ONE-PARAMETER CONTINUOUS FIELDS OF KIRCHBERG ALGEBRAS WITH RATIONAL K-THEORY

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ABSTRACT. We show that separable continuous fields over the unit interval whose fibers are stable Kirchberg algebras that satisfy the universal coefficient theorem in KK-theory (UCT) and have rational K-theory groups are classified up to isomorphism by filtrated K-theory.

1. INTRODUCTION

The purpose of this paper is to investigate the classification problem for continuous fields of Kirchberg algebras over the unit interval by K-theory invariants. It is natural to associate to a $C[0,1]$-algebra A the family of all exact triangles of $\mathbb{Z}/2$-graded K-theory groups

$$\begin{array}{ccc} \mathrm{K}_*(A(U)) & \longrightarrow & \mathrm{K}_*(A(Y)) \\ & \nwarrow \qquad \swarrow & \\ & \mathrm{K}_*(A(Y \setminus U)) & \end{array}$$

where Y is a subinterval of $[0,1]$ and U is a relatively open subinterval of Y. The family of these exact triangles are assembled into an invariant $\mathrm{FK}(A)$ called the filtrated K-theory of A, see Definition 3.4.

In this article we exhibit several classes of separable continuous fields over the unit interval whose fibers are stable UCT Kirchberg algebras and for which filtrated K-theory is a complete invariant. In particular, we show that this is the case for fields which are stable under tensoring with the universal UHF-algebra. A C^*-algebra D has rational K-theory if $\mathrm{K}_*(D) \cong \mathrm{K}_*(D) \otimes \mathbb{Q}$.

Theorem 1.1. *Let A and B be separable continuous fields over the unit interval whose fibers are stable Kirchberg algebras that satisfy the UCT and*

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have rational K*-theory groups. Then any isomorphism of filtrated* K*-theory* $\mathrm{FK}(A) \cong \mathrm{FK}(B)$ *lifts to a* $C[0,1]$*-linear* $*$*-isomorphism* $A \cong B$*.*

The continuous fields classified by this theorem include fields that are nowhere locally trivial. It is for this reason that one needs to include infinitely many subintervals of $[0,1]$ in any complete invariant. However it suffices to consider intervals whose endpoints belong to a countable dense subset of $[0,1]$. The result does not extend to continuous fields of Kirchberg algebras if torsion is allowed, as we will explain shortly.

Our approach relies on the following three crucial ingredients:

- Kirchberg's isomorphism theorem for non-simple nuclear strongly purely infinite C^*-algebras [15],
- the results from [10] which relate E-theory over a second countable space X with the corresponding version of KK-theory and with E-theory groups over finite approximating spaces of X,
- the universal coefficient theorem for accordion spaces from [2] (generalizing results from [4, 18, 20, 23]) including a description of projective and injective objects in the target category of filtrated K-theory.

The relevance of accordion spaces in this framework is due to the fact that sufficiently many finite approximating spaces of the unit interval are accordion spaces.

A major difficulty in any attempt to use the result of [15] is the computation of the group $\mathrm{KK}(X; A, B)$ or at least a quotient of this group which allows to detect $\mathrm{KK}(X)$-equivalences. In [10], the second named author and Meyer proved a universal *multi*-coefficient theorem (abbreviated UMCT) for separable $\mathrm{C}(X)$-algebras over a totally disconnected compact metrizable space X. As a consequence, by Kirchberg's isomorphism theorem [15], separable stable continuous fields over such spaces whose fibres are UCT Kirchberg algebras are classified by an invariant the authors call *filtrated* K*-theory with coefficients*. This result is also implicit in [11].

The filtrated K-theory with coefficients of [10] comprises the K-theory with coefficients (the Λ-modules defined in [9], also called *total* K*-theory*) of all distinguished subquotients of the given field, along with the action of all natural maps between these groups. It is demonstrated in [10], generalising a result from [7], that coefficients are necessary for such a classification result over any infinite metrisable compact space. This means that filtrated K-theory (without coefficients) can only be a classifying invariant on subclasses of fields with special K-theoretical properties and this explains the need for additional assumptions in our results. For comparison let us recall that the classification result of [8] is restricted to fields whose fibers have torsion free K_0-groups and vanishing K_1-groups or vice versa.

The construction of an effective filtrated K-theory with coefficients for C^*-algebras over the unit interval remains an open problem. In the final section 5 we describe some of the technical difficulties that are encountered in potential constructions of such an invariant.

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2. Preliminaries

In this section we summarize definitions and results by various authors which we shall use later. We make the convention $\mathbb{N} = \{1, 2, 3, \ldots\}$.

2.1. C^*-algebras over topological spaces.

Let X be any topological space. Recall from [19]:

Definition 2.1. A *C^*-algebra over X* is a C^*-algebra A equipped with a continuous map $\mathrm{Prim}(A) \to X$.

Definition 2.2. Let A be a C^*-algebra over X. Let $U \subseteq X$ be an open subset. Taking the preimage under the map $\mathrm{Prim}(A) \to X$, we may naturally associate the *distinguished ideal* $A(U) \subseteq A$ to U. A morphism of C^*-algebras over X is a *-homomorphism preserving all distinguished ideals.

A subset $Y \subset X$ is called *locally closed* if it can we written as a difference $U \setminus V$ of two open subsets $V \subseteq U \subseteq X$. It can be shown that the *distinguished subquotient* $A(Y) := A(U)/A(V)$ is well-defined.

We assume that X is locally compact Hausdorff in the following two definitions.

Definition 2.3. A $C_0(X)$-algebra is a C^*-algebra A equipped with a non-degenerate *-homomorphism from $C_0(X)$ to the center of the multiplier algebra of A. A morphism of $C_0(X)$-algebras is a $C_0(X)$-linear *-homomorphism.

The category of C^*-algebras over X and the category of $C_0(X)$-algebras are isomorphic (see [19, Proposition 2.11]). We denote the category of separable C^*-algebras over X by $\mathfrak{C}^*\mathfrak{sep}(X)$.

Definition 2.4. For $x \in X$ and a $C_0(X)$-algebra A, we denote the quotient map $A \twoheadrightarrow A(x)$ onto the fiber by π_x. The algebra A is called *continuous* if the function $x \mapsto \|\pi_x(a)\|$ is a continuous function on X for every $a \in A$.

2.2. **Bivariant K-theory for C^*-algebras over topological spaces.** Let X be a second countable topological space. Let us recall that $\mathfrak{KK}(X)$ is the triangulated category that extends KK-theory to separable C^*-algebras over X, see [19]. In [10], the second named author and Meyer define a version of E-theory for separable C^*-algebras over X and establish its basic properties. This construction yields a triangulated category $\mathfrak{E}(X)$ (if X is the one-point space, we simply write $\mathfrak{E}$) and a functor $\mathfrak{C}^*\mathfrak{sep}(X) \to \mathfrak{E}(X)$ which is characterized by a universal property. We recall two results which are of particular importance for us.

Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be an ordered basis for the topology on X. Denote by X_n the finite topological space, which arises as the T_0-quotient of X equipped with the topology generated by the set $\{U_1, \dots, U_n\}$. Observe that we have a projective system of spaces $\cdots \twoheadrightarrow X_2 \twoheadrightarrow X_1 \twoheadrightarrow X_0$ together with compatible maps $X \twoheadrightarrow X_n$. By functoriality in the space variable, we obtain a projective sequence of triangulated categories $\big(\mathfrak{E}(X_n)\big)_{n\in\mathbb{N}}$ together with compatible functors $\mathfrak{E}(X) \to \mathfrak{E}(X_n)$.

Proposition 2.5 ([10, Theorem 3.2])**.** *Let A and B be separable C^*-algebras over X. Then there is a natural short exact sequence of $\mathbb{Z}/2$-graded Abelian groups*

$$\varprojlim{}^1 \mathrm{E}_{*+1}(X_n; A, B) \rightarrowtail \mathrm{E}_*(X; A, B) \twoheadrightarrow \varprojlim \mathrm{E}_*(X_n; A, B).$$

Definition 2.6. The bootstrap class $\mathcal{B}_\mathrm{E}$ consists of all separable C^*-algebras that are equivalent in E-theory to a commutative C^*-algebra. The bootstrap class $\mathcal{B}_\mathrm{E}(X)$ consists of all separable C^*-algebras over X such that $A(U)$ belongs to $\mathcal{B}_\mathrm{E}$ for every open subset $U \subseteq X$.

Proposition 2.7 ([10, Theorem 4.6])**.** *Let A and B be separable C^*-algebras over X belonging to the bootstrap class $\mathcal{B}_\mathrm{E}(X)$. An element in $\mathrm{E}(X; A, B)$ is invertible if and only if the induced map $\mathrm{K}_*\big(A(U)\big) \to \mathrm{K}_*\big(B(U)\big)$ is invertible for every open subset U of X.*

2.3. **Continuous fields of Kirchberg algebras.** In this subsection we assume that X is a finite-dimensional, compact, metrizable topological space.

Proposition 2.8. *Let A be a separable continuous $C(X)$-algebra whose fibers are stable Kirchberg algebras. Then A is stable, nuclear and $\mathcal{O}_\infty$-absorbing.*

Proof. Bauval shows in [1, Théorème 7.2] that A, being continuous and having nuclear fibers, is nuclear (in fact $C(X)$-nuclear). A combination of results by Blanchard, Kirchberg and Rørdam in [3, 16, 17, 22] implies that $A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong A$, see [8, Theorem 7.4] and [14]. □

Corollary 2.9. *Let A and B be separable continuous $C(X)$-algebras whose fibers are stable Kirchberg algebras. Then every* $\mathrm{E}(X)$*-equivalence between A and B lifts to a $C(X)$-linear *-isomorphism.*

Proof. From [10, Theorem 5.4] we see that A is $\mathrm{KK}(X)$-equivalent to B. By Proposition 2.8, we can apply Kirchberg's isomorphism theorem [15]. □

Proposition 2.10. *Let A be a separable nuclear continuous $C(X)$-algebra whose fibers satisfy the* UCT. *Then A belongs to the* $\mathrm{E}(X)$*-theoretic bootstrap class $\mathcal{B}_{\mathrm{E}}(X)$.*

Proof. This follows from [5, Theorem 1.4] applied to every open subset of X. □

2.4. **Filtrated K-theory over finite spaces.** In this subsection we assume that X is a finite T_0-space.

Definition 2.11. Let $\mathfrak{Ab}^{\mathbb{Z}/2}$ be the category of $\mathbb{Z}/2$-graded Abelian groups and $\mathbb{Z}/2$-graded homomorphisms. We denote the collection of non-empty, connected, locally closed subsets of X by $\mathbb{LC}(X)^*$. For $Y \in \mathbb{LC}(X)^*$, we have a functor $\mathrm{FK}_Y^X \colon \mathfrak{E}(X) \to \mathfrak{Ab}^{\mathbb{Z}/2}$ taking A to $\mathrm{K}_*(A(Y))$. The $\mathcal{NT}^X$ be the $\mathbb{Z}/2$-graded pre-additive category whose object set is $\mathbb{LC}(X)^*$ and whose morphisms from Y to Z are the natural transformations from FK_Y^X to FK_Z^X regarded as functors from $\mathbb{Z}/2$-graded E-theory groups to $\mathbb{Z}/2$-graded Abelian groups with arbitrary group homomorphisms. The collection $\left(\mathrm{FK}_Y^X(A)\right)_{Y \in \mathbb{LC}(X)^*}$ has a natural graded module structure over $\mathcal{NT}^X$. This module is denoted by $\mathrm{FK}^X(A)$. Hence we have a functor $\mathrm{FK}^X \colon \mathfrak{E}(X) \to \mathfrak{Mod}(\mathcal{NT}^X)^{\mathbb{Z}/2}$.

Remark 2.12. If the space X is not too complicated, it is possible to describe the category $\mathcal{NT}^X$ in explicit terms. Suppose for instance that X is an accordion space in the sense of [2]. Then $\mathcal{NT}^X$ is generated by six-term sequence transformations corresponding to inclusions of distinguished subquotients and an explicit generating list of relations can be given, see [2].

Proposition 2.13 ([2, Theorem 8.9])**.** *Let X be an accordion space. Let A and B be separable C^*-algebras over X. Assume that A belongs to the* *bootstrap class* $\mathcal{B}_{\mathrm{E}}(X)$*. Then there is a natural short exact sequence of* $\mathbb{Z}/2$*-graded Abelian groups*

$$\mathrm{Ext}^1_{\mathcal{NT}^X}(\mathrm{FK}^X(A), \mathrm{FK}^X(SB)) \rightarrowtail \mathrm{E}_*(X; A, B) \twoheadrightarrow \mathrm{Hom}_{\mathcal{NT}^X}(\mathrm{FK}^X(A), \mathrm{FK}^X(B)).$$

Here we have replaced $\mathrm{KK}_*(X; A, B)$ by $\mathrm{E}_*(X; A, B)$ in the original statement. This is possible as we explain in the following remark.

Remark 2.14. It was shown in [2] that $\mathrm{FK}^X(A)$ has a projective resolution of length 1 for every separable C^*-algebra A over X. Regarding FK^X as a functor from $\mathfrak{E}(X)$, it is the universal $\mathcal{I}$-exact stable homological functor, where $\mathcal{I}$ is now the ideal in $\mathfrak{E}(X)$ consisting of all elements inducing zero maps in FK^X. This is because $\mathrm{KK}(X; R, R) \cong \mathrm{E}(X; R, R)$ by [10, Theorem 5.5], where $R \in \mathcal{B}(X)$ is the representing object for FK^X. Now the result follows from the general UCT of [18]. (Here $\mathcal{B}(X)$ denotes the KK-theoretic bootstrap class of C^*-algebras over X defined by Meyer–Nest in [19] as the smallest class of C^*-algebras containing all one-dimensional C^*-algebras over X and closed under certain operations. If A is a nuclear C^*-algebra over X, then A belongs to $\mathcal{B}(X)$ if and only if it belongs to $\mathcal{B}_{\mathrm{E}}(X)$.)

Not every $\mathcal{NT}^X$-module belongs to the range of the invariant FK^X. In particular, $\mathrm{FK}^X(A)$ is an *exact* $\mathcal{NT}^X$-module for every C^*-algebra A over X as defined in [18, Definition 3.5].

Proposition 2.15. *Let* X *be an accordion space and* M *an* $\mathcal{NT}^X$*-module. Then* M *is projective/injective if and only if* M *is exact and the* $\mathbb{Z}/2$*-graded Abelian group* $M(Y)$ *is projective/injective for every* $Y \in \mathbb{LC}(X)^*$*.*

Proof. The statement about projective modules is proven in [2]. The claim about injective modules follows from a dual argument. □

3. Finite approximations of the unit interval

Let $I = [0, 1]$ be the unit interval. Choose, once and for all, a dense sequence $(d_n)_{n\in\mathbb{N}}$ in I. For convenience, we may assume $d_m \neq d_n$ for $m \neq n$ and $d_n \notin \{0, 1\}$ for all $n \in \mathbb{N}$. Consider the ordered subbasis $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ for the topology on I given by $V_{2n-1} = [0, d_n)$ and $V_{2n} = (d_n, 1]$; denote by I_n the T_0-quotient of I equipped with the topology generated by the set $\{V_1, \ldots, V_{2n}\}$.

Let A and B be separable C^*-algebras over I. Since the spaces I_n form a cofinal family in the projective sequence of approximations corresponding to the basis generated by the subbasis $\mathcal{V}$ above, Proposition 2.5 yields a short exact sequence

$$\varprojlim{}^1 \mathrm{E}_{*+1}(I_n; A, B) \rightarrowtail \mathrm{E}_*(I; A, B) \twoheadrightarrow \varprojlim \mathrm{E}_*(I_n; A, B). \tag{3.1}$$

We are therefore interested in the computation of the groups $\mathrm{E}_*(I_n; A, B)$.

Lemma 3.2. *The spaces I_n are accordion spaces in the sense of* [2].

Proof. For a given natural number $n \in \mathbb{N}$, we order the set $\{d_1, \dots, d_n\}$ by writing $\{d_1, \dots, d_n\} = \{e_1, \dots, e_n\}$ where $e_k < e_{k+1}$ for $1 \le k < n$. Then we have

$$I_n = \{[0, e_1), \{e_1\}, (e_1, e_2), \{e_2\}, (e_2, e_3), \dots, \{e_n\}, (e_n, 1]\}.$$

Denoting $u_0 = [0, e_1)$, $u_k = (e_k, e_{k+1})$ for $1 \le k < n$, $u_n = (e_n, 1]$ and $c_k = \{e_k\}$ for $1 \le k \le n$, a basis for the topology on X_n given by the family of open subsets

$$\{\{u_k\} \mid 0 \le k \le n\} \cup \{\{u_k, c_k, u_{k+1}\} \mid 0 \le k < n\}.$$

Hence X_n is an accordion space of a specific form, the Hasse diagram of the specialization order of which is indicated in the diagram below.

$$\bullet \to \bullet \leftarrow \bullet \to \bullet \leftarrow \bullet \to \cdots \leftarrow \bullet \to \bullet \leftarrow \bullet \to \bullet \leftarrow \bullet$$ □

For $n \in \mathbb{N}$, we briefly write $\mathcal{NT}_n$ for $\mathcal{NT}^{I_n}$ and $\mathrm{FK}_n(A)$ for $\mathrm{FK}^{I_n}(A)$.

Assume that A belongs to the bootstrap class $\mathcal{B}_{\mathrm{E}}(I)$. By Proposition 2.13, for every $n \in \mathbb{N}$, we have a short exact sequence

$$\begin{aligned} \operatorname{Ext}^1_{\mathcal{NT}_n}(\mathrm{FK}_n(A), \mathrm{FK}_n(SB)) &\rightarrowtail \mathrm{E}_*(I_n; A, B) \\ &\twoheadrightarrow \operatorname{Hom}_{\mathcal{NT}_n}(\mathrm{FK}_n(A), \mathrm{FK}_n(B)). \end{aligned} \tag{3.3}$$

Definition 3.4. Let A be a C^*-algebra over $[0, 1]$. The *filtrated* K*-theory of* A consists of the $\mathbb{Z}/2$-graded Abelian groups $\mathrm{K}_*(A(Y))$ for all locally closed intervals $Y \subseteq I$ together with the graded group homomorphisms in the six-term exact sequence $\mathrm{K}_*(A(U)) \to \mathrm{K}_*(A(Y)) \to \mathrm{K}_*(A(Y \setminus U)) \to \mathrm{K}_{*+1}(A(U))$ for every relatively open subinterval U of a locally closed interval $Y \subseteq I$ with the property that the set $Y \setminus U$ is connected. A *homomorphism* from $\mathrm{FK}(A)$ to $\mathrm{FK}(B)$ is a family of graded group homomorphisms

$$\{\varphi_Y : \mathrm{K}_*(A(Y)) \to \mathrm{K}_*(B(Y))\}_Y$$

such that for all pairs $U \subset Y$ as above, all squares in the diagram

$$\begin{array}{ccccccc}
\mathrm{K}_*(A(U)) & \longrightarrow & \mathrm{K}_*(A(Y)) & \longrightarrow & \mathrm{K}_*(A(Y \setminus U)) & \longrightarrow & \mathrm{K}_{*+1}(A(U)) \\
\downarrow \varphi_U & & \downarrow \varphi_Y & & \downarrow \varphi_{Y\setminus U} & & \downarrow \varphi_U \\
\mathrm{K}_*(B(U)) & \longrightarrow & \mathrm{K}_*(B(Y)) & \longrightarrow & \mathrm{K}_*(B(Y \setminus U)) & \longrightarrow & \mathrm{K}_{*+1}(B(U))
\end{array}$$

commute.

The $\mathbb{Z}/2$-graded Abelian group of homomorphisms from $\mathrm{FK}(A)$ to $\mathrm{FK}(B)$ is denoted by $\mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B))$.

We note that one may consider a variation $\mathrm{FK}'(A)$ of $\mathrm{FK}(A)$ where only intervals with endpoints from the sequence $(d_n)_{n\in\mathbb{N}}$ and $0, 1$ are used. It is not hard to show that the restriction map $\mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B)) \to \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}'(A), \mathrm{FK}'(B))$ is bijective.

Remark 3.5. We can regard $\mathrm{FK}(A)$ as a $\mathbb{Z}/2$-graded module over a $\mathbb{Z}/2$-graded pre-additive category $\mathcal{NT}$ with objects the locally closed subintervals of I and morphisms generated by elements i_U^Y, $r_Y^{Y\setminus U}$, $\delta_{Y\setminus U}^U$ for every relatively open subinterval U of a locally closed interval $Y \subseteq I$ such that $Y \setminus U$ is connected. Regardless of the relations among these generators, homomorphisms from $\mathrm{FK}(A)$ to $\mathrm{FK}(B)$ would then simply be graded module homomorphisms. This justifies the notation $\mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B))$.

Lemma 3.6. *For C^*-algebras A and B over $[0, 1]$, we have a natural identification*

$$\mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B)) = \varprojlim \mathrm{Hom}_{\mathcal{NT}_n}(\mathrm{FK}_n(A), \mathrm{FK}_n(B)).$$

Proof. Restriction gives a natural map from $\mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B))$ to the inverse limit $\varprojlim \mathrm{Hom}_{\mathcal{NT}_n}(\mathrm{FK}_n(A), \mathrm{FK}_n(B))$, the bijectivity of which follows from continuity of K-theory. □

4. Classification results

We are now ready to put together the facts from the previous sections to derive classification results.

Applying inverse limits to the UCT-sequences (3.3), and using that $\varprojlim^1$ is a derived functor of $\varprojlim$, we obtain the exact sequence

$$(4.1) \quad 0 \to \varprojlim \mathrm{Ext}^1_{\mathcal{NT}_n}(\mathrm{FK}_n(A), \mathrm{FK}_n(SB)) \to \varprojlim \mathrm{E}_*(I_n; A, B) \\ \to \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B)) \xrightarrow{d} \varprojlim{}^1 \mathrm{Ext}^1_{\mathcal{NT}_n}(\mathrm{FK}_n(A), \mathrm{FK}_n(SB)).$$

Definition 4.2. Let $\mathcal{K}ir(I)$ denote the class of separable continuous $C(I)$-algebras whose fibers are stable Kirchberg algebras satisfying the UCT.

Theorem 4.3. *Let $\mathscr{C}$ be a subclass of $\mathcal{K}ir(I)$ such that for all A and B in $\mathscr{C}$, the map d in* (4.1) *vanishes. Then, for all A and B in $\mathscr{C}$, the map* $\mathrm{E}_*(I;A,B) \to \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B))$ *is surjective and every isomorphism* $\mathrm{FK}(A) \cong \mathrm{FK}(B)$ *lifts to a $C(I)$-linear *-isomorphism.*

Proof. Let $\alpha \in \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B))$. If $d = 0$, we can use the exact sequences (4.1) and (3.1) to lift α to an element $\tilde{\alpha} \in \mathrm{E}(I;A,B)$. If α was an isomorphism, then $\tilde{\alpha}$ is an $\mathrm{E}(I)$-equivalence by Proposition 2.7. We conclude the proof by applying Corollary 2.9. □

Remark 4.4. Theorem 4.3 does not hold for the whole class $\mathscr{C} = \mathcal{K}ir(I)$ as shown by Example 6.5 from [10]. If the fibers of A and B have torsion in K-theory, then the map $\mathrm{E}_*(I;A,B) \to \ker(d) \subset \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B))$ is typically not surjective.

We will now verify the hypotheses of Theorem 4.3 for certain classes of C^*-algebras over $[0,1]$. Our first example yields (in particular) a proof of Theorem 1.1.

Example 4.5. (*Proof of Theorem* 1.1.) By Proposition 2.15, the conclusion of Theorem 4.3 holds for the class $\mathscr{C}$ of $C(I)$-algebras A in $\mathcal{K}ir(I)$ for which $\mathrm{K}_*(A(Y))$ is a divisible Abelian group for every locally closed interval $Y \subseteq I$. By the Künneth formula for tensor products, the class $\mathscr{C}$ contains all objects in $\mathcal{K}ir(I)$ which are stable under tensoring with the universal UHF-algebra $M_{\mathbb{Q}}$. Let A be as in Theorem 1.1. Since $\mathrm{K}_*(A(x)) \cong \mathrm{K}_*(A(x)) \otimes \mathbb{Q}$ it follows that $A(x) \cong A(x) \otimes M_{\mathbb{Q}}$, for all $x \in I$, by the Kirchberg–Phillips classification theorem. We conclude the argument by noting that if each fiber of a $C(I)$-algebra A is stable under tensoring with the universal UHF-algebra $M_{\mathbb{Q}}$, then so is A itself by [14].

Example 4.6. Again by Proposition 2.15, the conclusion of Theorem 4.3 holds for the class $\mathscr{C}$ of $C(I)$-algebras A in $\mathcal{K}ir(I)$ for which $\mathrm{K}_*(A(Y))$ is a free Abelian group for every locally closed interval $Y \subseteq I$ because the Ext^1-terms in (4.1) vanish.

Example 4.7. Fix $i \in \{0,1\}$. Consider the class $\mathscr{C}$ of $C(I)$-algebras A in $\mathcal{K}ir(I)$ which satisfy $\mathrm{K}_i(A(Y)) = 0$ for every locally closed interval $Y \subseteq I$. For parity reasons, the Ext^1-terms in (4.1) vanish. Hence the class $\mathscr{C}$ satisfies the condition of Theorem 4.3.

Remark 4.8. Fix $i \in \{0,1\}$. It follows from the main result in [8] that the condition of Theorem 4.3 is also satisfied for the class $\mathscr{C}$ of $C(I)$-algebras A in $\mathcal{K}ir(I)$ whose fibers have vanishing K_d-groups and torsion-free K_{d+1}-groups. However, we have not been able to reprove this by an independent, purely K-theoretical argument.

5. A remark on coefficients

In order to get a classification result without any K-theoretical assumptions, one expects, as indicated in the introduction, to need some version of filtrated K-theory with coefficients for C^*-algebras over unit the interval. This requires, to begin with, the correct definition of filtrated K-theory with coefficients for C^*-algebras over accordion spaces. It was observed in [13] that, already over the two-point Sierpiński space S, the naïve candidate for such a definition—using the corresponding six-term sequence of Λ-modules—produces an invariant which lacks desired properties such as a UMCT.

We argue that, in order to give a fully satisfactory definition of filtrated K-theory with coefficients for C^*-algebras over S, one has to allow all finitely generated, indecomposable exact six-term sequences of Abelian groups as coefficients—just as all finitely generated, indecomposable Abelian groups as coefficients are needed in the UMCT of [9]. It is easy to see that there is a countable number of isomorphism classes of such six-term sequences. However, unlike in the case of Abelian groups, it follows from the main result in [21] that their classification is controlled $\mathbb{Z}/p$-wild for every prime p. This wildness phenomenon seems to make filtrated K-theory with (generalized) coefficients as sketched above very hard to compute explicitly, limiting its rôle in the theory to a rather theoretical one.

We conclude by remarking that recent results of Eilers, Restorff and Ruiz in [12] indicate that additional K-theoretical assumptions allow the usage of a smaller, more concrete invariant.

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