OBSTRUCTIONS TO MATRICIAL STABILITY OF DISCRETE GROUPS AND ALMOST FLAT K-THEORY

MARIUS DADARLAT

ABSTRACT. A discrete countable group G is matricially stable if the finite dimensional approximate unitary representations of G are perturbable to genuine representations in the point-norm topology. For large classes of groups G, we show that matricial stability implies the vanishing of the rational cohomology of G in all nonzero even dimensions. We revisit a method of constructing almost flat K-theory classes of BG which involves the dual assembly map and quasidiagonality properties of G. The existence of almost flat K-theory classes of BG which are not flat represents an obstruction to matricial stability of G due to continuity properties of the approximate monodromy correspondence.

1. INTRODUCTION

The realization that there are K-theory obstructions to perturbing approximate finite dimensional representations of C^*-algebras to genuine representations has emerged through work of Voiculescu [86], Connes, Gromov and Moscovici [19] and Connes and Higson [20].

In the realm of groups, it is natural to ask when an approximate representation is close to a genuine representation. The reader is referred to the survey papers by Arzhantseva [2] and Thom [80] for a background discussion and more context for this question. All norms considered in this paper refer to the uniform operator norm, $\|T\| = \sup_{\|\xi\| \le 1} \|T\xi\|$. M_n will denote the $n \times n$ complex matrices. We consider only countable discrete groups G. By a group representation we will always mean a unitary representation. A sequence of unital maps $\{\varphi_n : G \to U(k_n)\}_{n \in \mathbb{N}}$ is called an asymptotic homomorphism if

$$(1) \qquad \lim_{n \to \infty} \|\varphi_n(st) - \varphi_n(s)\varphi_n(t)\| = 0, \text{ for all } s, t \in G.$$

G is an MF-*group* [14] or a $(U(k_n), \|\cdot\|)_n$ approximated group (in the terminology of [33]) if there exists an asymptotic homomorphism that separates the elements of G in the sense that

$$(2) \qquad \limsup_{n \to \infty} \|\varphi_n(s) - 1\| > 0, \text{ for all } s \in G \setminus \{e\}.$$

It is an open problem to find examples of discrete countable groups which are not MF. Amenable groups, or more generally, groups that are locally embeddable in amenable groups (LEA) are MF by the breakthrough paper of Tikuisis, White and Winter [81].

A group G is called $(U(k_n), \|\cdot\|)_n$-stable [33] or *matricially stable* [37] if for any asymptotic homomorphism $\{\varphi_n : G \to U(k_n)\}_{n \in \mathbb{N}}$ (not necessarily a separating one) there is a sequence of homomorphisms $\{\pi_n : G \to U(k_n)\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} \|\varphi_n(s) - \pi_n(s)\| = 0$ for all $s \in G$. A systematic study of matricial stability was undertaken by Eilers, Shulman and Sørensen in [37].

In his seminal paper [86], Voiculescu showed that $\mathbb{Z}^2$ is not matricially stable by using a Fredholm index argument. However, he suggested that this phenomenon is due to the nonvanishing of the 2-cohomology of $\mathbb{T}^2$. Since $B\mathbb{Z}^2 = \mathbb{T}^2$ this amounts to the nonvanishing of $H^2(\mathbb{Z}^2, \mathbb{Z})$. The role of K-theory in Voiculescu's example was highlighted by Exel and Loring [39].

The aim of this note is to point out that the nonvanishing of even rational cohomology in positive dimensions is a first obstruction to matricial stability for large classes of discrete groups.

Theorem 1.1. *Let G be a countable discrete MF-group that admits a γ-element (e.g. G is uniformly embeddable in a Hilbert space). If $H^{2k}(G, \mathbb{Q}) \neq 0$ for some $k \geq 1$, then G is not matricially stable.*

For the proof, we use quasidiagonality and KK-theory methods [25], [65] to show that the rational K-theory of BG is almost flat. Once that is achieved, Theorem 1.1 is essentially a cohomological expression of the fact that a group G cannot be matricially stable if BG admits almost flat K-theory classes which are not flat. Indeed, the approximate monodromy representations associated to these nontrivial almost flat classes, [19], are not perturbable to genuine representations of G, due to continuity properties of the approximate monodromy correspondence [13, Thm.3.3].

We also consider completely positive (cp) versions of the properties MF and matricial stability and show that the same cohomological obstructions persists, by refining the argument outlined above. A unital map $\varphi : G \to M_n$ is completely positive definite if for any finite set of elements $s_1, ..., s_r$ of G, the matrix $(\varphi(s_i^{-1}s_j))$ is positive. By a theorem of Naimark, this is the case if and only if φ extends to a unital completely positive (ucp) map $C^*(G) \to M_n$ on the C^*-algebra of G.

A group G is *weakly quasidiagonal* if there is a sequence $\{\varphi_n : G \to M_{k_n}\}_{n \in \mathbb{N}}$ of unital completely positive definite maps which satisfies the conditions (1) and (2), see also Definition 3.10. A group G is *weakly matricially stable* if for any sequence $\{\varphi_n : G \to M_{k_n}\}_{n \in \mathbb{N}}$ of unital completely positive definite maps which satisfies the condition (1), there are two sequences of homomorphisms $\{\pi_n^{(i)} : G \to U(k_n^{(i)})\}_{n \in \mathbb{N}}$, $i = 0, 1$ such that $\lim_{n \to \infty} \|\varphi_n(s) \oplus \pi_n^{(0)}(s) - \pi_n^{(1)}(s)\| = 0$ for all $s \in G$.

It is clear from definitions that MF $\Rightarrow$ weak quasidiagonality and matricial stability $\Rightarrow$ weak matricial stability. A group G is *quasidiagonal* if it is isomorphic to a subgroup of the unitary group of a quasidiagonal C^*-algebra, see Definition 3.4. Quasidiagonal groups are weakly quasidiagonal. A group is maximally almost periodic (MAP) if its finite dimensional unitary representations separate the points. MAP groups and residually amenable groups are quasidiagonal by [81]. It is immediate from definition that a matricially stable MF-group must be MAP. We can then derive Theorem 1.1 from a more general statement which shows that nonvanishing of the rational even cohomology also obstructs weak matricial stability:

Theorem 1.2. *Let G be a countable discrete weakly quasidiagonal group that admits a γ-element. If G is weakly matricially stable, then G is MAP and $H^{2k}(G, \mathbb{Q}) = 0$ for all $k \geq 1$.*

In Example 2.7, we indicate a 3-dimensional Bieberbach group which is weakly matricially stable but not matricially stable. A linear group is a subgroup of $GL_n(F)$ where F is a field.

Corollary 1.3. *If G is a countable linear group such that $H^{2k}(G, \mathbb{Q}) \neq 0$ for some $k \geq 1$, then G is not weakly matricially stable.*

More example of groups which are not matricially stable are given in Section 2. It will follow from the proof of Theorem 1.2 that if the classifying space BG is a finite simplicial complex and G is weakly matricially stable then $K^0(BG)$ must be generated by flat bundles, Remark 4.8. If $\gamma = 1$ (as is the case for groups with Haagerup's property [48]), the Baum-Connes map is an isomorphism and so Theorem 1.2 implies that $K_0(C^*(G)) \otimes \mathbb{Q} \cong \mathbb{Q}$ whenever G is torsion free. In contrast, while any finite group G is matricially stable, $K^0(C^*(G)) \cong K_0(C^*(G)) \cong \mathbb{Z}^n$, where n is the number of equivalence classes of irreducible representations of G. Kasparov introduced the ring $KK_G(\mathbb{C}, \mathbb{C}) \cong K^0(C^*(G))$ as a generalization of the representation ring of finite (compact) groups. Let us denote by $R(G)_{fin}$ the subring of $K^0(C^*(G))$ generated by finite dimensional unitary representations. For the sake of simplicity, we will illustrate how one can obtain somewhat stronger obstructions to matrix stability in the case of groups with Haagerup's property.

Theorem 1.4. *Let G be a countable discrete weakly quasidiagonal group with Haagerup's property. Suppose that G is weakly matricially stable. Then G is MAP and:*

(i) *If $K_*(C^*(G))$ is finitely generated, then $K^0(C^*(G)) = R(G)_{fin}$*

(ii) *If G is torsion free, then $K_0(C^*(G)) \otimes \mathbb{Q} \cong \mathbb{Q}$.*

(iii) *If BG is a finite simplicial complex, then $K^0(BG)$ is generated by flat bundles.*

If a group G with Haagerup's property admits a G-compact model of $\underline{E}G$ then $K_*(C^*(G))$ is finitely generated, see [69]. If we assume moreover that BG is a finite simplicial complex, then $K^0(C^*(G)) = R(G)_{fin}$ if and only if $K^0(BG)$ is generated by flat bundles, see Remark 4.8.

Theorem 1.5. *Let G be an amenable countable discrete such that $K_*(C^*(G))$ is finitely generated. Then G is weakly matricially stable if and only if G is MAP and $K^0(C^*(G)) = R(G)_{fin}$.*

In a recent paper, Ioana, Spaas and Wiersma [49] use nonvanishing of 2-cohomology groups in conjunction with the relative Property (T) to exhibit obstructions to cp-lifting for full group C^*-algebras. In particular it follows from their arguments that if $G = \mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ (or if G is any other group that satisfies the assumptions of their Theorem A) then there is a unital $*$-homomorphism $C^*(G) \to \prod_n M_n / \bigoplus_n M_n$ which does not even have a ucp lifting and hence G is far from being matricially stable. However, it should be noted that the absence of a ucp lifting is only one aspect of nonstability. Indeed, Corollary 1.3 shows that there are $*$-homomorphisms $C^*(\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})) \to \prod_n M_n / \bigoplus_n M_n$ which have ucp liftings but do not lift to $*$-homomorphisms.

In Section 3 we discuss some basic aspects of MF, weakly quasidiagonal and quasidiagonal groups as these are the classes of groups that concern our main results. The proof of Theorem 1.2 is given in Section 4 and the proofs of Theorems 1.4 and 1.5 are given in Section 5. We rely on ideas and techniques developed in connection with the strong Novikov conjecture and the Baum-Connes conjecture [3], [54], [56], [52], [19], [90], [48], [15], [77], [84]. The statement of Theorem 1.1 for groups G with compact classifying space and $C^*(G)$ quasidiagonal can be derived from our earlier paper [25]. In view of [25, Prop.3.2], the papers [25], [24], [12], [31], [30] implicitly exhibit classes of groups which are not matricially stable. Just like in [25], we employ here the concept of quasidiagonality to produce local approximations of the dual assembly map, but additionally, we use a key idea of Kubota [65] and consider quasidiagonal C^*-algebras which are intermediate between the full and the reduced group C^*-algebras to enlarge the class of groups for which these approximation methods are applicable. In Section 6, we revisit briefly the KK-theoretic approach to almost flat K-theory that we introduced in [25]. In view of [65], this method extends now to the class of quasidiagonal groups. The problem of constructing almost flat classes on classifying spaces was considered first by Connes, Gromov and Moscovici [19] and Gromov in [41, 42]. For now, it seems that the use of quasidiagonal KK-theory classes remains the most effective approach to this problem.

The class of groups that admit a γ-element is very large as shown by work of Kasparov and Skandalis [56], [53]. Moreover, J.L.Tu has shown that if G is a discrete countable group that admits a uniform embedding in a Hilbert space, then G admits a γ-element [84]. The amenable groups, or more generally, the groups with Haagerup's property are uniformly embeddable in a Hilbert space [16] and so are the linear groups as shown by Guentner, Higson and Weinberger [45]. The class of groups that admit a uniform embedding in a Hilbert space is closed under subgroups and products, direct limits, free products with amalgam, and extensions by exact groups [27].

2. Examples

In this section we exhibit classes of groups which are not (weakly) matricially stable by virtue of Theorems 1.1 and 1.2. In view of our earlier discussion, it is not surprising that the list of

matricially stable groups cannot be too extensive. It was shown in [37] that all finitely generated virtually free groups are matricially stable. In particular it follows that $SL_2(\mathbb{Z})$ is matricially stable. Moreover, it was established in [37] that the only crystallographic groups that are matricially stable are the two line groups $\mathbb{Z}$ and $\mathbb{Z}/2 * \mathbb{Z}/2$ and the 12 wallpaper groups that contain at least one reflection or one glide reflection.

Proof of Corollary 1.3. Any linear group G is uniformly embeddable in a Hilbert space by [45]. Write G as the increasing union of a sequence $(G_n)_n$ of finitely generated subgroups. Each G_n is residually finite by a classic theorem of Malcev [11, 6.4.13]. It follows then that G is weakly quasidiagonal by Proposition 3.14. We conclude the proof by applying Theorem 1.2. $\qquad\square$

Corollary 2.1. *Let G be a hyperbolic residually finite group with $H^{2k}(G;\mathbb{Q}) \neq 0$ for some $k \geq 1$. Then G is not weakly matricially stable.*

Proof. Hyperbolic groups are exact [11] and hence they admit a γ-element by [84]. Thus the statement follows from Theorem 1.2. It is not known whether there exists a hyperbolic group which is not residually finite. $\qquad\square$

The following generalizes a result obtained in [37] for two-step nilpotent groups.

Corollary 2.2. $\mathbb{Z}$ *is the only nontrivial matricially stable torsion free finitely generated nilpotent group.*

Proof. Let G be a nontrivial torsion free finitely generated nilpotent group. Malcev has shown that G is linear and that one can associate to G a finite dimensional rational Lie algebra of dimension equal to the Hirsch number of G. By [70], $H^*(G;\mathbb{Q}) \cong H^*(L;\mathbb{Q})$. But $H^2(L,\mathbb{Q}) \neq 0$ if L has dimension > 1 by a result of Ado as explained in [17, p.86]. Thus if G is matricially stable, then $L = \mathbb{Q}$ and hence $G = \mathbb{Z}$. $\qquad\square$

If G is matricially stable and H is finite, then $G \times H$ is matricially stable since $C^*(G \times H) \cong C^*(G) \otimes C^*(H)$ and $C^*(H)$ is a finite dimensional C^*-algebra. The next corollary notes that up to rational cohomology these are the only possible examples of matricially stable direct products.

Corollary 2.3. *Let G_1, G_2 be countable discrete MF-groups that admit γ-elements. If $G_1 \times G_2$ is matricially stable, then one of the two groups must have the rational cohomology of a trivial group.*

Proof. It is immediate that both groups must be matricially stable and hence $H^{2k}(G_i;\mathbb{Q}) = 0$, $i = 1, 2$, for $k \geq 1$. Seeking a contradiction, suppose that there are $k_i \geq 0$ such that $H^{2k_i+1}(G_i;\mathbb{Q}) \neq 0$ for $i = 1, 2$. The Künneth formula implies that $H^{2k_1+2k_2+2}(G_1 \times G_2;\mathbb{Q}) \neq 0$ which contradicts Theorem 1.1. $\qquad\square$

One-relator groups. A one-relator group is a group with a presentation of the form $\langle S; r \rangle$, where r is single element in the free group $F(S)$ on the countable generating set S. An important example is the surface group

$$\Gamma_g = \pi_1(S_g) = \langle s_1, t_1, ..., s_g, t_g \,; \prod_{i=1}^{g} [s_i, t_i] \rangle,$$

where S_g is a connected closed orientable surface of genus g. Another important class of examples consists of the Baumslag-Solitar groups

$$BS(m, n) = \langle s, t \,; st^m s^{-1} t^{-n} \rangle, \quad m, n \in \mathbb{Z} \setminus \{0\}.$$

In general, one can always write $G = \langle S; s^n \rangle$ where s is not a proper power of an element of $F(S)$ and $n \geq 1$. By a theorem of Karrass, Magnus and Solitar [5, Thm.5.2], G is torsion free if

and only if $n = 1$. By a theorem of Wise [87], all torsion one-relator groups are residually finite (and hence MF). It is not known if all one-relator groups are MF or hyperlinear.

Corollary 2.4. *Let* $G = \langle S; s^n \rangle$ *be a one-relator group.*
   (i) *If* G *is MF (this is automatic if* $n > 1$*) and* $s \in [F(S), F(S)]$ *i.e. the image of* s *is trivial in* G_{ab}*, then* G *is not matricially stable.*
   (ii) *The surface groups* Γ_g *are not weakly matricially stable for* $g \geq 1$*.*
   (iii) *The Baumslag-Solitar groups* $BS(m, n)$ *are not weakly matricially stable if* $|m|, |n| \geq 2$ *and either* $|m| \neq |n|$ *or* $m = n$*.*

Proof. (i) It is known that $H^2(G; \mathbb{Q}) = \mathbb{Q}$ if $s \in [F(S), F(S)]$ and $H^2(G; \mathbb{Q}) = 0$ if $s \notin [F(S), F(S)]$. This is explained in [6, p.279] based on [5]. Since one-relator groups are exact by [83], [44] we can invoke Theorem 1.1.

   (ii) Γ_g are linear groups as a consequence of the uniformization theorem. A very short proof that surface groups are residually finite was given by Hempel [47]. The conclusion follows from (i). Previous proofs of (ii) are given (implicitly) in [24] and in [37].

   (iii) It was shown by Kropholler [63] that $BS(m, n)$ has the second derived group free and hence it is residually solvable as noted for example in [18, Ex.1.1]. Since $BS(m, n)$ is residually solvable, it is residually amenable and hence it is a quasidiagonal group by [81]. On the other hand, it was shown by Meskin [67] that $BS(m, n)$ is residually finite if and only if $|m| = 1$ or $|n| = 1$ or $|m| = |n|$. Thus $BS(m, n)$ is not residually finite if $|m|, |n| \geq 2$ and $|m| \neq |n|$ and hence it is not MAP. It follows that it cannot be weakly matricially stable. The case $m = n$ follows from (i) since $st^m s^{-1} t^{-n}$ is a commutator. The groups $B(m, m)$ and $B(2, 3)$ were already shown not to be matricially stable in [37]. $B(1, -1) \cong B(-1, 1)$ is matricially stable by [37]. The answer to matricial stability is unknown for the other groups in this class. $\square$

The mapping class group of S_g, denoted by $Mod(S_g)$, is defined as the group of the isotopy classes of orientation-preserving diffeomorphisms [40].

Corollary 2.5. *The mapping class group* $Mod(S_g)$ *is not weakly matricially stable for* $g \geq 3$*.*

Proof. The assumptions of Theorem 1.2 are verified due to the following important results. $Mod(S_g)$ is an exact (boundary amenable) group by work of Kida [57] and Hamenstädt [46] and it is residually finite by [40, Thm.6.1]. In addition, $H^2(Mod(S_g); \mathbb{Z}) \cong \mathbb{Z}$ for $g \geq 4$ by a result of Harer [40, Thm.5.8] and $H^2(Mod(S_3); \mathbb{Z})$ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$ by [62]. $\square$

Corollary 2.6. *The groups* $Aut(\mathbb{F}_n)$ *and* $Out(\mathbb{F}_n)$ *are not weakly matricially stable for* $n = 4, 6, 8$*.*

Proof. $Aut(\mathbb{F}_n)$ and $Out(\mathbb{F}_n)$ are residually finite by [4] and [43], respectively. The group $Out(\mathbb{F}_n)$ is exact by [68]. Since the center of $\mathbb{F}_n$ is trivial $Inn(\mathbb{F}_n) \cong \mathbb{F}_n$ and hence $Aut(\mathbb{F}_n)$ is exact being extension of an exact group by another exact group [59]. The cohomology of these groups is very hard to compute. Nevertheless it is known that $H^{4k}(Aut(\mathbb{F}_{2k+2}); \mathbb{Q})) \neq 0 \neq H^{4k}(Out(\mathbb{F}_{2k+2}); \mathbb{Q}))$ for $k = 1, 2, 3$. [85]. $\square$

Example 2.7. The vanishing of $H^{2k}(G; \mathbb{Q})$, $k \geq 1$, is only a first obstruction to matricial stability. Let G be the 3-dimensional Bieberbach group

$$G = \langle x, w, y \ : \ w^{-1} zw = x^{-1} zx = z^{-1}, \ w^{-1} xw = x^{-1} z \rangle.$$

G is not matricially stable, since as shown in [37] no crystallographic group of dimension ≥ 3 is matricially stable. We will argue that G is weakly matricially stable. One knows that $H_1(G; \mathbb{Z}) = G_{ab} = \mathbb{Z} \times \mathbb{Z}/4$ by [21] and $H^3(G, \mathbb{Z}) \cong \mathbb{Z}/2$ as the corresponding flat 3-manifold is not orientable

[88, p.120]. Since the Euler characteristic of a nontrivial torsion-free polycyclic-by-finite group G is zero by [35], one concludes that the rational cohomology of G is given by: $H^0(G;\mathbb{Q}) = H^1(G;\mathbb{Q}) = \mathbb{Q}$, $H^2(G;\mathbb{Q}) = H^3(G;\mathbb{Q}) = 0$. To show that G is weakly matricially stable, by Theorem 1.5 and Remark 4.8, it suffices to show that $K^0(BG)$ is generated by flat bundles. Since BG is 3-dimensional, $K^0(BG)$ is generated by line bundles and so $K^0(BG) \cong \mathbb{Z} \oplus H^2(BG,\mathbb{Z})$. By the universal coefficient theorem, $H^2(BG,\mathbb{Z}) \cong \text{Ext}(H_1(BG,\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}/4$ is a torsion group and hence its elements are represented by flat line bundles by [72, 2.6].

3. Quasidiagonal Groups

In this section we discuss some very basic aspects of the classes of groups (MF, quasidiagonal and weakly quasidiagonal) that appear in the statements of our main results: 1.1, 1.2 and 6.1. All the groups G that we consider are countable and discrete.

Recall that a C^*-algebra is MF if it embeds in $\prod_n M_{k_n}/\bigoplus_n M_{k_n}$, for some sequence (k_n), [8].

Definition 3.1 ([14]). A group G is MF if it is isomorphic to a subgroup of the unitary group of an MF-algebra.

Proposition 3.2. *The following assertion are equivalent.*

(i) *G is an MF group*
(ii) *G embeds in $\mathbf{U}/\mathbf{N}$ where $\mathbf{U} = \prod_{n=1}^{\infty} U(n)$ and $\mathbf{N} = \{(u_n)_n \in \mathbf{U} : \|u_n - 1_n\| \to 0\}$.*
(iii) *For each finite subset F of G and for any $\epsilon > 0$ there is a unital map $\varphi : G \to U(n)$ such that $\|\varphi(st) - \varphi(s)\varphi(t)\| < \varepsilon$ for all $s,t \in F$ and $\|\varphi(s) - 1\| \geq \sqrt{2}$ for all $s \in F \setminus \{e\}$.*

Proof. $(i) \Leftrightarrow (ii)$ is trivial whereas $(iii) \Leftrightarrow (i)$ is proved in [61]. See also Proposition 3.12 for a related property. $\qquad\square$

Remark 3.3. (i) Proposition 3.2 implies that a group that is locally embeddable in MF groups is MF. Since amenable groups are MF by [81], it follows that LEA groups are MF.

(ii) Thom [79] showed that a certain group K constructed by de Cornulier is hyperlinear but not LEA. The group K is the quotient of a property (T) subgroup of $SL_8(\mathbb{Z}[1/p])$ by a cyclic central subgroup. If N is a central subgroup of a residually finite group G, then G/N is MF by [14, Thm.2.17]. Thus K is also an example of an MF group which not LEA.

Definition 3.4. A countable discrete group G is *quasidiagonal* if it is isomorphic to a subgroup of the unitary group of a quasidiagonal C^*-algebra.

Equivalently, G is quasidiagonal if it admits a quasidiagonal faithful unitary representation on a separable Hilbert space. A representation $\pi : G \to U(H)$ is quasidiagonal if there is an increasing sequence $(p_n)_n$ of finite dimensional projections which converges strongly to 1_H and such that $\lim_{n\to\infty} \|[\pi(s), p_n]\| = 0$ for all $s \in G$. In other words, the C^*-algebra $C^*_\pi(G) = C^*(\pi(G))$ is quasidiagonal.

Remark 3.5. (i) If $G \subset \prod_n U(B_n)$ with B_n separable and quasidiagonal, then G is quasidiagonal since the C^*-algebra $\prod_n B_n$ is quasidiagonal. Thus the MAP groups are quasidiagonal. It also follows that the residually amenable groups are quasidiagonal by [81].

(ii) If H is quasidiagonal and G is amenable, then the wreath product $H \wr G$ is quasidiagonal. Indeed, by [32, Thm.4.2], if D is a unital separable quasidiagonal C^*-algebra and G is a countable discrete amenable group, then the crossed product $(\bigotimes_G D) \rtimes G$ is quasidiagonal (here we work with minimal tensor products and G acts via noncommutative Bernoulli shifts). Thus, if there is an embedding $\omega : H \to U(D)$, then the map $h \to 1_D \oplus \omega(h) \in U_2(D)$ induces an embedding

$\bigoplus_G H \to U(\bigotimes_G M_2(D))$. It follows that $H \wr G = (\bigoplus_G H) \rtimes G$ embeds in the unitary group of $(\bigotimes_G M_2(D)) \rtimes G$ and so $H \wr G$ is quasidiagonal.

(iii) The class of quasidiagonal groups is strictly larger than the class of residually amenable groups as noted in Example 3.6 below. We do not have examples of quasidiagonal groups which are not LEA.

Example 3.6. Let $O_n(\mathbb{Q}) \subset O_n(\mathbb{R})$ be the group of orthogonal matrices with rational entries. $O_n(\mathbb{Q})$ is MAP (since $O_n(\mathbb{R})$ is compact) and hence quasidiagonal. We will argue below that it is not residually amenable if $n \geq 5$. Let Ω_n denote the commutator subgroup of $O_n(\mathbb{Q})$. It was shown by Kneser [60] that the projective group $P\Omega_n$ is simple for $n \geq 5$. If n is odd, the center of $O_n(\mathbb{Q})$ is trivial and in particular $\Omega_5 = P\Omega_5$ is simple. Since Ω_5 is simple and infinite, it is not virtually solvable and hence by the Tits alternative [82] it contains a non-abelian free subgroup. It follows that any group that contains a subgroup isomorphic to Ω_5 is not residually amenable. In particular this applies to $O_n(\mathbb{Q})$, $n \geq 5$.

We denote by λ_G the left regular representation of G and by ι_G the trivial representation.

Proposition 3.7. *Let $\{\omega_n : G \to U(H_n)\}_{n\geq 1}$ be a sequence of group representations that separates the points of G. Then λ_G is weakly contained in $\{\iota_G\} \cup \{\omega_{i_1} \otimes \cdots \otimes \omega_{i_n} : 1 \leq i_1 \leq \cdots \leq i_n, n \geq 1\}$.*

Proof. Since $(\omega_n)_n$ separates the points of G, for each $s \in G \setminus \{e\}$ there is $i = i(s) \geq 1$ such that $\omega_i(s) \neq 1$. Using the spectral theorem we find a unit vector $\xi_i \in H_i$ such that the positive definite map associated to $\iota_G \oplus \omega_i$, $f_i(\cdot) = \frac{1}{2}(1 + \langle \omega_i(\cdot)\xi_i, \xi_i \rangle)$ satisfies $|f_i(s)| < 1$. For $I = (i_1, \cdots, i_n)$ with $1 \leq i_1 \leq \cdots \leq i_n$, the positive definite map $f_I = f_{i_1} \cdots f_{i_n}$ is associated to the representation $(\iota_G \oplus \omega_{i_1}) \otimes \cdots \otimes (\iota_G \oplus \omega_{i_n})$ which is unitarily equivalent to a direct sum of representations from $\Omega := \{\iota_G\} \cup \{\omega_{i_1} \otimes \cdots \otimes \omega_{i_n} : 1 \leq i_1 \leq \cdots \leq i_n, n \geq 1\}$. As I increases in size, for each $s \neq e$, f_I contains larger and larger powers of $f_{i(s)}$. Since $f_I(e) = 1$, it follows that $\lim_{|I|\to\infty} f_I(s) = \delta_e(s)$ for all $s \in G$. Since the positive definite map δ_e corresponds to a cyclic vector of λ_G and f_I is associated to Ω, it follows by [36, 18.1.4] that λ_G is weakly contained in Ω. $\qquad\square$

Proposition 3.8. *Let G be a countable discrete group. The following assertions are equivalent:*
(i) *G is quasidiagonal.*
(ii) *λ_G is weakly contained in a quasidiagonal representation π of G.*
(iii) *The canonical map $q_G : C^*(G) \to C_r^*(G)$ factors through a unital quasidiagonal C*-algebra.*

Proof. $(i) \Rightarrow (ii)$ By assumption, there is a faithful representation $\omega : G \to U(H)$ such that the C^*-algebra $B = C_\omega^*(G)$ is quasidiagonal. Proposition 3.7 shows that λ_G is weakly contained in the set $\{\omega^{\otimes n} : n \geq 0\}$, $\omega^{\otimes 0} = \iota_G$, and hence in $\pi = \bigoplus_{n\geq 0} \omega^{\otimes n}$. Since $C_\pi^*(G) \subset \prod_{n\geq 0} B^{\otimes n}$ (minimal tensor products) and B is quasidiagonal, the representation π is quasidiagonal.

$(ii) \Rightarrow (iii)$ If π is as in (ii), then q_G factors through the quasidiagonal C^*-algebra $C_\pi^*(G)$.

$(iii) \Rightarrow (i)$ By assumption, q_G factors through a unital quasidiagonal C*-algebra D. Let π be the composition $G \to U(C^*(G)) \to U(D)$. Since π is a lifting of $\lambda_G : G \to U(C_r^*(G))$, it follows that π is injective. Moreover, $\|\pi(s) - \pi(t)\| \geq \|\lambda_G(s) - \lambda_G(t)\| \geq \sqrt{2}$ for $s \neq t$. $\qquad\square$

Free products (without amalgamation) of quasidiagonal groups are quasidiagonal as a consequence of [9]. It is an open problem whether the amalgamated product of two amenable groups over a finite group is residually amenable or not [51, Question 2]. However, we show that it is quasidiagonal.

Proposition 3.9. *Let G_1, G_2 be residually amenable countable discrete groups with a common finite subgroup H. Then $G_1 \star_H G_2$ is quasidiagonal.*

Proof. Let $G = G_1 \star_H G_2$. As argued in [51, p291], since H is finite, there exist sufficiently many homomorphisms $G \to G'_1 \star_H G'_2$ with $H \subset G'_i$ and G'_i amenable to separate the points of G. Thus it suffices to assume that G_1 and G_2 are themselves amenable. Let τ_i be the canonical trace on $C^*(G_i) \cong C^*_r(G_i)$ and let τ be the unique tracial state of universal UHF algebra $\mathcal{Q} = \bigotimes_n M_n$. By Schafhauser's embedding theorem [74, Thm.B] there are unital $*$-monomorphisms $\varphi_i : C^*(G_i) \to \mathcal{Q}$, such that $\tau \circ \varphi_i = \tau_i$, $i = 1, 2$. Since $\tau_i|_{C^*(H)} = \tau_H$ (the canonical trace on $C^*(H)$), it follows that $\tau|_{\varphi_1(C^*(H))} = \tau|_{\varphi_2(C^*(H))}$. By standard perturbations arguments one can find two increasing sequences $(A_n^{(i)})_n$, $i = 1, 2$, of matrix subalgebras of $\mathcal{Q}$ such that $\varphi_i(C^*(H)) \subset A_n^{(i)}$ and $\bigcup_{n=1}^\infty A_n^{(i)}$ is dense in $\mathcal{Q}$, $i = 1, 2$. By Prop. 2.2 and Thm. 4.2 of [1] we deduce that $Q \star_{C^*(H)} Q$ is the closure of the union of an increasing sequence of residually finite dimensional C^*-subalgebras isomorphic to $A_n^{(1)} \star_{C^*(H)} A_n^{(2)}$ and hence it is quasidiagonal. One can also invoke [66, Cor.2] to derive the same conclusion directly. Appealing again to [1, Prop.2.2], we deduce that

$$C^*(G_1) \star_{C^*(H)} C^*(G_2) \subset Q \star_{C^*(H)} Q,$$

and hence $C^*(G_1 \star_H G_2)$ is quasidiagonal (if G_1 and G_2 are amenable). $\square$

Definition 3.10. A group G is weakly quasidiagonal if there is a ucp asymptotic homomorphism $\{\varphi_n : C^*(G) \to M_{k_n}\}_n$ which separates the points of G. In other words this sequence satisfies the conditions (1) and (2) from introduction.

Quasidiagonal groups are weakly quasidiagonal. We suspect that the two classes are distinct.

Proposition 3.11. *G is weakly quasidiagonal if and only it admits a unitary representation π on a separable Hilbert space H for which there is a sequence $(p_n)_n$ of finite dimensional projections such that $\lim_n \|[\pi(s), p_n]\| = 0$ and $\limsup_n \|p_n(\pi(s) - 1)p_n\| > 0$ for all $s \in G \setminus \{e\}$.*

Proof. In one direction, one observes that the maps $\varphi_n(s) = p_n \pi(s) p_n$ satisfy the conditions (1) and (2). Conversely, if $\pi_n : C^*(G) \to L(H_n)$ is the Naimark/Stinespring dilation of φ_n and p_n is the corresponding finite dimensional projection with $\varphi_n = p_n \pi_n p_n$, then $\varphi_n(st) - \varphi_n(s)\varphi_n(t) = p_n \pi_n(s)(1 - p_n)\pi_n(t)p_n$ and hence $\|1 - \varphi_n(s)\varphi_n(s^{-1})\| = \|p_n \pi_n(s)(1 - p_n)\|^2$. Setting $\pi = \oplus_n \pi_n$, we see that the asymptotic multiplicativity of φ_n implies that $\lim_n \|[\pi(s), p_n]\| = 0$ for all $s \in G$. $\square$

Proposition 3.12. *A group G is weakly quasidiagonal if and only if there is a ucp asymptotic homomorphism $\{\psi_n : C^*(G) \to M_{k_n}\}_{n \in \mathbb{N}}$ such that $\liminf_n \|\psi_n(s) - 1_{k_n}\| \geq \sqrt{2}$ for all $s \in G \setminus \{e\}$.*

Proof. Fix $s \in G \setminus \{e\}$. It suffices to find a ucp asymptotic homomorphism such that $\liminf_n \|\psi_n(s) - 1_{k_n}\| \geq \sqrt{2}$ for this fixed element. Indeed, by considering direct sums of tails of such sequences, we can then arrange to have the desired estimate for all nontrivial elements of G. Using the definition, we first find a ucp asymptotic morphism $\{\varphi_n : C^*(G) \to M_{k_n}\}_{n \in \mathbb{N}}$ such that $\lim_n \|\varphi_n(s) - 1_{k_n}\| = \delta > 0$. We will show that there is $m \geq 1$ such that the asymptotic homomorphism $\psi_n := \varphi_n^{\otimes m}$ satisfies $\liminf_n \|\psi_n(s) - 1_{mk_n}\| \geq \sqrt{2}$. If $\delta \geq \sqrt{2}$, then $m = 1$ will do. Thus we may assume that $\delta = |e^{i\alpha} - 1|$ for some $\alpha \in (0, \pi/2)$. By functional calculus there is a sequence of unitaries $u_n \in U(k_n)$ such that $\lim_n \|\varphi_n(s) - u_n\| = 0$. Since $\lim_n \|u_n - 1_{k_n}\| = |e^{i\alpha} - 1|$, u_n has an eigenvalue $e^{i\theta_n}$ with $\|u_n - 1_{k_n}\| = |e^{i\theta_n} - 1|$ and $\lim_n |\theta_n| = \alpha$. Let m be the smallest integer such that $m\alpha \geq \pi/2$. Then $\lim_n |e^{im\theta_n} - 1| = |e^{im\alpha} - 1| \geq \sqrt{2}$. Since $e^{im\theta_n}$ is an eigenvalue of $u_n^{\otimes m}$, it follows that $\liminf_n \|\psi_n(s) - 1_{mk_n}\| = \liminf_n \|u_n^{\otimes m} - 1_{mk_n}\| \geq \sqrt{2}$. $\square$

Remark 3.13. Let G be a weakly quasidiagonal group. If follows from Voiculescu's theorem that all unital faithful representations $\pi : C^*(G) \to L(H)$ satisfy the conditions from Proposition 3.11.

Proposition 3.14. *An increasing union of weakly quasidiagonal groups is weakly quasidiagonal.*

Proof. Suppose that $G = \bigcup_{n=1}^{\infty} G_n$, where $(G_n)_n$ is an increasing sequence of weakly quasidiagonal subgroups of G. Enumerate $G = \{s_n : n \geq 0\}$ with $s_0 = e$. We may assume that $F_n = \{s_i : i \leq n\} \subset G_n$. By Proposition 3.12 there is a ucp map $\varphi_n : C^*(G_n) \to M_{k_n}$ such that $\|\varphi_n(st) - \varphi_n(s)\varphi_n(t)\| < 1/n$ for all $s, t \in F_n$ and $\|\varphi_n(s) - 1_{k_n}\| \geq \sqrt{2} - 1/n$ for all $s \in F_n \setminus \{e\}$. By [71, Prop.8.8], the inclusion map $G_n \subset G$ induces an embedding $C^*(G_n) \subset C^*(G)$ and there is a conditional expectation $E_n : C^*(G) \to C^*(G_n)$ such that $E_n(s) = \chi_{G_n}(s)s$ for all $s \in G$. It is then clear that by setting $\psi_n := \varphi_n \circ E_n : C^*(G) \to M_{k_n}$ we obtain an asymptotic homomorphism consisting of ucp maps such that $\liminf_n \|\psi_n(s) - 1_{k_n}\| \geq \sqrt{2}$ for all $s \in G \setminus \{e\}$. $\qquad\square$

Example 3.15. (i) The group $SL_n(\mathbb{Q})$ ($n \geq 3$) is simple by the Jordan-Dickson theorem. It is nonamenable and hence not residually amenable. $SL_n(\mathbb{Q})$ is weakly quasidiagonal, since as argued in the proof of Corollary 1.3, all countable linear groups are weakly quasidiagonal. We don't know if $SL_n(\mathbb{Q})$ is quasidiagonal. (ii) Thom [78] (cf.Yamashita [89]) noted that infinite simple property (T) groups are not quasidiagonal, and in fact they are not weakly quasidiagonal, see Proposition 3.19.

The notion of quasidiagonal trace was introduced in [10, 3.3.1]. Suppose that the trace τ_G of $C^*(G)$ induced by the canonical trace of $C_r^*(G)$ is quasidiagonal. Then G is weakly quasidiagonal. Moreover, since quasidiagonal traces are amenable [10], it also follows that G has Kirchberg's factorization property [11, 6.4.2]. The next proposition indicates a class of quasidiagonal groups G for which τ_G is quasidiagonal.

Proposition 3.16. *If G embeds in the unitary group of a unital C^*-algebra D that admits a faithful quasidiagonal trace (e.g. D is a countable product of simple quasidiagonal C^*-algebras), then τ_G is quasidiagonal.*

Proof. This is verified by an argument similar to the proof of the implication $(ii) \Rightarrow (i)$ of Corollary 1.2 in [58]. One uses basic permanence properties of quasidiagonal traces established in [10, 3.5]. By assumption, D has a faithful quasidiagonal trace state τ and there is a unital $*$-homomorphism $\pi : C^*(G) \to D$ which separates the points of G. Let $s \in G \setminus \{e\}$. Since $\pi(s) \neq 1$ and τ is faithful, $\tau((\pi(s) - 1)^*(\pi(s) - 1)) = 2(1 - Re\,\tau(\pi(s))) > 0$ and hence $\tau(\pi(s)) \neq 1$. Define $\rho : C^*(G) \to M_2(D)$ by $\rho(s) = 1_D \oplus \pi(s)$. Let τ_2 be the tracial state of $M_2(D)$ that extends τ. Then τ_2 is quasidiagonal and $|\tau_2(\rho(s))| < 1$ for all $s \in G \setminus \{e\}$. The trace $\tau_2^{\otimes n} : M_2(D)^{\otimes n} \to \mathbb{C}$ is quasidiagonal since τ_2 is so, [10, 3.5.7] . It follows that for each $n \geq 1$ the trace $tr_n : C^*(G) \to \mathbb{C}$, $tr_n(s) = \tau_2^{\otimes n}(\rho(s)^{\otimes n}) = \tau_2(\rho(s))^n$ is quasidiagonal. Since $lim_n\, tr_n(s) = \delta_e(s) = \tau_G(s)$ for all $s \in G$, we conclude that τ_G is quasidiagonal as a weak*-limit of quasidiagonal traces, [10, 3.5.1]. $\qquad\square$

Lemma 3.17. *(i) A matricially stable MF-group is MAP. (ii) If a weakly quasidiagonal group G is weakly matricially stable, then G is MAP.*

Proof. Both (i) and (ii) are immediate from definitions. $\qquad\square$

We need the following well-known lemma, see [73], [78], [75, Thm.4.3.4].

Lemma 3.18. *Let $\pi : G \to U(H)$ be a unitary representation for which there is a sequence $(p_n)_n$ of nonzero finite dimensional projections such that $\lim_n \|[\pi(s), p_n]\| = 0$ for all $s \in G$. Then the trivial representation ι_G is weakly contained $\pi \otimes \bar{\pi}$.*

Proof. For $P \in L(H)$ denote by $\|P\|_2$ its Hilbert-Schmidt norm. Since

$$\|\pi(s)p_n - p_n\pi(s)\|_2 \leq \|(\pi(s)p_n - p_n\pi(s))p_n\|_2 + \|p_n(\pi(s)p_n - p_n\pi(s))\|_2 \leq 2\|p_n\|_2\|[\pi(s), p_n]\|$$

it follows that $\left\|\pi(s)\left(\frac{p_n}{\|p_n\|_2}\right)\pi(s)^* - \left(\frac{p_n}{\|p_n\|_2}\right)\right\|_2 \leq 2\|[\pi(s), p_n]\|$, $s \in G$. $\qquad\square$

The following fact was essentially pointed out by Thom on MathOverflow [78] . It shows that an infinite simple property (T) group G is not weakly quasidiagonal. As explained in [34, p.93] any lattice in $Sp(n,1)$, $n \geq 2$ has uncountably many infinite quotients which are simple (and torsion).

Proposition 3.19 (Ozawa-Thom). *If an infinite property* (T) *group G is weakly quasidiagonal, then G has an infinite residually finite quotient.*

Proof. Let F be the set of all equivalence classes of finite dimensional irreducible representations of G. If F is infinite, then G has an infinite residually finite quotient (whether or not it is weak quasidiagonal). Let $N = \cap_{\sigma \in F} \ker(\sigma)$. If F is infinite, then G/N is infinite, since finite groups have only finitely many non-equivalent irreducible representations. Moreover, G/N is MAP since it embeds in $\prod_{\sigma \in F} U(H_\sigma)$. Property (T) groups are finitely generated [7] and hence G/N is residually finite by Malcev's theorem.

If the set F is finite, we show that G cannot be weakly quasidiagonal. Let $f \in C^*(G)$ be sum of the central Kazhdan projections corresponding to the elements of F. Then $\|\sigma(f)\| = 1$ for any nonzero finite dimensional representation σ of G. Write $C^*(G) = fC^*(G)f \oplus (1-f)C^*(G)(1-f)$ (direct sum of C^*-algebras) where $C = fC^*(G)f$ is finite dimensional. The canonical morphism $G \to U(C)$ is not injective. Otherwise G would be residually finite and hence finite, since F is finite. Thus there is $s \in G \setminus \{e\}$ whose image in $U(C)$ is 1. Let $\pi : C^*(G) \to L(H)$ and (p_n) be as in Proposition 3.11. Then π decomposes as $\pi = \pi_F \oplus \rho$ where $\pi_F = \pi(f)\pi$ and $\rho = (1 - \pi(f))\pi$. Since $\lim_{n\to\infty}[\pi(f), p_n] = 0$, by functional calculus we find finite dimensional projections $r_n \leq \pi(f)$ and $q_n \leq 1 - \pi(f)$ such that $\lim_{n\to\infty} \|p_n - (r_n + q_n)\| = 0$. Since $\pi_F(s) = 1$ as π_F factors through C, it follows that $\limsup_{n\to\infty} \|q_n(\rho(s)-1)q_n\| = \limsup_{n\to\infty} \|p_n(\pi(s)-1)p_n\| > 0$. Thus (q_n) must have a subsequence q_{n_k} consisting of nonzero projections. Since $\|[\rho(s), q_{n_k}]\| \to 0$ for $s \in G$, it follows by Lemma 3.18 that ι_G is weakly contained in $\rho \otimes \bar{\rho}$. Since G has property (T), it follows that ι_G is contained in $\rho \otimes \bar{\rho}$, and hence, by a classic argument [7, A.1.12], ρ has a nonzero finite dimensional subrepresentation σ. Since $\rho(f) = 0$, this implies that $\sigma(f) = 0$ which is a contradiction. $\square$

4. The dual assembly map and quasidiagonality

In this section we prove Theorem 1.2 and prepare the ground for Theorem 6.1.

Let $\underline{E}G$ be the classifying space for proper actions of G, [3]. It is known that $\underline{E}G$ admits a locally compact model, [53]. Let us recall that G has a γ-element if there exists a $G - C_0(\underline{E}G)$-algebra A in the sense of Kasparov [54] and two elements $d \in KK_G(A, \mathbb{C})$ and $\eta \in KK_G(\mathbb{C}, A)$ (called Dirac and dual-Dirac elements, respectively) such that the element $\gamma = \eta \otimes_A d \in KK_G(\mathbb{C}, \mathbb{C})$ has the property that $p^*(\gamma) = 1 \in \mathcal{R}KK_G(\underline{E}G; C_0(\underline{E}G), C_0(\underline{E}G))$ where $p : \underline{E}G \to$ point, [84]. We refer the reader to [54] for the definitions and the basic properties of these groups and we will freely employ the notation from there.

Let B be a separable C^*-algebra endowed with the trivial G-action. Consider the dual assembly map with coefficients in B:

$$\alpha : KK_G(\mathbb{C}, B) \to RKK_G^0(\underline{E}G; \mathbb{C}, B)$$

defined by $\alpha(y) = p^*(y)$ where $p : \underline{E}G \to$ point. As in [54], we write $RKK_G^0(\underline{E}G; \mathbb{C}, B)$ for $\mathcal{R}KK_G(\underline{E}G; C_0(\underline{E}G), C_0(\underline{E}G) \otimes B)$. We need the following result of Kasparov [54, Th.6.5], see also the proofs of [56, Thm. 3.1] and [52, Thm. 2.3] for general coefficients.

Theorem 4.1 (Kasparov). *If G is a countable discrete group that admits a γ-element, then the dual assembly map $\alpha : KK_G(\mathbb{C}, B) \to RKK_G^0(\underline{E}G; \mathbb{C}, B)$ is split surjective with kernel $(1 - \gamma)KK_G(\mathbb{C}, B)$.*

Proof. Since the two canonical projections $q_1, q_2 : \underline{E}G \times \underline{E}G \to \underline{E}G$ are G-homotopic by [3, Prop.8], one can simply repeat the arguments from the proof of [54, Thm. 6.5] (with $X = \underline{E}G$ in place of EG). $\square$

By universality of $\underline{E}G$, there is a G-equivariant map (unique up to homotopy) $\sigma : EG \to \underline{E}G$. It induces a map $\sigma^* : RKK_G^0(\underline{E}G; \mathbb{C}, B) \to RKK_G^0(EG; \mathbb{C}, B)$. Recall that $\mathcal{Q} = \bigotimes_n M_n$ is universal UHF algebra and $K_0(\mathcal{Q}) = \mathbb{Q}$ and $K_1(\mathcal{Q}) = 0$. We view $\mathcal{Q}$ as a trivial G-algebra.

Corollary 4.2. *Let G be a countable discrete group that admits a γ-element. Then the composition*

$$\gamma KK_G(\mathbb{C}, \mathcal{Q}) \hookrightarrow KK_G(\mathbb{C}, \mathcal{Q}) \xrightarrow{\alpha} RKK_G^0(\underline{E}G; \mathbb{C}, \mathcal{Q}) \xrightarrow{\sigma^*} RKK_G^0(EG; \mathbb{C}, \mathcal{Q})$$

is a surjective map. If moreover G is torsion free, then $\underline{E}G = EG$ and the composition $\gamma KK_G(\mathbb{C}, \mathbb{C}) \hookrightarrow KK_G(\mathbb{C}, \mathbb{C}) \xrightarrow{\alpha} RKK_G^0(EG; \mathbb{C}, \mathbb{C})$ is an isomorphism.

Proof. It was shown in [3, p.275-6] that σ induces a rationally injective homomorphism

$$\sigma_* : RK_0^G(EG) \to RK_0^G(\underline{E}G).$$

It follows that the map $(\sigma_*)^* : \mathrm{Hom}(RK_0^G(\underline{E}G), \mathbb{Q}) \to \mathrm{Hom}(RK_0^G(EG), \mathbb{Q})$ is surjective. By the universal coefficient theorems stated as Lemma 2.3 of [52] and Lemma 3.4 of [56] applied for the coefficient algebra $\mathcal{Q}$, the horizontal maps in the commutative diagram

$$
\begin{array}{ccc}
RKK_G^0(\underline{E}G; \mathbb{C}, \mathcal{Q}) & \longrightarrow & \mathrm{Hom}(RK_0^G(\underline{E}G), \mathbb{Q}) \\
\sigma^* \downarrow & & \downarrow (\sigma_*)^* \\
RKK_G^0(EG; \mathbb{C}, \mathcal{Q}) & \longrightarrow & \mathrm{Hom}(RK_0^G(EG), \mathbb{Q})
\end{array}
$$

are bijections. It follows that the restriction map $\sigma^* : RKK_G^0(\underline{E}G; \mathbb{C}, \mathcal{Q}) \to RKK_G^0(EG; \mathbb{C}, \mathcal{Q})$ is surjective. The first part of the statement follows now from Theorem 4.1. The second part follows directly from the same theorem since if G is torsion free then $EG = \underline{E}G$. $\square$

Let j_G and $j_{G,r}$ be the descent maps of Kasparov [54, Thm.3.11]. Thus $\gamma \in KK_G(\mathbb{C}, \mathbb{C})$ gives an element $j_G(\gamma) \in KK(C^*(G), C^*(G))$ which induces a map

$$j_G(\gamma)^* = j_G(\gamma) \otimes_{C^*(G)} - : KK(C^*(G), B) \to KK(C^*(G), B).$$

The image of $j_G(\gamma)^*$ is $j_G(\gamma) \otimes_{C^*(G)} KK(C^*(G), B)$, and as it is customary, we will denote it by $\gamma KK(C^*(G), B)$, $(\gamma_r KK(C_r^*(G), B)$ is defined similarly as the image of $j_{G,r}(\gamma)^*)$. Since G is discrete and acts trivially on B, there is a canonical isomorphism, [54],

$$\kappa : KK_G(\mathbb{C}, B) \xrightarrow{\cong} KK(C^*(G), B)$$

which is compatible with the module structure over the group ring of G. Thus, by [69, Lemma 11], for every $x \in KK_G(\mathbb{C}, \mathbb{C})$, the following diagram is commutative.

$$
\begin{array}{ccc}
(3) & & \\
KK_G(\mathbb{C}, B) & \xrightarrow{\kappa} & KK(C^*(G), B) \\
x \otimes - \downarrow & & \downarrow j_G(x) \otimes - \\
KK_G(\mathbb{C}, B) & \xrightarrow{\kappa} & KK(C^*(G), B)
\end{array}
$$

Kasparov [54, 3.12] has shown that the canonical surjection $q_G : C^*(G) \to C_r^*(G)$ induces an isomorphism of γ-parts $q_G^* : \gamma_r KK(C_r^*(G), B) \xrightarrow{\cong} \gamma KK(C^*(G), B)$. In particular:

Proposition 4.3 (Kasparov)**.** *If G is a discrete countable group that admits a γ-element, then $\gamma KK(C^*(G), B) \subset q_G^*(KK(C_r^*(G), B))$.*

Proof. We review the argument as it plays an important role in the paper. Let A, d and η be as in the definition of the γ-element. The following diagram is commutative in the category with objects separable C^*-algebras and morphisms KK-elements [54].

$$
\begin{array}{ccccc}
C^*(G) & \xrightarrow{j_G(\eta)} & A \rtimes G & \xrightarrow{j_G(d)} & C^*(G) \\
\downarrow{\scriptstyle q_G} & & \downarrow{\scriptstyle q_A} & & \downarrow{\scriptstyle q_G} \\
C_r^*(G) & \xrightarrow[j_{G,r}(\eta)]{} & A \rtimes_r G & \xrightarrow[j_{G,r}(d)]{} & C_r^*(G)
\end{array}
$$

Since the action of G is proper, the natural map $q_A : A \rtimes G \to A \rtimes_r G$ is an isomorphism by [22, 3.4.16]. We obtain then a commutative diagram of abelian groups

$$
\begin{array}{ccccc}
KK(C^*(G), B) & \xleftarrow{j_G(\eta)^*} & KK(A \rtimes G, B) & \xleftarrow{j_G(d)^*} & KK(C^*(G), B) \\
{\scriptstyle q_G^*}\uparrow & & \| & & \\
KK(C_r^*(G), B) & \xleftarrow[j_{G,r}(\eta)^*]{} & KK(A \rtimes_r G, B) & &
\end{array}
$$

which shows that $\mathrm{Im}(j_G(\gamma)^*) = \mathrm{Im}(j_G(\eta)^* \circ j_G(d)^*) \subset \mathrm{Im}(j_G(\eta)^*) \subset \mathrm{Im}(q_G^*)$. Note that for $B = \mathbb{C}$ this shows that $\gamma \in K^0(C^*(G))$ is in the image of q_G^* and this means that it can be represented by a class whose underlying G-representation is weakly contained in the left regular representation. This property of γ is part of the hypotheses of [54, 3.12]. $\qquad\square$

Let A, B be separable C^*-algebras. Any class $x \in KK(A, B)$ is represented by some Cuntz pair, i.e. a pair of $*$-homomorphisms $\varphi, \psi : A \to M(K(H) \otimes B)$, such that $\varphi(a) - \psi(a) \in K(H) \otimes B$, for all $a \in A$. The following notion was introduced in [25] for K-homology classes. Assume that B is unital.

Definition 4.4. An element $x \in KK(A, B)$ is quasidiagonal if it is represented by a Cuntz pair $\varphi, \psi : A \to M(K(H) \otimes B)$, $\varphi(a) - \psi(a) \in K(H) \otimes B$, for all $a \in A$, with the property that there exists an approximate unit of projections $(p_n)_n$ of $K(H)$ such that $\lim_{n\to\infty} \|[\psi(a), p_n \otimes 1_B]\| = 0$, for all $a \in A$. The quasidiagonal elements form a subgroup of $KK(A, B)$, denoted by $KK(A, B)_{qd}$. The C^*-algebra A is called K-quasidiagonal if $K^0(A) = K^0(A)_{qd}$.

Remark 4.5. (a) If $\theta : A \to D$ is a $*$-homomorphism, then $\theta^*[\varphi, \psi] = [\varphi \circ \theta, \psi \circ \theta]$ and hence

$$\theta^*(KK(D, B)_{qd}) \subset KK(A, B)_{qd}.$$

(b) Let D, B be separable C^*-algebras with B nuclear and unital. Fix a faithful representation $\psi_0 : D \to M(K(H))$ such that $\psi_0(D) \cap K(H) = \{0\}$ and $(\psi_0(D)H)^\perp$ is infinite dimensional. Then any element $x \in KK(D, B)$ is represented by a Cuntz pair (φ, ψ) where $\psi = \psi_0 \otimes 1_B : A \to M(K(H) \otimes B)$, [76]. Therefore, if D is quasidiagonal, then $\psi_0(D)$ is a quasidiagonal subset of $M(K(H))$ and hence $KK(D, B) = KK(D, B)_{qd}$.

By [54, Thm.3.4], there is natural descent isomorphism

$$\lambda^G : RKK_G^0(EG; \mathbb{C}, B) \xrightarrow{\cong} RKK^0(BG; \mathbb{C}, B).$$

In particular, $RKK_G^0(EG; \mathbb{C}, \mathcal{Q}) \cong RKK^0(BG; \mathbb{C}, \mathcal{Q}) =: RK^0(BG; \mathbb{Q})$, [55]. Let ν be the map

$$\nu = \lambda^G \circ \sigma^* \circ \alpha \circ \kappa^{-1} : KK^0(C^*(G), B) \cong KK_G(\mathbb{C}, B) \to RKK_G^0(EG; \mathbb{C}, B) \cong RKK^0(BG; \mathbb{C}, B).$$

Note that $\sigma^* \circ \alpha = p^*$ where $p : EG \to point$.

Kubota's idea [65] of using a quasidiagonal C^*-algebra intermediate between $C_r^*(G)$ and $C^*(G)$ has strengthened significantly the approach to almost flat K-theory based on K-quasidiagonality of $C^*(G)$, introduced in [25]. By Proposition 3.8, the quasidiagonal groups are exactly those which admit quasidiagonal intermediate C^*-algebras. This enables us to extend a result from [65] to the class of quasidiagonal groups (we consider only the non-relative, single group case). Moreover, the direct use of Theorem 4.1 allows for integral coefficients in the case of torsion free groups.

Theorem 4.6. *Let G be a countable discrete quasidiagonal group and let B be a separable nuclear unital C^*-algebra. If G admits a γ-element, then $\gamma KK(C^*(G), B)) \subset KK(C^*(G), B)_{qd}$. It follows that $\nu(KK(C^*(G), \mathcal{Q})_{qd}) = RK^0(BG; \mathbb{Q})$ and that $\nu(K^0(C^*(G))_{qd}) = RK^0(BG)$ if we also assume that G is torsion free.*

Proof. The factorization $C^*(G) \xrightarrow{q_D} D \to C_r^*(G)$ of q_G with D unital and quasidiagonal, given by Proposition 3.8, in conjunction with Remark 4.5 implies that

$$q_G^*(KK(C_r^*(G), B)) \subset q_D^*(KK(D, B)) = q_D^*(KK(D, B)_{qd}) \subset KK(C^*(G), B)_{qd}.$$

From this and Proposition 4.3 we obtain that $\gamma KK(C^*(G), B)) \subset KK(C^*(G), B)_{qd}$.

By Theorem 4.1, α vanishes on $(1 - \gamma)KK_G(\mathbb{C}, B))$. Since the diagram (3) is commutative, this group is mapped to $(1 - \gamma)KK^0(C^*(G), B)$ by κ and hence

$$\nu(KK^0(C^*(G), B)) = \nu(\gamma KK^0(C^*(G), B)) = \nu(KK(C^*(G), B)_{qd}).$$

By Corollary 4.2, the map ν is surjective if either $B = \mathcal{Q}$ or if G is torsion free and $B = \mathbb{C}$. $\square$

Remark 4.7. Suppose that $\{\pi_n : A \to D_n\}_n$ is a ucp asymptotic homomorphism of unital C*-algebras. Thus $\lim_{n \to \infty} \|\pi_n(aa') - \pi_n(a)\pi_n(a')\| = 0$ for all $a, a' \in A$. The sequence $\{\pi_n\}_n$ induces a unital $*$-homomorphism $A \to \prod_n D_n / \bigoplus_n D_n$ and hence a group homomorphism $K_0(A) \to \prod_n K_0(D_n) / \bigoplus_n K_0(D_n)$. This gives a canonical way to push forward an element $x \in K_0(A)$ to a sequence $(\pi_{n\sharp}(x))_n$ with components in $K_0(D_n)$ which is well-defined up to tail equivalence: two sequences are tail equivalent, $(y_n) \equiv (z_n)$, if there is m such that $x_n = y_n$ for all $n \geq m$. Note that $\pi_{n\sharp}(x + x') \equiv \pi_{n\sharp}(x) + \pi_{n\sharp}(x')$. Of course, if π_n are genuine $*$-homomorphisms then $\pi_{n\sharp}(x) = \pi_{n*}(x)$. It is convenient to extend the notation above for the corresponding maps on K_1 and K-theory with coefficients $K_*(A; \mathbb{Z}/p) \to \prod_n K_*(D_n; \mathbb{Z}/p) / \bigoplus_n K_*(D_n; \mathbb{Z}/p)$.

If BG is written as the union of an increasing sequence $(Y_i)_i$ of finite CW complexes, then as explained in the proof of Lemma 3.4 from [56], there is a Milnor $\varprojlim^1$ exact sequence which gives

$$(4) \qquad RK^0(BG; \mathbb{Q}) \cong \varprojlim RK^0(Y_i; \mathbb{Q}).$$

We denote by ν_i the composition of the map ν defined above with the restriction map $RK^0(BG; \mathbb{Q}) \to RK^0(Y_i; \mathbb{Q})$. If $Y = Y_i$ is fixed we will write ν_Y for ν_i.

Proof of Theorem 1.2:

If G is weakly quasidiagonal and weakly matricially stable, then by Lemma 3.17, G is MAP and in particular quasidiagonal. It follows by Theorem 4.6 that $\nu(KK(C^*(G), \mathcal{Q})_{qd}) = RK^0(BG; \mathbb{Q})$.

Consider the flat line-bundle ℓ with fiber $C^*(G)$ defined by $\widetilde{EG} \times_G C^*(G) \to BG$, where $G \subset C^*(G)$ acts diagonally. Let Y be a connected finite CW complex $Y \subset BG$. The restriction of ℓ to Y, denoted ℓ_Y, yields a self-adjoint projection $P = P_Y$ in $M_m(\mathbb{C}) \otimes C(Y) \otimes C^*(G)$ for some $m \geq 1$. It was shown by Kasparov [54, Lemma 6.2], [55], that the map

$$\nu_Y : KK(C^*(G), \mathcal{Q}) \xrightarrow{\nu} K^0(BG; \mathbb{Q}) \to K^0(Y; \mathbb{Q}) \cong K_0(C(Y) \otimes \mathcal{Q})$$

is given by $\nu_Y(x) = [P] \otimes_{C^*(G)} x$. Let $(U_i)_{i \in I}$ be finite covering of Y by open sets such that ℓ is trivial on each U_i and $U_i \cap U_j$ is connected. Using trivializations of ℓ to U_i one obtains group elements $s_{ij} \in G$ which define a 1-cocycle that is constant on each nonempty set $U_i \cap U_j$ and which represents ℓ_Y. Thus $s_{ij}^{-1} = s_{ji}$ and $s_{ij} \cdot s_{jk} = s_{ik}$ whenever $U_i \cap U_j \cap U_k \neq \emptyset$. Let $(\chi_i)_{i \in I}$ be positive continuous functions with χ_i supported in U_i and such that $\sum_{i \in I} \chi_i^2 = 1$. Set $m = |I|$ and let (e_{ij}) be the canonical matrix unit of $M_m(\mathbb{C})$. Then ℓ_Y is represented by the selfadjoint projection

$$P = \sum_{i,j \in I} e_{ij} \otimes \chi_i \chi_j \otimes s_{ij} \in M_m(\mathbb{C}) \otimes C(Y) \otimes C^*(G).$$

We now take advantage of the following realization of ν_Y on quasidiagonal KK-classes introduced in [25]. Let $x \in KK(C^*(G), \mathcal{Q})_{qd}$. Then x is represented by a pair of nonzero $*$-representations $\varphi, \psi : C^*(G) \to M(K(H) \otimes \mathcal{Q})$, such that $\varphi(a) - \psi(a) \in K(H) \otimes \mathcal{Q}$, $a \in C^*(G)$, and with property that there is an increasing approximate unit $(p_n)_n$ of $K(H)$ consisting of projections such that $(p_n \otimes 1_\mathcal{Q})_n$ commutes asymptotically with both $\varphi(a)$ and $\psi(a)$, for all $a \in C^*(G)$. It is then clear that $\varphi_n^{(0)} = (p_n \otimes 1_\mathcal{Q})\varphi(p_n \otimes 1_\mathcal{Q})$ and $\varphi_n^{(1)} = (p_n \otimes 1_\mathcal{Q})\psi(p_n \otimes 1_\mathcal{Q})$ are cp asymptotic homomorphisms $\varphi_n^{(r)} : C^*(G) \to K(H) \otimes \mathcal{Q}$. Let 1 denote the unit of $C^*(G)$. It is routine to further perturb these maps to cp asymptotic homomorphisms such that $\varphi_n^{(r)}(1)$, $r = 0, 1$, are projections. If $[\varphi_n^{(r)}(1)] = k_n^{(r)}/q_n^{(r)} \in \mathbb{Q} = K_0(\mathcal{Q})$, then after conjugating by unitaries in $\mathcal{Q}$, we can arrange that $\varphi_n^{(r)} : C^*(G) \to M_{k_n^{(r)}}(\mathbb{C}e_0) \subset M_{k_n^{(r)}}(\mathcal{Q}_n^{(r)})$, where $\mathcal{Q}_n^{(r)} = (1 \otimes \cdots \otimes 1 \otimes M_{q_n^{(r)}} \otimes 1 \cdots) \subset \bigotimes_{q=1}^\infty M_q = \mathbb{Q}$, and e_0 is a rank one projection in a copy of $M_{q_n^{(r)}} \cong \mathcal{Q}_n^{(r)}$. As argued in [25, Prop.2.5]:

$$(5) \qquad \nu_Y(x) = [P] \otimes_{C^*(G)} x \equiv (\mathrm{id}_m \otimes \mathrm{id}_{C(Y)} \otimes \varphi_n^{(0)})_\sharp(P) - (\mathrm{id}_m \otimes \mathrm{id}_{C(Y)} \otimes \varphi_n^{(1)})_\sharp(P).$$

Since G is assumed to be weakly matricially stable, there exist sequences of genuine group representations $\sigma_n^{(r)}$ and $\pi_n^{(r)}$, $r = 0, 1$, such that

$$(6) \qquad \lim_n \|\varphi_n^{(r)}(s) \oplus \sigma_n^{(r)}(s) - \pi_n^{(r)}(s)\| = 0$$

for all $s \in G$. It is clear that we can view $\sigma_n^{(r)}$ and $\pi_n^{(r)}$ as maps into matrices over $\mathcal{Q}_n^{(r)}$. Let x_n denote the class of $[\sigma_n^{(0)}, \sigma_n^{(1)}] \in KK(C^*(G), \mathcal{Q})$. Then

$$(7) \qquad \nu_Y(x_n) = [P] \otimes_{C^*(G)} (x_n) \equiv (\mathrm{id}_m \otimes \mathrm{id}_{C(Y)} \otimes \sigma_n^{(0)})_*[P] - (\mathrm{id}_m \otimes \mathrm{id}_{C(Y)} \otimes \sigma_n^{(1)})_*[P].$$

It follows from (5), (6) and (7) that

$$(8) \qquad \nu_Y(x + x_n) = [P] \otimes_{C^*(G)} (x + x_n) \equiv (\mathrm{id}_m \otimes \mathrm{id}_{C(Y)} \otimes \pi_n^{(0)})_*[P] - (\mathrm{id}_m \otimes \mathrm{id}_{C(Y)} \otimes \pi_n^{(1)})_*[P].$$

On the other hand we observe that the projections $e_n^{(r)}$, $f_n^{(r)}$, $r = 0, 1$, defined by

$$e_n^{(r)} = (\mathrm{id}_m \otimes \mathrm{id}_{C(Y)} \otimes \pi_n^{(r)})(P) = \sum_{i,j \in I} e_{ij} \otimes \chi_i \chi_j \otimes \pi_n^{(r)}(s_{ij}) \in M_m(\mathbb{C}) \otimes C(Y) \otimes M_{K(n)}(\mathcal{Q}_n^{(r)}),$$

$$f_n^{(r)} = (\mathrm{id}_m \otimes \mathrm{id}_{C(Y)} \otimes \sigma_n^{(r)})(P) = \sum_{i,j \in I} e_{ij} \otimes \chi_i \chi_j \otimes \sigma_n^{(r)}(s_{ij}) \in M_m(\mathbb{C}) \otimes C(Y) \otimes M_{K(n)}(\mathcal{Q}_n^{(r)}),$$

correspond to flat finite rank complex bundles since they are realized via the constant cocycles $\pi_n^{(r)}(s_{ij})$ and $\sigma_n^{(r)}(s_{ij})$. By an extension of a result of Milnor given in [50], all the rational Chern classes of flat complex bundles vanish. It follows that the bundles corresponding to $m_n \cdot e_n^{(r)}$ and $m_n \cdot f_n^{(r)}$ will be trivial for suitable integers m_n and hence

$$\nu_Y(x) = \nu_Y(x + x_n) - \nu_Y(x_n) = [e_n^{(0)}] - [e_n^{(1)}] - [f_n^{(0)}] + [f_n^{(1)}] \in K^0(point; \mathbb{Q}).$$

Since x was arbitrary, it follows that $\nu(KK(C^*(G), \mathcal{Q})_{qd}) \subset \mathbb{Q} = RK^0(point; \mathbb{Q}) \subset RK^0(BG; \mathbb{Q})$. On the other hand we know that ν is surjective by Theorem 4.6. Thus $RK^0(BG; \mathbb{Q}) = \mathbb{Q}$ and hence $H^{even}(BG; \mathbb{Q}) = \mathbb{Q}$, since the Chern character $Ch : RK^0(BG; \mathbb{Q}) \to H^{even}(BG; \mathbb{Q})$ is an isomorphism. □

Remark 4.8. (i) Let G be as in Theorem 1.2 and assume in addition that BG admits a finite simplicial complex model. If G is weakly matricially stable it follows that $K^0(BG)$ is generated by flat bundles. Indeed, if BG is a finite simplicial complex, then G is torsion free and so $\nu : K^0(C^*(G)) \to K^0(BG)$ is surjective. Given $y \in K^0(BG)$ we lift y to quasidiagonal class $x = [\varphi, \psi]$ and reasoning as above we see that $y = [e_n^{(0)}] - [e_n^{(1)}] - [f_n^{(0)}] + [f_n^{(1)}]$ for sufficiently large n.

(ii) If G is a group with Haagerup's property and BG admits a finite simplicial complex model, then $K^0(C^*(G)) = R(G)_{fin}$ if and only if $K^0(BG)$ is generated by flat bundles. This is explained by the fact that ν is a generalization of the Atiyah-Segal map and that it is a bijection under the present assumptions. If $\rho : G \to U(k)$ is a unitary representation, then $\nu[\rho] = [E_\rho]$ where $E_\rho = EG \times_G \mathbb{C}^k$ (G acts on C^k via ρ) is a flat hermitian bundle. Conversely, if E is a flat hermitian bundle of rank k over BG, it is well-known that $E \cong E_\rho$ where $\rho : \pi_1(G) \cong G \to U(k)$ is the monodromy representation of E, see (for example)[13].

5. MATRICIALLY STABLE C^*-ALGEBRAS

In this section we prove Theorems 1.4 and 1.5 by establishing first their C^*-algebraic versions.

Definition 5.1. A unital C^*-algebra A is weakly matricially stable if for any ucp asymptotic homomorphism $\{\varphi_n : A \to M_{k_n}\}_n$ there are two sequences of unital finite dimensional $*$-representations of A, denoted $(\pi_n)_n$ and $(\sigma_n)_n$, such that $\lim_n \|\varphi_n(a) \oplus \sigma_n(a) - \pi_n(a)\| = 0$ for all $a \in A$. Note that a group G is weakly matricially stable if and only if $C^*(G)$ is so.

Recall that $\mathrm{Inf} K_0(A) = \{x \in K_0(A) : \forall k \in \mathbb{Z}, \exists m > 0, m([1_A] + kx) \geq 0\}$ is the subgroup of infinitesimal elements of $K_0(A)$. If $\varphi : A \to B$ is a unital $*$-homomorphism, then $\varphi_*(\mathrm{Inf} K_0(A)) \subset \mathrm{Inf} K_0(B)$.

Theorem 5.2. *Let A be a separable unital K-quasidiagonal C^*-algebra that satisfies the UCT and such that $K_*(A)$ is finitely generated. Suppose that A is weakly matricially stable. Then $K^0(A)$ is generated by finite dimensional representations of A and $\mathrm{Inf} K_0(A)$ is a torsion group.*

Proof. A satisfies the UCT if and only if it satisfies the UMCT of [28]. A concise account of the UMCT is included in [29]. Let $\mathcal{P} \subseteq \mathbb{N}$ be the set consisting of 0 and all prime powers. The total K-theory group of a C^*-algebra A is defined as $\underline{K}(A) = \bigoplus_{p \in \mathcal{P}} K_*(A; \mathbb{Z}/p)$. $\underline{K}(A)$ is a module over the ring Λ of Böckstein operations. Since $K_1(A)$ is finitely generated, $\mathrm{Pext}(K_1(A), K_0(\mathbb{C})) = 0$ and by the UMCT we have an isomorphism

$$(9) \qquad \theta : K^0(A) \xrightarrow{\cong} \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(\mathbb{C}))$$

induced by the Kasparov product $K_*(A; \mathbb{Z}/p) \times K^0(A) \to K_*(\mathbb{C}; \mathbb{Z}/p)$, $\theta(x)(y) = y \otimes_A x$. Since $K_*(A)$ is finitely generated, $\underline{K}(A)$ is a finitely generated Λ-module. Moreover, if $\mathrm{Tor} K_*(A)$ has order N and $\mathcal{P}_N = \{p \in \mathcal{P} : p|N\} \cup \{0\}$, then by [28, Cor.2.11] we can replace $\underline{K}(-)$ and Λ in (9) by $\underline{K}'(-)$ and Λ', where $\underline{K}'(A) = \bigoplus_{p \in \mathcal{P}_N} K_*(A; \mathbb{Z}/p)$ and Λ' is the corresponding subset of Böckstein operations acting on $\underline{K}'(A)$. Since A is quasidiagonal, any element $x \in K^0(A)$ is represented by a Cuntz pair $(\varphi^{(0)}, \varphi^{(1)})$ for which there is an increasing approximate unit of $K(H)$ consisting of projections, denoted $(p_n)_n$, such that $\lim_n \|[\varphi^{(i)}(a), p_n]\| = 0$, $i = 0, 1$, for all $a \in A$.

Then $p_n \varphi^{(i)}(\cdot) p_n$ are finite rank cp asymptotic morphisms which we can perturb to ucp asymptotic homomorphisms $\varphi_n^{(i)} : A \to M_{k_n^{(i)}}$. It follows by [25, Prop.2.5] that for each $y \in \underline{K}(A)$:

$$(\varphi_n^{(0)})_\sharp(y) - (\varphi_n^{(1)})_\sharp(y) \equiv y \otimes_A x.$$

Suppose now that A is weakly matricially stable. Then there are sequences of finite dimensional representations of A, $(\sigma_n^{(i)})$ and $(\pi_n^{(i)})_n$, $i = 0, 1$, such that $\lim_n \|\varphi_n^{(i)}(a) \oplus \sigma_n^{(i)}(a) - \pi_n^{(i)}(a)\| = 0$, $i = 0, 1$, for all $a \in A$. Let x_n denote the class of $[\sigma_n^{(0)}, \sigma_n^{(1)}] \in K^0(A)$. Since K-theory is stable under small perturbations, we obtain that for all $y \in \underline{K}(A)$:

$$(\pi_n^{(0)})_*(y) - (\pi_n^{(1)})_*(y) \equiv (\varphi_n^{(0)} \oplus \sigma_n^{(0)})_\sharp(x) - (\varphi_n^{(1)} \oplus \sigma_n^{(1)})_\sharp(x) \equiv y \otimes_A (x + x_n).$$

Since the maps $\pi_{n*}^{(i)}$ are Λ'-linear and since the Λ'-module $\underline{K}'(A)$ is finitely generated, there is a sufficiently large m such that $(\pi_m^{(0)})_*(y) - (\pi_m^{(1)})_*(y) = y \otimes_A (x + x_m) = y \otimes_A x + (\sigma_m^{(0)})_*(y) - (\sigma_m^{(1)})_*(y)$ for all $y \in \underline{K}'(A)$. Since θ is an isomorphism, it follows that $x = [\pi_m^{(0)}] - [\pi_m^{(1)}] - [\sigma_m^{(0)}] + [\sigma_m^{(1)}]$. This finishes the proof of the first part of the statement.

We prove the second part by contradiction. If $y \in \operatorname{Inf} K_0(A)$ is a non-torsion element, then there is a homomorphism $h : K_0(A) \to \mathbb{Z}$ such that $h(y) \neq 0$. By the UCT, θ induces a surjection $K^0(A) \to Hom(K_0(A), \mathbb{Z})$. Thus there is $x \in K^0(A)$ such that $\theta(x) = h$. By the first part of the proof, we find two finite dimensional representations $\pi_i : A \to M_{k_i}$ of A such that $x = [\pi_0] - [\pi_1]$. It follows that $h(y) = \theta(x)(y) = (\pi_0)_*(y) - (\pi_1)_*(y) = 0$ since $\operatorname{Inf} K_0(\mathbb{C}) = 0$. $\qquad\square$

Eilers, Loring and Pedersen [38] introduced and studied natural noncommutative analogues of CW complexes (NCCW). They showed that if A is a 2-dimensional NCCW such that $\operatorname{Inf} K_0(A)$ is a torsion group, then A is matricially stable. We show below that the reverse implication also holds. The C^*-algebras associated with wallpapers groups are examples of 2-dimensional NCCW complexes. The list of all matricially stable crystallographic groups was obtained [37]. In addition to Theorem 1.2, the following corollary sheds new lights on the results of [37] concerning these groups, see [37, Remarks 4.4-4.5].

Corollary 5.3. *A 2-dimensional NCCW A is matricially stable if and only $\operatorname{Inf} K_0(A)$ is a torsion group.*

Proof. The implication $(\Rightarrow)$ was proven in [38, Cor.8.2.2]. The converse follows from Theorem 5.2 since A is type I, it is quasidiagonal and it has finitely generated K-theory. $\qquad\square$

Theorem 5.4. *Let A be a separable unital exact quasidiagonal C^*-algebra satisfying the UCT and such that $K_*(A)$ is finitely generated. Then A is weakly matricially stable if and only if A is residually finite dimensional and $K^0(A)$ is generated by finite dimensional representations of A.*

Proof. $(\Rightarrow)$ Since A is quasidiagonal, there is a ucp asymptotic homomorphism $\{\varphi_n : A \to M_{k_n}\}_n$ which is asymptotically isometric. If $(\sigma_n)_n$ and $(\pi_n)_n$ are as in Definition 5.1, then (π_n) is also asymptotically isometric and hence A is residually finite dimensional. The required property of $K^0(A)$ follows from Theorem 5.2. $(\Leftarrow)$ In order to prove that A is matricially stable is suffices to show that for any ucp asymptotic homomorphism $\{\varphi_n : A \to M_{k_n}\}_n$, for any finite subset F of A and $\varepsilon > 0$, there are two sequences of finite dimensional representations, (α_n) and (β_n), such that

$$(10) \qquad \limsup_n \|\varphi_n(a) \oplus \alpha_n(a) - \beta_n(a)\| < \varepsilon.$$

for all $a \in F$. Since $K_*(A)$ is finitely generated, as explained in the proof of Theorem 5.2, we can reduce the Λ-module $\underline{K}(A)$ in the UMCT to a finite array of groups and a finite number of Böckstein operations denoted by $\underline{K}'(A)$ and Λ'. Thus the UMCT reads: $KK(A, B) \cong \operatorname{Hom}_{\Lambda'}(\underline{K}'(A), \underline{K}'(B))$.

Since $\underline{K}'(A)$ is a finitely generated Λ' module and since the set Λ' is finite we can associate to each φ_n an element $h_n \in \operatorname{Hom}_{\Lambda'}(\underline{K}'(A), \underline{K}'(M_{r_n}))$ such that $(\varphi_n)_\sharp(x) \equiv h_n(x)$, for all $x \in \underline{K}'(A)$. Since $K^0(A) \cong \operatorname{Hom}_{\Lambda'}(\underline{K}'(A), \underline{K}'(\mathbb{C}))$, it follows from our assumption on $K^0(A)$ that for each n there are finite dimensional representations σ_n and ψ_n of A such that $h_n = (\psi_n)_* - (\sigma_n)_*$. Thus, $(\varphi_n)_\sharp(x) \equiv (\psi_n)_*(x) - (\sigma_n)_*(x)$, for all $x \in \underline{K}'(A)$. Setting $\phi_n = \varphi_n \oplus \sigma_n : A \to M_{r_n}$ it follows that $(\phi_n)_\sharp(x) \equiv (\psi_n)_\sharp(x)$ for all $x \in \underline{K}'(A)$. Let $D = \prod_n M_{r_n}$, $I = \bigoplus_n M_{r_n}$ and $B = D/I$. Consider the unital $*$-homomorphisms $\Phi, \Psi : A \to B$ induced by the asymptotic homomorphisms $(\phi_n)_n$ and $(\psi_n)_n$. One verifies that $\Phi_* = \Psi_* : \underline{K}'(A) \to \underline{K}'(B)$ and hence that $[\Phi] = [\Psi]$ in $KK(A, B)$. We substantiate this claim as follows. Let $B_n = \prod_{k \geq n} M_{r_k}$. Then $B = \varinjlim B_n$ where $B_n \to B_{n+1}$ is the natural restriction map. We are going to show that the map

$$J : \underline{K}(B) \cong \varinjlim \underline{K}(\prod_{k \geq n} M_{r_k}) \to \varinjlim \prod_{k \geq n} \underline{K}(M_{r_k}) \cong \prod_n \underline{K}(M_{r_n}) / \bigoplus_n \underline{K}(M_{r_n})$$

is injective. This will follow if one shows that the natural map $j_n : \underline{K}(\prod_{k \geq n} M_{r_k}) \to \prod_{k \geq n} \underline{K}(M_{r_k})$ is injective. To that purpose we note that $K_1(B_n) = 0$ and that

$$K_0(B_n) = \{(x_k)_{k \geq n} : x_k \in \mathbb{Z}, \exists m \text{ with } |x_k| \leq m r_k, \forall k \geq n\}.$$

Since the map $K_0(B_n) \xrightarrow{\times p} K_0(B_n)$ is injective, we deduce that $K_1(B_n; \mathbb{Z}/p) = 0$ for $p \geq 2$ and all n. Using the exactness and the naturality of the Böckstein sequence, we have the following commutative diagram which is easily checked to have injective vertical maps:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(\prod_{k \geq n} M_{r_k}) & \xrightarrow{\times p} & K_0(\prod_{k \geq n} M_{r_k}) & \longrightarrow & K_0(\prod_{k \geq n} M_{r_k}; \mathbb{Z}/p) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{k \geq n} K_0(M_{r_k}) & \xrightarrow{\times p} & \prod_{k \geq n} K_0(M_{r_k}) & \longrightarrow & \prod_{k \geq n} K_0(M_{r_k}; \mathbb{Z}/p) & \longrightarrow & 0
\end{array}
$$

Therefore all the maps j_n are injective and hence J is injective. We have seen earlier that $(\phi_n)_\sharp(x) \equiv (\psi_n)_\sharp(x)$ for all $x \in \underline{K}(A)$. This means precisely that $J \circ \Phi_* = J \circ \Psi_*$. Since J is injective, it follows that $\Phi_* = \Psi_*$ as desired. Let $(\eta_k)_k$ be a sequence of finite dimensional representations of A that separates the points of A and such that each η_k occurs infinitely many times. Set $\gamma_k = \eta_1 \oplus \cdots \oplus \eta_k$ and view it as a map $\gamma_k : A \to M_{m_k}(\mathbb{C}1_B) \subset M_{m_k}(B)$. Let us note that A is K-nuclear since it is equivalent to a commutative C^*-algebra and hence $KK(A, B) \cong KK_{nuc}(A, B)$. Moreover, since A is exact, the maps Φ and Ψ are nuclear by [23, Prop.3.3]. By [26, Thm.4.3], since $[\Phi] = [\Psi]$ in $KK(A, B)$, there is a sequence of unitaries $u_k \in M_{m_k+1}(B)$ such that $\lim_n \|\Psi(a) \oplus \gamma_k(a) - u_k(\Psi(a) \oplus \gamma_k(a))u_k^*\| = 0$ for all $a \in A$. Thus, for any finite subset F of A and $\varepsilon > 0$ there is a k such that if we set $\gamma := \gamma_k$, $m := m_k$, and $u := u_k$, then $\|\Psi(a) \oplus \gamma(a) - u(\Psi(a) \oplus \gamma(a))u^*\| < \varepsilon$ for all $a \in F$. We can view γ as a map into $M_m(\mathbb{C}1_D)$ with components (γ_n). Lift u to a unitary $v \in M_{m+1}(D)$ with components (v_n). It follows that for all $a \in F$,

$$(11) \qquad \limsup_n \|\varphi_n(a) \oplus \sigma_n(a) \oplus \gamma_n(a) - v_n(\psi_n(a) \oplus \gamma_n(a))v_n^*\| < \varepsilon.$$

This proves (10) with $\alpha_n = \sigma_n \oplus \gamma_n$ and $\beta_n = v_n(\psi_n \oplus \gamma_n)v_n^*$. $\qquad\square$

Remark 5.5. Two ucp asymptotic homomorphism $\{\varphi_n^{(i)} : A \to M_{k_n^{(i)}}\}_n$, $i = 0, 1$, are called stably unitarily equivalent if there are two sequences of unital $*$-homomorphisms $\{\pi_n^{(i)} : A \to M_{r_n^{(i)}}\}_n$, $i = 0, 1$, and a sequence of unitaries (u_n) such that

$$\lim_n \|\varphi_n^{(0)}(a) \oplus \pi_n^{(0)}(a) - u_n(\varphi_n^{(1)}(a) \oplus \pi_n^{(1)}(a))u_n^*\| = 0$$

for all $a \in A$. Let A be a separable exact residually finite dimensional C^*-algebra satisfying the UCT. Arguing as in the proof of Theorem 5.4 one shows that two ucp asymptotic homomorphism $\{\varphi_n^{(i)} : A \to M_{k_n^{(i)}}\}_n$, $i = 0, 1$, are stably unitarily equivalent if and only if $(\varphi_n^{(0)})_\sharp(x) \equiv (\varphi_n^{(1)})_\sharp(x)$ for all $x \in K_0(A; \mathbb{Z}/p)$, $p \in \mathcal{P}$.

Proof of Theorem 1.4:

(i) Higson and Kasparov [48] proved that the Baum-Connes map is an isomorphism and that the canonical map $q_G : C^*(G) \to C_r^*(G)$ is a KK-equivalence. Moreover, Tu [84] showed that $C^*(G)$ satisfies the UCT. Suppose now that G is weakly matricially stable. Then G is MAP by Lemma 3.17 and hence quasidiagonal. By Proposition 3.8 the canonical map $q_G : C^*(G) \to C_r^*(G)$ factors as: $C^*(G) \xrightarrow{q_D} D \to C_r^*(G)$ with D unital and quasidiagonal. Since q_G is a KK-equivalence, q_D^* is surjective. Using Remark 4.5 we see that

$$K^0(C^*(G)) = q_G^*(K^0(C_r^*(G))) = q_D^*(K^0(D)) = q_D^*(K^0(D)_{qd}) \subseteq K^0(C^*(G))_{qd}.$$

Thus $C^*(G)$ is K-quasidiagonal. We conclude the argument by applying Theorem 5.2.

(ii) Suppose now that G is torsion free. By Theorem 1.2, $H^{even}(G; \mathbb{Q}) \cong \mathbb{Q}$. Since the Baum-Connes map is an isomorphism, we have $K_0(C^*(G)) \otimes \mathbb{Q} \cong RK_0(BG) \otimes \mathbb{Q} \cong H_{even}(G, \mathbb{Q}) \cong \mathbb{Q}$.

(iii) This was discussed in Remark 4.8.   □

Proof of Theorem 1.5: One direction follows Theorem 1.4 since amenable groups have Haagerup's property. The converse follows by applying Theorem 5.4 to the nuclear C^*-algebra $C^*(G)$.   □

6. Almost flat K-theory

Connes, Gromov and Moscovici [19] used the approximate monodromy correspondence which associates a group quasi-representation to an almost flat bundle to prove the Novikov conjecture for large classes of groups. Gromov indicates in [41, 42] how one constructs nontrivial almost flat K-theory classes for residually finite groups that are fundamental groups of even dimensional non-positively curved compact manifolds. We introduced in [24], [25] an approach to almost flat K-theory via quasidiagonality of KK-elements. This topic was further explored in [13] where we discussed continuity properties of the approximate modronomy correspondence. The results of [25] and [13] were extended in [64, 65] to a relative setting for pair of groups which are residually amenable.

Let Y be a compact Hausdorff space and let $(U_i)_{i \in I}$ be a fixed finite open cover of Y. A complex vector bundle on Y of rank m is called ε-flat if is represented by a cocycle $v_{ij} : U_i \cap U_j \to U(m)$ such that $\|v_{ij}(y) - v_{ij}(y')\| < \varepsilon$ for all $y, y' \in U_i \cap U_j$ and all $i, j \in I$. An element $x \in K^0(Y)$ is called almost flat if for any $\varepsilon > 0$ there are ε-flat vector bundles E, F such that $x = [E] - [F]$. This property does not depend on the cover $(U_i)_{i \in I}$.

An element $x \in RK^0(BG)$ is called locally almost flat, if for any finite CW-subcomplex $j : Y \hookrightarrow BG$, the element $j^*(x) \in K^0(Y)$ is almost flat. We say that $x \in RK^0(BG)$ has almost flat local multiples if for any Y as above there is $m \geq 1$ such that $m \cdot j^*(x) \in K^0(Y)$ is almost flat. For a discussion of almost flatness, see (for example) [25], [13].

By combining the methods of [25] and [64],[65] one obtains the following general result on the existence of almost flat K-theory classes. A rational version of this result for residually amenable groups (including the relative case of pairs of groups) is due to Kubota [65, Cor.5.14].

Theorem 6.1. *Let G be a countable discrete quasidiagonal group that admits a γ-element. Then all the elements of $RK^0(BG)$ have almost flat local multiples. If in addition G is torsion free, then all the elements of $RK^0(BG)$ are locally almost flat.*

Proof. Suppose first that G is torsion free. Then $\nu(KK(C^*(G), \mathbb{C})_{qd}) = RK^0(BG)$ by Theorem 4.6. By [25, Cor.4.4], for any finite CW-subcomplex $j : Y \hookrightarrow BG$, $j^*(\nu(KK(C^*(G), \mathbb{C})_{qd})$ consists of entirely of almost flat elements. This last property is explained by the fact that the classes $\nu_Y(x)$ in (5) are almost flat as discussed in [25] and with more details in [13]. The degree of flatness of the right hand side of (5) is given by the degree to which the maps $\varphi_n^{(i)}$ are multiplicative. The first part of the statement is proved similarly using the fact that $\nu(KK(C^*(G), \mathcal{Q})_{qd}) = RK^0(BG; \mathbb{Q})$ as seen in the proof of Theorem 1.2 and the observation that if Y is a finite CW complex, then $RK^0(Y; \mathbb{Q}) \cong RK^0(Y) \otimes \mathbb{Q}$. $\qquad\square$

References

[1] S. Armstrong, K. Dykema, R. Exel, and H. Li. On embeddings of full amalgamated free product C^*-algebras. *Proc. Amer. Math. Soc.*, 132(7):2019–2030, 2004. 8

[2] G. Arzhantseva. Asymptotic approximations of finitely generated groups. In *Extended abstracts Fall 2012—automorphisms of free groups*, volume 1 of *Trends Math. Res. Perspect. CRM Barc.*, pages 7–15. Springer, Cham, 2014. 1

[3] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and K-theory of group C^*-algebras. In C^*-*algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994. 3, 10, 11

[4] G. Baumslag. Automorphism groups of residually finite groups. *J. London Math. Soc.*, 38:117–118, 1963. 5

[5] G. Baumslag. A survey of groups with a single defining relation. In *Proceedings of groups—St. Andrews 1985*, volume 121 of *London Math. Soc. Lecture Note Ser.*, pages 30–58. Cambridge Univ. Press, Cambridge, 1986. 4, 5

[6] C. Béguin, H. Bettaieb, and A. Valette. K-theory for C^*-algebras of one-relator groups. *K-Theory*, 16(3):277–298, 1999. 5

[7] B. Bekka, P. de la Harpe, and A. Valette. *Kazhdan's property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008. 10

[8] B. Blackadar and E. Kirchberg. Generalized inductive limits of finite-dimensional C^*-algebras. *Math. Ann.*, 307(3):343–380, 1997. 6

[9] F. P. Boca. A note on full free product C^*-algebras, lifting and quasidiagonality. In *Operator theory, operator algebras and related topics (Timişoara, 1996)*, pages 51–63. Theta Found., Bucharest, 1997. 7

[10] N. P. Brown. Invariant means and finite representation theory of C^*-algebras. *Mem. Amer. Math. Soc.*, 184(865):viii+105, 2006. 9

[11] N. P. Brown and N. Ozawa. C^*-*algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008. 4, 9

[12] J. R. Carrión and M. Dadarlat. Quasi-representations of surface groups. *J. Lond. Math. Soc. (2)*, 88(2):501–522, 2013. 3

[13] J. R. Carrión and M. Dadarlat. Almost flat K-theory of classifying spaces. *J. Noncommut. Geom.*, 12(2):407–438, 2018. 2, 15, 18, 19

[14] J. R. Carrión, M. Dadarlat, and C. Eckhardt. On groups with quasidiagonal C^*-algebras. *J. Funct. Anal.*, 265(1):135–152, 2013. 1, 6

[15] J. Chabert and S. Echterhoff. Permanence properties of the Baum-Connes conjecture. *Doc. Math.* 3

[16] P.-A. Cherix, M. Cowling, P. Jolissaint, P. Julg, and A. Valette. *Groups with the Haagerup property*, volume 197 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2001. Gromov's a-T-menability. 3

[17] C. Chevalley and S. Eilenberg. Cohomology theory of Lie groups and Lie algebras. *Trans. Amer. Math. Soc.*, 63:85–124, 1948. 4

[18] B. Clair. Residual amenability and the approximation of L^2-invariants. *Michigan Math. J.*, 46(2):331–346, 1999. 5

[19] A. Connes, M. Gromov, and H. Moscovici. Conjecture de Novikov et fibrés presque plats. *C. R. Acad. Sci. Paris Sér. I Math.*, 310(5):273–277, 1990. 1, 2, 3, 18

[20] A. Connes and N. Higson. Déformations, morphismes asymptotiques et K-théorie bivariante. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(2):101–106, 1990. 1

[21] J. H. Conway and J. P. Rossetti. Describing the platycosms. Preprint 2003, arXiv:math/0311476v1. 5

[22] J. Cuntz, S. Echterhoff, X. Li, and G. Yu. *K-theory for group C^*-algebras and semigroup C^*-algebras*, volume 47 of *Oberwolfach Seminars*. Birkhäuser/Springer, Cham, 2017. 12

[23] M. Dadarlat. Quasidiagonal morphisms and homotopy. *J. Funct. Anal.*, 151(1):213–233, 1997. 17

[24] M. Dadarlat. Group quasi-representations and index theory. *J. Topol. Anal.*, 4(3):297–319, 2012. 3, 5, 18

[25] M. Dadarlat. Group quasi-representations and almost flat bundles. *J. Noncommut. Geom.*, 8(1):163–178, 2014. 2, 3, 12, 13, 14, 16, 18, 19

[26] M. Dadarlat and S. Eilers. Asymptotic unitary equivalence in KK-theory. *K-Theory*, 23(4):305–322, 2001. 17

[27] M. Dadarlat and E. Guentner. Constructions preserving Hilbert space uniform embeddability of discrete groups. *Trans. Amer. Math. Soc.*, 355(8):3253–3275 (electronic), 2003. 3

[28] M. Dadarlat and T. A. Loring. A universal multicoefficient theorem for the Kasparov groups. *Duke Math. J.*, 84(2):355–377, 1996. 15

[29] M. Dadarlat and R. Meyer. E-theory for C*-algebras over topological spaces. *J. Funct. Anal.*, 263(1):216–247, 2012. 15

[30] M. Dadarlat and U. Pennig. Connective C^*-algebras. *J. Funct. Anal.*, 272(12):4919–4943, 2017. 3

[31] M. Dadarlat and U. Pennig. Deformations of nilpotent groups and homotopy symmetric C^*-algebras. *Math. Ann.*, 367(1-2):121–134, 2017. 3

[32] M. Dadarlat, U. Pennig, and A. Schneider. Deformations of wreath products. *Bull. Lond. Math. Soc.*, 49(1):23–32, 2017. 6

[33] M. De Chiffre, L. Glebsky, A. Lubotzky, and A. Thom. Stability, cohomology vanishing, and nonapproximable groups. *Forum Math. Sigma*, 8:e18, 2020. 1

[34] P. de la Harpe, A. G. Robertson, and A. Valette. On the spectrum of the sum of generators for a finitely generated group. *Israel J. Math.*, 81(1-2):65–96, 1993. 10

[35] K. Dekimpe and N. Petrosyan. Homology of Hantzsche-Wendt groups. In *Discrete groups and geometric structures*, volume 501 of *Contemp. Math.*, pages 87–102. Amer. Math. Soc., Providence, RI, 2009. 6

[36] J. Dixmier. *C*-algebras*. North Holland, Amsterdam, 1982. 7

[37] S. Eilers, T. Shulman, and A. P. Sørensen. *C*-stability of discrete groups. 1, 4, 5, 16

[38] S. r. Eilers, T. A. Loring, and G. K. Pedersen. Stability of anticommutation relations: an application of noncommutative CW complexes. *J. Reine Angew. Math.*, 499:101–143, 1998. 16

[39] R. Exel and T. A. Loring. Invariants of almost commuting unitaries. *J. Funct. Anal.*, 95(2):364–376, 1991. 1

[40] B. Farb and D. Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012. 5

[41] M. Gromov. Geometric reflections on the Novikov conjecture. In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, volume 226 of *London Math. Soc. Lecture Note Ser.*, pages 164–173. Cambridge Univ. Press, Cambridge, 1995. 3, 18

[42] M. Gromov. Positive curvature, macroscopic dimension, spectral gaps and higher signatures. In *Functional analysis on the eve of the 21st century, Vol. II (New Brunswick, NJ, 1993)*, volume

132 of *Progr. Math.*, pages 1–213. Birkhäuser Boston, Boston, MA, 1996. 3, 18

[43] E. K. Grossman. On the residual finiteness of certain mapping class groups. *J. London Math. Soc. (2)*, 9:160–164, 1974/75. 5

[44] E. Guentner. Exactness of the one relator groups. *Proc. Amer. Math. Soc.*, 130(4):1087–1093, 2002. 5

[45] E. Guentner, N. Higson, and S. Weinberger. The Novikov conjecture for linear groups. *Publ. Math. Inst. Hautes Études Sci.*, (101):243–268, 2005. 3, 4

[46] U. Hamenstädt. Geometry of the mapping class groups. I. Boundary amenability. *Invent. Math.*, 175(3):545–609, 2009. 5

[47] J. Hempel. Residual finiteness of surface groups. *Proc. Amer. Math. Soc.*, 32:323, 1972. 5

[48] N. Higson and G. Kasparov. E-theory and KK-theory for groups which act properly and isometrically on Hilbert space. *Invent. Math.*, 144(1):23–74, 2001. 2, 3, 18

[49] A. Ioana, P. Spaas, and M. Wiersma. Cohomological obstructions to lifting properties for full group C*-algebras. https://arxiv.org/abs/2006.01874 [math.OA], 2020. 3

[50] F. Kamber and P. Tondeur. The characteristic homomorphism of flat bundles. *Topology*, 6:153–159, 1967. 14

[51] A. Kar and N. Nikolov. A non-LEA sofic group. *Proc. Indian Acad. Sci. Math. Sci.*, 127(2):289–293, 2017. 7, 8

[52] G. Kasparov and G. Skandalis. Groupes "boliques" et conjecture de Novikov. *C. R. Acad. Sci. Paris Sér. I Math.*, 319(8):815–820, 1994. 3, 10, 11

[53] G. Kasparov and G. Skandalis. Groups acting properly on "bolic" spaces and the Novikov conjecture. *Ann. of Math. (2)*, 158(1):165–206, 2003. 3, 10

[54] G. G. Kasparov. Equivariant KK-theory and the Novikov conjecture. *Invent. Math.*, 91(1):147–201, 1988. 3, 10, 11, 12, 13

[55] G. G. Kasparov. K-theory, group C^*-algebras, and higher signatures (conspectus). In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, volume 226 of *London Math. Soc. Lecture Note Ser.*, pages 101–146. Cambridge Univ. Press, Cambridge, 1995. 12, 13

[56] G. G. Kasparov and G. Skandalis. Groups acting on buildings, operator K-theory, and Novikov's conjecture. *K-Theory*, 4(4):303–337, 1991. 3, 10, 11, 13

[57] Y. Kida. The mapping class group from the viewpoint of measure equivalence theory. *Mem. Amer. Math. Soc.*, 196(916):viii+190, 2008. 5

[58] E. Kirchberg. Discrete groups with Kazhdan's property T and factorization property are residually finite. *Math. Ann.*, 299(3):551–563, 1994. 9

[59] E. Kirchberg and S. Wassermann. Permanence properties of C^*-exact groups. *Doc. Math.*, 4:513–558 (electronic), 1999. 5

[60] M. Kneser. Orthogonale Gruppen über algebraischen Zahlkörpern. *J. Reine Angew. Math.*, 196:213–220, 1956. 7

[61] A. I. Korchagin. The MF-property for countable discrete groups. *Mat. Zametki*, 102(2):231–246, 2017. 6

[62] M. Korkmaz and A. I. Stipsicz. The second homology groups of mapping class groups of oriented surfaces. *Math. Proc. Cambridge Philos. Soc.*, 134(3):479–489, 2003. 5

[63] P. H. Kropholler. Baumslag-Solitar groups and some other groups of cohomological dimension two. *Comment. Math. Helv.*, 65(4):547–558, 1990. 5

[64] Y. Kubota. The relative Mishchenko–Fomenko higher index and almost flat bundles I: The relative Mishchenko–Fomenko index. 18

[65] Y. Kubota. The relative Mishchenko–Fomenko higher index and almost flat bundles II: Almost flat index pairing. 2, 3, 13, 18

[66] Li, Quihi and Hadwin, Don and Ma, Xiujuan and Shen, Junhao . On unital full amalgamated free products of quasidiagonal C*-algebras. *Funktsional. Anal. i Prilozhen.*, 50(1), 2016. 8

[67] S. Meskin. Nonresidually finite one-relator groups. *Trans. Amer. Math. Soc.*, 164:105–114, 1972. 5

[68] C. H. Mladen Bestvina, Vincent Guirardel. Boundary amenability of $Out(\mathbb{F}_n)$. arXiv:1705.07017 [math.GR]. 5

[69] S. Nishikawa and V. Proietti. Groups with Spanier-Whitehead duality. preprint 2019, arXiv:1908.03749 [math.KT]. 3, 11

[70] P. F. Pickel. Rational cohomology of nilpotent groups and Lie algebras. *Comm. Algebra*, 6(4):409–419, 1978. 4

[71] G. Pisier. *Introduction to operator space theory*, volume 294 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003. 9

[72] A. Putman. The Picard group of the moduli space of curves with level structures. *Duke Math. J.*, 161(4):623–674, 2012. 6

[73] J. Rosenberg. Quadiagonality and nuclearity. *J. Operator Th.*, 18:15–18, 1987. Appendix to Strongly quasidiagonal C*-algebras by D. Hadwin. 9

[74] C. Schafhauser. Subalgebras of simple AF-algebras. https://arxiv.org/abs/1807.07381 [math.OA], 2019. 8

[75] A. J. Schneider. *Finite dimensional approximations and deformations of group C*-algebras*. ProQuest LLC, Ann Arbor, MI, 2016. Thesis (Ph.D.)–Purdue University. 9

[76] G. Skandalis. Une Notion de Nucléarité en K-théorie. *K-theory*, 1(5):549–573, 1988. 12

[77] G. Skandalis, J. L. Tu, and G. Yu. The coarse Baum-Connes conjecture and groupoids. *Topology*, 41(4):807–834, 2002. 3

[78] A. Thom. Discrete groups G whose full C^*-algebra $C^*(G)$ is not quasidiagonal? MathOverflow. URL:https://mathoverflow.net/q/67608 (version: 2011-06-12). 9, 10

[79] A. Thom. Examples of hyperlinear groups without factorization property. *Groups Geom. Dyn.*, 4(1):195–208, 2010. 6

[80] A. Thom. Finitary approximations of groups and their applications. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures*, pages 1779–1799. World Sci. Publ., Hackensack, NJ, 2018. 1

[81] A. Tikuisis, S. White, and W. Winter. Quasidiagonality of nuclear C^*-algebras. *Ann. of Math. (2)*, 185(1):229–284, 2017. 1, 2, 5, 6

[82] J. Tits. Free subgroups in linear groups. *J. Algebra*, 20:250–270, 1972. 7

[83] J.-L. Tu. Remarks on Yu's "property A" for discrete metric spaces and groups. *Bull. Soc. Math. France*, 129(1):115–139, 2001. 5

[84] J.-L. Tu. The gamma element for groups which admit a uniform embedding into Hilbert space. In *Recent advances in operator theory, operator algebras, and their applications*, volume 153 of *Oper. Theory Adv. Appl.*, pages 271–286. Birkhäuser, Basel, 2005. 3, 4, 10, 18

[85] K. Vogtmann. The topology and geometry of automorphism groups of free groups. In *European Congress of Mathematics*, pages 181–202. Eur. Math. Soc., Zürich, 2018. 5

[86] D. Voiculescu. Asymptotically commuting finite rank unitary operators without commuting approximats. *Acta Sci. Math. (Szeged)*, 45:429–431, 1983. 1

[87] D. T. Wise. Research announcement: the structure of groups with a quasiconvex hierarchy. *Electron. Res. Announc. Math. Sci.*, 16:44–55, 2009. 5

[88] J. A. Wolf. *Spaces of constant curvature*. McGraw-Hill Book Co., New York-London-Sydney, 1967. 6

[89] M. Yamashita. Discrete groups G whose full C^*-algebra $C^*(G)$ is not quasidiagonal? MathOverflow. URL:https://mathoverflow.net/q/67646 (version: 2011-06-13). 9

[90] G. Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.*, 139(1):201–240, 2000. 3

MD: Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA

Email address: mdd@purdue.edu