UNIT SPECTRA OF K-THEORY
FROM STRONGLY SELF-ABSORBING C^*-ALGEBRAS

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ABSTRACT. We give an operator algebraic model for the first group of the unit spectrum $gl_1(KU)$ of complex topological K-theory, i.e. $[X, BGL_1(KU)]$, by bundles of stabilized infinite Cuntz C^*-algebras $\mathcal{O}_\infty \otimes \mathbb{K}$. We develop similar models for the localizations of KU at a prime p and away from p. Our work is based on the $\mathcal{I}$-monoid model for the units of K-theory by Sagave and Schlichtkrull and it was motivated by the goal of finding connections between the infinite loop space structure of the classifying space of the automorphism group of stabilized strongly self-absorbing C^*-algebras that arose in our generalization of the Dixmier-Douady theory and classical spectra from algebraic topology.

CONTENTS

1. INTRODUCTION

Suppose $E^\bullet$ is a multiplicative generalized cohomology theory represented by a commutative ring spectrum R. The units $GL_1(E^0(X))$ of $E^0(X)$ provide an abelian group functorially associated to the space X. From the point of view of algebraic topology it is therefore a natural question, whether we can lift GL_1 to spectra, i.e. whether there is a spectrum of units $gl_1(R)$ such that $gl_1(R)^0(X) = GL_1(E^0(X))$.

It was realized by Sullivan in [22] that $gl_1(R)$ is closely connected to questions of orientability in algebraic topology. In particular, the units of K-theory act on the K-orientations of PL-bundles. Segal [21] proved that the classifying space $\{1\} \times BU \subset \mathbb{Z} \times BU$ for virtual vector bundles of virtual dimension 1 equipped with the H-space structure from the tensor product is in fact a Γ-space, which in turn yields a spectrum of a connective generalized cohomology theory $bu_{\otimes}^*(X)$. His method is easily extended to include the virtual vector bundles of virtual dimension -1 to obtain a generalized cohomology theory $gl_1(KU)^*(X) \supset bu_{\otimes}^*(X)$ answering the above question affirmatively: $GL_1(K^0(X)) \cong gl_1(KU)^0(X)$. Later May, Quinn, Ray and Tornehave [15] came up with the notion of E_∞-ring spectra, which always have associated unit spectra.

Since $gl_1(R)$ is defined via stable homotopy theory, there is in general no nice geometric interpretation of the higher groups even though R may have one. In particular, no geometric interpretation was known for $gl_1(KU)^k(X)$. In this article we give an operator algebra interpretation of $gl_1(KU)^1(X)$ as the group of isomorphism classes of locally trivial bundles of C*-algebras with fiber isomorphic to the stable Cuntz algebra $\mathcal{O}_\infty \otimes \mathbb{K}$ with the group operation induced by the tensor product. In fact one can also recover Segal's original infinite loop space $BBU_\otimes$ as $B\mathrm{Aut}(\mathcal{Z} \otimes \mathbb{K})$, where $\mathcal{Z}$ is the ubiquitous Jiang-Su algebra [23]. For localizations of KU we obtain that $gl_1(KU_{(p)})^1(X)$ is the group of isomorphism classes of locally trivial bundles with fiber isomorphic to the C*-algebra $M_{(p)} \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ with the group operation induced by the tensor product. Here $M_{(p)}$ is a C*-algebra with $K_0(M_{(p)}) \cong \mathbb{Z}_{(p)}$, $K_1(M_{(p)}) = 0$ that can be obtained as an infinite tensor product of matrix algebras.

Our approach is based on the work of Sagave and Schlichtkrull [18, 19], who developed a representation of $gl_1(R)$ for a commutative symmetric ring spectrum R as a commutative $\mathcal{I}$-monoid. Motivated by the definition of twisted cohomology theories, we study the following situation, which appears to be a natural setup beyond the case where R is K-theory: Suppose G is an $\mathcal{I}$-space, such that each $G(\mathbf{n})$ is a topological group acting on R_n. To formulate a sensible compatibility condition between the group action κ and the multiplication μ^R on R, we need to demand that G itself carries an additional $\mathcal{I}$-monoid structure μ^G and the following diagram commutes:

$$
\begin{array}{ccc}
G(\mathbf{m}) \times R_m \times G(\mathbf{n}) \times R_n & \xrightarrow{\kappa_m \times \kappa_n} & R_m \times R_n \\
{\scriptstyle (\mu_{m,n}^G \times \mu_{m,n}^R) \circ \tau} \downarrow & & \downarrow {\scriptstyle \mu_{m,n}^R} \\
G(\mathbf{m} \sqcup \mathbf{n}) \times R_{m+n} & \xrightarrow{\kappa_{m+n}} & R_{m+n}
\end{array}
$$

where τ switches the two middle factors. Associativity of the group action suggests that the analogous diagram, which has $G(\mathbf{n})$ in place of R_n and μ^G instead of μ^R, should also commute. This condition can be seen as a homotopy theoretic version of the property needed for the Eckmann-Hilton trick, which is why we will call such a G an Eckmann-Hilton $\mathcal{I}$-group (EH-$\mathcal{I}$-group for short). Commutativity of the above diagram has the following important implications:

- the $\mathcal{I}$-monoid structure of G is commutative (Lemma 3.2),
- the classifying spaces $B_\nu G(\mathbf{n})$ with respect to the group multiplication ν of G form a commutative $\mathcal{I}$-monoid $\mathbf{n} \mapsto B_\nu G(\mathbf{n})$,
- if G is convergent and $G(\mathbf{n})$ has the homotopy type of a CW-complex, then the Γ-spaces associated to G and $B_\nu G$ satisfy $B_\mu \Gamma(G) \simeq \Gamma(B_\nu G)$, where $B_\mu \Gamma(X)$ for a commutative $\mathcal{I}$-monoid X denotes the Γ-space delooping of $\Gamma(X)$ (Theorem 3.6).

Let $\Omega^\infty(R)^*(\mathbf{n})$ be the commutative $\mathcal{I}$-monoid with associated spectrum $gl_1(R)$. If G acts on R and the inverses of G with respect to both multiplicative structures μ^G and ν are compatible in the sense of Definition 3.1, then the action induces a map of Γ-spaces $\Gamma(G) \to \Gamma(\Omega^\infty(R)^*)$. This deloops to a map $B_\mu \Gamma(G) \to B_\mu \Gamma(\Omega^\infty(R)^*)$ and we give sufficient conditions for this to be a strict equivalence of (very special) Γ-spaces.

In the second part of the paper, we consider the EH-$\mathcal{I}$-group $G_A(\mathbf{n}) = \mathrm{Aut}((A \otimes \mathbb{K})^{\otimes n})$ associated to the automorphisms of a (stabilized) strongly self-absorbing C^*-algebra A. This class of C^*-algebras was introduced by Toms and Winter in [23]. It contains the algebras $\mathcal{O}_\infty$ and $M_{(p)}$ alluded to above as well as the Jiang-Su algebra $\mathcal{Z}$ and the Cuntz algebra $\mathcal{O}_2$. It is closed with respect to tensor product and plays a fundamental role in the classification theory of nuclear C^*-algebras.

For a strongly self-absorbing C^*-algebra A, $X \mapsto K_0(C(X) \otimes A)$ turns out to be a multiplicative cohomology theory. In fact, this structure can be lifted to a commutative symmetric ring spectrum KU^A along the lines of [10, 12, 7]. The authors showed in [5] that $B\mathrm{Aut}(A \otimes \mathbb{K})$ is an infinite loop space and the first space in the spectrum of a generalized cohomology theory $E_A^*(X)$ such that $E_A^0(X) \cong K_0(C(X) \otimes A)_+^\times$, in particular $E_{\mathcal{O}_\infty}^0(X) \cong GL_1(K^0(X))$, which suggests that $E_{\mathcal{O}_\infty}^*(X) \cong gl_1(KU)^*(X)$. In fact, we can prove:

Theorem 1.1. *Let $A \neq \mathbb{C}$ be a separable strongly self-absorbing C^*-algebra.*

(a) *The EH-$\mathcal{I}$-group G_A associated to A acts on the commutative symmetric ring spectrum $KU_\bullet^A$ inducing a map $\Gamma(G_A) \to \Gamma(\Omega^\infty(KU^A)^*)$.*

(b) *The induced map on spectra is an isomorphism on all homotopy groups π_n with $n > 0$ and the inclusion $K_0(A)_+^\times \to K_0(A)^\times$ on π_0.*

(c) *In particular $B\mathrm{Aut}(A \otimes \mathcal{O}_\infty \otimes \mathbb{K}) \simeq BGL_1(KU^A)$ and $gl_1(KU)^1(X) \cong \mathcal{B}un_X(A \otimes \mathcal{O}_\infty \otimes \mathbb{K})$, where the right hand side denotes the group of isomorphism classes of C^*-algebra bundles with fiber $A \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ with respect to the tensor product $\otimes$.*

We also compare the spectrum defined by the Γ-space $\Gamma(B_\nu G_A)$ with the one obtained from the infinite loop space construction used in [5] and show that they are equivalent. The group $gl_1(KU^A)^1(X)$ alias $E_A^1(X)$ is a natural receptacle for invariants of not necessarily locally trivial continuous fields of C^*-algebras with stable strongly self-absorbing fibers that satisfy a Fell condition. This provides a substantial extension of results by Dixmier and Douady with $gl_1(KU^A)$ replacing ordinary cohomology, [5]. The above theorem lays the ground for an operator algebraic interpretation of the "higher" twists of K-theory. Twisted K-theory as defined first by Donovan

and Karoubi [8] and later in increased generality by Rosenberg [17] and Atiyah and Segal [2] has a nice interpretation in terms of bundles of compact operators [17]. From the point of view of homotopy theory, it is possible to define twisted K-theory with more than just the $K(\mathbb{Z}, 3)$-twists [1, 14] and the present paper suggests an interpretation of these more general invariants in terms of bundles with fiber $\mathcal{O}_\infty \otimes \mathbb{K}$. We will pursue this idea in upcoming work.

2. Preliminaries

2.1. **Symmetric ring spectra, units and $\mathcal{I}$-spaces.** Since our exposition below is based on symmetric ring spectra and their units, we will recall their definition in this section. The standard references for this material are [11] and [13]. Let Σ_n be the symmetric group on n letters and let $S^n = S^1 \wedge \cdots \wedge S^1$ be the smash product of n circles. This space carries a canonical Σ_n-action. Define S^0 to be the two-point space. Let $\mathcal{T}op$ be the category of compactly generated Hausdorff spaces and denote by $\mathcal{T}op_*$ its pointed counterpart.

Definition 2.1. A *commutative symmetric ring spectrum $R_\bullet$* consists of a sequence of pointed topological spaces R_n for $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ with a basepoint preserving action by Σ_n together with a sequence of pointed equivariant maps $\eta_n \colon S^n \to R_n$ and a collection $\mu_{m,n}$ of pointed $\Sigma_m \times \Sigma_n$-equivariant maps

$$\mu_{m,n} \colon R_m \wedge R_n \to R_{m+n}$$

such that the following conditions hold:

(a) associativity: $\mu_{p+q,r} \circ (\mu_{p,q} \wedge \mathrm{id}_{R_r}) = \mu_{p,q+r} \circ (\mathrm{id}_{R_p} \wedge \mu_{q,r})$,

(b) compatibility: $\mu_{p,q} \circ (\eta_p \wedge \eta_q) = \eta_{p+q}$ and $\mu_{p,0} \circ (\mathrm{id}_{R_p} \wedge \eta_0) = \mathrm{id}_{R_p}$, $\mu_{0,q} \circ (\eta_0 \wedge \mathrm{id}_{R_q}) = \mathrm{id}_{R_q}$
where we identify $S^0 \wedge R_p$ and $R_p \wedge S^0$ with R_p via the canonical maps,

(c) commutativity: The following diagram commutes

$$
\begin{array}{ccc}
R_m \wedge R_n & \xrightarrow{\mu_{m,n}} & R_{m+n} \\
{\scriptstyle \mathrm{tw}} \downarrow & & \downarrow {\scriptstyle \tau_{m,n}} \\
R_n \wedge R_m & \xrightarrow{\mu_{n,m}} & R_{n+m}
\end{array}
$$

where tw is the flip map and $\tau_{m,n}$ is the block permutation exchanging the first m letters with the last n letters preserving their order.

In order to talk about units in a symmetric ring spectrum with respect to its graded multiplication $\mu_{m,n} \colon R_m \wedge R_n \to R_{m+n}$ we need to deal with homotopy colimits. Given a small (discrete) indexing category $\mathcal{J}$ and a functor $F \colon \mathcal{J} \to \mathcal{T}op$, i.e. a diagram in spaces, we define its *homotopy colimit* $\mathrm{hocolim}_{\mathcal{J}} F$ to be the geometric realization of its (topological) *transport category* $\mathcal{T}_F$ with

object space $\coprod_{j \in \mathrm{obj}(\mathcal{J})} F(j)$ and morphism space $\coprod_{j,j' \in \mathrm{obj}(\mathcal{J})} F(j) \times \hom_{\mathcal{J}}(j, j')$, where the source map is given by the projection to the first factor and the value of the target map on a morphism $(x, f) \in F(j) \times \hom_{\mathcal{J}}(j, j')$ is $F(f)(x)$ [25, Proposition 5.7].

To define inverses for the graded multiplication μ of R, we need a bookkeeping device that keeps track of the degree, i.e. the suspension coordinate. We follow the work of Sagave and Schlichtkrull [18, 19], in particular [19, section 2.2], which tackles this issue using $\mathcal{I}$-spaces and $\mathcal{I}$-monoids.

Definition 2.2. Let $\mathcal{I}$ be the category, whose objects are the finite sets $\mathbf{n} = \{1, \ldots, n\}$ and whose morphisms are injective maps. The empty set $\mathbf{0}$ is an initial object in this category. Concatenation $\mathbf{m} \sqcup \mathbf{n}$ and the block permutations $\tau_{m,n} \colon \mathbf{m} \sqcup \mathbf{n} \to \mathbf{n} \sqcup \mathbf{m}$ turn $\mathcal{I}$ into a symmetric monoidal category. An $\mathcal{I}$-*space* is a functor $X \colon \mathcal{I} \to \mathcal{T}op_*$.

Moreover, an $\mathcal{I}$-space X is called an $\mathcal{I}$-*monoid* if it comes equipped with a natural transformation $\mu \colon X \times X \to X \circ \sqcup$ of functors $\mathcal{I}^2 \to \mathcal{T}op_*$, i.e. a family of continuous maps

$$\mu_{m,n} \colon X(\mathbf{m}) \times X(\mathbf{n}) \to X(\mathbf{m} \sqcup \mathbf{n}) \ ,$$

which is associative in the sense that $\mu_{l,m+n} \circ (\mathrm{id}_{X(\mathbf{l})} \times \mu_{m,n}) = \mu_{l+m,n} \circ (\mu_{l,m} \times \mathrm{id}_{X(\mathbf{n})})$ for all $\mathbf{l}, \mathbf{m}, \mathbf{n} \in \mathrm{obj}(\mathcal{I})$ and unital in the sense that the diagrams

$$
\begin{array}{ccc}
X(\mathbf{0}) \times X(\mathbf{n}) & \xrightarrow{\mu_{0,n}} & X(\mathbf{n}) \\
\uparrow & \nearrow_{\mathrm{id}_{X(\mathbf{n})}} & \\
X(\mathbf{n}) & &
\end{array}
\qquad
\begin{array}{ccc}
X(\mathbf{n}) \times X(\mathbf{0}) & \xrightarrow{\mu_{n,0}} & X(\mathbf{n}) \\
\uparrow & \nearrow_{\mathrm{id}_{X(\mathbf{n})}} & \\
X(\mathbf{n}) & &
\end{array}
$$

commute for every $\mathbf{n} \in \mathrm{obj}(\mathcal{I})$, where the two upwards arrows are the inclusion with respect to the basepoint in $X(\mathbf{0})$. Likewise we call an $\mathcal{I}$-monoid X *commutative*, if

$$
\begin{array}{ccc}
X(\mathbf{m}) \times X(\mathbf{n}) & \xrightarrow{\mu_{m,n}} & X(\mathbf{m} \sqcup \mathbf{n}) \\
\mathrm{sw} \downarrow & & \downarrow \tau_{m,n*} \\
X(\mathbf{n}) \times X(\mathbf{m}) & \xrightarrow{\mu_{n,m}} & X(\mathbf{n} \sqcup \mathbf{m})
\end{array}
$$

commutes. We denote the homotopy colimit of X over $\mathcal{I}$ by $X_{h\mathcal{I}} = \mathrm{hocolim}_{\mathcal{I}}(X)$. If X is an $\mathcal{I}$-monoid, then $(X_{h\mathcal{I}}, \mu)$ is a topological monoid as explained in [19, p. 652]. We call X *grouplike*, if $\pi_0(X_{h\mathcal{I}})$ is a group with respect to the multiplication induced by the monoid structure.

Let $\mathcal{N}$ be the category associated to the directed poset $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Note that there is an inclusion functor $\mathcal{N} \to \mathcal{I}$, which sends a map $n \to m$ to the standard inclusion $\mathbf{n} \to \mathbf{m}$. This way, we can associate a space called the *telescope* to an $\mathcal{I}$-space X: $\mathrm{Tel}(X) := \mathrm{hocolim}_{\mathcal{N}} X$. If X is convergent, then $\mathrm{Tel}(X) \to X_{h\mathcal{I}}$ is a weak homotopy equivalence. Any space Y together with a continuous self-map $f \colon Y \to Y$ yields a functor $F \colon \mathcal{N} \to \mathcal{T}op$ with $F(n) = Y$, $F(n \to m) = f^{(m-n)}$, where $f^{(m-n)}$ denotes the composition of $m - n$ copies of f. We will denote the associated telescope by $\mathrm{Tel}(Y; f)$ or $\mathrm{Tel}(Y)$ if the map is clear.

As described in [19, section 2.3], a symmetric ring spectrum $R_\bullet$ yields an $\mathcal{I}$-monoid $\Omega^\infty(R)$ as follows: Let $\Omega^\infty(R)(\mathbf{n}) = \Omega^n(R_n)$ with basepoint η_n. A morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ uniquely defines a permutation $\bar{\alpha}\colon \mathbf{n} = \mathbf{l} \sqcup \mathbf{m} \to \mathbf{n}$, which is order preserving on the first l elements and given by α on the last m entries. Mapping $f \in \Omega^m(R_m)$ to

$$S^n \xrightarrow{\bar{\alpha}^{-1}} S^n \xrightarrow{\eta_l \wedge f} R_l \wedge R_m \xrightarrow{\mu_{l,m}} R_n \xrightarrow{\bar{\alpha}} R_n \ .$$

yields the functoriality with respect to injective maps. The monoid structure is induced by the multiplication of $R_\bullet$ as follows

$$\mu_{m,n}(f,g)\colon \ S^m \wedge S^n \xrightarrow{f \wedge g} R_m \wedge R_n \xrightarrow{\mu_{m,n}} R_{m+n}$$

for $f \in \Omega^m(R_m)$ and $g \in \Omega^n(R_n)$. If R is commutative, then $\Omega^\infty(R)$ is a commutative $\mathcal{I}$-monoid.

Let $\Omega^\infty(R)^*$ be the $\mathcal{I}$-monoid of units of R given as follows $\Omega^\infty(R)^*(\mathbf{n})$ is the union of those components of $\Omega^\infty(R)(\mathbf{n})$ that have stable inverses in the sense that for each $f \in \Omega^\infty(R)^*(\mathbf{n})$ there exists $g \in \Omega^\infty(R)(\mathbf{m})$, such that $\mu_{n,m}(f,g)$ and $\mu_{m,n}(g,f)$ are homotopic to the basepoint of $\Omega^\infty(R)(\mathbf{n} \sqcup \mathbf{m})$ and $\Omega^\infty(R)(\mathbf{m} \sqcup \mathbf{n})$ respectively. Define the space of units by $GL_1(R) = \operatorname{hocolim}_{\mathcal{I}}(\Omega^\infty(R)^*)$.

If R is commutative, the spectrum of units associated to the Γ-space $\Gamma(\Omega^\infty(R)^*)$ will be denoted by $gl_1(R)$. If R is convergent, then $\pi_0((\Omega^\infty(R)^*)_{h\mathcal{I}}) = GL_1(\pi_0(R))$, $\Gamma(\Omega^\infty(R)^*)$ is very special and $gl_1(R)$ is an Ω-spectrum.

3. Eckmann-Hilton $\mathcal{I}$-groups

As motivated in the introduction, we study the following particularly nice class of $\mathcal{I}$-monoids.

Definition 3.1. Let $\mathcal{G}rp_*$ be the category of topological groups (which we assume to be *well-pointed* by the identity element) and continuous homomorphisms. A functor $G\colon \mathcal{I} \to \mathcal{G}rp_*$ is called an *$\mathcal{I}$-group*. An $\mathcal{I}$-group G is called an *Eckmann-Hilton $\mathcal{I}$-group* (or EH-$\mathcal{I}$-group for short) if it is an $\mathcal{I}$-monoid in $\mathcal{G}rp_*$ with multiplication $\mu_{m,n}$, such that the following diagram of natural transformations between functors $\mathcal{I}^2 \to \mathcal{G}rp_*$ commutes,

$$
\begin{array}{ccc}
G(\mathbf{m}) \times G(\mathbf{m}) \times G(\mathbf{n}) \times G(\mathbf{n}) & \xrightarrow{(\mu_{m,n} \times \mu_{m,n}) \circ \tau} & G(\mathbf{m} \sqcup \mathbf{n}) \times G(\mathbf{m} \sqcup \mathbf{n}) \\
{\scriptstyle \nu_m \times \nu_n} \downarrow & & \downarrow {\scriptstyle \nu_{m+n}} \\
G(\mathbf{m}) \times G(\mathbf{n}) & \xrightarrow{\mu_{m,n}} & G(\mathbf{m} \sqcup \mathbf{n})
\end{array}
$$

(where $\nu_n\colon G(\mathbf{n}) \times G(\mathbf{n}) \to G(\mathbf{n})$ denotes the group multiplication and τ is the map that switches the two innermost factors). We call G *convergent*, if it is convergent as an $\mathcal{I}$-space in the sense of [19, section 2.2]. If all morphisms in $\mathcal{I}$ except for the maps $\mathbf{0} \to \mathbf{n}$ are mapped to homotopy equivalences, the EH-$\mathcal{I}$-group G is called *stable* (this implies convergence).

Let $\iota_m \colon \mathbf{0} \to \mathbf{m}$ be the unique morphism. We say that an EH-$\mathcal{I}$-group has *compatible inverses*, if there is a path from $(\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(g) \in G(\mathbf{m} \sqcup \mathbf{m})$ to $(\mathrm{id}_{\mathbf{m}} \sqcup \iota_m)_*(g)$ for all $\mathbf{m} \in \mathrm{obj}(\mathcal{I})$ and $g \in G(\mathbf{m})$.

The above diagram is easily recognized as a graded version of the Eckmann-Hilton compatibility condition, where the group multiplication and the monoid structure provide the two operations. Thus, the following comes as no surprise.

Lemma 3.2. *Let G be an EH-$\mathcal{I}$-group. Then the $\mathcal{I}$-monoid structure of G is commutative.*

Proof. Let $1_m \in G(\mathbf{m})$ be the identity element. If $\iota_m \colon \mathbf{0} \to \mathbf{m}$ is the unique morphism, then $\iota_{m*}(1_0) = 1_m$. For $g \in G(\mathbf{n})$

$$\mu_{m,n}(1_m, g) = \mu_{m,n}(\iota_{m*}(1_0), g) = (\iota_m \sqcup \mathrm{id}_{\mathbf{n}})_* \mu_{0,n}(1_0, g) = (\iota_m \sqcup \mathrm{id}_{\mathbf{n}})_*(g)$$

by naturality. Let $g \in G(\mathbf{n})$, $h \in G(\mathbf{m})$, then

$$\begin{aligned}
\mu_{m,n}(g, h) &= \mu_{m,n}(\nu_m(1_m, g), \nu_n(h, 1_n)) = \nu_{m+n}(\mu_{m,n}(1_m, g), \mu_{m,n}(h, 1_n)) \\
&= \nu_{m+n}((\iota_m \sqcup \mathrm{id}_{\mathbf{n}})_*(g), (\mathrm{id}_{\mathbf{m}} \sqcup \iota_n)_*(h)) \\
&= \nu_{m+n}(\tau_{m,n*}(\mathrm{id}_{\mathbf{n}} \sqcup \iota_m)_*(g), \tau_{m,n*}(\iota_n \sqcup \mathrm{id}_{\mathbf{m}})_*(h)) \\
&= \tau_{m,n*} \nu_{n+m}((\mathrm{id}_{\mathbf{n}} \sqcup \iota_m)_*(g), (\iota_n \sqcup \mathrm{id}_{\mathbf{m}})_*(h)) = \tau_{m,n*} \mu_{n,m}(h, g)
\end{aligned}$$

In the last step we have used the fact that $\tau_{m,n*}$ is a group homomorphism. $\square$

Lemma 3.3. *Let G be an EH-$\mathcal{I}$-group with compatible inverses, let $g \in G(\mathbf{m})$, then there is a path connecting $\mu_{m,m}(g, g^{-1}) \in G(\mathbf{m} \sqcup \mathbf{m})$ and $1_{m \sqcup m} \in G(\mathbf{m} \sqcup \mathbf{m})$.*

Proof. Just as in the proof of Lemma 3.2 we see that

$$\mu_{m,m}(g, g^{-1}) = \nu_{m+m}((\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(g), (\mathrm{id}_{\mathbf{m}} \sqcup \iota_m)_*(g^{-1})) \ .$$

But by assumption $(\mathrm{id}_{\mathbf{m}} \sqcup \iota_m)_*(g^{-1})$ is homotopic to $(\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(g^{-1})$. We get

$$\nu_{m+m}((\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(g), (\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(g^{-1})) = (\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(\nu_m(g, g^{-1})) = (\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(1_m)$$

proving the claim. $\square$

Let G be an EH-$\mathcal{I}$-group and let $\Delta_r \colon \mathcal{I} \to \mathcal{I}^r$ be the multidiagonal functor that maps $\mathbf{n}$ to $(\mathbf{n}, \dots, \mathbf{n}) \in \mathrm{obj}(\mathcal{I}^r)$. Then we obtain an $\mathcal{I}$-monoid $G^{(r)}$ from this via

$$G^{(r)}(\mathbf{n}) = \underbrace{(G \times \cdots \times G)}_{r \text{ times}} \circ \Delta_r(\mathbf{n}) = G(\mathbf{n})^r \ ,$$

where we define $G^{(0)}(\mathbf{n})$ to be the trivial group and $G^{(1)}(\mathbf{n}) = G(\mathbf{n})$. The multiplication is given by

$$\mu_{m,n}^{(r)} \colon G(\mathbf{m})^r \times G(\mathbf{n})^r \to (G(\mathbf{m}) \times G(\mathbf{n}))^r \to G(\mathbf{m} \sqcup \mathbf{n})^r$$

where the first map is reshuffling the factors in an order-preserving way and the second map is $\mu_{m,n} = \mu_{m,n}^{(1)}$. We can rephrase the fact that $\varphi_* \colon G(\mathbf{m}) \to G(\mathbf{n})$ is a group homomorphism for every

$\varphi \in \mathrm{mor}(\mathcal{I})$ by saying that $\nu_\bullet \colon (G \times G) \circ \Delta_2 \to G$ is a natural transformation. Thus, we obtain face maps of the form

$$d_{i,n} \colon G^{(r)}(\mathbf{n}) \to G^{(r-1)}(\mathbf{n}) \quad ; \quad (g_1, \ldots, g_r) \mapsto \begin{cases} (g_2, \ldots, g_r) & \text{if } i = 0 \\ (g_1, \ldots, \nu_n(g_i, g_{i+1}), \ldots, g_r) & \text{if } 0 < i < r \\ (g_1, \ldots, g_{r-1}) & \text{if } i = r \end{cases}$$

and corresponding degeneracy maps $s_{i,n} \colon G^{(r)} \to G^{(r+1)}$, which insert the identity of $G(\mathbf{n})$ after the ith element. Altogether, we see that $G^{(r)}$ is a simplicial $\mathcal{I}$-space. If we fix $\mathbf{n}$, then $G^{(\bullet)}(\mathbf{n})$ is a simplicial space. Its geometric realization is the classifying space of the group $G(\mathbf{n})$, which we will denote by $B_\nu G(\mathbf{n})$. Any morphism $\varphi \colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ induces a simplicial map $G^{(\bullet)}(\mathbf{m}) \to G^{(\bullet)}(\mathbf{n})$ and therefore a map on the corresponding classifying spaces: $B_\nu G(\mathbf{m}) \to B_\nu G(\mathbf{n})$. This way $\mathbf{n} \mapsto B_\nu G(\mathbf{n})$ becomes an $\mathcal{I}$-space, of which we can form the homotopy colimit $(B_\nu G)_{h\mathcal{I}}$.

Alternatively, we can first form the homotopy colimit of the r-simplices to obtain the simplicial space $G_{h\mathcal{I}}^{(r)} = \mathrm{hocolim}_{\mathcal{I}} G^{(r)}$. Let $B_\nu(G_{h\mathcal{I}}) := |G_{h\mathcal{I}}^{(\bullet)}|$. As the notation already suggests there is not much of a difference between these two spaces.

Lemma 3.4. *Let $\mathcal{J}$ be a small (discrete) category, let X be a simplicial $\mathcal{J}$-space, then $c \mapsto |X^{(\bullet)}(c)|$ is a $\mathcal{J}$-space, $r \mapsto \mathrm{hocolim}_{\mathcal{J}} X^{(r)}$ is a simplicial space and there is a homeomorphism $|\mathrm{hocolim}_{\mathcal{J}} X^{(\bullet)}| \cong \mathrm{hocolim}_{\mathcal{J}} |X^{(\bullet)}|$. In particular, we have $(B_\nu G)_{h\mathcal{I}} \cong B_\nu(G_{h\mathcal{I}}) =: B_\nu G_{h\mathcal{I}}$. Given another category $\mathcal{J}'$, a functor $F \colon \mathcal{J} \to \mathcal{J}'$, a simplicial $\mathcal{J}'$-space X' and a natural transformation $\kappa \colon X \Rightarrow X' \circ F$, the following diagram, in which the vertical maps are induced by F and κ, commutes*

$$\begin{array}{ccc} |\mathrm{hocolim}_{\mathcal{J}} X^{(\bullet)}| & \xrightarrow{\cong} & \mathrm{hocolim}_{\mathcal{J}} |X^{(\bullet)}| \\ \downarrow & & \downarrow \\ |\mathrm{hocolim}_{\mathcal{J}'} X'^{(\bullet)}| & \xrightarrow{\cong} & \mathrm{hocolim}_{\mathcal{J}'} |X'^{(\bullet)}| \end{array}$$

Proof. The first two statements are clear, since the face and degeneracy maps are maps of $\mathcal{J}$-spaces. In particular, the space $X^{(r)}$ of r-simplices is a $\mathcal{J}$-space. Let $\mathcal{C}_X^{(r)}$ be the transport category of $X^{(r)}$ and let $N_s \mathcal{C}_X^{(r)}$ be the sth space of the nerve of $\mathcal{C}_X^{(r)}$. This is a bisimplicial space and we have

$$|\mathrm{hocolim}_{\mathcal{J}} X^{(\bullet)}| = ||N_s \mathcal{C}_X^{(r)}|_s|_r \cong ||N_s \mathcal{C}_X^{(r)}|_r|_s \cong \mathrm{hocolim}_{\mathcal{J}} |X^\bullet| .$$

The functor F in combination with the natural transformation yield a map of bisimplicial spaces $N_s \mathcal{C}_X^{(r)} \to N_s \mathcal{C}_{X'}^{(r)}$. So the last claim follows from the fact that the homeomorphism $||N_s \mathcal{C}_X^{(r)}|_s|_r \cong ||N_s \mathcal{C}_X^{(r)}|_r|_s$ is natural in the bisimplicial space. $\square$

Lemma 3.5. *Let G be a stable EH-$\mathcal{I}$-group, such that $G(\mathbf{n})$ has the homotopy type of a CW-complex for each $\mathbf{n}$. The inclusion map $G(\mathbf{1})^m \to G_{h\mathcal{I}}^{(m)}$ induces a homotopy equivalence for every $m \in \mathbb{N}$. In particular, $B_\nu G(\mathbf{1}) \to B_\nu G_{h\mathcal{I}}$ is a homotopy equivalence.*

Proof. Since we assumed all maps $G(\mathbf{m}) \to G(\mathbf{n})$ to be homotopy equivalences for every $m \geq 1$, the Lemma follows from [19, Lemma 2.1] together with [21, Proposition A.1 (ii)]. $\qquad\square$

Due to the Eckmann-Hilton condition the following diagram commutes

$$
\begin{array}{ccc}
G^{(r)}(\mathbf{m}) \times G^{(r)}(\mathbf{n}) & \xrightarrow{\;\mu_{m,n}^{(r)}\;} & G^{(r)}(\mathbf{m} \sqcup \mathbf{n}) \\
{\scriptstyle d_{i,m} \times d_{i,n}} \Big\downarrow & & \Big\downarrow {\scriptstyle d_{i,m \sqcup n}} \\
G^{(r-1)}(\mathbf{m}) \times G^{(r-1)}(\mathbf{n}) & \xrightarrow[\;\mu_{m,n}^{(r-1)}\;]{} & G^{(r-1)}(\mathbf{m} \sqcup \mathbf{n})
\end{array}
$$

Thus, $G^{(\bullet)}$ is a simplicial $\mathcal{I}$-monoid, which is commutative by Lemma 3.2. This implies that $\mathbf{n} \mapsto B_\nu G(\mathbf{n})$ is a commutative $\mathcal{I}$-monoid. Let $B_\nu G^{(s)} \colon \mathcal{I}^s \to \mathcal{T}op_*$ be the $\mathcal{I}^s$-space given by $B_\nu G^{(s)}(\mathbf{n}_1, \ldots, \mathbf{n}_s) = B_\nu G(\mathbf{n}_1) \times \cdots \times B_\nu G(\mathbf{n}_s)$. As explained in [19, section 5.2] there is a Γ-space $\Gamma(B_\nu G)$ associated to $B_\nu G$ constructed as follows: For a pointed set $(S, *)$ let $\bar{S} = S \setminus \{*\}$ and let $\mathcal{P}(\bar{S})$ be the power set of $\bar{S}$ considered as a category with respect to inclusion. We define $D(S)$ to be the category of those functors $\mathcal{P}(\bar{S}) \to \mathcal{I}$ that map disjoint unions of sets to coproducts of finite sets. The morphisms in $D(S)$ are natural transformations. Then

$$\Gamma(B_\nu G)(S) = \mathrm{hocolim}_{D(S)}\, B_\nu G^{(s)} \circ \pi_S \ ,$$

where $\pi_S \colon D(S) \to \mathcal{I}^{\bar{S}}$ is the canonical projection functor, which restricts to one-point subsets. All $G^{(r)}$ are commutative $\mathcal{I}$-monoids. Therefore we have analogous Γ-spaces $\Gamma(G^{(r)})$ and $\Gamma(G)$. Following [21, Definition 1.3], we can deloop $\Gamma(G)$: Let $Y^{(k)}(S) = \Gamma(G)([k] \wedge S)$. This is a simplicial space and $B_\mu \Gamma(G)(S) = |Y^{(\bullet)}(S)|$ is another Γ-space.

In [3] Bousfield and Friedlander discussed two model category structures on Γ-spaces: a strict and a stable one. The strict homotopy category of very special Γ-spaces is equivalent to the stable homotopy category of connective spectra [3, Theorem 5.1]. Instead of topological spaces, Bousfield and Friedlander considered (pointed) simplicial sets as a target category. The discussion in [20, Appendix B] shows that this difference is not essential. Recall from [3, Theorem 3.5] that a *strict equivalence* between Γ-spaces X and Y is a natural transformation $f \colon X \to Y$, such that $f_S \colon X(S) \to Y(S)$ is a weak equivalence for every pointed set S with $|S| \geq 2$. This agrees with the notion of strict equivalence given in [20].

Theorem 3.6. *Let G be a convergent EH-$\mathcal{I}$-group, such that each $G(\mathbf{n})$ has the homotopy type of a CW-complex. Let $B_\mu G_{h\mathcal{I}}$ be the classifying space of the topological monoid $(G_{h\mathcal{I}}, \mu)$. There is a strict equivalence of very special Γ-spaces $B_\mu \Gamma(G) \simeq \Gamma(B_\nu G)$ inducing a stable equivalence of the induced spectra in the stable homotopy category.*

In particular, $B_\nu G_{h\mathcal{I}} \simeq B_\mu G_{h\mathcal{I}}$. If G is stable, then $B_\mu G_{h\mathcal{I}}$ is a classifying space for principal $G(\mathbf{1})$-bundles and $B_\nu G(\mathbf{1})$ is an infinite loop space.

Proof. Let $G^{(r,s)}\colon \mathcal{I}^s \to \mathcal{T}op_*$ be the simplicial $\mathcal{I}^s$-space given by

$$G^{(r,s)}(\mathbf{n}_1,\dots,\mathbf{n}_s) = G^{(r)}(\mathbf{n}_1) \times \cdots \times G^{(r)}(\mathbf{n}_s) \ .$$

Observe that $G^{(r,s)} \circ \pi_S$ is a simplicial $D(S)$-space with $|G^{(\bullet,s)} \circ \pi_S| = B_\nu G^{(s)} \circ \pi_S$, therefore Lemma 3.4 yields a homeomorphism $\Gamma(B_\nu G)(S) \cong |\Gamma(G^{(\bullet)})(S)|$. By the commutativity of the diagram in Lemma 3.4, this is natural in S. Let $|\Gamma(G^{(\bullet)})|(S) := |\Gamma(G^{(\bullet)})(S)|$. We obtain a levelwise homeomorphism of Γ-spaces $\Gamma(B_\nu G) \cong |\Gamma(G^{(\bullet)})|$. In particular, this is a strict equivalence. The projection maps $\pi_k\colon G^{(r)}(\mathbf{n}) = G(\mathbf{n})^r \to G(\mathbf{n}) = G^{(1)}(\mathbf{n})$ induce

$$\pi\colon \Gamma(G^{(r)})(S) \to \Gamma(G)(S) \times \cdots \times \Gamma(G)(S) \ .$$

For $S = S^0$ we have $\Gamma(G^{(r)})(S) = G^{(r)}_{h\mathcal{I}}$, $\Gamma(G)(S) = G_{h\mathcal{I}}$. π fits into the commutative diagram

$$
\begin{array}{ccc}
G^{(r)}_{h\mathcal{I}} & \xrightarrow{\ \ \pi\ \ } & G_{h\mathcal{I}} \times \cdots \times G_{h\mathcal{I}} \\
{\scriptstyle\simeq}\uparrow & & \uparrow{\scriptstyle\simeq} \\
\mathrm{Tel}(G^{(r)}) & \xrightarrow{\ \ \widehat{\pi}\ \ } & \mathrm{Tel}(G) \times \cdots \times \mathrm{Tel}(G)
\end{array}
$$

in which the homotopy equivalences follow from [19, Lemma 2.1], since G is convergent. Note that $\pi_k(\mathrm{Tel}(G)^{(r)}) = \varinjlim(\pi_k(G(\mathbf{n})) \times \cdots \times \pi_k(G(\mathbf{n})))$, where the limit runs over the directed poset $\mathbb{N}_0$. Moreover, $\pi_k(\mathrm{Tel}(G)^r) = \varinjlim(\pi_k(G(\mathbf{n}_1)) \times \cdots \times \pi_k(G(\mathbf{n}_r)))$, where the limit runs over the poset $\mathbb{N}_0^r$. The map $\widehat{\pi}$ is induced by the diagonal $\Delta_r\colon \mathbb{N}_0 \to \mathbb{N}_0^r$. Since the subset $\Delta_r(\mathbb{N}_0)$ is cofinal in $\mathbb{N}_0^r$, we obtain an isomorphism on all homotopy groups and therefore – by our assumption on G – a homotopy equivalence.

Let now S be arbitrary, let $s = |S|$. Let $G^{(r,s)}_{h\mathcal{I}} = \mathrm{hocolim}_{\mathcal{I}^s} G^{(r,s)}$. By [19, Lemma 5.1], the natural map $\Gamma(G^{(r)})(S) \to G^{(r,s)}_{h\mathcal{I}}$ is an equivalence. From the above, we obtain that $\widetilde{\pi}\colon G^{(r,s)}_{h\mathcal{I}} \to G^{(1,s)}_{h\mathcal{I}} \times \cdots \times G^{(1,s)}_{h\mathcal{I}}$ is an equivalence. The diagram

$$
\begin{array}{ccc}
\Gamma(G^{(r)})(S) & \xrightarrow{\ \pi\ } & \Gamma(G)(S) \times \cdots \times \Gamma(G)(S) \\
{\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\
G^{(r,s)}_{h\mathcal{I}} & \xrightarrow[\simeq]{\ \widetilde{\pi}\ } & G^{(1,s)}_{h\mathcal{I}} \times \cdots \times G^{(1,s)}_{h\mathcal{I}}
\end{array}
$$

shows that π is an equivalence for arbitrary S, which in turn implies that the induced map $\pi\colon B_\mu\Gamma(G^{(r)})(S) \to B_\mu\Gamma(G)(S) \times \cdots \times B_\mu\Gamma(G)(S)$ is an equivalence as well. Now observe that $B_\mu|\Gamma(G^{(\bullet)})|(S) \cong |B_\mu\Gamma(G^{(\bullet)})|(S)$ and $[k] \mapsto B_\mu\Gamma(G^{(k)})(S)$ is a simplicial space satisfying the properties of [21, Proposition 1.5]. In particular, $\pi_0(B_\mu\Gamma(G)(S)) \cong \pi_0((B_\mu G_{h\mathcal{I}})^s)$ is trivial. Therefore $B_\mu\Gamma(G)(S) \to \Omega|B_\mu\Gamma(G^{(\bullet)})|(S) \cong \Omega B_\mu|\Gamma(G^{(\bullet)})|(S)$ is a homotopy equivalence, which is natural in S. Altogether we obtain a sequence of strict equivalences

$$\Gamma(B_\nu G) \simeq \Omega B_\mu\Gamma(B_\nu G) \simeq \Omega B_\mu|\Gamma(G^{(\bullet)})| \simeq B_\mu\Gamma(G) \ .$$

We used that $B_\nu G$ is connected (and therefore $\Gamma(B_\nu G)$ is a very special Γ-space) in the first step. $S = S^0$ together with Lemma 3.5 yields $B_\nu G(\mathbf{1}) \simeq B_\nu G_{h\mathcal{I}} \simeq B_\mu G_{h\mathcal{I}}$ in case G is stable. $\qquad\square$

3.1. Actions of Eckmann-Hilton $\mathcal{I}$-groups on spectra.

As was alluded to in the introduction, the compatibility diagram of an EH-$\mathcal{I}$-group G from Definition 3.1 enables us to talk about the action of G on a commutative symmetric ring spectrum R.

Definition 3.7. Let G be an EH-$\mathcal{I}$-group and let R be a commutative symmetric ring spectrum. Then G is said to *act on* R, if $G(\mathbf{n})$ acts on R_n via $\kappa_n \colon G(\mathbf{n}) \times R_n \to R_n$ such that

(i) κ_n preserves the basepoint of R_n and is Σ_n-equivariant, where Σ_n acts on $G(\mathbf{n}) \times R_n$ diagonally and $G(\mathbf{0})$ acts trivially on R_0,

(ii) the action is compatible with the multiplication in the sense that the following diagram commutes

$$
\begin{array}{ccc}
G(\mathbf{m}) \times R_m \times G(\mathbf{n}) \times R_n & \xrightarrow{\ \kappa_m \times \kappa_n\ } & R_m \wedge R_n \\
\scriptstyle (\mu^G_{m,n} \times \mu^R_{m,n}) \circ \tau \downarrow & & \downarrow \scriptstyle \mu^R_{m,n} \\
G(\mathbf{m} \sqcup \mathbf{n}) \times R_{m+n} & \xrightarrow{\ \kappa_{m+n}\ } & R_{m+n}
\end{array}
$$

where τ denotes the map that switches the two inner factors.

(iii) the action is compatible with stabilization in the sense that the following diagram commutes for $l, m, n \in \mathbb{N}_0$ with $l + m = n$, $\sigma_{l,m} = \mu_{l,m} \circ (\eta_l \wedge \mathrm{id}_{R_m})$ and the order preserving inclusion $\iota_{m,n} \colon \mathbf{m} \to \mathbf{n}$ onto the last m elements

$$
\begin{array}{ccc}
G(\mathbf{m}) \times (S^l \wedge R_m) & \xrightarrow{\ \mathrm{id}_{S^l} \wedge \kappa_m\ } & S^l \wedge R_m \\
\scriptstyle \iota_{m,n*} \times \sigma_{l,m} \downarrow & & \downarrow \scriptstyle \sigma_{l,m} \\
G(\mathbf{n}) \times R_n & \xrightarrow{\ \kappa_n\ } & R_n
\end{array}
$$

Theorem 3.8. *An action of an EH-$\mathcal{I}$-group G on a commutative symmetric ring spectrum R defines a map of $\mathcal{I}$-monoids $G \to \Omega^\infty(R)$, which sends $g \in G(\mathbf{n})$ to $g \cdot \eta_n := \kappa_n(g, \eta_n) \in \Omega^\infty(R)(\mathbf{n}) = \Omega^n R_n$. If G has compatible inverses, this factors over a morphism $G \to \Omega^\infty(R)^*$ of commutative $\mathcal{I}$-monoids, which deloops to a map $B_\mu \Gamma(G) \to B_\mu \Gamma(\Omega^\infty(R)^*)$. In particular, we obtain $B_\mu G_{h\mathcal{I}} \to BGL_1(R)$.*

Proof. To see that $G \to \Omega^\infty(R)$ really defines a natural transformation, observe that each morphism $\alpha \colon \mathbf{m} \to \mathbf{n}$ factors as $\alpha = \bar{\alpha} \circ \iota_{m,n}$ with $\bar{\alpha} \in \Sigma_n$ as explained in the paragraph after Definition 2.2. Now note that

$$
\alpha_*(g \cdot \eta_m) = \bar{\alpha} \circ \mu^R_{l,m}(\eta_l \wedge (g \cdot \eta_m)) \circ \bar{\alpha}^{-1} = \bar{\alpha} \circ \iota_{m,n*}(g) \cdot \mu^R_{l,m}(\eta_l \wedge \eta_m) \circ \bar{\alpha}^{-1}
$$

$$
= \alpha_*(g) \cdot \bar{\alpha} \circ \eta_n \circ \bar{\alpha}^{-1} = \alpha_*(g) \cdot \eta_n
$$

where we used (i) and (iii) of Definition 3.7. That $G \to \Omega^\infty(R)$ is a morphism of $\mathcal{I}$-monoids is a consequence of (ii). Indeed, for $g \in G(\mathbf{m})$, $h \in G(\mathbf{n})$

$$\mu^R_{m,n}(g \cdot \eta_m, h \cdot \eta_n) = \mu^G_{m,n}(g,h) \cdot \mu^R_{m,n}(\eta_m, \eta_n) = \mu^G_{m,n}(g,h) \cdot \eta_{m+n} \ .$$

If G has compatible inverses, then g^{-1} provides a stable inverse of $g \in G(\mathbf{n})$ by Lemma 3.3. Forming homotopy colimits, we obtain a morphism of topological monoids $G_{h\mathcal{I}} \to GL_1(R)$, which deloops. $\qquad\square$

3.2. Eckmann-Hilton $\mathcal{I}$-groups and permutative categories.

Given a permutative category $(\mathcal{C}, \otimes)$ and an object $x \in \mathrm{obj}(\mathcal{C})$, there is a canonical commutative $\mathcal{I}$-monoid E_x associated to it: Let $E_x(\mathbf{n}) = \mathrm{End}_{\mathcal{C}}(x^{\otimes n})$ (with $x^{\otimes 0} = 1_{\mathcal{C}}$), let $\bar\alpha \in \Sigma_n$ be the permutation associated to a morphism $\alpha \colon \mathbf{m} \to \mathbf{n}$ as above and define $\alpha_* \colon E_x(\mathbf{m}) \to E_x(\mathbf{n})$ by sending an endomorphism f to $\bar\alpha \circ (\mathrm{id}_{x^{\otimes n-m}} \otimes f) \circ \bar\alpha^{-1}$, where the permutation group Σ_n acts on $x^{\otimes n}$ using the symmetry of $\mathcal{C}$. The monoid structure of E_x is given by

$$\mu_E \colon E_x(\mathbf{m}) \times E_x(\mathbf{n}) \to E_x(\mathbf{m} \sqcup \mathbf{n}) \quad ; \quad (f,g) \mapsto f \otimes g \ .$$

Let $A_x(\mathbf{n}) = \mathrm{Aut}_{\mathcal{C}}(x^{\otimes n})$ together with the analogous structures as described above. This is an Eckmann-Hilton $\mathcal{I}$-group. Let $\mathcal{C}_x$ be the full permutative subcategory of $\mathcal{C}$ containing the objects $x^{\otimes n}$, $n \in \mathbb{N}_0$.

Definition 3.9. A strict symmetric monoidal functor $\theta \colon \mathcal{I} \to \mathcal{C}_x$ will be called a *stabilization of* x, if $\theta(\mathbf{1}) = x$ and for each morphism $\alpha \colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ and each $f \in E_x(\mathbf{m})$ the following diagram commutes:

$$\begin{array}{ccc} \theta(\mathbf{m}) & \xrightarrow{\ f\ } & \theta(\mathbf{m}) \\ {\scriptstyle \theta(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle \theta(\alpha)} \\ \theta(\mathbf{n}) & \xrightarrow[\alpha_*(f)]{} & \theta(\mathbf{n}) \end{array}$$

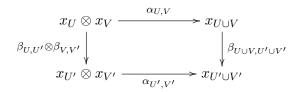**Lemma 3.10.** *Let $\iota_{\mathbf{1}} \colon \mathbf{0} \to \mathbf{1}$ be the unique morphism in $\mathcal{I}$. The map that associates to a stabilization θ the morphism $\theta(\iota_{\mathbf{1}}) \in \mathrm{hom}_{\mathcal{C}}(1_{\mathcal{C}}, x)$ yields a bijection between stabilizations and elements in $\mathrm{hom}_{\mathcal{C}}(1_{\mathcal{C}}, x)$.*

Proof. Note that a stabilization is completely fixed by knowing $\theta(\iota_{\mathbf{1}})$, therefore the map is injective. Let $\varphi \in \mathrm{hom}_{\mathcal{C}}(1_{\mathcal{C}}, x)$. Define $\theta(\mathbf{n}) = x^{\otimes n}$ on objects. Let $\alpha \colon \mathbf{m} \to \mathbf{n}$ be a morphism in $\mathcal{I}$ and let $\alpha = \bar\alpha \circ \iota_{m,n}$ be the factorization as explained after Definition 2.2. Define $\theta(\iota_{m,n}) = \varphi^{\otimes(n-m)} \otimes \mathrm{id}_{x^{\otimes m}} \colon x^{\otimes m} = 1_{\mathcal{C}}^{\otimes(n-m)} \otimes x^{\otimes m} \to x^{\otimes n}$ and let $\theta(\bar\alpha)$ be the permutation of the tensor factors. The decomposition $\alpha = \bar\alpha \circ \iota_{m,n}$ is *not* functorial, due to the fact that $\overline{\beta \circ \alpha}$ and $\bar\beta \circ (\mathrm{id} \sqcup \bar\alpha)$ differ by a permutation. Nevertheless $\theta(\alpha) = \theta(\bar\alpha) \circ \theta(\iota_{m,n})$ turns out to be functorial due to the permutation invariance of $\varphi^{\otimes k}$. It is straightforward to check that this is also strict symmetric monoidal. Let $f \colon x^{\otimes m} \to x^{\otimes m}$ be in $E_x(\mathbf{m})$. We have

$$\theta(\alpha) \circ f = \bar\alpha \circ (\varphi^{\otimes(n-m)} \otimes \mathrm{id}_{x^{\otimes m}}) \circ f = \bar\alpha \circ (\mathrm{id}_{x^{\otimes n-m}} \otimes f) \circ \bar\alpha^{-1} \circ \theta(\alpha) = \alpha_*(f) \circ \theta(\alpha) \ .$$

This shows that the map is also surjective.   □

As sketched in [21] there is a Γ-space $\Gamma(\mathcal{C})$ associated to a Γ-category $\mathcal{A}_\mathcal{C}$, which is constructed as follows: Let S be a finite pointed set and denote by $\bar{S}$ the complement of the basepoint of S, then the objects of $\mathcal{A}_\mathcal{C}(S)$ are families $\{x_U \in \mathrm{obj}(\mathcal{C}) \mid U \subset \bar{S}\}$ together with isomorphisms $\alpha_{U,V} \colon x_U \otimes x_V \to x_{U \cup V}$, whenever $U \cap V = \emptyset$, compatible with the symmetry of $\mathcal{C}$ and such that $x_\emptyset = 1_\mathcal{C}$ and $x_U \otimes x_\emptyset \to x_U$ is the identity. The morphisms of $\mathcal{A}_\mathcal{C}(S)$ are families of morphisms $\beta_{U,U'} \colon x_U \to x_{U'}$ in $\mathcal{C}$ such that

$$
\begin{array}{ccc}
x_U \otimes x_V & \xrightarrow{\ \alpha_{U,V}\ } & x_{U \cup V} \\
{\scriptstyle \beta_{U,U'} \otimes \beta_{V,V'}} \downarrow & & \downarrow {\scriptstyle \beta_{U \cup V, U' \cup V'}} \\
x_{U'} \otimes x_{V'} & \xrightarrow[\ \alpha_{U',V'}\ ]{} & x_{U' \cup V'}
\end{array}
$$

commutes. We define $\Gamma(\mathcal{C})(S) = |\mathcal{A}_\mathcal{C}(S)|$. Let $\mathcal{C}_x$ be the full (permutative) subcategory of $\mathcal{C}$ containing the objects $x^{\otimes n}$ for $n \in \mathbb{N}_0$.

In the next lemma we will introduce a technical tool from simplicial homotopy theory: The nerve of a topological category $\mathcal{D}$ is the simplicial space $N_n \mathcal{D} = \mathrm{Fun}([n], \mathcal{D})$, where $[n]$ is the category of the directed poset $\{0, \ldots, n\}$. We define the *double nerve* of $\mathcal{D}$ as the *bi*simplicial space $N_\bullet N_\bullet \mathcal{D}$ with $N_m N_n \mathcal{D} = \mathrm{Fun}([m] \times [n], \mathcal{D})$. It consists of an $m \times n$-array of commuting squares in $\mathcal{D}$. The diagonal functors $\mathrm{diag}_n \colon [n] \to [n] \times [n]$ induce a simplicial map $\varphi_n \colon N_n N_n \mathcal{D} \to N_n \mathcal{D}$.

Lemma 3.11. *The simplicial map $\varphi_\bullet$ induces a homotopy equivalence $|N_\bullet N_\bullet \mathcal{D}| \to |N_\bullet \mathcal{D}|$.*

Proof. To construct the homotopy inverse, let $\mathrm{max}_n \colon [n] \times [n] \to [n]$ be given by $\mathrm{max}_n(k, \ell) = \max\{k, \ell\}$, which completely determines its value on morphisms. We have $\mathrm{max}_n \circ \mathrm{diag}_n = \mathrm{id}_{[n]}$. Let $(k, \ell) \in [n] \times [n]$. There is a unique morphism $\kappa_{(k,\ell)} \colon (k, \ell) \to (\max\{k, \ell\}, \max\{k, \ell\})$. Therefore there is a natural transformation $\kappa \colon \mathrm{id}_{[n] \times [n]} \to \mathrm{diag}_n \circ \mathrm{max}_n$. From κ, we can construct a functor $h \colon [n] \times [n] \times [1] \to [n] \times [n]$, which induces

$$H \colon N_n N_n \mathcal{D} \times \hom([n] \times [n], [1]) \to N_n N_n \mathcal{D} \; ; \; (F, f) \mapsto F \circ h \circ (\mathrm{id}_{[n] \times [n]} \times f) \circ \mathrm{diag}_{[n] \times [n]} \; .$$
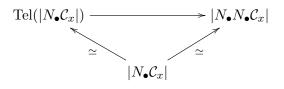
After geometric realization, H yields a homotopy inverse of $\mathrm{diag}_n \circ \mathrm{max}_n$.   □

Let $S_\otimes \colon \mathcal{C}_x \to \mathcal{C}_x$ be the functor $x^{\otimes n} \mapsto x^{\otimes(n+1)}$ and $f \mapsto f \otimes \mathrm{id}_x$. Observe that $|\mathrm{Tel}(N_k \mathcal{C}_x; N_k S_\otimes)|_k \cong \mathrm{Tel}(|N_\bullet \mathcal{C}_x|; |N_\bullet S_\otimes|) = \mathrm{Tel}(|N_\bullet \mathcal{C}_x|)$ by Lemma 3.4 and that θ provides a natural transformation $\mathrm{id}_{\mathcal{C}_x} \to S_\otimes$. Therefore we obtain a map $\mathrm{Tel}(|N_\bullet \mathcal{C}_x|) \to |N_\bullet N_\bullet \mathcal{C}_x|$.

Lemma 3.12. *The map $\mathrm{Tel}(|N_\bullet \mathcal{C}_x|) \to |N_\bullet N_\bullet \mathcal{C}_x|$ constructed in the last paragraph is a homotopy equivalence.*

Proof. θ yields a natural transformation $\mathrm{id}_{\mathcal{C}_x} \to S_\otimes$. Therefore $|N_\bullet S_\otimes|$ is homotopic to the identity and the map $|N_\bullet \mathcal{C}_x| \to \mathrm{Tel}(|N_\bullet \mathcal{C}_x|)$ onto the 0-skeleton is a homotopy equivalence. Likewise, $|N_\bullet \mathcal{C}_x| \to |N_\bullet N_\bullet \mathcal{C}_x|$ induced by the simplicial map $N_\ell \mathcal{C}_x \to N_\ell N_\ell \mathcal{C}_x$ sending a diagram to the

corresponding square that has ℓ copies of the diagram in its rows and only identities as vertical maps is a homotopy equivalence by Lemma 3.11 since composition with $\varphi_\bullet$ yields the identity on $N_\ell \mathcal{C}_x$. The statement now follows from the commutative triangle

$$\text{Tel}(|N_\bullet \mathcal{C}_x|) \longrightarrow |N_\bullet N_\bullet \mathcal{C}_x|$$

$$\simeq \qquad |N_\bullet \mathcal{C}_x| \qquad \simeq$$

which finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.13. *Let $x \in \mathrm{obj}(\mathcal{C})$ and let $\theta\colon \mathcal{I} \to \mathcal{C}_x$ be a stabilization of x. Let $G = A_x$. There is a map of Γ-spaces*

$$\Phi_\theta\colon \Gamma(B_\nu G) \to \Gamma(\mathcal{C}_x) \ .$$

If G is convergent and there exists an unbounded non-decreasing sequence of natural numbers λ_n, such that $B_\nu G(\mathbf{m}) \to |N_\bullet \mathcal{C}_x|$ is λ_n-connected for all $m > n$, then Φ_θ is an equivalence as well.

Proof. Lemma 3.4 yields a homeomorphism $\Gamma(B_\nu G)(S) \cong |\Gamma(G^{(\bullet)})(S)|$ with $\Gamma(G^{(r)})(S) = \mathrm{hocolim}_{D(S)} G^{(r,s)} \circ \pi_S$. The last space is the geometric realization of the transport category $\mathcal{T}_G^{(r)}(S)$ with object space $\coprod_{d \in \mathrm{obj}(D(S))} G^{(r,s)}(\pi_S(d))$ and morphism space

$$\mathrm{mor}(\mathcal{T}_G^{(r)}(S)) = \coprod_{d,d' \in \mathrm{obj}(D(S))} G^{(r,s)}(\pi_S(d)) \times \mathrm{hom}_{D(S)}(d, d') \ .$$

Given a diagram $d \in \mathrm{obj}(D(S))$, we obtain $\mathbf{n}_U = d(U) \in \mathrm{obj}(\mathcal{I})$ for every $U \subset \bar{S}$ and define $\mathbf{n}_i = d(\{i\})$. Let $(g_1, \ldots g_s) \in G^{(r)}(\mathbf{n}_1) \times \cdots \times G^{(r)}(\mathbf{n}_s)$. We may interpret g_i as an r-tupel $(g_{j,i})_{j \in \{1,\ldots,r\}}$ of automorphisms $g_{j,i} \in G(\mathbf{n}_i) = \mathrm{Aut}_{\mathcal{C}}(\theta(\mathbf{n}_i))$. If $U, V \subset \bar{S}$ are two subsets with $U \cap V = \emptyset$, then d yields two morphisms $\mathbf{n}_U \to \mathbf{n}_{U \cup V}$ and $\mathbf{n}_V \to \mathbf{n}_{U \cup V}$, which form an isomorphism $\iota_{U,V}\colon \mathbf{n}_U \sqcup \mathbf{n}_V \to \mathbf{n}_{U \cup V}$ in $\mathcal{I}$. Let $\alpha_{U,V} = \theta(\iota_{U,V})\colon \theta(\mathbf{n}_U) \otimes \theta(\mathbf{n}_V) \to \theta(\mathbf{n}_{U \cup V})$. Let $\iota_U^i\colon \mathbf{n}_i \to \mathbf{n}_U$ be induced by the inclusion $\{i\} \subset U$ and define $g_{j,U} = \prod_{i \in U} \iota_{U*}^i(g_{j,i})$. This does not depend on the order, in which the factors are multiplied. The elements $g_{j,U}$ fit into a commutative diagram

$$
\begin{array}{ccc}
\theta(\mathbf{n}_U) \otimes \theta(\mathbf{n}_V) & \xrightarrow{\alpha_{U,V}} & \theta(\mathbf{n}_{U \cup V}) \\
{\scriptstyle g_{j,U} \otimes g_{j,V}} \downarrow & & \downarrow {\scriptstyle g_{j,U \cup V}} \\
\theta(\mathbf{n}_U) \otimes \theta(\mathbf{n}_V) & \xrightarrow{\alpha_{U,V}} & \theta(\mathbf{n}_{U \cup V})
\end{array}
$$

Thus, we can interpret the families $g_{j,U}$ for $j \in \{1, \ldots, r\}$ as an r-chain of automorphisms of the object $(\theta(\mathbf{n}_U), \alpha_{U,V})$ in $\mathcal{A}_{\mathcal{C}_x}(S)$.

Let d' be another diagram, let $\mathbf{m}_U = d'(U)$, $\iota'_{U,V}\colon \mathbf{m}_U \sqcup \mathbf{m}_V \to \mathbf{m}_{U \cup V}$ and let $\beta_{U,V} = \theta(\iota'_{U,V})$. A natural transformation $d \to d'$ consists of morphisms $\varphi_U\colon \mathbf{n}_U \to \mathbf{m}_U$ for every $U \in \mathcal{P}(\bar{S})$. Let

$f_U = \theta(\varphi_U)$, then the following diagrams commute:

$$
\begin{array}{ccc}
\theta(\mathbf{n}_U) \xrightarrow{\;g_{j,U}\;} \theta(\mathbf{n}_U) & \qquad & \theta(\mathbf{n}_U) \otimes \theta(\mathbf{n}_V) \xrightarrow{\;\alpha_{U,V}\;} \theta(\mathbf{n}_{U\cup V}) \\
\;\;\downarrow{\scriptstyle f_U} \qquad \downarrow{\scriptstyle f_U} & & \;\;\downarrow{\scriptstyle f_U\otimes f_V} \qquad\qquad\quad \downarrow{\scriptstyle f_{U\cup V}} \\
\theta(\mathbf{m}_U) \xrightarrow[\varphi_{U*}(g_{j,U})]{} \theta(\mathbf{m}_U) & & \theta(\mathbf{m}_U) \otimes \theta(\mathbf{m}_V) \xrightarrow[\;\beta_{U,V}\;]{} \theta(\mathbf{m}_{U\cup V})
\end{array}
$$

the first since θ is a stabilization, the second by naturality of the transformation. This is compatible with respect to composition of natural transformations of diagrams. Altogether we have constructed a bisimplicial map $N_\ell \mathcal{T}_G^{(r)}(S) \to N_\ell N_r \mathcal{A}_{\mathcal{C}_x}(S)$. Combining this with $N_\ell N_\ell \mathcal{A}_{\mathcal{C}_x}(S) \to N_\ell \mathcal{A}_{\mathcal{C}_x}(S)$ from Lemma 3.11, we obtain $\Gamma(B_\nu G)(S) \to \Gamma(\mathcal{C}_x)(S)$ after geometric realization.

It remains to be proven that this map is functorial with respect to morphisms $\kappa\colon S \to T$ in Γ^{op}. Recall that

$$
N_\ell \mathcal{T}_G^{(r)}(S) = \coprod_{d_1,\ldots,d_\ell \in \mathrm{obj}(D(S))} \prod_{s\in S} G^{(r)}(d_1(\{s\})) \times \hom_{D(S)}(d_1,d_2) \times \cdots \times \hom_{D(S)}(d_{\ell-1},d_\ell)
$$

Let $\kappa_*\colon N_\ell \mathcal{T}_G^{(r)}(S) \to N_\ell \mathcal{T}_G^{(r)}(T)$ be the induced map as defined in [19, section 5.2]. We have

$$
\kappa_*\left((g_{j,i})_{i\in S, j\in\{0,\ldots,r\}}, \varphi^1,\ldots,\varphi^{\ell-1} \right) = \left((h_{j,k})_{k\in T, j\in\{0,\ldots,r\}}, \kappa_*\varphi^1,\ldots,\kappa_*\varphi^{\ell-1} \right)
$$

where $h_{j,k} = \prod_{i\in\kappa^{-1}(k)} \iota^i_{\kappa^{-1}(k)*}(g_{j,i})$. If the left hand side lies in the component $(d_1,\ldots,d_\ell)$, then the right hand side is in $(\kappa_* d_1,\ldots,\kappa_* d_\ell)$ with $(\kappa_* d_m)(V) = d_m(\kappa^{-1}(V))$ for $V \subset \bar{T}$. Likewise $(\kappa_*\varphi^m)_V = \varphi^m_{\kappa^{-1}(V)}$. The functor $\kappa_*\colon \mathcal{A}_{\mathcal{C}_x}(S) \to \mathcal{A}_{\mathcal{C}_x}(T)$ sends the object $(x_U, \alpha_{U,V})_{U,V\subset\bar{S}}$ to $(x_{\kappa^{-1}(\tilde{U})}, \alpha_{\kappa^{-1}(\tilde{U}),\kappa^{-1}(\tilde{V})})_{\tilde{U},\tilde{V}\subset\bar{T}}$ and is defined analogously on morphisms. Observe that the composition $N_\ell \mathcal{T}_G^{(r)}(S) \to N_\ell \mathcal{T}_G^{(r)}(T) \to N_\ell N_r \mathcal{A}_{\mathcal{C}_x}(T)$ maps $(g_{j,i})_{i\in S, j\in\{0,\ldots,r\}}$ to the r-chain of automorphisms given by $h_{j,V}\colon \theta(\kappa_* d(V)) \to \theta(\kappa_* d(V))$ with

$$
h_{j,V} = \prod_{k\in V} \prod_{i\in\kappa^{-1}(k)} (\kappa_* d)(\iota^k_V) \circ d(\iota^i_{\kappa^{-1}(k)})(g_{j,i}) = \prod_{i\in\kappa^{-1}(V)} d(\iota^i_{\kappa^{-1}(V)})(g_{j,i}) = g_{j,\kappa^{-1}(V)}
$$

for $V \subset \bar{T}$. The transformations φ^m are mapped to $\theta(\varphi^m_{\kappa^{-1}(V)}) = f^m_{\kappa^{-1}(V)}$. This implies the commutativity of

$$
\begin{array}{ccc}
N_\ell \mathcal{T}_G^{(r)}(S) & \xrightarrow{\;\kappa_*\;} & N_\ell \mathcal{T}_G^{(r)}(T) \\
{\scriptstyle \Phi_\theta}\downarrow & & \downarrow{\scriptstyle \Phi_\theta} \\
N_\ell N_r \mathcal{A}_{\mathcal{C}_x}(S) & \xrightarrow[\;\kappa_*\;]{} & N_\ell N_r \mathcal{A}_{\mathcal{C}_x}(T)
\end{array}
$$

and therefore functoriality after geometric realization.

To see that Φ_θ is an equivalence for each S it suffices to check that $B_\nu G_{h\mathcal{I}} \to |N_\bullet \mathcal{C}_x|$ is a homotopy equivalence due to the following commutative diagram

$$
\begin{array}{ccc}
\Gamma(B_\nu G)(S) & \longrightarrow & \Gamma(\mathcal{C}_x)(S) \\
\simeq \downarrow & & \downarrow \simeq \\
(B_\nu G_{h\mathcal{I}})^s & \longrightarrow & |N_\bullet \mathcal{C}_x|^s
\end{array}
$$

Let $\mathrm{Tel}(|N_\bullet \mathcal{C}_x|)$ be the telescope of $|N_\bullet S_\otimes|$ as defined above. The conditions on the maps $B_\nu G(\mathbf{n}) \to |N_\bullet \mathcal{C}_x|$ ensure that $\mathrm{Tel}(B_\nu G) \to \mathrm{Tel}(|N_\bullet \mathcal{C}_x|)$ is a homotopy equivalence. Now we have the following commutative diagram

$$
\begin{array}{ccccc}
B_\nu G_{h\mathcal{I}} & \longrightarrow & |N_\bullet N_\bullet \mathcal{C}_x| & \xrightarrow{\;\simeq\;} & |N_\bullet \mathcal{C}_x| \\
\simeq \uparrow & & \uparrow \simeq & & \\
\mathrm{Tel}(B_\nu G) & \xrightarrow{\;\simeq\;} & \mathrm{Tel}(|N_\bullet \mathcal{C}_x|) & &
\end{array}
$$

in which the upper horizontal map is the one we are looking for. This finishes the proof. $\qquad\square$

4. Strongly self-absorbing C^*-algebras and $gl_1(KU^A)$

A C^*-algebra A is called *strongly self-absorbing* if it is separable, unital and there exists a $*$-isomorphism $\psi \colon A \to A \otimes A$ such that ψ is approximately unitarily equivalent to the map $l \colon A \to A \otimes A$, $l(a) = a \otimes 1_A$ [23]. This means that there is a sequence of unitaries (u_n) in A such that $\|u_n \varphi(a) u_n^* - l(a)\| \to 0$ as $n \to \infty$ for all $a \in A$. In fact, it is a consequence of [6, Theorem 2.2] and [26] that ψ, l and $r \colon A \to A \otimes A$ with $r(a) = 1_A \otimes a$ are homotopy equivalent and in fact the group $\mathrm{Aut}(A)$ is contractible [5]. The inverse isomorphism ψ^{-1} equips $K_*(C(X) \otimes A)$ with a ring structure induced by the tensor product. By homotopy invariance of K-theory, the K_0-class of the constant map on X with value $1 \otimes e$ for a rank 1-projection $e \in \mathbb{K}$ is the unit of this ring structure. Given a separable, unital, strongly self-absorbing C^*-algebra A, the functor $X \mapsto K_*(C(X) \otimes A)$ is a multiplicative cohomology theory on finite CW-complexes with respect to this ring structure.

4.1. A commutative symmetric ring spectrum representing K-theory.

A C^*-algebra B is *graded*, if it comes equipped $\mathbb{Z}/2\mathbb{Z}$-action, i.e. a $*$-automorphism $\alpha \colon B \to B$, such that $\alpha^2 = \mathrm{id}_B$. A *graded homomorphism* $\varphi \colon (B, \alpha) \to (B', \alpha')$ has to satisfy $\varphi \circ \alpha = \alpha' \circ \varphi$. The algebraic tensor product $B \mathbin{\widehat{\odot}} B'$ can be equipped with the multiplication and $*$-operation

$$
(a \mathbin{\widehat{\odot}} b)(a' \mathbin{\widehat{\odot}} b') = (-1)^{\partial b \cdot \partial a'}(aa' \mathbin{\widehat{\odot}} bb') \quad \text{and} \quad (a \mathbin{\widehat{\odot}} b)^* = (-1)^{\partial a \cdot \partial b}(a^* \mathbin{\widehat{\odot}} b^*)
$$

where $a, a' \in B$ and $b, b' \in B'$ are homogeneous elements and ∂a denotes the degree of a. It is graded via $\partial(a \mathbin{\widehat{\odot}} b) = \partial a + \partial b$ modulo 2. The (minimal) graded tensor product $B \mathbin{\widehat{\otimes}} B'$ is the completion of $B \mathbin{\widehat{\odot}} B'$ with respect to the tensor product of faithful representations of B and B' on graded Hilbert spaces. For details we refer the reader to [4, section 14.4].

We define $\widehat{S} = C_0(\mathbb{R})$ with the grading by even and odd functions. The Clifford algebra $\mathbb{C}\ell_1$ will be spanned by the even element 1 and the odd element c with $c^2 = 1$. The algebra $\widehat{\mathbb{K}}$ will denote the

graded compact operators on a graded Hilbert space $H = H_0 \oplus H_1$ with grading Ad_u, $u = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$, whereas we will use $\mathbb{K}$ for the *trivially* graded compact operators. If we take tensor products between a graded C^*-algebra and a trivially graded one, e.g. $\mathbb{C}\ell_1 \widehat{\otimes} \mathbb{K}$, we will write $\otimes$ instead of $\widehat{\otimes}$. It is a consequence of [4, Corollary 14.5.5] that $(\mathbb{C}\ell_1 \otimes \mathbb{K}) \widehat{\otimes} (\mathbb{C}\ell_1 \otimes \mathbb{K}) \cong M_2(\mathbb{C}) \otimes \mathbb{K} \cong \widehat{\mathbb{K}}$, where $M_2(\mathbb{C})$ is graded by Ad_u with $u = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$.

As is explained in [7, section 3.2], [12, section 4], the algebra $\widehat{S}$ carries a counital, coassociative and cocommutative coalgebra structure. This arises from a $1:1$-correspondence between essential graded $*$-homomorphisms $\widehat{S} \to A$ and odd, self-adjoint, regular unbounded multipliers of A [24, Proposition 3.1]. Let X be the multiplier corresponding to the identity map on $\widehat{S}$, then the comultiplication $\Delta \colon \widehat{S} \to \widehat{S} \otimes \widehat{S}$ is given by $1 \widehat{\otimes} X + X \widehat{\otimes} 1$, whereas the counit $\epsilon \colon \widehat{S} \to \mathbb{C}$ corresponds to $0 \in \mathbb{C}$, i.e. it maps $f \mapsto f(0)$.

Definition 4.1. Let A be a separable, unital, strongly self-absorbing and trivially graded C^*-algebra. Let $KU_\bullet^A$ be the following sequence of spaces

$$KU_n^A = \hom_{\mathrm{gr}}(\widehat{S}, (\mathbb{C}\ell_1 \otimes A \otimes \mathbb{K})^{\widehat{\otimes}n}) \, ,$$

where the graded homomorphisms are equipped with the point-norm topology.

KU_n^A is pointed by the 0-homomorphism and carries a basepoint preserving Σ_n-action by permuting the factors of the graded tensor product (this involves signs!). We set $B^{\widehat{\otimes}0} = \mathbb{C}$ and observe that KU_0^A is the two-point space consisting of the 0-homomorphism and the evaluation at 0, which is the counit of the coalgebra structure on $\widehat{S}$.

Let $\mu_{m,n}$ be the following family of maps

$$\mu_{m,n} \colon KU_m^A \wedge KU_n^A \to KU_{m+n}^A \quad ; \quad \varphi \wedge \psi \mapsto (\varphi \widehat{\otimes} \psi) \circ \Delta$$

To construct the maps $\eta_n \colon S^n \to KU_n^A$, note that $t \mapsto t c$ is an odd, self-adjoint, regular unbounded multiplier on $C_0(\mathbb{R}, \mathbb{C}\ell_1)$. Therefore the functional calculus for this multiplier is a graded $*$-homomorphism $\widehat{S} \to C_0(\mathbb{R}, \mathbb{C}\ell_1)$. This in turn can be seen as a basepoint preserving map $S^1 \to \hom_{\mathrm{gr}}(\widehat{S}, \mathbb{C}\ell_1)$. Now consider

$$C_0(\mathbb{R}, \mathbb{C}\ell_1) \to C_0(\mathbb{R}, \mathbb{C}\ell_1 \otimes A \otimes \mathbb{K}) \quad ; \quad f \mapsto f \otimes (1 \otimes e) \, ,$$

where e is a rank 1-projection in $\mathbb{K}$. After concatenation, we obtain a graded $*$-homomorphism $\hat{\eta}_1 \colon \widehat{S} \to C_0(\mathbb{R}, \mathbb{C}\ell_1 \otimes A \otimes \mathbb{K})$ and from this a continuous map $\eta_1 \colon S^1 \to \hom_{\mathrm{gr}}(\widehat{S}, \mathbb{C}\ell_1 \otimes A \otimes \mathbb{K})$. We define $\eta_0 \colon S^0 \to KU_0^A$ by sending the non-basepoint to the counit of $\widehat{S}$. Now let

$$\hat{\eta}_n \colon \widehat{S} \to C_0(\mathbb{R}^n, (\mathbb{C}\ell_1 \otimes A \otimes \mathbb{K})^{\widehat{\otimes}n}) \quad \text{with } \hat{\eta}_n = (\hat{\eta}_1 \widehat{\otimes} \ldots \widehat{\otimes} \hat{\eta}_1) \circ \Delta_n \, ,$$

where $\Delta_n \colon \widehat{S} \to \widehat{S}^{\widehat{\otimes}n}$ is defined recursively by $\Delta_1 = \mathrm{id}_{\widehat{S}}$, $\Delta_2 = \Delta$ and $\Delta_n = (\Delta \widehat{\otimes} \mathrm{id}) \circ \Delta_{n-1}$, for $n \geq 3$. This yields a well-defined map $\eta_n \colon S^n \to KU_n^A$.

Theorem 4.2. *Let A be a separable, unital, strongly self-absorbing C^*-algebra. The spaces $KU_\bullet^A$ together with the maps $\mu_{\bullet,\bullet}$ and $\eta_\bullet$ form a commutative symmetric ring spectrum with coefficients*

$$\pi_n(KU_\bullet^A) = K_n(A) \ .$$

Moreover, all structure maps $KU_n^A \to \Omega KU_{n+1}^A$ are weak homotopy equivalences for $n \geq 1$.

Proof. It is a consequence of the cocommutativity of Δ that Δ_n is Σ_n-invariant, i.e. for a permutation $\tau \in \Sigma_n$ and its induced action $\tau_* \colon \widehat{S}^{\widehat{\otimes} n} \to \widehat{S}^{\widehat{\otimes} n}$ we have $\tau_* \circ \Delta_n = \Delta_n$. If we let τ act on

$$C_0(\mathbb{R}^n, (\mathbb{C}\ell_1 \otimes A \otimes \mathbb{K})^{\widehat{\otimes} n}) \cong (C_0(\mathbb{R}, \mathbb{C}\ell_1 \otimes A \otimes \mathbb{K}))^{\widehat{\otimes} n}$$

by permuting the tensor factors, then we have that $\hat{\eta}_n$ is invariant under the action of Σ_n since

$$\tau_* \circ \hat{\eta}_n = \tau_* \circ (\hat{\eta}_1 \widehat{\otimes} \ldots \widehat{\otimes} \hat{\eta}_1) \circ \Delta_n = (\hat{\eta}_1 \widehat{\otimes} \ldots \widehat{\otimes} \hat{\eta}_1) \circ \tau_* \circ \Delta_n = \hat{\eta}_n \ ,$$

which proves that η_n is Σ_n-*equivariant*. The $\Sigma_m \times \Sigma_n$-equivariance of $\mu_{m,n}$ is clear from the definition of the Σ_{m+n}-action on KU_{m+n}^A and the symmetry of the graded tensor product. Associativity of $\mu_{\bullet,\bullet}$ (see Definition 2.1 (a)) is a direct consequence of the coassociativity of $\widehat{S}$ and the associativity of the graded tensor product. The map $\mu_{m,n}(\eta_m \wedge \eta_n)$ corresponds to the $*$-homomorphism

$$(\hat{\eta}_m \widehat{\otimes} \hat{\eta}_n) \circ \Delta = \underbrace{\hat{\eta}_1 \widehat{\otimes} \ldots \widehat{\otimes} \hat{\eta}_1}_{n+m \text{ times}} \circ (\Delta_m \widehat{\otimes} \Delta_n) \circ \Delta \ .$$

By coassociativity we have $(\Delta_m \widehat{\otimes} \Delta_n) \circ \Delta = \Delta_{m+n}$. Therefore $(\hat{\eta}_m \widehat{\otimes} \hat{\eta}_n) \circ \Delta = \hat{\eta}_{m+n}$, which translates into the first compatibility condition of Definition 2.1. The other compatibility conditions are consequences of the fact that $\widehat{S}$ is counital. The commutativity of $\mu_{\bullet,\bullet}$ follows from the definition of the permutation action and the cocommutativity of the coalgebra structure on $\widehat{S}$. Thus, the spaces $KU_\bullet^A$ indeed form a commutative symmetric ring spectrum.

To see that the structure maps induce weak equivalences, observe that for $k \geq 1$

$$\pi_k(KU_n^A) = \pi_0(\Omega^k KU_n^A) = \pi_0(\hom_{\mathrm{gr}}(\widehat{S}, C_0(\mathbb{R}^k, (\mathbb{C}\ell_1 \otimes \mathbb{K} \otimes A)^{\widehat{\otimes} n})))$$

The algebra $\mathbb{C}\ell_1 \otimes \mathbb{K}$ with trivially graded $\mathbb{K}$ is isomorphic as a graded C^*-algebra to $\mathbb{C}\ell_1 \widehat{\otimes} \widehat{\mathbb{K}}$, where $\widehat{\mathbb{K}} = \widehat{\mathbb{K}}(H_+ \oplus H_-)$ is equipped with its standard even grading. Therefore

$$C_0(\mathbb{R}^k, (\mathbb{C}\ell_1 \otimes \mathbb{K} \otimes A)^{\widehat{\otimes} n}) \cong C_0(\mathbb{R}^k, (\mathbb{C}\ell_1 \otimes A)^{\widehat{\otimes} n}) \widehat{\otimes} \widehat{\mathbb{K}}^{\widehat{\otimes} n} \cong C_0(\mathbb{R}^k, (\mathbb{C}\ell_1 \otimes A)^{\widehat{\otimes} n}) \widehat{\otimes} \widehat{\mathbb{K}}$$

and by [24, Theorem 4.7]

$$\pi_0(\hom_{\mathrm{gr}}(\widehat{S}, C_0(\mathbb{R}^k, (\mathbb{C}\ell_1 \otimes A)^{\widehat{\otimes} n}) \widehat{\otimes} \widehat{\mathbb{K}}) \cong KK(\mathbb{C}, C_0(\mathbb{R}^k, (\mathbb{C}\ell_1 \otimes A)^{\widehat{\otimes} n})) \cong K_{k-n}(A)$$

where we used Bott periodicity and the isomorphism $A^{\otimes n} \cong A$ in the last map. The structure map $KU_n^A \to \Omega KU_{n+1}^A$ is now given by exterior multiplication with the class in $KK(\mathbb{C}, C_0(\mathbb{R}, \mathbb{C}\ell_1 \otimes A))$ represented by $\hat{\eta}_1$. But by definition this is a combination of the Bott element together with the map $A \otimes \mathbb{K} \to (A \otimes \mathbb{K})^{\otimes 2}$ that sends a to $a \otimes (1 \otimes e)$, which is homotopic to an isomorphism, since A is *strongly* self-absorbing. Thus, both operations induce isomorphisms on K-theory and therefore

also on all homotopy groups. Finally, we have for $n \in \mathbb{Z}$

$$\pi_n(KU_\bullet^A) = \operatorname{colim}_{k \to \infty} \pi_{n+k}(KU_k^A) \cong K_n(A) \ . \qquad \square$$

Remark 4.3. As can be seen from the above proof, the fact that A is *strongly* self-absorbing (and not just self-absorbing, which would be $A \cong A \otimes A$) is important for the spectrum to be positive fibrant.

Remark 4.4. Note that the spectrum $KU_\bullet^A$ is not connective. However, the $\mathcal{I}$-monoid $\Omega^\infty(R)^*$ of a symmetric ring spectrum R only "sees" the connective cover. In fact, $\Omega^\infty(R)_{h\mathcal{I}}$ is its underlying infinite loop space. This was already remarked in [18, Section 1.6] and was the motivation for the introduction of $\mathcal{J}$-spaces to also capture periodic phenomena.

4.2. **The Eckmann-Hilton $\mathcal{I}$-group G_A.** Let A be a separable, unital, strongly self-absorbing algebra and let $G_A(\mathbf{n}) = \operatorname{Aut}((A \otimes \mathbb{K})^{\otimes n})$ (with $(A \otimes \mathbb{K})^{\otimes 0} := \mathbb{C}$, such that $G_A(\mathbf{0})$ is the trivial group). Note that $\sigma \in \Sigma_n$ acts on $G_A(\mathbf{n})$ by mapping $g \in G_A(\mathbf{n})$ to $\sigma \circ g \circ \sigma^{-1}$, where σ permutes the tensor factors of $(A \otimes \mathbb{K})^{\otimes n}$. Let $g \in G_A(\mathbf{m})$ and $\alpha \colon \mathbf{m} \to \mathbf{n}$. We can enhance G_A to a functor $G_A \colon \mathcal{I} \to \mathcal{G}rp_*$ via

$$\alpha_*(g) = \bar{\alpha} \circ (\operatorname{id}_{(A \otimes \mathbb{K})^{\otimes (n-m)}} \otimes g) \circ \bar{\alpha}^{-1}$$

with $\bar{\alpha}$ as explained after Definition 2.2. Moreover, G_A is an $\mathcal{I}$-monoid via

$$\mu_{m,n} \colon G_A(\mathbf{m}) \times G_A(\mathbf{n}) \to G_A(\mathbf{m} \sqcup \mathbf{n}) \quad ; \quad (g, h) \mapsto g \otimes h \ .$$

Note that $\mu_{m,0}$ and $\mu_{0,n}$ are induced by the canonical isomorphisms $\mathbb{C} \otimes (A \otimes \mathbb{K})^{\otimes n} \cong (A \otimes \mathbb{K})^{\otimes n}$ and $(A \otimes \mathbb{K})^{\otimes m} \otimes \mathbb{C} \cong (A \otimes \mathbb{K})^{\otimes m}$ respectively.

Theorem 4.5. *G_A as defined above is a stable EH-$\mathcal{I}$-group with compatible inverses in the sense of Definition 3.1.*

Proof. The commutativity of the Eckmann-Hilton diagram in Definition 3.1 is a consequence of $(g \otimes h) \cdot (g' \otimes h') = (g \cdot g') \otimes (h \cdot h')$ for $g, g' \in G_A(\mathbf{m})$ and $h, h' \in G_A(\mathbf{n})$. Well-pointedness of $G_A(\mathbf{n})$ was proven in [5, Prop.2.6]. To see that non-initial morphisms are mapped to homotopy equivalences, first observe that G_A maps permutations to homeomorphisms, therefore it suffices to check that

$$\operatorname{Aut}(A \otimes \mathbb{K}) \to \operatorname{Aut}((A \otimes \mathbb{K})^{\otimes n}) \quad ; \quad g \mapsto g \otimes \operatorname{id}_{(A \otimes \mathbb{K})^{\otimes (n-1)}}$$

is a homotopy equivalence. But the target space is homeomorphic to $\operatorname{Aut}(A \otimes \mathbb{K})$ via conjugation with a suitable isomorphism. Altogether it is enough to check that we can find an isomorphism $\psi \colon A \otimes \mathbb{K} \to (A \otimes \mathbb{K})^{\otimes 2}$ such that

$$\operatorname{Aut}(A \otimes \mathbb{K}) \to \operatorname{Aut}(A \otimes \mathbb{K}) \quad ; \quad g \mapsto \psi^{-1} \circ (g \otimes \operatorname{id}_{A \otimes \mathbb{K}}) \circ \psi$$

is homotopic to $\mathrm{id}_{\mathrm{Aut}(A\otimes\mathbb{K})}$. Let $\hat{\psi}\colon I\times(A\otimes\mathbb{K})\to(A\otimes\mathbb{K})^{\otimes 2}$ be the homotopy of [5, Thm.2.5] and let $\psi=\hat{\psi}(\frac{1}{2})$. Then

$$H\colon I\times\mathrm{Aut}(A\otimes\mathbb{K})\to\mathrm{Aut}(A\otimes\mathbb{K})\;;\;\;g\mapsto\begin{cases}\hat{\psi}(\frac{t}{2})^{-1}\circ(g\otimes\mathrm{id}_{A\otimes\mathbb{K}})\circ\hat{\psi}(\frac{t}{2})&\text{for }0<t\leq1\\g&\text{for }t=0\end{cases}$$

satisfies the conditions.

Let $g\in G_A(\mathbf{m})$. To see that there is a path connecting $(\iota_m\sqcup\mathrm{id}_{\mathbf{m}})_*(g)=\mathrm{id}_{(A\otimes\mathbb{K})^{\otimes m}}\otimes g$ to $g\otimes\mathrm{id}_{(A\otimes\mathbb{K})^{\otimes m}}$, observe that $\tau_A\colon A\otimes A\to A\otimes A$ with $\tau_A(a\otimes b)=b\otimes a$ is homotopic to the identity by the contractibility of $\mathrm{Aut}(A\otimes A)\cong\mathrm{Aut}(A)$ [5, Thm.2.3]. Similarly, there is a homotopy between $\tau_\mathbb{K}\colon\mathbb{K}\otimes\mathbb{K}\to\mathbb{K}\otimes\mathbb{K}$ with $\tau_\mathbb{K}(S\otimes T)=T\otimes S$ and the identity since $\mathrm{Aut}(\mathbb{K}\otimes\mathbb{K})\cong PU(H)$ is path-connected.

Thus, we obtain a homotopy between the identity on $(A\otimes\mathbb{K})^{\otimes m}\otimes(A\otimes\mathbb{K})^{\otimes m}$ and the corresponding switch automorphism $\tau_m\colon(A\otimes\mathbb{K})^{\otimes m}\otimes(A\otimes\mathbb{K})^{\otimes m}\to(A\otimes\mathbb{K})^{\otimes m}\otimes(A\otimes\mathbb{K})^{\otimes m}$. But, $\tau_m\circ(\mathrm{id}_{(A\otimes\mathbb{K})^{\otimes m}}\otimes g)\circ\tau_m^{-1}=g\otimes\mathrm{id}_{(A\otimes\mathbb{K})^{\otimes m}}$. $\qquad\square$

Recall that $KU_n^A=\hom_{\mathrm{gr}}(\widehat{S},(\mathbb{C}\ell_1\otimes A\otimes\mathbb{K})^{\widehat{\otimes}n})$. There is a unique isomorphism, which preserves the order of factors of the same type:

$$\theta_n\colon(\mathbb{C}\ell_1\otimes A\otimes\mathbb{K})^{\widehat{\otimes}n}\to(\mathbb{C}\ell_1)^{\widehat{\otimes}n}\otimes(A\otimes\mathbb{K})^{\otimes n}\;,$$

where – as above – the grading on both sides arises from the grading of $\mathbb{C}\ell_1$ and $A\otimes\mathbb{K}$ is trivially graded. In particular, if $\sigma\in\Sigma_n$ is a permutation and $\sigma_*\colon(\mathbb{C}\ell_1\otimes A\otimes\mathbb{K})^{\widehat{\otimes}n}\to(\mathbb{C}\ell_1\otimes A\otimes\mathbb{K})^{\widehat{\otimes}n}$ is the operation permuting the factors of the graded tensor product, then

$$(1)\qquad\qquad\qquad\theta_n\circ\sigma_*=(\sigma_*^{\mathbb{C}\ell_1}\otimes\sigma_*^{A\otimes\mathbb{K}})\circ\theta_n$$

where $\sigma_*^{\mathbb{C}\ell_1}\colon(\mathbb{C}\ell_1)^{\widehat{\otimes}n}\to(\mathbb{C}\ell_1)^{\widehat{\otimes}n}$ and $\sigma_*^{A\otimes\mathbb{K}}\colon(A\otimes\mathbb{K})^{\otimes n}\to(A\otimes\mathbb{K})^{\otimes n}$ are the corresponding permutations. Thus, we can define an action of $G_A(\mathbf{n})$ on KU_n^A via

$$\kappa_n\colon G_A(\mathbf{n})\times KU_n^A\to KU_n^A\quad;\quad(g,\varphi)\mapsto\theta_n^{-1}\circ(\mathrm{id}_{\mathbb{C}\ell_1^{\widehat{\otimes}n}}\otimes g)\circ\theta_n\circ\varphi$$

Theorem 4.6. *Let A be a separable, unital, strongly self-absorbing C^*-algebra. Then the EH-$\mathcal{I}$-group G_A associated to A acts on the commutative symmetric ring spectrum $KU_\bullet^A$ via $\kappa_\bullet$ as defined above. We obtain a map of very special Γ-spaces $\Gamma(G_A)\to\Gamma(\Omega^\infty(KU^A)^*)$, which induces an isomorphism on all homotopy groups π_n of the associated spectra with $n>0$ and the inclusion $K_0(A)_+^\times\to K_0(A)^\times$ on π_0. If A is purely infinite [16, Def.4.1.2], this is an equivalence in the stable homotopy category.*

Proof. It is clear that κ_n preserves the basepoint of KU_n^A. The Σ_n-equivariance is a direct consequence of (1). Indeed, we have for a permutation $\sigma \in \Sigma_n$, $g \in G_A(\mathbf{n})$ and $\varphi \in KU_n^A$:

$$\kappa_n(\sigma^{A\otimes\mathbb{K}} \circ g \circ (\sigma^{A\otimes\mathbb{K}})^{-1}, \sigma \circ \varphi)$$
$$= \theta_n^{-1} \circ (\mathrm{id} \otimes (\sigma^{A\otimes\mathbb{K}} \circ g \circ (\sigma^{A\otimes\mathbb{K}})^{-1})) \circ \theta_n \circ \sigma \circ \varphi$$
$$= \theta_n^{-1} \circ (\sigma^{\mathbb{C}\ell_1} \otimes \sigma^{A\otimes\mathbb{K}}) \circ (\mathrm{id} \otimes g) \circ (\sigma^{\mathbb{C}\ell_1} \otimes \sigma^{A\otimes\mathbb{K}})^{-1} \circ \theta_n \circ \sigma \circ \varphi$$
$$= \sigma \circ \theta_n^{-1} \circ (\mathrm{id} \otimes g) \circ \theta_n \circ \varphi = \sigma \circ \kappa_n(g, \varphi) ,$$

where we omitted the stars from the notation.

Let $\tau \colon \mathbb{C}\ell_1^{\widehat{\otimes}m} \otimes (A\otimes\mathbb{K})^{\otimes m} \widehat{\otimes} \mathbb{C}\ell_1^{\widehat{\otimes}n} \otimes (A\otimes\mathbb{K})^{\otimes n} \to \mathbb{C}\ell_1^{\widehat{\otimes}m+n} \widehat{\otimes} (A\otimes\mathbb{K})^{\otimes m+n}$ be the permutation of the two middle factors, then $\tau \circ (\theta_m \widehat{\otimes} \theta_n) = \theta_{m+n}$. This implies that

$$(\kappa_m(g, \varphi) \widehat{\otimes} \kappa_n(h, \psi)) \circ \Delta = \kappa_{m+n}(g \otimes h, (\varphi \widehat{\otimes} \psi) \circ \Delta)$$

for $g \in G_A(\mathbf{m}), h \in G_A(\mathbf{n})$, $\varphi \in KU_m^A, \psi \in KU_n^A$, which is the compatibility condition in Definition 3.7 (ii). The same argument shows that for $l + m = n$, $g \in G(\mathbf{m})$ and $\varphi \in KU_m^A$

$$(\widehat{\eta}_l \widehat{\otimes} \kappa_m(g, \varphi)) \circ \Delta = \kappa_n(\mathrm{id} \otimes g, (\widehat{\eta}_l \widehat{\otimes} \varphi) \circ \Delta) ,$$
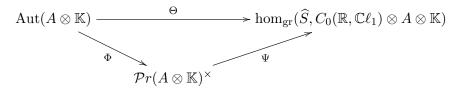
which is the crucial observation to see that diagram (iii) in Definition 3.7 commutes.

Thus, we have proven that $G = G_A$ acts on the spectrum KU^A. By Theorem 3.8 together with Theorem 3.6 we obtain a map of Γ-spaces $\Gamma(B_\nu G_A) \to B_\mu \Gamma(G_A) \to B_\mu \Gamma(\Omega^\infty(KU^A)^*)$ where the first map is a strict equivalence. We see from Lemma 3.3 that $G_{h\mathcal{I}}$ is in fact a grouplike topological monoid, i.e. $G_{h\mathcal{I}} \to \Omega B_\mu G_{h\mathcal{I}}$ is a homotopy equivalence. Thus, to finish the proof, we only need to check that $G_{h\mathcal{I}} \to GL_1(KU^A) = \Omega^\infty(KU^A)^*_{h\mathcal{I}}$ has the desired properties. Consider the diagram

$$\begin{array}{ccc} G_{h\mathcal{I}} & \longrightarrow & GL_1(KU^A) \\ {\scriptstyle\simeq}\uparrow & & \uparrow{\scriptstyle\simeq} \\ G(\mathbf{1}) & \longrightarrow & \Omega^\infty(KU^A)^*(\mathbf{1}) \end{array}$$

where the vertical maps are given by the inclusions into the zero skeleton of the homotopy colimit. The latter are equivalences by [19, Lemma 2.1]. By this lemma, we also see that $\pi_0(GL_1(KU^A)) = GL_1(\pi_0(KU^A)) = GL_1(K_0(A)) = K_0(A)^\times$. It remains to be seen that

$$\Theta \colon \mathrm{Aut}(A \otimes \mathbb{K}) \to \Omega KU_1^A = \hom_{\mathrm{gr}}(\widehat{S}, C_0(\mathbb{R}, \mathbb{C}\ell_1) \otimes A \otimes \mathbb{K}) \quad ; \quad g \mapsto \kappa_1(g, \widehat{\eta}_1)$$

induces an isomorphism on all homotopy groups π_n with $n > 0$ and the inclusion $K_0(A)_+^\times \to K_0(A)$ on π_0. The basepoint of the target space is now given by $\widehat{\eta}_1$ instead of the zero homomorphism. Θ

fits into a commutative diagram

$$\mathrm{Aut}(A \otimes \mathbb{K}) \xrightarrow{\quad\Theta\quad} \hom_{\mathrm{gr}}(\widehat{S}, C_0(\mathbb{R}, \mathbb{C}\ell_1) \otimes A \otimes \mathbb{K})$$
$$\Phi \searrow \qquad \nearrow \Psi$$
$$\mathcal{P}r(A \otimes \mathbb{K})^{\times}$$

with $\Phi(g) = g(1 \otimes e)$ and $\Psi(p) = \widetilde{\eta}_1 \otimes p$, where $\widetilde{\eta}_1 \in \hom_{\mathrm{gr}}(\widehat{S}, C_0(\mathbb{R}, \mathbb{C}\ell_1))$ arises from the functional calculus of the operator described after Definition 4.1. It was shown in [5, Thm.2.16, Thm.2.5] that Φ is a homotopy equivalence. Let $\epsilon \in \hom_{\mathrm{gr}}(\widehat{S}, \mathbb{C})$ be the counit of $\widehat{S}$. The map Ψ factors as

$$\Psi \colon \mathcal{P}r(A \otimes \mathbb{K})^{\times} \to \hom_{\mathrm{gr}}(\widehat{S}, A \widehat{\otimes} \widehat{\mathbb{K}}) \to \hom_{\mathrm{gr}}(\widehat{S}, C_0(\mathbb{R}, \mathbb{C}\ell_1) \otimes A \otimes \mathbb{K})$$

where the first map sends a projection p to $\epsilon \cdot \left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and the second sends φ to $\varphi \widehat{\otimes} \widetilde{\eta}_1 \circ \Delta$ and applies the graded isomorphism to shift the grading to $C_0(\mathbb{R}, \mathbb{C}\ell_1)$ only. Since the second map induces multiplication with the Bott element, it is an isomorphism on π_0. It was proven in [5, Cor.2.17] that $\pi_0(\mathcal{P}r(A \otimes \mathbb{K})^{\times}) \cong K_0(A)^{\times}_+$. The discussion after the proof of [24, Thm.4.7] shows that the first map induces the inclusion $K_0(A)^{\times}_+ \to K_0(A)$ on π_0.

Let B be a graded, σ-unital C^*-algebra and define $K'_n(B)$ to be the kernel of the map $K'(C(S^n) \otimes B) \to K'(B)$ induced by evaluation at the basepoint. Here, we used the notation $K'(B) = \pi_0(\hom_{\mathrm{gr}}(\widehat{S}, B \widehat{\otimes} \widehat{K}))$ introduced in [24]. The five lemma shows that $K'_n(B)$ is in fact isomorphic to $K_n(B)$, if we identify the latter with the kernel $K_0(C(S^n) \otimes B) \to K_0(B)$. For $n > 0$ we have the commutative diagram

$$\pi_n(\mathcal{P}r(A \otimes \mathbb{K})^{\times}, 1 \otimes e) \xrightarrow{\Psi_*} \pi_n(\Omega KU_1^A, \widehat{\eta}_1)$$
$$\cong \big\downarrow \qquad\qquad\qquad\qquad \big\downarrow \cong$$
$$K_n(A) \xrightarrow{\quad\cong\quad} K'_n(C_0(\mathbb{R}, \mathbb{C}\ell_1) \otimes A)$$

where the lower horizontal map sends $[p] - [q] \in K_n(A) = \ker(K_0(C(S^n) \otimes A) \to K_0(A))$ to $\left[\widetilde{\eta}_1 \widehat{\otimes} \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)\right] \in K'_n(C_0(\mathbb{R}, \mathbb{C}\ell_1) \otimes A)$. The same argument as for π_0 above shows that this is an isomorphism. Every element $\gamma \colon (S^n, x_0) \to (\mathcal{P}r(A \otimes \mathbb{K}), 1 \otimes e)$ induces a projection $p_\gamma \in C(S^n) \otimes A \otimes \mathbb{K}$. The vertical map on the left sends $[\gamma]$ to $[p_\gamma] - [1_{C(S^n)} \otimes 1 \otimes e] \in K_0(C(S^n, x_0) \otimes A) \cong K_n(A)$. This map is an isomorphism by Bott periodicity. Finally, we can consider $\gamma' \colon S^n \to \Omega KU_1^A$ as an element $\varphi_{\gamma'} \in \hom_{\mathrm{gr}}(\widehat{S}, C(S^n) \otimes C_0(\mathbb{R}, \mathbb{C}\ell_1) \otimes A \otimes \mathbb{K})$ ($\mathbb{K}$ trivially graded!). The vertical map on the right hand side sends $[\gamma']$ to $\left[\left(\begin{smallmatrix} \varphi_{\gamma'} & 0 \\ 0 & 1_{C(S^n)} \otimes \widehat{\eta}_1 \end{smallmatrix}\right)\right] \in K'_n(C_0(\mathbb{R}, \mathbb{C}\ell_1) \otimes A)$. It corresponds to the 'subtraction of 1', i.e. it corrects the basepoint by shifting back to the component of the zero homomorphism. Its inverse is given by $\psi \mapsto \psi \oplus \left(\begin{smallmatrix} 1_{C(S^n)} \otimes \widehat{\eta}_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where $\oplus$ is the addition operation described in [24].

If A is purely infinite, we have $K_0(A)^{\times}_+ = K_0(A)^{\times}$, which implies the last statement. This finishes the proof. $\qquad\square$

4.3. **Applications.** In this section we apply the above results to some examples of strongly self-absorbing C^*-algebras starting with $\mathcal{O}_\infty$ and $\mathcal{Z}$. It is a consequence of [23, 26] that any unital $*$-homomorphism between two strongly self-absorbing C^*-algebras is unique up to asymptotic unitary equivalence. From a categorical point of view, the Jiang-Su algebra $\mathcal{Z}$ is characterized by the property that it is the unique infinite dimensional strongly self-absorbing C^*-algebra which maps into any other infinite dimensional strongly self-absorbing C^*-algebra by a unital $*$-homomorphism. In this sense it is initial among those [26]. Likewise the infinite Cuntz algebra $\mathcal{O}_\infty$ is the unique purely infinite strongly self-absorbing C^*-algebra that maps unitally into any other purely infinite strongly self-absorbing C^*-algebra. Alternatively, $\mathcal{O}_\infty$ is the universal unital C^*-algebra generated by countably infinitely many generators s_i that satisfy the relations $s_i^* s_j = \delta_{ij} 1$. For any locally compact Hausdorff space X the unit homomorphisms $\mathbb{C} \to \mathcal{O}_\infty$ and $\mathbb{C} \to \mathcal{Z}$ induce natural isomorphisms of multiplicative cohomology theories $K^0(X) = K_0(C(X)) \to K_0(C(X) \otimes \mathcal{Z})$ and $K^0(X) \to K_0(C(X) \otimes \mathcal{O}_\infty)$.

Theorem 4.7. *The very special Γ-spaces $\Gamma(G_{\mathcal{O}_\infty})$ and $\Gamma(\Omega^\infty(KU)^*)$ are strictly equivalent, which implies that the spectrum associated to $\Gamma(G_{\mathcal{O}_\infty})$ is equivalent to $gl_1(KU)$ in the stable homotopy category. In particular, $B\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathbb{K})$ is weakly homotopy equivalent to $BBU_\otimes \times B(\mathbb{Z}/2\mathbb{Z})$. Likewise, the spectrum associated to $\Gamma(G_{\mathcal{Z}})$ is equivalent to $sl_1(KU)$, the 0-connected cover of $gl_1(KU)$, and $B\mathrm{Aut}(\mathcal{Z} \otimes \mathbb{K})$ is weakly homotopy equivalent to $BBU_\otimes$.*

Proof. The unit homomorphism $\mathbb{C} \to \mathcal{O}_\infty$ yields $\Gamma(\Omega^\infty(KU^\mathbb{C})^*) \to \Gamma(\Omega^\infty(KU^{\mathcal{O}_\infty})^*)$. To see that this is a strict equivalence, it suffices to check that $GL_1(KU) \to GL_1(KU^{\mathcal{O}_\infty})$ is a weak equivalence. Let X be a finite CW-complex. The isomorphism $[X, \Omega^1(KU_1^{\mathcal{O}_\infty})] \cong K_0(C(X) \otimes \mathcal{O}_\infty)$ constructed above restricts to $[X, GL_1(KU^{\mathcal{O}_\infty})] \cong GL_1(K_0(C(X) \otimes \mathcal{O}_\infty))$ and similarly with $\mathbb{C}$ instead of $\mathcal{O}_\infty$. The composition

$$GL_1(K^0(X)) \cong [X, GL_1(KU)] \to [X, GL_1(KU^{\mathcal{O}_\infty})] \cong GL_1(K_0(C(X) \otimes \mathcal{O}_\infty))$$

is the restriction of the ring isomorphism $K^0(X) \to K_0(C(X) \otimes \mathcal{O}_\infty)$ to the invertible elements.

By Theorem 4.6 we obtain a strict equivalence $\Gamma(G_{\mathcal{O}_\infty}) \to \Gamma(\Omega^\infty(KU^{\mathcal{O}_\infty})^*)$. Therefore the zig-zag $\Gamma(G_{\mathcal{O}_\infty}) \to \Gamma(\Omega^\infty(KU^{\mathcal{O}_\infty})^*) \leftarrow \Gamma(\Omega^\infty(KU)^*)$ proves the first claim. From this, we get a weak equivalence between $BGL_1(KU) \simeq BBU_\otimes \times B(\mathbb{Z}/2\mathbb{Z})$ and $B_\nu G_A(\mathbf{1}) = B\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathbb{K})$ using Lemma 3.5 and the stability of G_A.

The unit homomorphism $\mathbb{C} \to \mathcal{Z}$ yields $\Gamma(\Omega^\infty(KU^\mathbb{C})^*) \to \Gamma(\Omega^\infty(KU^{\mathcal{Z}})^*)$, which is again a strict equivalence by the same reasoning as above. By Theorem 4.6 the map $\Gamma(G_{\mathcal{Z}}) \to \Gamma(\Omega^\infty(KU^{\mathcal{Z}})^*)$ induces an isomorphism on π_k for $k > 0$ and corresponds to the inclusion $K_0(\mathcal{Z})_+^\times \to K_0(\mathcal{Z})^\times$ on π_0. But since $K_0(\mathcal{Z})_+^\times \cong (\mathbb{Z})_+^\times$ is trivial, the spectrum associated to $\Gamma(G_{\mathcal{Z}})$ corresponds to the 0-connected cover of the one associated to $\Gamma(\Omega^\infty(KU^{\mathcal{Z}})^*)$, which is $sl_1(KU)$. This implies $B\mathrm{Aut}(\mathcal{Z} \otimes \mathbb{K}) \simeq BSL_1(KU) \simeq BBU_\otimes$ by Lemma 3.5. $\square$

The UHF-algebra M_{p^∞} is constructed as an infinite tensor product of matrix algebras $M_p(\mathbb{C})$. It is separable, unital, strongly self-absorbing with $K_0(M_{p^\infty}) \cong \mathbb{Z}[\frac{1}{p}]$, $K_1(M_{p^\infty}) = 0$. Likewise, if we

fix a prime p and choose a sequence $(d_j)_{j\in\mathbb{N}}$ such that each prime number except p appears in the sequence infinitely many times, we can recursively define $A_{n+1} = A_n \otimes M_{d_n}$, $A_0 = \mathbb{C}$. The direct limit $M_{(p)} = \lim A_n$ is a separable, unital, strongly self-absorbing C^*-algebra with $K_0(M_{(p)}) \cong \mathbb{Z}_{(p)}$ – the localization of the integers at p – and $K_1(M_{(p)}) = 0$.

Theorem 4.8. *The very special Γ-spaces $\Gamma(G_{M_{(p)}\otimes\mathcal{O}_\infty})$ and $\Gamma(\Omega^\infty(KU_{(p)})^*)$ are strictly equivalent, which implies that the spectrum associated to $\Gamma(G_{M_{(p)}\otimes\mathcal{O}_\infty})$ is equivalent to $gl_1(KU_{(p)})$ in the stable homotopy category. In particular, $B\mathrm{Aut}(M_{(p)} \otimes \mathcal{O}_\infty \otimes \mathbb{K})$ is weakly homotopy equivalent to $GL_1(KU_{(p)})$. The analogous statement for the localization away from p is also true if $M_{(p)}$ is replaced by M_{p^∞}.*

Proof. Consider the commutative symmetric ring spectrum $KU_\bullet^{M_{(p)}\otimes\mathcal{O}_\infty}$ with homotopy groups $\pi_{2k}(KU^{M_{(p)}\otimes\mathcal{O}_\infty}) = \mathbb{Z}_{(p)}$, $\pi_{2k+1}(KU^{M_{(p)}\otimes\mathcal{O}_\infty}) = 0$. Note that the unit homomorphism $\mathbb{C} \to M_{(p)} \otimes \mathcal{O}_\infty$ induces a map of spectra $KU^\mathbb{C} \to KU^{M_{(p)}\otimes\mathcal{O}_\infty}$, which is the localization map $\mathbb{Z} \to \mathbb{Z}_{(p)}$ on the non-zero coefficient groups. Let $S_{(p)}$ be the Moore spectrum, i.e. the commutative symmetric ring spectrum with $\widetilde{H}^1(S_{(p)};\mathbb{Z}) \cong \mathbb{Z}_{(p)}$ and $\widetilde{H}^k(S_{(p)};\mathbb{Z}) = 0$ for $k \neq 1$. We have $KU_{(p)} = KU \wedge S_{(p)}$ and $\pi_k(KU) \to \pi_k(KU_{(p)}) = \pi_k(KU) \otimes \mathbb{Z}_{(p)}$ is the localization map. It follows that $KU_{(p)} \to KU^{M_{(p)}\otimes\mathcal{O}_\infty} \wedge S_{(p)} \leftarrow KU^{M_{(p)}\otimes\mathcal{O}_\infty}$ is a zig-zag of π_*-equivalences. Just as in Theorem 4.7 we show that it induces a zig-zag of strict equivalences $\Gamma(\Omega^\infty(KU_{(p)})^*) \to \Gamma(\Omega^\infty(KU^{M_{(p)}\otimes\mathcal{O}_\infty} \wedge S_{(p)})^*) \leftarrow \Gamma(\Omega^\infty(KU^{M_{(p)}\otimes\mathcal{O}_\infty})^*)$, which shows that $gl_1(KU_{(p)})$ is stably equivalent to $gl_1(KU^{M_{(p)}\otimes\mathcal{O}_\infty})$. But by Theorem 4.6 we have a strict equivalence $\Gamma(G_{M_{(p)}\otimes\mathcal{O}_\infty}) \to \Gamma(\Omega^\infty(KU^{M_{(p)}\otimes\mathcal{O}_\infty})^*)$. The proof for the localization away from p is completely analogous; therefore we omit it. $\square$

In [5] the authors used a permutative category $\mathcal{B}_\otimes$ to show that $B\mathrm{Aut}(A \otimes \mathbb{K})$ carries an infinite loop space structure: The objects of $\mathcal{B}_\otimes$ are the natural numbers $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $\hom(m, n) = \{\alpha \in \hom((A\otimes\mathbb{K})^{\otimes m}, (A\otimes\mathbb{K})^{\otimes n}) \mid \alpha((1\otimes e)^{\otimes m}) \in GL_1(K_0((A\otimes\mathbb{K})^{\otimes n}))\}$, where $e \in \mathbb{K}$ is a rank 1-projection and $(1 \otimes e)^{\otimes 0} = 1 \in \mathbb{C}$. Since $\hom(0, 1)$ is non-empty, there is a stabilization θ of the object $1 \in \mathrm{obj}(\mathcal{B}_\otimes)$ by Lemma 3.10. In fact, we may choose $\theta(\iota_\mathbf{1}) \in \hom(\mathbb{C}, A \otimes \mathbb{K})$ to be $\theta(\lambda) = \lambda(1 \otimes e)$.

Theorem 4.9. *Let A be a separable, strongly self-absorbing C^*-algebra. Then there is a strict equivalence of Γ-spaces $\Gamma(B_\nu G_A) \to \Gamma(\mathcal{B}_\otimes)$. In particular, the induced infinite loop space structures on $B\mathrm{Aut}(A \otimes \mathbb{K})$ agree.*

Proof. Note that $\mathrm{Aut}_{\mathcal{B}_\otimes}(1) = \mathrm{Aut}(A \otimes \mathbb{K})$ and that the maps $B_\nu G_A(\mathbf{m}) = B\mathrm{Aut}((A \otimes \mathbb{K})^{\otimes m}) \to |N_\bullet\mathcal{B}_\otimes|$ are homotopy equivalences for $m > 0$. Thus, the statement follows from Theorem 3.13. $\square$

The above identification may be used to prove theorems about bundles (and continuous fields) of strongly self-absorbing C^*-algebras using what is known about the unit spectrum of K-theory. As an example let X be a compact metrizable space and consider the cohomology group $[X, B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})]$, where we use the notation of [5]. Note that the third Postnikov section of $B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})$ is a $K(\mathbb{Z}, 3)$, let $B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K}) \to K(\mathbb{Z}, 3)$ be the corresponding map and denote

by F its homotopy fiber. The composition $B\mathrm{Aut}(\mathbb{K}) \to B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K}) \to K(\mathbb{Z}, 3)$, where the first map is induced by the unit homomorphism $\mathbb{C} \to \mathcal{O}_\infty$, is a homotopy equivalence. Therefore we obtain a homotopy splitting

$$B\mathrm{Aut}(\mathbb{K}) \times F \xrightarrow{\simeq} B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})$$

and a corresponding fibration $B\mathrm{Aut}(\mathbb{K}) \to B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K}) \to F$. The weak equivalence between $B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})$ and $BBU_\otimes \simeq BBU(1) \times BBSU_\otimes$ identifies F with the corresponding homotopy fiber of the third Postnikov section of $BBU_\otimes$, which is $BBSU_\otimes$. Thus, we obtain a short exact sequence

$$0 \to [X, B\mathrm{Aut}(\mathbb{K})] \to [X, B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})] \to [X, F] \cong bsu_\otimes^1(X) \to 0$$

The following was proven by Gómez in [9, Theorem 5].

Theorem 4.10. *Let G be a compact Lie group. Then $bsu_\otimes^1(BG) = 0$.*

In light of our previous results, we obtain the Corollary:

Corollary 4.11. *Let X be a compact metrizable space, let G be a compact Lie group and let $P \to X$ be a principal G-bundle. Let $\alpha\colon G \to \mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})$ be a continuous homomorphism. Then the associated $\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})$-bundle $Q \to X$ given by $Q = P \times_\alpha \mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})$ is isomorphic to $\widetilde{Q} \times_{\mathrm{Ad}} \mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})$ for a principal $PU(H)$-bundle $\widetilde{Q}$.*

Proof. The class $[Q] \in [X, B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})]$ is the pullback of $[B\alpha] \in [BG, B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})]$ via the classifying map $f_P\colon X \to BG$ of P. The diagram

$$
\begin{array}{ccc}
[BG, B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})] & \longrightarrow & [BG, F] \cong bsu_\otimes^1(BG) \\
\downarrow & & \downarrow \\
\end{array}
$$

$$
0 \longrightarrow [X, B\mathrm{Aut}(\mathbb{K})] \longrightarrow [X, B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})] \longrightarrow [X, F] \longrightarrow 0
$$

shows that $[Q]$ is mapped to 0 in $[X, F] \cong bsu_\otimes^1(X)$, which implies the statement. $\qquad\square$

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