banach right B-module. Let (e_0, f_0) and (e_1, f_1) be pairs of idempotents in $L_B(E)$ such that $e_0 - f_0 \in K_B(E)$ and $e_1 - f_1 \in K_B(E)$. If there is $M \geq 0$ such that $\|e_i\| \leq M$, $\|f_i\| \leq M$, $i = 0, 1$, $\|e_0 - e_1\| < 1/(8M + 4)$ and $\|f_0 - f_1\| < 1/(8M + 4)$, then $[e_0, f_0] = [e_1, f_1]$ in $KK^{ban}(\mathbb{C}, B) \cong K_0(B)$.*

Proof. Let $\varepsilon = 1/(8M + 4)$. Define $x, y \in C([0, 1], L_B(E))$ by $x(t) = (1 - t)e_0 + te_1$ and $y(t) = (1 - t)f_0 + tf_1$. Then $\|x - e_0\| \leq \|e_1 - e_0\| < \varepsilon$ and $\|y - f_0\| \leq \|f_1 - f_0\| < \varepsilon$. Moreover

$$\|x^2 - x\| \leq \|x\|\|x - e_0\| + \|x - e_0\|\|e_0\| + \|e_0 - x\| < (2M + 1)\varepsilon \leq 1/4.$$

Similarly one checks that $\|y^2 - y\| < 1/4$. As observed earlier, it follows that $Sp(x)$ and $Sp(y)$ are contained in $V(1/2)$. Then $e := \chi(x)$ and $f := \chi(y)$ are idempotents in $L_{B[0,1]}(E[0, 1])$ such that $e - f \in K_{B[0,1]}(E[0, 1])$ since $x - y \in K_{B[0,1]}(E[0, 1])$ and the Riesz calculus is functorial. It follows that the pair (e, f) gives a homotopy between (e_0, f_0) and (e_1, f_1). $\square$

Lemma 2.3. *For any $M \geq 0$ and $0 < \varepsilon < 1/2$ there is $\delta = \delta(M, \varepsilon) > 0$ with the following property. If B is a unital Banach algebra and $x, y \in B$ satisfy $\|x\| \leq M$, $\|x^2 - x\| \leq \delta$, $\|x - y\| \leq \delta$, then $Sp(x) \cup Sp(y) \subset V(\varepsilon)$ and $\|\chi(y) - x\| < \varepsilon$, $\|\chi(x) - x\| < \varepsilon$, $\|\chi(y) - y\| < \varepsilon$, $\|\chi(x) - \chi(y)\| < \varepsilon$.*

Proof. Given $0 < \varepsilon < 1/2$, suppose that $\delta > 0$ is sufficiently small so that $2M\delta + \delta^2 + 2\delta < \varepsilon^2$. Then $\|x^2 - x\| \leq \delta < \varepsilon^2 < 1/4$ and $\|y^2 - y\| \leq \|y\|\|y - x\| + \|y - x\|\|x\| + \|x^2 - x\| + \|x - y\| \leq 2M\delta + \delta^2 + 2\delta < \varepsilon^2 < 1/4$. As observed earlier, from these estimates it follows that $Sp(x), Sp(y) \subset V(\varepsilon)$. Seeking a contradiction, suppose now that there are $M \geq 0$ and $0 < \varepsilon < 1/2$ for which the lemma is false. It follows that we can find a decreasing sequence (δ_n) converging to 0, with $2M\delta_n + \delta_n^2 + 2\delta_n < \varepsilon^2/2$ for all $n \geq 1$, and such that the conclusion of the lemma is false for each δ_n. Thus for each $n \geq 1$, there is a unital Banach B_n and there exists elements $x_n, y_n \in B_n$ which satisfy $\|x_n\| \leq M$, $\|x_n^2 - x_n\| \leq \delta_n$, $\|x_n - y_n\| \leq \delta_n$ and such that for each $n \geq 1$,

(6) $\max\{\|\chi(y_n) - x_n\|, \|\chi(x_n) - x_n\|, \|\chi(y_n) - y_n\|, \|\chi(x_n) - \chi(y_n)\|\} \geq \varepsilon.$

Consider the unital Banach algebra $A := \prod_n B_n$ consisting of all bounded sequences $b = (b_n)$ with $b_n \in B_n$ and norm $\|b\| = \sup_n \|b_n\|$. Let J be the two-sided closed ideal of A consisting of sequences $b = (b_n)$ such that $\lim_n \|b_n\| = 0$ and let $q : A \rightarrow A/J$ denote the canonical quotient morphism. Then A/J is a unital Banach algebra . Let us consider the elements $x = (x_n)$ and $y = (y_n)$ of A. By arguing as above we see that $Sp(x) \cup Sp(y) \subset V(\varepsilon)$

and that $x^2 - x, y^2 - y \in J$. It is then clear that $q(x)^2 = q(x) = q(y) = q(y)^2$ and by the functoriality of the Riesz calculus, $q(x) = \chi(q(x)) = q(\chi(x))$ and $q(y) = \chi(q(y)) = q(\chi(y))$. This shows that $\{x - \chi(x), y - \chi(y), x - \chi(y), \chi(x) - \chi(y)\} \subset J$. It follows that $\lim_n \|x_n - \chi(x_n)\| = \lim_n \|y_n - \chi(y_n)\| = \lim_n \|x_n - \chi(y_n)\| = \lim_n \|\chi(x_n) - \chi(y_n)\| = 0$. This contradicts (6). $\qquad\square$

Lemma 2.4. *For any $M \geq 1$ and $0 < \varepsilon < 1/2$ there is $\theta = \theta(M, \varepsilon) > 0$ with the following property. Let B be a unital Banach algebra and let E be a Banach right B-module. Let $e, f \in L_B(E)$ and $h \in K_B(E)$ be idempotents such that*

  (i) $e - f \in K_B(E)$,
  (ii) $\|e\| \leq M$, $\|f\| \leq M$, $\|h\| \leq M$,
  (iii) $\|[e, h]\| \leq \theta$, $\|[f, h]\| \leq \theta$, $\|(1 - h)(e - f)(1 - h)\| \leq \theta$.

Then

  (a) $Sp(heh) \cup Sp(hfh) \subset V(\varepsilon)$,
  (b) $\|heh - \chi(heh)\| < \varepsilon$, $\quad \|hfh - \chi(hfh)\| < \varepsilon$
  (c) $[e, f] = [\chi(heh), \chi(hfh)] \in KK^{ban}(\mathbb{C}, B) \cong K_0(B)$.

Proof. We may assume that $\varepsilon < 1/(8(M+1)^3 + 4)$. Let δ be the two-variable function given by Lemma 2.3. We show that if

$$(7) \qquad 0 < \theta < \min\left\{\varepsilon, \ \frac{\delta(M^3, \varepsilon)}{M^3}, \ \frac{\delta((M+1)^3, \varepsilon)}{(M+1)^3}, \ \frac{\delta(2(M+1)^3, \varepsilon)}{2M}\right\}$$

then θ satisfies the statement of the Lemma 2.4. Set

$$x = heh, \ x' = (1 - h)e(1 - h), \ y = hfh, \ y' = (1 - h)f(1 - h).$$

We have a series of straightforward estimates:

$$\|x\| \leq M^3, \quad \|x'\| \leq (M+1)^2 M, \quad \|y\| \leq M^3, \quad \|y'\| \leq (M+1)^2 M,$$

$$\|x + x'\| \leq 2(M+1)^3, \quad \|y + y'\| \leq 2(M+1)^3$$

$$\|e - (x + x')\| = \|h(he - eh) + (eh - he)h\| \leq 2M\theta < \delta(2(M+1)^3, \varepsilon),$$

$$\|f - (y + y')\| < \delta(2(M+1)^3, \varepsilon)$$

$$\|x' - y'\| = \|(1 - h)(e - f)(1 - h)\| \leq \theta < \varepsilon,$$

$$\|x^2 - x\| = \|he(he - eh)h\| \leq M^3\theta < \delta(M^3, \varepsilon),$$

$$\|x'^2 - x'\| = \|((1 - h)e(he - eh)(1 - h)\| \leq (M+1)^3\theta < \delta((M+1)^3, \varepsilon),$$

$$\|y^2 - y\| < \delta(M^3, \varepsilon), \quad \|y'^2 - y'\| < \delta((M+1)^3, \varepsilon).$$

These estimates enable us to apply Lemma 2.3 and obtain that both $Sp(x)$ an $Sp(y)$ are contained in $V(\varepsilon)$ and moreover

$$\|x - \chi(x)\| < \varepsilon, \quad \|x' - \chi(x')\| < \varepsilon, \quad \|y - \chi(y)\| < \varepsilon, \quad \|y' - \chi(y')\| < \varepsilon;$$

$$\|\chi(x') - \chi(y')\| < \varepsilon, \quad \|e - \chi(x + x')\| < \varepsilon, \quad \|f - \chi(y + y')\| < \varepsilon.$$

Note that $\chi(x), \chi(y) \in K_B(E)$, $x' - y' \in K_B(E)$ and hence $\chi(x') - \chi(y') \in K_B(E)$. Recall that $x \in hBh$ and $x' \in (1 - h)B(1 - h)$ and hence $\chi(x + x') = \chi(x) + \chi(x')$ and $\chi(x)\chi(x') = \chi(x')\chi(x) = 0$. Similar properties hold for y and y' and we will use these in the sequel. Since $\|e\| \leq M$, $\|\chi(x + x')\| \leq \|e\| + \varepsilon < M + 1$, $\|f\| \leq M$, $\|\chi(y + y')\| \leq \|f\| + \varepsilon < M + 1$, and $\varepsilon < 1/(8(M + 1) + 4)$, we deduce from Lemma 2.2 that

$$[e, f] = [\chi(x + x'), \chi(y + y')] = [\chi(x) + \chi(x'), \chi(y) + \chi(y')].$$

On the other hand,

$$[\chi(x) + \chi(x'), \chi(y) + \chi(y')] = [\chi(x), \chi(y)] + [\chi(x'), \chi(y')] = [\chi(x), \chi(y)],$$

since $[\chi(x'), \chi(y')] = [\chi(x'), \chi(x')] = 0$, also by Lemma 2.2 which we can apply since we have $\|\chi(x')\| \leq \|x'\| + \varepsilon < (M + 1)^3$, $\|\chi(y')\| \leq \|y'\| + \varepsilon < (M + 1)^3$ and

$$\|\chi(x') - \chi(y')\| < \varepsilon < 1/(8(M + 1)^3 + 4).$$

We conclude that $[e, f] = [\chi(x), \chi(y)]$.   □

The following description of a special case of the cap product (3) plays an important role in the paper.

Proposition 2.5. *Let A, B be unital Banach algebras. Suppose that an element $\alpha \in KK^{ban}(A, \mathbb{C})$ is given by the class of a triplet*

$$\left(H \oplus H, \varphi = \begin{pmatrix} \varphi_0 & 0 \\ 0 & \varphi_1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),$$

where H is a Banach space. Let $p \in A \otimes B$ be an idempotent. Let $\widetilde{\varphi}_i := \varphi_i \otimes id_B : A \otimes B \to L(H) \otimes B$. Let $\widetilde{e} \in M_r(\mathbb{C}) \otimes B \subset L(H) \otimes B$ be an idempotent ($M_r(\mathbb{C}) \cong L(\mathbb{C}^m) \subset L(H)$ is induced by a choice of embedding $\mathbb{C}^m \subset H$). Suppose that there is $M > 0$ and $0 < \varepsilon < 1/2$ such that

    (i) $\|\widetilde{e}\| \leq M, \quad \|\widetilde{\varphi}_i(p)\| \leq M, \quad i = 0, 1$,
    (ii) $\|[\widetilde{\varphi}_i(p), \widetilde{e}]\| < \theta(M, \varepsilon), \quad i = 0, 1$,
    (iii) $\|(1 - \widetilde{e})(\widetilde{\varphi}_0(p) - \widetilde{\varphi}_1(p))(1 - \widetilde{e})\| < \theta(M, \varepsilon)$,

where $\theta(M, \varepsilon)$ is given by Lemma 2.4.

Then the idempotents $q_i := \chi(\widetilde{e}\,\widetilde{\varphi}_i(p)\,\widetilde{e}) \in M_r(B)$ are well defined, they satisfy

$$\|q_i - \widetilde{e}\,\widetilde{\varphi}_i(p)\widetilde{e}\| < \varepsilon, \quad i = 0, 1,$$

and the cap product

$$KK^{ban}(\mathbb{C}, A \otimes B) \times KK^{ban}(A, \mathbb{C}) \to KK^{ban}(\mathbb{C}, B)$$

satisfies

$$[p] \circ (\alpha \otimes 1_B) = [q_0] - [q_1],$$

under the isomorphism $KK^{ban}(\mathbb{C}, B) \cong K_0(B)$. More generally, an entirely similar statement holds for idempotents $p \in M_m(\mathbb{C}) \otimes A \otimes B$.

Proof. Let j be the linear contractive map $j : L(H) \otimes B \to L_B(H \otimes B)$ and let $\widetilde{\widetilde{\varphi}}_i = j \circ \widetilde{\varphi}_i$, $i = 0, 1$. From the definition of the product (2) as described in (5), it follows that $[p] \circ (\alpha \otimes 1_B) = [\widetilde{\widetilde{\varphi}}_0(p), \widetilde{\widetilde{\varphi}}_1(p)]$. On the other hand, since j is contractive, we deduce from the hypotheses of the proposition that

$$\|\widetilde{\widetilde{e}}\| \le M, \quad \|\widetilde{\widetilde{\varphi}}_i(p)\| \le M, \quad i = 0, 1$$

$$\|[\widetilde{\widetilde{\varphi}}_i(p), \widetilde{\widetilde{e}}]\| < \theta(M, \varepsilon), \quad i = 0, 1$$

$$\|(1 - \widetilde{\widetilde{e}})(\widetilde{\widetilde{\varphi}}_0(p) - \widetilde{\widetilde{\varphi}}_1(p))(1 - \widetilde{\widetilde{e}})\| < \theta(M, \varepsilon).$$

These estimates enable us to apply Lemma 2.4 and obtain that

$$
\begin{aligned}
(8) \quad [\widetilde{\widetilde{\varphi}}_0(p), \widetilde{\widetilde{\varphi}}_1(p)] &= [\chi(\widetilde{\widetilde{e}}\,\widetilde{\widetilde{\varphi}}_0(p)\,\widetilde{\widetilde{e}}), \chi(\widetilde{\widetilde{e}}\,\widetilde{\widetilde{\varphi}}_1(p)\,\widetilde{\widetilde{e}})] \\
&= [j(\chi(\widetilde{e}\,\widetilde{\varphi}_0(p)\,\widetilde{e})), j(\chi(\widetilde{e}\,\widetilde{\varphi}_1(p)\,\widetilde{e}))] = [j(q_0), j(q_1)]
\end{aligned}
$$

in $KK^{ban}(\mathbb{C}, B)$. By [16, Thm. 1.2.8], the image of $[j(q_0), j(q_1)]$ in $K_0(B)$ is equal to $[q_0] - [q_1]$. $\qquad\square$

3. Group quasi-representations and index theorems

Let G be a discrete countable group. Consider the action of G on $\ell^1(G)$ by multiplication, by viewing G as a set of units of $\ell^1(G)$. Let $\widetilde{Y} \to Y$ be a principal G-bundle over a compact metrizable space Y. Let ℓ_Y be the "line bundle" $\widetilde{Y} \times_G \ell^1(G) \to Y$ induced by the diagonal action of G. One can cover Y by a finite family of open sets $(U_\alpha)_{\alpha \in I}$ in such a way that the bundle ℓ_Y is obtained by gluing the trivial bundles $U_\alpha \times \ell^1(G)$ via a constant cocycle $s_{\alpha\beta} \in G \subset \ell^1(G)$. Thus $s_{\alpha\beta}s_{\beta\gamma} = s_{\alpha\gamma}$ whenever $U_\alpha \cap U_\beta \cap U_\gamma \ne \emptyset$. If $(\chi_\alpha)_{\alpha \in I}$ are positive continuous functions with χ_α supported in U_α and such that $\sum_{\alpha \in I} \chi_\alpha^2 = 1$, then the finitely generated

projective right Banach $C(Y) \otimes \ell^1(G)$-module of continuous sections in ℓ_Y is represented by an idempotent

$$(9) \qquad P = \sum_{\alpha, \beta \in I} e_{\alpha\beta} \otimes \chi_\alpha \chi_\beta \otimes s_{\alpha\beta} \in M_m \otimes C(Y) \otimes \ell^1(G).$$

Here $m = |I|$ and $(e_{\alpha\beta})$ is the canonical matrix unit of $M_m(\mathbb{C})$.

Definition 3.1. *Let G be a discrete countable group and let B be a unital Banach algebra. Let $\mathcal{G} \subset G$ be a finite subset and let $\omega > 0$. A map $\pi : G \to B$ is called a $(\mathcal{G}, \omega)$-representation if it satisfies the following conditions: (i) $\pi(1) = 1$ and $\|\pi(s)\| \leq 1$ for all $s \in G$; (ii) $\|\pi(st) - \pi(s)\pi(t)\| < \omega$ for all $s, t \in \mathcal{G}$.*

If B is a unital C*-algebra, a map $\pi : G \to B$ is called a $(\mathcal{G}, \omega)$-unitary representation if it satisfies (i), (ii) from above and (iii)$\|\pi(s^{-1}) - \pi(s)^*\| < \omega$ for all $s \in \mathcal{G}$. Note that if $\omega < 1$ and $\{s, s^{-1}\} \subset \mathcal{G}$, then $\pi(s) \in GL(B)$.

One can extend π to a linear contraction $\pi : \ell^1(G) \to B$ by $\pi(\sum_{s \in G} \lambda_s s) = \sum_{s \in G} \lambda_s \pi(s)$.

Let B be a unital Banach algebra. An element $z \in KK(C(Y), \mathbb{C})$ gives rise to an element $z \otimes 1_B \in KK^{ban}(C(Y) \otimes B, B)$ which induces via the cap product (3) a map $K_0(C(Y) \otimes B) \to K_0(B)$, $x \mapsto x \circ (z \otimes 1_B)$. We work with the following ℓ^1-version of the assembly map:

$$(10) \qquad \mu : KK(C(Y), \mathbb{C}) \to K_0(\ell^1(G)), \qquad \mu(z) = [P] \circ (z \otimes 1_{\ell^1(G)}).$$

If $\pi : \ell^1(G) \to B$ is a unital contractive morphism of Banach algebras, then there is a commutative diagram

$$
\begin{array}{ccc}
K_0(C(Y) \otimes \ell^1(G)) \times KK(C(Y), \mathbb{C}) & \longrightarrow & K_0(\ell^1(G)) \\
{\scriptstyle (id \otimes \pi)_*} \downarrow & \quad\| & \downarrow {\scriptstyle \pi_*} \\
K_0(C(Y) \otimes B) \times KK(C(Y), \mathbb{C}) & \longrightarrow & K_0(B)
\end{array}
$$

The following result extends this property to quasi-representations (starting from scratch), see Definitions 2.1 and 3.1.

Theorem 3.2. *Let $z \in KK(C(Y), \mathbb{C})$ and let q_0, q_1 be idempotents in matrices over $\ell^1(G)$ such that $\mu(z) = [P] \circ (z \otimes 1_{\ell^1(G)}) = [q_0] - [q_1]$. Then there exist a finite subset $\mathcal{G} \subset G$ and $\omega > 0$ (which depend on the representatives P, q_0 and q_1), such that if B is a unital Banach algebra and $\pi : G \to B$ is any $(\mathcal{G}, \omega)$-representation, then*

$$\pi_\sharp(q_0) - \pi_\sharp(q_1) = (\mathrm{id}_{C(Y)} \otimes \pi)_\sharp(P) \circ (z \otimes 1_B).$$

With the convention that follows Definition 2.1, we could have written $\pi_\sharp(\mu(z))$ in place of $\pi_\sharp(q_0) - \pi_\sharp(q_1)$.

Proof. Given two pairs of idempotents q_0, q_1 and q'_0, q'_1 in matrices over $\ell^1(G)$ such that $[q_0] - [q_1] = [q'_0] - [q'_1]$, there exist $\mathcal{G}_0$ and $\omega_0 > 0$ such that if $\mathcal{G}_0 \subset \mathcal{G}$ and $\omega < \omega_0$, then for any $(\mathcal{G}, \omega)$-representation $\pi : G \to B$, one has $\pi_\sharp(q_0) - \pi_\sharp(q_1) = \pi_\sharp(q'_0) - \pi_\sharp(q'_1)$. Thus it suffices to prove the theorem for a pair of idempotents q_0, q_1 of our choice with $\mu(z) = [q_0] - [q_1]$.

Before we can exhibit $\mathcal{G}$ and ω we need to choose carefully three constants ε_1, ε_2 and η. Let us regard the numbers $\delta(M, \varepsilon)$ and $\theta(M, \varepsilon)$ given by the Lemma 2.3 and respectively Lemma 2.4 as functions of the variables M and ε. For $m > 0$ as in the definition of P and $0 < \varepsilon < 1/3$ we choose $0 < \varepsilon_1 < 1/2$ such that

$$\varepsilon_1 < \delta(m^2, \varepsilon). \tag{11}$$

Next we choose $\varepsilon_2 > 0$ such that

$$\varepsilon_2 < \min\left\{\varepsilon, \frac{1}{4}\theta(m^2 + 1, \varepsilon)\right\}. \tag{12}$$

Then we fix a number $0 < \eta < 1$ such that

$$\eta < \min\left\{\frac{1}{2m^3}\delta(m^2, \varepsilon), \ \frac{1}{m^2}\theta(m^2, \varepsilon_1), \ \frac{1}{2m^2}\theta(m^2 + 1, \varepsilon)\right\}. \tag{13}$$

We are going to describe the construction of the idempotents q_0 and q_1 that are convenient to work with. By the Cuntz-picture of the KK-groups [3], any element $z \in KK(C(Y), \mathbb{C})$ is represented by a pair of $*$-representations $\varphi_0, \varphi_1 : C(Y) \to L(H)$ such that $\varphi_0(a) - \varphi_1(a) \in K(H)$ for all $a \in C(Y)$. Correspondingly, we have Banach algebra morphisms

$$\widetilde{\varphi}_i = \mathrm{id}_m \otimes \varphi_i \otimes \mathrm{id}_{\ell^1(G)} : M_m \otimes C(Y) \otimes \ell^1(G) \longrightarrow M_m \otimes L(H) \otimes \ell^1(G).$$

By Voiculescu's theorem, φ_1 is a compact perturbation of a $*$-representation of $C(Y)$ which is a direct sum of one-dimensional representations. Therefore, there is an approximate unit $(e_n)_n$ of $K(H)$ consisting of projections such that $\lim_n \|[\varphi_1(a), e_n]\| = 0$ for all $a \in C(Y)$. Since $\varphi_0(a) - \varphi_1(a) \in K(H)$ for all $a \in C(Y)$, it follows that there is a projection $e \in M_r(\mathbb{C}) = L(\mathbb{C}^r) \subset L(H)$ such that

$$\| [\varphi_i(a), e] \| < \eta, \quad \| (1-e)(\varphi_0(a) - \varphi_1(a))(1-e) \| < \eta, \tag{14}$$

for all $a \in \{\chi_\alpha \chi_\beta; \alpha, \beta \in I\}$ and $i = 0, 1$.

Let $\widetilde{e} := 1_m \otimes e \otimes 1_{\ell^1(G)}$. Then $\|\widetilde{e}\| \leq 1$. Let us note that

$$\|\widetilde{\varphi}_i(P)\| = \| \sum_{\alpha, \beta \in I} e_{\alpha\beta} \otimes \varphi_i(\chi_\alpha \chi_\beta) \otimes s_{\alpha\beta}\| \leq m^2,$$

since $\|s_{\alpha\beta}\| \leq 1$. Next we verify that

(15) $\quad \|[\widetilde{\varphi}_i(P), \widetilde{e}]\| < m^2\eta, \quad \text{and} \quad \|(1 - \widetilde{e})(\widetilde{\varphi}_0(P) - \widetilde{\varphi}_1(P))(1 - \widetilde{e})\| < m^2\eta.$

Using (14) we have

$$\|[\widetilde{\varphi}_i(P), \widetilde{e}]\| = \| \sum_{\alpha,\beta \in I} e_{\alpha\beta} \otimes [\varphi_i(\chi_\alpha \chi_\beta), e] \otimes s_{\alpha\beta}\| < m^2\eta.$$

For the other estimate, we write $\|(1 - \widetilde{e})(\widetilde{\varphi}_0(P) - \widetilde{\varphi}_1(P))(1 - \widetilde{e})\|$ as

$$\| \sum_{\alpha,\beta \in I} e_{\alpha\beta} \otimes (1 - e)(\varphi_0(\chi_\alpha \chi_\beta) - \varphi_1(\chi_\alpha \chi_\beta))(1 - e) \otimes s_{\alpha\beta}\| < m^2\eta.$$

Since $m^2\eta < \theta(m^2, \varepsilon_1)$ by (13), it follows by Proposition 2.5 that if we set $x_i := \widetilde{e}\,\widetilde{\varphi}_i(P)\widetilde{e}$, then $Sp(x_i) \subset V(1/2)$ and $q_i := \chi(x_i) \in M_m \otimes M_r \otimes \ell^1(G)$ are idempotents such that

(16) $\qquad\qquad\qquad\qquad \|x_i - q_i\| < \varepsilon_1, \quad i = 0, 1,$

(17) $\qquad\quad \mu(z) = [P] \circ (z \otimes 1_{\ell^1(G)}) = [q_0] - [q_1] \in K_0(\ell^1(G)).$

We now fix these idempotents $q_i = \chi(x_i)$, $i = 0, 1$ and we will prove that if $\mathcal{G} = \{s_{\alpha\beta} : \alpha, \beta \in I\}$ and $\omega > 0$ is such that

(18) $\qquad\qquad \omega < \min\left\{\frac{1}{2m^3}\delta(m^2, \varepsilon), \ \frac{1}{m^3}\delta(m^2, \varepsilon_2)\right\},$

then $\mathcal{G}$ and ω satisfy the statement of the theorem.

First we want to compute $\pi_\sharp(q_0) - \pi_\sharp(q_1)$ where $\pi : \ell^1(G) \to B$ is the canonical extension of a unital map $\pi : G \to B$ which satisfies $\|\pi(s)\| \leq 1$ for all $s \in G$ and

(19) $\qquad\qquad \|\pi(st) - \pi(s)\pi(t)\| < \omega \quad \text{for all} \quad s, t \in \mathcal{G}.$

Let $\widetilde{\pi} := \mathrm{id}_m \otimes \mathrm{id}_r \otimes \pi : M_m \otimes M_r \otimes \ell^1(G) \to M_m \otimes M_r \otimes B$ and recall that

(20) $\qquad\quad x_i = \widetilde{e}\,\widetilde{\varphi}_i(P)\widetilde{e} = \sum_{\alpha,\beta \in I} e_{\alpha\beta} \otimes e\varphi_i(\chi_\alpha \chi_\beta)e \otimes s_{\alpha\beta}.$

Since $\widetilde{\pi}$ is contractive, $\|\widetilde{\pi}(x_i)\| \leq m^2$. Moreover, it follows from (16) and (11) that

(21) $\qquad\qquad\qquad \|\widetilde{\pi}(q_i) - \widetilde{\pi}(x_i)\| < \varepsilon_1 < \delta(m^2, \varepsilon).$

We are going to verify that

(22) $\qquad\qquad \|\widetilde{\pi}(x_i)^2 - \widetilde{\pi}(x_i)\| < \delta(m^2, \varepsilon), \quad i = 0, 1.$

To this purpose let us consider the following expressions:

$$U_i = \sum_{\alpha,\beta,\gamma} e_{\alpha\gamma} \otimes e\varphi_i(\chi_\alpha \chi_\beta)e\varphi_i(\chi_\beta \chi_\gamma)e \otimes \pi(s_{\alpha\beta})\pi(s_{\beta\gamma}),$$

$$V_i = \sum_{\alpha,\beta,\gamma} e_{\alpha\gamma} \otimes e\varphi_i(\chi_\alpha\chi_\beta)\varphi_i(\chi_\beta\chi_\gamma)e \otimes \pi(s_{\alpha\beta})\pi(s_{\beta\gamma}),$$

$$W_i = \widetilde{\pi}(x_i) = \sum_{\alpha,\beta,\gamma} e_{\alpha\gamma} \otimes e\varphi_i(\chi_\alpha\chi_\beta^2\chi_\gamma)e \otimes \pi(s_{\alpha\gamma}),$$

and observe that $\widetilde{\pi}(x_i)^2 - \widetilde{\pi}(x_i) = (U_i - V_i) + (V_i - W_i)$, that $\|U_i - V_i\| < m^3\eta$ by (14) and that $\|V_i - W_i\| < m^3\omega$ by (19). It follows from (13) and (18) that

$$\|\widetilde{\pi}(x_i)^2 - \widetilde{\pi}(x_i)\| \leq m^3\eta + m^3\omega < \delta(m^2, \varepsilon).$$

Thus we have verified the inequality (22).

Due to the estimates (21), (22) and $\|\widetilde{\pi}(x_i)\| \leq m^2$, we can apply Lemma 2.3 to deduce that for any map $\pi : G \to B$ as in the statement, we have

$$(23) \qquad \|\chi(\widetilde{\pi}(q_i)) - \widetilde{\pi}(x_i)\| < \varepsilon,$$

$$(24) \qquad \pi_\sharp(q_0) - \pi_\sharp(q_1) = [\chi(\widetilde{\pi}(q_0))] - [\chi(\widetilde{\pi}(q_1))].$$

In the second part of the proof we are going to compute $(\mathrm{id}_{C(Y)} \otimes \pi)_\sharp(P) \circ (z \otimes 1_B)$ by using again Lemma 2.3 and Proposition 2.5. The result of this calculation is given in (31). The proof will be completed by comparing the expressions from (24) and (31). If we set

$$x := (\mathrm{id}_m \otimes \mathrm{id}_{C(Y)} \otimes \pi)(P) = \sum_{\alpha,\beta\in I} e_{\alpha\beta} \otimes \chi_\alpha\chi_\beta \otimes \pi(s_{\alpha\beta}) \in M_m \otimes C(Y) \otimes B,$$

then $\|x\| \leq m^2$. Since

$$x^2 - x = \sum_{\alpha,\delta} e_{\alpha\delta} \otimes \sum_\beta \chi_\alpha\chi_\beta^2\chi_\delta \otimes \left(\pi(s_{\alpha\beta})\pi(s_{\beta,\delta}) - \pi(s_{\alpha\delta})\right),$$

we obtain from (18) and (19) the estimate

$$(25) \qquad \|x^2 - x\| \leq m^3\omega < \delta(m^2, \varepsilon_2).$$

By definition, $(\mathrm{id}_{C(Y)} \otimes \pi)_\sharp(P) = [h]$, where h is the idempotent $h = \chi(x)$ given by Lemma 2.3. Thus $(\mathrm{id}_{C(Y)} \otimes \pi)_\sharp(P) \circ (z \otimes 1_B) = [h] \circ (z \otimes 1_B)$, where

$$(26) \qquad \|h - x\| = \|\chi(x) - x\| < \varepsilon_2.$$

In order to compute $[h] \circ (z \otimes 1_B)$, we need to consider the maps

$$\Phi_i = \mathrm{id}_m \otimes \varphi_i \otimes \mathrm{id}_B : M_m \otimes C(Y) \otimes B \to M_m \otimes L(H) \otimes B.$$

Since φ_i is contractive, so is Φ_i and hence by (26) we obtain that

$$(27) \qquad \|\Phi_i(h) - \Phi_i(x)\| < \varepsilon_2, \quad i = 0, 1,$$

and hence $\|\Phi_i(h)\| \leq \|\Phi_i(x)\| + \varepsilon_2 < m^2 + 1$, $i = 0, 1$.

We are going to show that the idempotents $\Phi_i(h)$ and $E := 1_m \otimes e \otimes 1_B$ satisfy the assumptions of Proposition 2.5 with $\theta = \theta(m^2 + 1, \varepsilon)$. Thus we must verify that

$$\|[\Phi_i(h), E]\| < \theta(m^2+1, \varepsilon) \quad \text{and} \quad \|(1-E)(\Phi_0(h) - \Phi_1(h))(1-E)\| < \theta(m^2, \varepsilon).$$

We first estimate $\|[\Phi_i(x), E]\|$ and $\|(1 - E)(\Phi_0(x) - \Phi_1(x))(1 - E)\|$. Since

$$(28) \qquad \Phi_i(x) = \sum_{\alpha, \beta \in I} e_{\alpha\beta} \otimes \varphi_i(\chi_\alpha \chi_\beta) \otimes \pi(s_{\alpha\beta}),$$

using (14) we have

$$(29) \qquad [\Phi_i(x), E] = \| \sum_{\alpha, \beta \in I} e_{\alpha\beta} \otimes [e, \varphi_i(\chi_\alpha \chi_\beta)] \otimes \pi(s_{\alpha\beta})\| < m^2 \eta, \quad \text{and}$$

$$(30) \qquad \|(1 - E)(\Phi_0(x) - \Phi_1(x))(1 - E)\| < m^2 \eta,$$

as it becomes clear after writing it in the form

$$\| \sum_{\alpha, \beta \in I} e_{\alpha\beta} \otimes (1 - e)(\varphi_0(\chi_\alpha \chi_\beta) - \varphi_1(\chi_\alpha \chi_\beta))(1 - e) \otimes \pi(s_{\alpha\beta})\|.$$

From (27), (29) and (30) we deduce that $\|(1 - E)(\Phi_0(h) - \Phi_1(h))(1 - E)\|$ and $\|[\Phi_i(h), E]\|$, $i = 0, 1$ are dominated by $m^2 \eta + 2\varepsilon_2$ and hence they are smaller than $\theta(m^2 + 1, \varepsilon)$ by (12) and (13). Applying Proposition 2.5 we obtain that

$$(31) \qquad [h] \circ (z \otimes 1_B) = [\chi(E\Phi_0(h)E)] - [\chi(E\Phi_1(h)E)],$$

as well as the estimate

$$(32) \qquad \|\chi(E\Phi_i(h)E) - E\Phi_i(h)E\| < \varepsilon.$$

On the other hand, from (26) and (32) we see that

$$(33) \qquad \|\chi(E\Phi_i(h)E) - E\Phi_i(x)E\| < \varepsilon + \varepsilon_2 < 2\varepsilon.$$

It is important to note that $E\Phi_i(x)E = \tilde{\pi}(x_i)$, as it follows by comparing (20) and (28). Consequently, from (23) and (33) we deduce that

$$\|\chi(\tilde{\pi}(q_i)) - \chi(E\Phi_i(h)E)| < 3\varepsilon < 1.$$

This shows that $[\chi(\tilde{\pi}(q_i))] = [\chi(E\Phi_i(h)E)]$ in $K_0(B)$, $i = 0, 1$. Therefore from (24) and (31) we obtain that $[h] \circ (z \otimes 1_B) = \pi_\sharp(q_0) - \pi_\sharp(q_1)$. $\qquad \square$

Let G be a discrete countable group and let $j : \ell^1(G) \to C^*(G)$ be the canonical $*$-homomorphism.

Lemma 3.3. *Let $x, y \in K_0(\ell^1(G))$ be such that $j_*(x) = j_*(y)$ in $K_0(C^*(G))$. Then there exist a finite subset $\mathcal{G}$ of G and $\omega > 0$ with the property that for any unital C*-algebra B and any $(\mathcal{G}, \omega)$-unitary representation, it follows that $\pi_\sharp(x) = \pi_\sharp(y)$.*

Proof. The definitions of $\pi_\sharp(x)$, $\pi_\sharp(y)$ depend on the choices of representatives for x and y as as explained in the comments following Definition 2.1. The set $\mathcal{G}$ and $\omega > 0$ will also depend on these representatives.

For any given $(\mathcal{G}, \omega)$-unitary representation $\pi : G \to B$ we will denote also by π its canonical linear extension $\pi : \ell^1(G) \to B$.

Fix idempotents $e_i \in M_k(\ell^1(G))$, $i = 1, 2, 3, 4$ such that $x = [e_1] - [e_2]$ and $y = [e_3] - [e_4]$. Seeking a contradiction, assume that the conclusion of the lemma is false. Then there exists a sequence of $(\mathcal{G}_n, \omega_n)$-unitary representations $\pi_n : G \to U(B_n)$ such that $\cup_n \mathcal{G}_n = G$, $0 < \omega_n \leq 1/n$ with $\|x_n(i)^2 - x_n(i)\| < 1/4$, where $x_n(i) = (id_k \otimes \pi_n)(e_i)$, $i = 1, 2, 3, 4$, and such that $\pi_{n\sharp}(x) \neq \pi_{n\sharp}(y)$, or equivalently $[\chi(x_n(1))] - [\chi(x_n(2))] \neq [\chi(x_n(3))] - [\chi(x_n(4))]$ in $K_0(B_n)$ for all $n \geq 1$. We can assemble the sequence (π_n) to form a linear contractive map $\Phi : \ell^1(G) \to \prod_n B_n$. Let $q : \prod_n B_n \to B_\infty := \prod_n B_n / \sum_n B_n$ be the quotient map. Then $q \circ \Phi$ is a unital $*$-homomorphism. It follows that $q \circ \Phi$ factors through the universal group C*-algebra $C^*(G)$. In other words, there is a unital $*$-homomorphism $\Psi : C^*(G) \to B_\infty$ such that $q \circ \Phi = \Psi \circ j$. Since $j_*(x) = j_*(y)$ in $K_0(C^*(G)$ by assumption, it follows that $(q \circ \Phi)_*(x) = (q \circ \Phi)_*(y)$ in $K_0(B_\infty)$. Let us note that by naturality of functional calculus, the idempotent $E_i = (\chi(x_n(i)))_n \in M_k(\prod_n B_n)$ is a lifting of $(id_k \otimes (q \circ \Phi))(e_i)$. It follows that $q_*([E_1] - [E_2]) = q_*([E_3] - [E_4])$ and hence $[E_1] - [E_2]$ and $[E_3] - [E_4]$ have the same image under the map

$$K_0(\prod_n B_n) \to \prod_n K_0(B_n) / \sum_n K_0(B_n).$$

Therefore the sequences $([\chi(x_n(1))] - [\chi(x_n(2))])_n$ and $([\chi(x_n(3))] - [\chi(x_n(4))])_n$ could differ only by an element of $\sum_n K_0(B_n)$. This implies that $\pi_{n\sharp}(x) = \pi_{n\sharp}(y)$ for all sufficiently large n, leading to the desired contradiction. $\square$

Definition 3.4. *Let G be a discrete countable group. Let $\widetilde{Y} \to Y$ be a principal G-bundle over a compact metrizable space Y. Following Lafforgue, we consider the map $\mu_{\ell^1} : K_0(Y) \to K_0(\ell^1(G))$ which is defined as follows. Let $p \in \ell^1(G, C_0(\widetilde{Y}))$ be an idempotent constructed in the usual way from a cutoff function c, i.e. $p(s)(y) = \sqrt{c(y)c(sy)}$, $s \in G$, $y \in \widetilde{Y}$. Let $j_G : KK_G^{ban}(C_0(\widetilde{Y}), \mathbb{C}) \to KK^{ban}(\ell^1(G, C_0(\widetilde{Y})), \ell^1(G))$ be the descent map of [16]. Let $z' \in KK_G^{ban}(C_0(\widetilde{Y}), \mathbb{C})$ be the image of $z \in K_0(Y) = KK(C(Y), \mathbb{C})$*

under the composition of the maps

$$KK(C(Y), \mathbb{C}) \cong KK_G(C_0(\widetilde{Y}), \mathbb{C}) \to KK_G^{ban}(C_0(\widetilde{Y}), \mathbb{C}).$$

Then by definition $\mu_{\ell^1}(z) = [p] \circ j_G(z')$.

We do not know whether or not μ coincide with μ_{ℓ^1}. We show that if B is a C*-algebra and if π is a sufficiently multiplicative quasi-representation of G in B, then one can replace μ by μ_{ℓ^1} in Theorem 3.2.

Corollary 3.5. *Let $z \in K_0(Y)$ and let q_0, q_1 be idempotents in matrices over $\ell^1(G)$ such that $\mu_{\ell^1}(z) = [q_0] - [q_1]$. Then there exist a finite subset $\mathcal{G} \subset G$ and $\omega > 0$ such that if B is a unital C*-algebra and $\pi : G \to B$ is any $(\mathcal{G}, \omega)$-unitary representation, then*

$$\pi_\sharp(\mu_{\ell^1}(z)) = \pi_\sharp(q_0) - \pi_\sharp(q_1) = (\mathrm{id}_{C(Y)} \otimes \pi)_\sharp(P) \circ (z \otimes 1_B).$$

Proof. We use the fact that the natural map $K_0(Y) \to K_0(C^*(G))$ (this is called the assembly map if $\widetilde{Y} \to Y$ is the universal covering of Y) has two equivalent descriptions μ' and μ''. The definition of μ' is similar to (10):

$$(34) \qquad \mu' : KK(C(Y), \mathbb{C}) \to K_0(C^*(G)), \qquad \mu'(z) = [P] \circ (z \otimes 1_{C^*(G)}).$$

The definition of μ'' involves an idempotent $p \in C_0(\widetilde{Y}) \rtimes G$ (constructed from a cutoff function) and the image of $z \in KK(C(Y), \mathbb{C}) \cong KK_G(C_0(\widetilde{Y}), \mathbb{C})$ under the descent map $j_G : KK_G(C_0(\widetilde{Y}), \mathbb{C}) \to KK(C_0(\widetilde{Y}) \rtimes G, C^*(G))$.

$$(35) \qquad \mu'' : KK(C(Y), \mathbb{C}) \to K_0(C^*(G)), \qquad \mu''(z) = [p] \circ j_G(z).$$

It is known that $\mu'' = \mu'$, see [14], [26]. The definition of μ'' is similar to that of μ_{ℓ^1} and it is shown in [16] that if $j : \ell^1(G) \to C^*(G)$ is the canonical $*$-homomorphism, then $j_* \circ \mu_{\ell^1} = \mu''$. On the other hand by comparing equations (10) and (34) we see that $j_* \circ \mu = \mu'$ and hence $j_* \circ \mu_{\ell^1} = j_* \circ \mu$. It follows then by Lemma 3.3 that $\pi_\sharp(\mu_{\ell^1}(z)) = \pi_\sharp(\mu(z))$. We conclude the proof by applying Theorem 3.2. $\qquad \square$

Let M^n be a closed connected smooth manifold and let G be a discrete countable group. Let $f : M \to BG$ be a continuous map to the classifying space of G and let $\widetilde{M}$ be the pull-back of the universal cover $EG \to BG$. The diagonal action of G on $\widetilde{M} \times \ell^1(G)$ defines a flat bundle $\widetilde{M} \times_G \ell^1(G) \to M$, to which one can associate (as seen earlier (9)) an idempotent P in matrices over $C(M) \otimes \mathbb{C}[G] \subset C(M) \otimes \ell^1(G)$. The construction of the analytic index $Ind_{f,C^*(G)}(D)$ of an elliptic pseudodifferential operator D on M of Mishchenko and Kasparov has been refined by Connes and Moscovici and Lafforgue. The covering index of D associated to the cover $\widetilde{M}$, denoted by $Ind_{f,G}(D)$, was constructed in [6] as an element of

$K_0(\mathbb{C}[G] \otimes \mathcal{R})$, where $\mathcal{R}$ is the algebra of rapid decay infinite matrices. We denote its image in $K_0(\ell^1(G))$ by $Ind_{f,\ell^1(G)}(D)$. On the other hand the map $\mu_{\ell^1} : K_0(M) \to K_0(\ell^1(G))$ described in Definition 3.4 gives an element $\mu_{\ell^1}[D] \in K_0(\ell^1(G))$. It is known that the images of $Ind_{f,\ell^1(G)}(D)$ and $\mu_{\ell^1}[D]$ in $K_0(C^*(G))$ coincide and they are both equal to $Ind_{f,C^*(G)}(D)$ by [6], [4, Lemma 8, p.243] and [16]. If h is the K-theory class of an idempotent in matrices over $C(M)$ we denote by $Ind(D_h) = h \circ [D] \in \mathbb{Z}$ the index of D twisted by h given by the pairing $K^0(M) \times K_0(M) \to \mathbb{Z}$. The principal symbol of D is denoted by $\sigma(D)$.

Let B be a unital C*-algebra which has a tracial state τ. We shall use the Chern character $ch_\tau(E)$ defined for a smooth bundle E on M with fibers finitely generated projective modules over B, [20], [24].

Let us fix idempotents q_0, q_1 in $M_r(\ell^1(G))$ such that $\mu_{\ell^1}[D] = [q_0] - [q_1]$. Given a $(\mathcal{G}, \omega)$-unitary representation $\pi : G \to B$, where B is a unital C*-algebra, we will write $\pi_\sharp(\mu_{\ell^1}[D]) := \pi_\sharp(q_0) - \pi_\sharp(q_1)$. Fix a smooth finitely generated projective B-module bundle E_π over M such that

$$(36) \qquad\qquad [E_\pi] = (1 \otimes \pi)_\sharp(P)$$

in $K_0(C(M) \otimes B)$.

Theorem 3.6. *Let M^n be a closed connected smooth manifold and let $f : M \to BG$ be a continuous map to the classifying space of a discrete countable group G. Let D be an elliptic pseudodifferential operator on M. There exist a finite set $\mathcal{G} \subset G$ and $\omega > 0$ (which depend on the choice of the representatives q_0, q_1 and P) such that if B is a unital C*-algebra endowed with a tracial state τ and $\pi : G \to B$ is a $(\mathcal{G}, \omega)$-unitary representation, then*

$$\tau(\pi_\sharp(\mu_{\ell^1}[D])) = (-1)^n(ch(\sigma(D)) \cup Td(TM \otimes \mathbb{C}) \cup ch_\tau(E_\pi)[T^*M].$$

Proof. We apply Corollary 3.5 with $Y = M$ and $z = [D]$ to obtain that

$$\pi_\sharp(q_0) - \pi_\sharp(q_1) = (1 \otimes \pi)_\sharp(P) \circ ([D] \otimes 1_B) = [E_\pi] \circ ([D] \otimes 1_B) =: Ind(D_{E_\pi})$$

and hence $\tau(\pi_\sharp(q_0) - \pi_\sharp(q_1)) = \tau(Ind(D_{E_\pi}))$. The latter expression is computed by the cohomological version of the Mishchenko-Fomenko index theorem, see [20], [24, Thm. 5.9]:

$$\tau(Ind(D_E)) = (-1)^n(ch(\sigma(D)) \cup Td(TM \otimes \mathbb{C}) \cup ch_\tau(E))[T^*M]. \qquad \square$$

Remark 3.7. *Since the images of $Ind_{f,\ell^1(G)}(D)$, $\mu[D]$ and $\mu_{\ell^1}[D]$ coincide in $K_0(C^*(G))$, they are all equal to $Ind_{f,C^*(G)}(D)$, by Lemma 3.3 one may replace $\mu_{\ell^1}[D]$ by $Ind_{f,\ell^1(G)}(D)$ or by $\mu[D]$ in the statement of Theorem 3.6. Just as above, the definitions of $\pi_\sharp(Ind_{f,\ell^1(G)}(D))$ and $\pi_\sharp[D]$ involve representative idempotents e_i in matrices over $\ell^1(G)$ and $\mathcal{G} \subset G$ and*

$\omega > 0$ *depend on the choice of* e_0, e_1 *and* P. *A similar remark applies to* $Ind_{f,G}(D) \in K_0(\mathbb{C}[G] \otimes \mathcal{R})$ *since one can pair idempotents in matrices over* $\mathbb{C}[G] \otimes \mathcal{R}$ *with sufficiently multiplicative quasi-representations.*

Corollary 3.8. *If* M^n *is oriented, then by integration over the fibers of* $p :$ $TM \to M$ *one obtains under the assumptions and the setup of Theorem 3.6,*

$$\tau(\pi_\sharp(\mu_{\ell^1}[D]) = (-1)^{n(n+1)/2}(p_!ch(\sigma(D)) \cup Td(TM \otimes \mathbb{C}) \cup ch_\tau(E_\pi)[M].$$

Let M be a closed manifold with universal cover $\widetilde{M}$ and classifying map $f : M \to BG$. Let $P = P^{(M,f)}$ be a projection in matrices over $C(M) \otimes \mathbb{C}[G]$ that represents the Mishchenko line bundle $\widetilde{M} \times_G \ell^1(G) \to M$ constructed as above. The corresponding "assembly" map is denoted by $\mu^{(M,f)} : K_0(M) \to K_0(\ell^1(G))$. If π is a quasi-representation of G we define $E_\pi = E_\pi^{(M,f)}$ as in (36). The following is a variation of a result of Hilsum and Skandalis [12], see also [10]. In this context, similar results were obtained by Hanke and Schick [11] using sequences of almost-flat bundles. Let $\mathcal{L}(M)$ denote the stable L-class of M.

Corollary 3.9. *Let* M, N *be two closed connected oriented manifolds and let* $h : N \to M$ *be an oriented homotopy equivalence. There exist a finite subset* $\mathcal{G} \subset G$ *and* $\varepsilon > 0$ *such that for any* $(\mathcal{G}, \varepsilon)$-*unitary representation* $\pi : G \to B$ *into a unital C*-algebra* B *with a trace* τ, *one has* $(\mathcal{L}(N) \cup h^*ch_\tau(E_\pi))[N] = (\mathcal{L}(M) \cup ch_\tau(E_\pi))[M]$.

Proof. Let D_M be the signature operator of M. Then $Ind_{f,C^*(G)}(D_M)$ is an oriented homotopy invariant of M by results of Mishchenko and Kasparov, and hence $Ind_{f,C^*(G)}(D_M) = Ind_{f \circ h, C^*(G)}(D_N)$. By Lemma 3.3, $\pi_\sharp(\mu^{(M,f)}[D_M]) = \pi_\sharp(\mu^{(N,f \circ h)}[D_N])$ if π is sufficiently multiplicative. On the other hand, $h^*(P^{(M,f)}) = P^{(N,f \circ h)}$ and hence $h^*(E_\pi^{(M,f)}) \cong E_\pi^{(N,f \circ h)}$. We conclude the proof by applying Corollary 3.8. $\square$

Using Theorem 3.6 in the case when $G = \pi_1(M)$ and $B = M_m(\mathbb{C})$, one can give a new proof of the following theorem from [5], see also [21]. Let $\widetilde{M}$ be the universal cover of a closed connected Riemannian manifold M with classifying map $f : M \to BG$ and let D be as above.

Theorem 3.10 (Connes, Gromov and Moscovici). *There is* $\varepsilon_0 > 0$ *such that if* E *is any* ε-*flat smooth Hermitian vector bundle on* M *with* $\varepsilon < \varepsilon_0$ *and* π *is the associated quasi-representation of* $G = \pi_1(M)$, *then*

$$\pi_\sharp(Ind_{f,G}(D)) = (-1)^n(ch(\sigma(D)) \cup Td(TM \otimes \mathbb{C}) \cup ch(E))[T^*M].$$

Proof. We only give a sketch of the proof. The argument relies on the correspondence between almost flat vector bundles on M and finite dimensional quasi-representations of the fundamental group of M. By definition of ε-flatness, E is a Hermitian vector bundle over M which admits a metric preserving connection ∇ such that if $\Theta = \nabla^2$ is the curvature of ∇, then

$$\|(E, \nabla)\| := \sup\{\|\Theta_x(X, Y)\| \; : \; \|X \wedge Y\| \leq 1, \, X, Y \in T_x M, \, x \in M\} < \varepsilon.$$

One constructs a quasi-representation $\pi : G \to L(\mathbb{C}^m)$ using parallel transport as in [5], where m is the rank of E. By Lemma 4 of [5], for any finite subset $\mathcal{G} \subset G$, there is $C > 0$ such that $\|\pi(st) - \pi(s)\pi(t)\| \leq C\|(E, \nabla)\|$ for all couples (E, ∇) and $s, t \in \mathcal{G}$. Therefore, for any given $\omega > 0$, π will be a $(\mathcal{G}, \omega)$-unitary representation provided that ε is sufficiently small. Conversely, for any given $\varepsilon > 0$, there exist a finite set $\mathcal{G}$ of G and $\omega > 0$ such that one can associate an ε-flat vector bundle E_π to any $(\mathcal{G}, \omega)$-unitary representation of G via a construction of [23, Sect. 3, and Appendix], see also [19]. Moreover, one verifies that in agreement with our earlier notation one has in fact that $[E_\pi] = (1 \otimes \pi)_\sharp(P)$, where P is the projection corresponding to the $\widetilde{M} \times_G \ell^1(G) \to M$. As noted in the remark on page 312 of [25], if a $(\mathcal{G}, \omega)$-unitary representation π is associated to an ε-flat bundle E, one can take E as a particular model of E_π ($E_\pi \cong E$). One concludes the proof by applying Theorem 3.6 and Remark 3.7 for the quasi-representation π associated to E. $\square$

In order to derive very explicit formulas from Corollary 3.8 it would be desirable to compute $ch_\tau(E_\pi)$. In the next section we work out this computation for a Riemaniann surface of genus $n > 0$ in the scalar case $B = M_k(\mathbb{C})$.

4. Quasi-representations of surface groups

Given two unitaries $u, v \in U(m)$ such that $\|uv - vu\| < \varepsilon$, Loring has introduced in [17] an integral invariant $\kappa(u, v)$ defined for small ε. One can view the pair u, v as associated with a quasi-representation $\varphi : C(\mathbb{T}^2) \to M_m(\mathbb{C})$, such that $\varphi(z_1) = u$ and $\varphi(z_2) = v$, where z_i are the canonical unitary generators of $C(\mathbb{T}^2)$. Then one defines $\kappa(u, v) := \varphi_\sharp(\beta)$, where $K_0(C(\mathbb{T}^2)) \cong \mathbb{Z} \oplus \mathbb{Z}\beta$ and β is the Bott element. On the other hand, Loring and Exel [9] introduced another invariant $\omega(u, v)$ defined as the winding number in $\mathbb{C} \setminus \{0\}$ of the loop $t \mapsto det((1 - t)uv + tvu)$ and they proved the equality $\kappa(u, v) = \omega(u, v)$. Exel pointed out that $\omega(u, v) = -\frac{1}{2\pi i}Tr \log(uvu^{-1}v^{-1})$ and reproved the equality $\kappa(u, v) = \omega(u, v)$ using the soft torus C*-algebra, see [8].

In this section we use the results from Section 3 to generalize the result of Loring and Exel as follows. Two almost commuting unitaries can be viewed as corresponding to a quasi-representation of $\mathbb{Z}^2 = \pi_1(\mathbb{T}^2)$. More generally, we are going to consider quasi-representations of surface groups, $\Gamma_n = \pi_1(Y_n)$, where Y_n is a compact oriented Riemann surface of genus $n \geq 1$. It is well-known that Γ_n has a standard presentation as a one-relator group

$$\Gamma_n = \langle s_1, t_1, ..., s_n, t_n \, ; \, \prod_{i=1}^{n} [s_i, t_i] \, \rangle,$$

(where $[s,t] = sts^{-1}t^{-1}$) which we will review in the sequel. But first let us note the following consequence of Corollary 3.5 (or Theorem 3.6).

The assembly map $\mu_{\ell^1} : K_0(Y_n) \to K_0(\ell^1(\Gamma_n))$ (to be denoted in the remainder of this section by μ) is an isomorphism [16], since Γ_n has the Haagerup property. Thus $K_0(\ell^1(\Gamma_n)) \cong \mathbb{Z} \oplus \mathbb{Z}\mu[Y_n]$ where $z = [Y_n]$ is the fundamental class of Y_n in K-homology. It is known that $[Y_n] = [\bar{\partial}_n] + (n - 1)[1]$, where $\bar{\partial}_n$ is the Dolbeault operator for some (any) complex structure on Y_n and [1] denotes the class of a character $C(Y_n) \to \mathbb{C}$. Moreover, the Chern character of $[Y_n]$ is equal to the fundamental class of Y_n in $H_2(Y_n, \mathbb{Z})$, which we denote again by $[Y_n]$, see [2, Lemma 7.9].

Corollary 4.1. *There exist a finite set $\mathcal{G} \subset \Gamma_n$ and $\omega > 0$ such that if $\pi : \Gamma_n \to U(m)$ is $(\mathcal{G}, \omega)$-representation, then $\pi_\sharp(\mu[Y_n]) = \langle c_1(E_\pi), [Y_n] \rangle$.*

Proof. By Corollary 3.5 applied to $B = M_m(\mathbb{C})$ and $z = [Y_n]$ we have that

$$\pi_\sharp(\mu[Y_n]) = \langle [E_\pi], [Y_n] \rangle = \langle ch[E_\pi], ch[Y_n] \rangle = \langle c_1(E_\pi), [Y_n] \rangle. \qquad \square$$

Consider the free group $\mathbb{F}_{2n}$ with generators $\hat{s}_1, \hat{t}_1, ..., \hat{s}_n, \hat{t}_n$ and the canonical morphism $\rho : \mathbb{F}_{2n} \to \Gamma_n$, $\rho(\hat{s}_i) = s_i$, $\rho(\hat{t}_i) = t_i$. Let $\sigma : \Gamma_n \to \mathbb{F}_{2n}$ be a fixed set theoretic section of ρ with the property that $\sigma(s_i^{\pm 1}) = \hat{s}_i^{\pm 1}$ and $\sigma(t_i^{\pm 1}) = \hat{t}_i^{\pm 1}$. Using σ, we can associate to each representation $\hat{\pi} : \mathbb{F}_{2n} \to U(m)$ a quasi-representation $\pi : \Gamma_n \to U(m)$ defined by $\pi = \hat{\pi} \circ \sigma$.

Any set $\{u_1, v_1, ..., u_n, v_n\} \subset U(m)$ determines a unique representation $\hat{\pi} : \mathbb{F}_{2n} \to U(m)$ such that $\hat{\pi}(\hat{s}_i) = u_i$ and $\hat{\pi}(\hat{t}_i) = v_i$, $i = 1, ..n$. Define $\pi : \Gamma_n \to U(m)$ as above, $\pi = \hat{\pi} \circ \sigma$. It is then easy to verify that for any given finite subset $\mathcal{G}$ of Γ_n and any $\omega > 0$ there is $\varepsilon > 0$ such that if the unitaries $\{u_1, v_1, ..., u_n, v_n\} \subset U(m)$ satisfy $\| \prod_{i=1}^{n} [u_i, v_i] - 1 \| < \varepsilon$, then π is a $(\mathcal{G}, \omega)$-unitary representation. Let $h : \Gamma_n \to \Gamma_1$ be the homomorphism defined by $h(s_1) = s_1$, $h(t_1) = t_1$, $h(s_i) = 1$ and $h(t_i) = 1$ for $i \geq 2$. We denote also by h the induced map $h : Y_n \cong B\Gamma_n \to Y_1 \cong B\Gamma_1$. All the maps

in the following diagram are known to be bijections.

$$\begin{array}{ccc} K_0(Y_n) & \xrightarrow{\ \mu\ } & K_0(\ell^1(\Gamma_n)) \\ \downarrow h_* & & \downarrow h_* \\ K_0(Y_1) & \xrightarrow{\ \mu\ } & K_0(\ell^1(\Gamma_1)) \end{array}$$

Let $\beta \in K_0(\ell^1(\Gamma_n))$ be the unique element with the property that the Chern character of image of $h_*(\beta)$ under the canonical map $K_0(\ell^1(\Gamma_1)) \to K_0(C^*(\Gamma_1)) \cong K^0(\mathbb{T}^2)$ is equal to the fundamental class $[\mathbb{T}^2] \in H^2(\mathbb{T}^2)$. The element β is equal to $\mu[Y_n]$ up to sign, so that $K_0(\ell^1(\Gamma_n)) \cong \mathbb{Z} \oplus \mathbb{Z}\beta$.

We will apply Corollary 4.1 to obtain the following formula.

Theorem 4.2. *There exists $\epsilon > 0$ such that for any $m \geq 1$ and any set of unitaries $\{u_1, v_1, ..., u_n, v_n\} \subset U(m)$ with $\| \prod_{i=1}^n [u_i, v_i] - 1 \| < \varepsilon$,*

$$\pi_\sharp(\beta) = -\frac{1}{2\pi i} \operatorname{Tr} \log([u_1, v_1] \cdots [u_n, v_n]).$$

The proof requires some preparation. Let us view Y_n as obtained by identifying pairs of edges in a $4n$-gon Q_n in the hyperbolic plane, as illustrated in Figure 1. We follow closely the presentation of this construction given in [13]. An edge of Q_n is an oriented side of Q_n. For each edge e we denote by e^{-1} the result of changing the orientation. The polygon Q_n comes with a side pairing. This is an involution $e \mapsto *e$ on the set of edges $\mathcal{E}_n$ of Q_n which satisfies $*(e^{-1}) = (*e)^{-1}$. For each edge $e \in \mathcal{E}_n$ we denote by τ_e the isometry of the hyperbolic plane which maps $*e$ to e in such a way that locally the half-plane bounded by $*e$ and containing Q_n is mapped to the half-plane bounded by e but opposite Q_n. One refers to τ_e as the transformation determined by e and one has $\tau_{e^{-1}} = \tau_e$ and $\tau_{*e} = \tau_e^{-1}$. We denote by $\mathcal{E}_n = \{*a_1, a_1, *b_1, b_1, ..., *a_n, a_n, *b_n, b_n\}$ the set of edges of Q_n oriented as shown in Figure 1 and set $\alpha_k = \tau_{a_k}$, $\beta_k = \tau_{b_k}$. It is shown in [13] that the group Γ_n generated by these isometries is isomorphic to the one-relator group

$$\Gamma_n = \langle \alpha_1, \beta_1, ..., \alpha_n, \beta_n : [\beta_1^{-1}, \alpha_1] \cdots [\beta_n^{-1}, \alpha_n] \rangle.$$

Thus s_i corresponds to β_i^{-1} and t_i corresponds to α_i from our earlier presentation of Γ_n.

We construct orientation preserving parametrizations for all edges as follows. First, we fix orientation preserving homeomorphisms $[0, 1] \to *a_k$ and $[0, 1] \to *b_k$, $1 \leq k \leq n$. Then, by composing these maps with α_k and β_k we obtain orientation preserving parametrizations for a_k and b_k. It is convenient to use the same notation for a point $t \in [0, 1]$ and its images $t \in *a_k$ and respectively $t \in *b_k$.
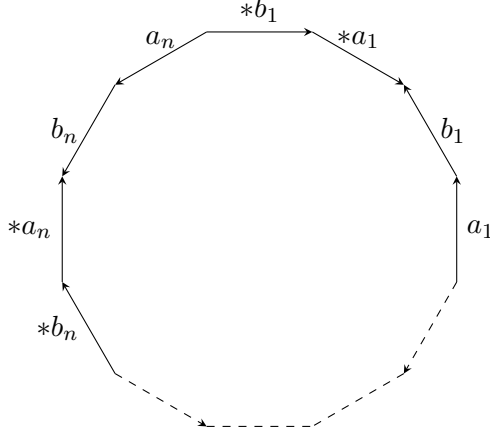
FIGURE 1. 4n-gon Q_n

Let $u_k, v_k \in U(m)$, $k = 1, ..., n$ with the property that

$$\|\prod_{k=1}^{n}[v_k^{-1}, u_k] - 1_m\| < \varepsilon < 1.$$

Set $W_0 = 1_m$ and $W_k := \prod_{i=1}^{k}[v_i^{-1}, u_i]$, $1 \le k \le n$. We are going to construct a complex vector bundle E of rank m over the surface Y_n that is a model for E_π. This bundle is obtained by gluing together pairs of fibers $(t \times \mathbb{C}^m)$, $(\tau_e t \times \mathbb{C}^m)$, $t \in *e$, on the edges of the base Q_n of the trivial bundle $Q_n \times \mathbb{C}^m$ according to the following equivalence relation.

(1) $(t, \xi) \sim (\beta_k t, v_k \xi)$, $t \in *b_k$, $\xi \in \mathbb{C}^m$, $1 \le k \le n$.
(2) $(t, \xi) \sim (\alpha_k t, u_k \xi)$, $t \in *a_k$, $\xi \in \mathbb{C}^m$, $1 \le k \le n - 1$.
(3) $(t, \xi) \sim (\alpha_n t, y_n(t)\xi)$, $t \in *a_n$, $\xi \in \mathbb{C}^m$, where

$$y_n(t) = (t1_m + (1 - t)W_n)u_n.$$

Let us note also that $\|u_n - y_n(t)\| < \varepsilon$ since $\|1_m - W_n\| < \varepsilon$. One verifies immediately that $\sim$ is indeed an equivalence relation. This property would not hold for $y_n = u_n$ if $W_n \ne 1$. To explain why we consider the bundle E, let us note that the projection P from Theorem 3.2 corresponds to the bundle on Y_n obtained as the quotient of $Q_n \times \ell^1(\Gamma_n)$ by the equivalence relation:

(1') $(t, \xi) \sim (\beta_k t, \beta_k \xi)$, $t \in *b_k$, $\xi \in \ell^1(\Gamma_n)$, $1 \le k \le n$.
(2') $(t, \xi) \sim (\alpha_k t, \alpha_k \xi)$, $t \in *a_k$, $\xi \in \ell^1(\Gamma_n)$, $1 \le k \le n - 1$.
(3') $(t, \xi) \sim (\alpha_n t, \alpha_n \xi)$, $t \in *a_n$, $\xi \in \ell^1(\Gamma_n)$.

Using a standard perturbation argument, one verifies that if ε is sufficiently small, then the projection associated to $(1 \otimes \pi)(P)$ corresponds to a vector

bundle isomorphic to E and hence $[E] = [E_\pi]$. Indeed, this is seen by comparing the equivalence relation defined by (1),(2), (3) with the one defined by (1'), (2'), (3'). If ε is sufficiently small, then $(1 \otimes \pi)_\sharp(P) = [E]$, since $\pi(\beta_k) = v_k$, $1 \le k \le n$, $\pi(\alpha_k) = u_k$, $1 \le k \le n-1$ and $\|\pi(\alpha_n) - y_n(t)\| = \|u_n - y_n(t)\| < \varepsilon$. It follows that $c_1((1 \otimes \pi)_\sharp(P)) = c_1(E_\pi) = c_1(E)$.

We compute $c_1(E)$, by realizing it as the first obstruction for the existence of a cross-section in the principal $GL(m,\mathbb{C})$-bundle F associated to E.

Proposition 4.3. *The first Chern class $c_1(E) \in H^2(Y_n, \mathbb{Z}) \cong \mathbb{Z}$ is given (up to a sign) by the winding number of $t \mapsto det(t1_m + (1-t)W_n)$ and hence it is equal to $\pm\frac{1}{2\pi i}Tr(\log W_n)$.*

Proof. We begin by constructing a cross-section Φ in F over the 1-skeleton of Y_n. Choose continuous paths $t \mapsto V_k(t)$ and $t \mapsto U_k(t)$, $t \in [0,1]$ in $U(m)$ such that $V_k(0) = U_k(0) = 1_m$ and $V_k(1) = v_k$, $U_k(1) = u_k$. The map Φ will given by a continuous map $\phi : \mathcal{E}_n \to U(m)$, defined on the boundary $\mathcal{E}_n$ of Q_n as follows.

For $1 \le k \le n$ and $t \in *b_k$, $\beta_k t \in b_k$,

$$\phi(t) = V_k(t)^{-1}U_k(t)^{-1}V_k(t)W_{k-1}^{-1}$$

and $\phi(\beta_k t) = v_k\phi(t) = V_k(1)V_k(t)^{-1}U_k(t)^{-1}V_k(t)W_{k-1}^{-1}$.

For $1 \le k \le n-1$ and $t \in *a_k$, $\alpha_k t \in a_k$,

$$\phi(t) = V_k(t)V_k(1)^{-1}U_k(1)^{-1}V_k(1)W_{k-1}^{-1}$$

and $\phi(\alpha_k t) = u_k\phi(t) = U_k(1)V_k(t)V_k(1)^{-1}U_k(1)^{-1}V_k(1)W_{k-1}^{-1}$.

For $t \in *a_n$ and $\alpha_n t \in a_n$,

$$\phi(t) = V_n(t)V_n(1)^{-1}U_n(1)^{-1}V_n(1)W_{n-1}^{-1}, \quad \text{and}$$

$$\phi(\alpha_n t) = y_n(t)\phi(t) = (t1_m + (1-t)W_n)U_n(1)V_n(t)V_n(1)^{-1}U_n(1)^{-1}V_n(1)W_{n-1}^{-1}.$$

One verifies that ϕ is well-defined and that $\phi(\beta_k t) = v_k\phi(t)$ for $t \in *b_k$, $1 \le k \le n$, $\phi(\alpha_k t) = u_k\phi(t)$ for $t \in *a_k$, $1 \le k \le n-1$, and $\phi(\alpha_n t) = y_n(t)\phi(t)$ for $t \in *a_n$. Since ϕ satisfies these relations, it defines a cross-section Φ in F over the 1-skeleton of Y_n. The first Chern class of E (or F) is represented by the obstruction to extending ϕ from the boundary of Q_n to Q_n. This obstruction lies in $H^2(Y_n, \pi_1(GL(m,\mathbb{C})) \cong \pi_1(GL(m,\mathbb{C})) \cong \mathbb{Z}$. The map $det : GL(m,\mathbb{C}) \to \mathbb{C} \setminus \{0\}$ induces an isomorphism of fundamental groups. The restriction of $det\,\phi(-)$ to four successive edges $*b_k, *a_k, b_k^{-1}, a_k^{-1}$, $1 \le k \le n-1$, gives a loop in $\mathbb{C} \setminus \{0\}$ based at 1. This loop is obtained by the concatenation of the paths $det(U_k(t)^{-1})$, $det(V_k(t)U_k(1)^{-1})$,

$det(V_k(1)U_k(1-t)^{-1})$ and $det(V_k(1-t))$. We note that this loop is trivial in $\pi_1(\mathbb{C} \setminus \{0\}, 1)$ via the homotopy of loops which for $s \in [0,1]$ is obtained by the concatenation of the paths $det(U_k(st)^{-1})$, $det(V_k(st)U_k(s)^{-1})$, $det(V_k(s)U_k(s(1-t))^{-1})$ and $det(V_k(s(1-t)))$ respectively. By a similar reasoning, the restriction of $det\,\phi(-)$ to the edges $*b_n, *a_n, b_n^{-1}, a_n^{-1}$ is homotopic to $t \mapsto det((1-t)1_m + tW_n)$. This shows that $c_1(E) = c_1(det(E))$ is given (up to a sign) by the winding number of $t \mapsto det(t1_m + (1-t)W_n)$. Finally, it is known and immediate that there is $\theta > 0$ with the following property. Let $w \in GL(m, \mathbb{C})$ be such that $\|w - 1\| < \theta$ and $det(w) = 1$. The winding number of $t \mapsto det((1-t)1_m + tw)$ is equal to $\frac{1}{2\pi i}Tr(\log w)$, where log is the branch of the natural logarithm with $\log 1 = 0$.

$\square$

Conclusion of proof of Theorem 4.2. Since we have seen that $[E_\pi] = [E]$ the result follows (up to a sign) from Corollary 4.1 and Proposition 4.3. In order to determine the appropriate sign it suffices to consider the case of a quasi-representation $\pi : \Gamma_n \to U(m)$ of the form $\pi = \pi' \circ h$, where $h : \Gamma_n \to \Gamma_1$ is the morphism and π' is a quasi-representation of Γ_1 described earlier, corresponding to Voiculescu's unitaries [27]. In other words it suffices to determine the sign in the case of the 2-torus. From the calculations done in [17] for Voiculescu's unitaries, one concludes that $\pi_\sharp(\beta) = -\frac{1}{2\pi i}Tr\log(uvu^{-1}v^{-1})$.

$\square$

Remark 4.4. *Another possible approach to the computation of $c_1(E_\pi)$ is to appeal to a formula from a survey paper of Manuilov and Mishchenko [19]. That entails to working with a triangularization of Y_n and becomes quite laborious. For instance, the minimum number of triangles in a triangulation of the torus is 14. Manuilov [18] has results on homotopy properties of quasi-representations of group surfaces that can be used to deduce Theorem 4.2 from the original Exel-Loring formula.*

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