NON-STABLE GROUPS

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ABSTRACT. In this article we discuss cohomological obstructions to two kinds of group stability. In the first part, we show that residually finite groups Γ which arise as fundamental groups of compact Riemannian manifolds with strictly negative sectional curvature are not uniform-to-local stable with respect to the operator norm if their even Betti numbers $b_{2i}(\Gamma)$ do not vanish. In the second part, we show that non-vanishing of Betti numbers $b_i(\Gamma)$ in dimension $i > 1$ obstructs C^*-algebra stability for groups approximable by unitary matrices that admit a coarse embedding in a Hilbert space.

1. INTRODUCTION

For a countable discrete group Γ we consider several natural stability properties relative to unitary groups $U(n)$ equipped with the uniform norm. We also consider stability properties of Γ with respect to unitary groups of C^*-algebras. The reader is referred to the survey papers by Arzhantseva [1] and Thom [54] for an introduction to the approximation and stability properties of groups. Just like in our earlier paper on this subject [17], we rely on ideas due to Kasparov [35], Connes, Gromov and Moscovici [14], Gromov [26,27], Tu [56] and Kubota [43]. In addition we use results of Hanke and Schick [30], [31], Hunger [32] and Baird and Ramras [4].

Definition 1.1. For a countable discrete group Γ we consider sequences $\{\rho_n\}$ of unital maps and sequences $\{\pi_n\}$ of unitary group representations with $\rho_n, \pi_n : \Gamma \to U(n)$.

The sequence $\{\pi_n\}$ approximates $\{\rho_n\}$ *locally* if

$$\lim_{n\to\infty} \|\rho_n(s) - \pi_n(s)\| = 0, \text{ for all } s \in \Gamma.$$

The sequence $\{\pi_n\}$ approximates $\{\rho_n\}$ *uniformly* if

$$\lim_{n\to\infty} \sup_{s\in\Gamma} \|\rho_n(s) - \pi_n(s)\| = 0.$$

(a) Γ is *locally stable* if any sequence $\{\rho_n\}$ which satisfies

$$\lim_{n\to\infty} \|\rho_n(st) - \rho_n(s)\rho_n(t)\| = 0, \text{ for all } s, t \in \Gamma$$

can be approximated *locally* by a sequence $\{\pi_n\}$ of unitary representations.

(b) Γ is *uniformly stable* if any sequence $\{\rho_n\}$ which satisfies

$$\lim_{n\to\infty} \sup_{s,t\in\Gamma} \|\rho_n(st) - \rho_n(s)\rho_n(t)\| = 0$$

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can be approximated *uniformly* by a sequence $\{\pi_n\}$ of unitary representations.

(c) Γ is *uniform-to-local stable* if any sequence $\{\rho_n\}$ which satisfies the assumption from (b) can be approximated *locally* by a sequence $\{\pi_n\}$ of unitary representations.

(d) Γ is *local-to-uniform stable* if any sequence $\{\rho_n\}$ which satisfies the assumption from (a) can be approximated *uniformly* by a sequence $\{\pi_n\}$ of unitary representations.

One may visualize the conditions (a), (b) and (c) in a diagram:

$$\begin{array}{ccc}
\left(\sup_{s,t\in\Gamma}\|\rho_n(st)-\rho_n(s)\rho_n(t)\|\to 0\right) & \xRightarrow{\text{uniformly stable}} & \left(\sup_{s\in\Gamma}\|\rho_n(s)-\pi_n(s)\|\to 0\right) \\
 & \searrow^{\text{uniform-to-local stable}} & \\
\|\rho_n(st)-\rho_n(s)\rho_n(t)\|\to 0\ {}_{\forall s,t} & \xRightarrow[\text{locally stable}]{} & \|\rho_n(s)-\pi_n(s)\|\to 0\ {}_{\forall s}
\end{array}$$

The stability properties considered in Definition 1.1 are quite different in nature. Local-to-uniform stability is not too interesting for it is satisfied only by finite groups. While it is clear that both uniform stability and local stability imply uniform-to-local stability, the study of these three properties is more challenging. Local stability is referred to as *matricial stability* in the paper of Eilers, Shulman and Sørensen [21]. We showed that the nonvanishing of rational cohomology in even dimensions is an obstruction to local stability for large classes of groups, including amenable groups, linear groups and residually finite hyperbolic groups, [17] and [18]. Bader, Lubotzky, Sauer and Weinberger used the result of showed that lattices in semisimple real Lie groups are typically not locally stable [3], using [17].

Uniform stability is called *Ulam stability* in the article of Burger, Ozawa and Thom [9]. By a classical result of Kazhdan discrete amenable groups are uniformly stable, [39]. In contrast, an amenable group Γ is not locally stable if $H^{2i}(\Gamma,\mathbb{R})\neq 0$ for some $i>1$, [17]. A recent paper of Glebsky, Lubotzky, Monod and Rangarajan [25] studies uniform stability for cocompact lattices in semisimple Lie groups by means of asymptotic cohomology.

Uniform stability is a stronger condition than uniform-to-local stability. Indeed, if Γ has a finite index subgroup isomorphic to a free group $\mathbb{F}_k$, $k\geq 2$, then Γ is locally stable [21], [3] and hence uniform-to-local stable, but Γ is not uniformly stable. More generally, it was shown in [9] that if the comparison map $j: H^2_b(\Gamma,\mathbb{R})\to H^2(\Gamma,\mathbb{R})$ is not injective, then Γ is not uniformly stable, and consequently, the non-elementary hyperbolic groups are not uniformly stable, since j is not injective for such groups [24].

The surface groups Γ_g of genus $g>1$ were the first groups shown not be uniformly stable, [39]. One observes that Kazhdan's proof, which exploits the nonvanishing of $H^2(\Gamma_g,\mathbb{R})$, shows that, in fact, the groups Γ_g of genus $g>1$ are not even uniform-to-local stable. Motivated by this observation, in the first part of this paper, we point out that many of the cocompact lattices in the Lorentz group $SO_0(n,1)$, $n>1$ are not uniform-to-local stable. This will follow from the following theorem inspired by an idea of Gromov [28, p.166]:

Theorem 1.2. *Let M be a closed connected Riemannian manifold with strictly negative sectional curvature and residually finite fundamental group. If $b_{2i}(M) \neq 0$ for some $i > 0$, then $\pi_1(M)$ is not uniform-to-local stable.*

Theorem 1.2 is a direct consequence of Theorem 4.2 from Section 4. Concerning the assumption on Betti numbers, observe that if M is orientable and $\dim M = 2m$ then $b_{2m}(M) = 1$ while if M is orientable and $\dim M = 2m+1$, then it suffices to require that $b_i(M) > 0$ for some $1 \leq i \leq 2m$, since $b_i(M) = b_{2m+1-i}(M)$ by Poincaré duality and either i or $2m+1-i$ must be even.

Recall that a cocompact lattice in a semisimple real Lie group G is a discrete subgroup Γ of G such that the quotient space G/Γ is compact. The n-dimensional hyperbolic space $\mathbb{H}^n$, $n \geq 2$, has constant sectional curvature equal to -1. The connected component of the identity of the group of orientation preserving isometries of $\mathbb{H}^n$ is the Lorentz group $SO_0(n,1)$. Since $\mathbb{H}^n$ is isometric to the symmetric space $SO_0(n,1)/SO(n)$, if Γ is a torsion free cocompact lattice in $SO_0(n,1)$, then $M = \Gamma \setminus SO_0(n,1)/SO(n)$ is an orientable closed connected Riemannian manifold with sectional curvature $= -1$. Moreover, $\Gamma = \pi_1(M)$ is finitely generated by co-compactness and hence it is residually finite by Malcev's theorem since $SO_0(n,1) \subset \mathrm{GL}(n,\mathbb{R})$. Thus one obtains the following:

Corollary 1.3. *Let Γ be a torsion free cocompact lattice in $SO_0(n,1)$.*

(i) *If n is even then Γ is not uniform-to-local stable.*
(ii) *If n is odd and $b_i(\Gamma) > 0$ for some $i > 0$ then Γ is not uniform-to-local stable.*

The corollary reproves Kazhdan's result since $\Gamma_g \subset SO_0(2,1)$, $g > 1$. In order to apply the corollary to other examples, let us note that by Selberg's lemma, any cocompact discrete subgroup Λ of $SO_0(n,1)$ has a finite index torsion free subgroup Γ. Thus, in order to apply Corollary 1.3(ii) to Γ, it remains to realize the condition on Betti numbers.

It was shown in [3] that if Γ is a cocompact lattice in a real semisimple Lie group G which is not locally isomorphic to either $SO(n,1)$ for n odd or $SL_3(\mathbb{R})$, then $b_{2i}(\Gamma) > 0$ for some $i > 0$, so that Γ is not locally stable by [17]. Concerning lattices in $SO(n,1)$, the following nonvanishing result is established in [3].

Theorem 1.4 (Cor. 3.13 of [3])**.** *Let Λ be a cocompact lattice in $SO(n,1)$ with $n > 1$ odd. Suppose either (i) $n = 3$ or (ii) $n = 4m+1$ or (iii) $n = 4m+3$ and Λ is arithmetic but not of the form ${}^6D_4$ if $n = 7$. Then there is a finite index subgroup $\Lambda_1 \leq \Lambda$ such that for any finite index subgroup $\Gamma \leq \Lambda_1$ there is $i > 0$ such that $b_{2i}(\Gamma) > 0$. In particular, the group Γ is not locally stable.*

By Theorem 1.2, we deduce:

Corollary 1.5. *The groups Γ as in Theorem 1.4 are not uniform-to-local stable.*

For concrete examples one may consider $G = SO_0(x_1^2 + \cdots + x_n^2 - \sqrt{p}\, x_{n+1}^2, \mathbb{R}) \cong SO_0(n,1)$, where p is a square free integer. Let $\mathcal{O}$ be ring of integers of $\mathbb{Q}\sqrt{p}$. Then $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\sqrt{p}$ if $p \not\equiv 1 \pmod 4$ and $\mathcal{O} = \{\frac{a+b\sqrt{p}}{2} \colon a,b \in \mathbb{Z},\ a-b \equiv 0 \pmod 2\}$ if $p \equiv 1 \pmod 4$ and $G_{\mathcal{O}}$ is a cocompact arithmetic lattice in G, [6]. By a result of Li and Millson [44], any arithmetic lattice in $SO_0(n,1)$, $n \neq 3, 7$ contains a congruence subgroup Γ such that $b_1(\Gamma) > 0$.

In the second part of the paper we revisit local stability and discuss C^*-stability of discrete groups [21], a property which can be viewed as local stability relative to C^*-algebras.

Definition 1.6. A group Γ is *C^*-stable* if for any sequence of unital maps $\{\rho_n : \Gamma \to U(B_n)\}_n$, where B_n are unital C^*-algebras such that

$$\lim_{n\to\infty} \|\rho_n(st) - \rho_n(s)\rho_n(t)\| = 0, \quad \text{for all } s, t \in \Gamma,$$

there exists a sequence of group homomorphisms $\{\pi_n : \Gamma \to U(B_n)\}_n$ satisfying

$$\lim_{n\to\infty} \|\rho_n(s) - \pi_n(s)\| = 0, \quad \text{for all } s \in \Gamma.$$

We note that local stability corresponds to C^*-stability relative to finite dimensional C^*-algebras. As discussed earlier, nonvanishing of even-dimensional rational cohomology is an obstruction to matricial stability for many groups. Prompted by a question of Dima Shlyakhtenko concerning the possible role of odd-dimensional cohomology in group stability, we show the following:

Theorem 1.7. *Let Γ be a countable discrete MF-group that admits a γ-element. If $H^k(\Gamma, \mathbb{Q}) \neq 0$ for some $k > 1$, then Γ is not C^*-stable.*

Let us recall that a group Γ is MF if it is isomorphic to a subgroup of the unitary group of the corona C^*-algebra $\prod_n M_n / \bigoplus_n M_n$, [12]. Equivalently, Γ embeds in $\mathbf{U}/\mathbf{N}$ where $\mathbf{U} = \prod_{n=1}^\infty U(n)$ and $\mathbf{N} = \{(u_n)_n \in \mathbf{U} : \|u_n - 1_n\| \to 0\}$. In other words a group is MF if it admits sufficiently many approximate unitary representations to effectively separate its elements. In the terminology of [20] these are the $\big(U(n), \|\cdot\|\big)_{n=1}^\infty$-approximated groups. It is an open problem to find examples of discrete countable groups which are not MF. The groups that are locally embeddable in amenable groups are MF as a consequence of [55].

The class of groups that admit a γ-element is large [38], [34]. It includes the groups that admits a uniform embedding in a Hilbert space [56]. The amenable groups, or more generally, the groups with Haagerup's property are uniformly embeddable in a Hilbert space [13] and so are the linear groups [29]. Hilbert space uniform embeddability passes to subgroups and products, direct limits, free products with amalgam, and extensions by exact groups [19].

Theorem 1.7 will be established as a consequence of a result which assumes weaker forms of stability, see Theorem 5.15 and Theorem 5.17. More precisely, Theorem 1.7 follows from Theorem 5.17(ii), where only stability with respect to C^*-algebras of the form $M_n(C(\mathbb{T}))$ is assumed. The proof proceeds by an adaptation of the arguments from [17] (and we repeat many of them here for the sake of completion) with the novelty that one employs a theorem of Baird and Ramras [4] on vanishing of Chern classes for families of flat bundles in place of a result of Milnor [46].

Moreover, we show in Theorem 5.15 that if a quasidiagonal groups Γ admits a γ-element and has a nonvanishing Betti number $b_{2k}(\Gamma)$, then there are sequences $\{\rho_n\}$ as in Definition 1.1(a) which cannot be locally approximated even if we allow for non-unitary representations $\pi_n : \Gamma \to GL_n(\mathbb{C})$. Similarly, for a finitely generated group Γ as in Theorem 1.7, there is a sequence of unital maps $\{\rho_n : \Gamma \to U_n(C(\mathbb{T}))\}$ as in Definition 1.6 which cannot be approximated locally by a sequence of homomorphisms $\{\pi_n : \Gamma \to GL_n(C(\mathbb{T}))\}$.

For the sake of accessibility, we include in our exposition several facts well-known to the experts. In Sections 2 and 3 we revisit the topics of flat bundles and almost flat bundles. The proof of Theorem 1.2 which relies on the approximate monodromy correspondence for almost flat bundles is given in Section 4. The proof of Theorem 1.7 is given in Section 5.

2. Flat bundles

Let A be a unital C^*-algebra and let V be a finitely generated (projective) right Hilbert A-module. Let $\mathcal{L}(V)$ be the C^*-algebra of adjointable A-linear operators acting on V. The unitary group of $\mathcal{L}(V)$ will be denoted by $U(V)$. For a compact Hausdorff space M we denote by $\mathrm{Bdl}_A^V(M)$ the set of isomorphism classes of locally trivial bundles with fiber V and structure group $U(V)$. If $V = A$ we write $\mathrm{Bdl}_A(M)$ for $\mathrm{Bdl}_A^A(M)$. If $E \in \mathrm{Bdl}_A^V(M)$ we say that E is a Hilbert A-module bundle with typical fiber V. If M is a smooth manifold, then every $E \in \mathrm{Bdl}_A^V(M)$ admits a smooth structure which unique up to isomorphism [51, Thm.3.14]. The $C^\infty(M, A)$-module of smooth sections of E is denoted by $C^\infty(E)$. If $A = \mathbb{C}$ and $V = \mathbb{C}^r$, we write $\mathrm{Bdl}_{\mathbb{C}}^r(M)$ for $\mathrm{Bdl}_{\mathbb{C}}^{\mathbb{C}^r}(M)$, the set of isomorphism classes of hermitian complex vector bundles of rank r.

Definition 2.1. A flat structure on a smooth bundle $E \in \mathrm{Bdl}_A^V(M)$ is given by a finite cover $\mathcal{U} = (U_i)_{i\in I}$ of M together with smooth trivializations $U_i \times V \to E_{U_i}$ such that the corresponding transitions functions $v_{ij} : U_i \cap U_j \to U(V)$ are constant functions.

The classic theory of connections on vector bundles extends to smooth Hilbert A-module bundles as discussed in [51, Sec.3]. Let E be smooth Hilbert A-module bundle over a Riemannian manifold M. A connection on E is a $\mathbb{C}$-linear map

$$\nabla : C^\infty(TM \otimes \mathbb{C}) \otimes C^\infty(E) \to C^\infty(E), \quad X \otimes s \mapsto \nabla_X(s)$$

which satisfies the conditions

(i) $\nabla_X(s \cdot \mathbf{f}) = s \cdot (X\mathbf{f}) + \nabla_X(s) \cdot \mathbf{f}$, (Leibnitz formula)
(ii) $\nabla_{fX}(s) = f\nabla_X(s)$

for every $X \in C^\infty(TM \otimes \mathbb{C})$, $f \in C^\infty(M, \mathbb{C})$, $\mathbf{f} \in C^\infty(M, A)$ and $s \in C^\infty(E)$. The connection ∇ is compatible with the metric of E if

$$X\langle s, s'\rangle = \langle \nabla_X s, s'\rangle + \langle s, \nabla_X s'\rangle.$$

The importance of having a connection is that allows one to lift smooth paths $\gamma(t)$ between points $p, q \in M$ to isomorphisms between fibers E_p and E_q via parallel transport $P_\gamma : E_p \to E_q$. Recall that a section s along γ is parallel if it satisfies the differential equation $\nabla_{\partial_t\gamma} s = 0$. This equation has a unique solution for each initial value $s(p) \in E_p$ and thus determines uniquely the value of $s(q) \in E_q$ so that one defines $P_\gamma(s(p)) = s(q)$. Compatibility of ∇ with the metric implies that P_γ is a unitary operator, this is why such a connection ∇ is also called a unitary connection.

The curvature of ∇ is the tensor $R^\nabla \in \Omega^2(M, \mathrm{End}\, E)$ defined by the equation

$$R^\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Definition 2.2. A couple (E, ∇) consisting of a bundle $E \in \mathrm{Bdl}_A^V(M)$ and a unitary connection is flat if the curvature of the unitary connection vanishes, $R^\nabla = 0$.

For our discussion of flat bundles, it is convenient to adopt a setup from [30, Sec.3]. Let $\mathcal{P}_1(M)$ be the path groupoid of M with objects points of M and morphisms $\mathcal{P}_1(M)(p, q)$ the piecewise smooth paths $[0,1] \to M$ connecting p to q. The product $\gamma \cdot \gamma'$ of two paths is the path obtained by first traversing γ' and then γ, thus it is given by the concatenation $\gamma' * \gamma$. One endows $\mathcal{P}_1(M)$ with its natural topology. As it is usual, we let $\Omega_1(M,p)$ stand for $\mathcal{P}_1(M)(p,p)$. Denote by $\Pi_1(M)$ the fundamental groupoid of M obtained from $\mathcal{P}_1(M)$ by taking homotopy classes of paths with fixed endpoints and let $\Gamma = \pi_1(M,p)$ be the fundamental group of M.

For a bundle $E \in \mathrm{Bdl}_A^V(M)$, we denote by $\mathcal{T}(E)$ the transport groupoid of E with objects the points of M and morphisms $\mathcal{T}(E)(p,q) = \mathrm{Isom}_A(E_p, E_q)$. Following [30, p.288], we endow the groupoid $\mathcal{T}(E)$ with its natural topology, where the set of morphisms is topologized by using local trivializations in order to identify nearby fibers of E and $\mathrm{Isom}_A(E_p, E_q)$ is given the uniform norm. A holonomy representation on the bundle E is a continuous morphism of groupoids

$$h : \mathcal{P}_1(M) \to \mathcal{T}(E) \tag{1}$$

By a classic result, if E is smooth and it is endowed with a unitary connection ∇, then the corresponding parallel transport satisfies $P_{\gamma\cdot\gamma'} = P_\gamma \circ P_{\gamma'}$ and hence it defines a holonomy representation $\gamma \mapsto h(\gamma) = P_\gamma$, in the sense discussed above, [41, Thm.9.8].

Definition 2.3. Let $\pi : \Gamma = \pi_1(M,p) \to U(V)$ be a group homomorphism. The universal cover of M is denoted by $\widetilde{M}$. One realizes $\widetilde{M}$ as a space of homotopy classes of curves $\eta : [0,1] \to M$ with $\eta(0) = p$ and homotopies preserving the endpoints. The left action of G on $\widetilde{M}$ is defined as follows. If $s \in \pi_1(M,p)$ is represented by a loop $\gamma \in \Omega_1(M,p)$, then $s \cdot [\eta] = [\eta \cdot \gamma^{-1}]$ is represented by the path given by traversing γ in opposite direction followed by traversing η. The orbit space of the left action of Γ on $\widetilde{M} \times V$, defined by $s \cdot ([\eta], v) = (s \cdot [\eta], \pi(s)v)$, is the total space of a (flat) Hilbert A-module bundle $\widetilde{M} \times_\pi V \to M$ denoted by $L_\pi \in \mathrm{Bdl}_A^V(M)$. The map

$$\pi \mapsto L_\pi$$

was introduced by Atiyah [2] in the context of finite group representations.

If $A = C^*(\Gamma)$ and $\jmath : \Gamma \hookrightarrow U(C^*(\Gamma))$ is the natural inclusion, then $L_\jmath$ is a bundle of free rank-one Hilbert $C^*(\Gamma)$-modules. The bundle $L_\jmath \in \mathrm{Bdl}_{C^*(\Gamma)}(M)$ is called Mishchenko's flat bundle.

Proposition 2.4 ([40]). *For a smooth bundle $E \in \mathrm{Bdl}_A^V(M)$ the following are equivalent:*

- (i) *E admits a flat structure.*
- (ii) *E admits a unitary connection ∇ with zero curvature, $R^\nabla = 0$, i.e. (E, ∇) is flat.*
- (iii) *E is defined by a representation $\pi : \pi_1(M) \to U(V)$ in the sense that $E \cong L_\pi$.*
- (iv) *There is a holonomy representation $h : \mathcal{P}_1(M) \to \mathcal{T}(E)$ which descends to a morphism of groupoids $\mathbf{h} : \Pi_1(M) \to \mathcal{T}(E)$*

Proof. This is proved in [40] (see Prop.1.2.5 and 1.4.21) for complex vector bundles. The same arguments work without change in the present context. The holonomy representation h is given by parallel transport with respect to ∇. Restriction of $\mathbf{h}$ to $\pi_1(M,p)$ gives a representation π as in (iii). ☐

It is also convenient to work with selfadjoint idempotents e in matrices $m \times m$ over the C^*-algebra $C(M) \otimes A$ that represent bundles $E \in \mathrm{Bdl}_A^V(M)$, where $V \cong e(p)A^m$, $p \in M$.

Notation 2.5. Fix a flat structure for the Mishchenko's bundle $L_\jmath$ given by some finite cover $\mathcal{U} = (U_i)_{i\in I}$ of M together with smooth trivializations $U_i \times C^*(\Gamma) \to (L_\jmath)_{U_i}$ such that all the corresponding transitions functions $s_{ij} : U_i \cap U_j \to \Gamma \subset U(C^*(\Gamma))$ are constant. Thus one obtains group elements $s_{ij} \in \Gamma$ which form a 1-cocycle: $s_{ij}^{-1} = s_{ji}$ and $s_{ij}\cdot s_{jk} = s_{ik}$ whenever $U_i\cap U_j\cap U_k \neq \emptyset$. Let $(\chi_i)_{i\in I}$ be positive smooth functions with χ_i supported in U_i and such that $\sum_{i\in I}\chi_i^2 = 1$. Set $m = |I|$ and let (e_{ij}) be the canonical matrix unit of $M_m(\mathbb{C})$. Then $L_\jmath \in \mathrm{Bdl}_{C^*(\Gamma)}(M)$ is represented by the selfadjoint projection

$$\ell_\jmath = \sum_{i,j\in I} e_{ij} \otimes \chi_i\chi_j \otimes s_{ij} \in M_m(\mathbb{C}) \otimes C(M) \otimes C^*(\Gamma). \tag{2}$$

Moreover, the bundle $L_\pi \in \mathrm{Bdl}_A^V(M)$, corresponding to the group homomorphism $\pi : \Gamma \to U(V)$ (see Definition 2.3), is represented by the selfadjoint projection

$$\ell_\pi = \sum_{i,j\in I} e_{ij} \otimes \chi_i\chi_j \otimes \pi(s_{ij}) \in M_m(\mathbb{C}) \otimes C(M) \otimes \mathcal{L}(V). \tag{3}$$

3. Almost flat bundles

The goal of this section is to present a proof of Theorem 3.10 on approximate monodromy correspondence. The corresponding result was noted without proof by Skandalis in a remark on page 313 of [53]. It generalizes a result from [14] according to which ε-flat bundles on simply connected manifolds are trivial if ε is sufficiently small. In [15], we had used the content of Theorem 3.10 in conjunction with the Mishchenko-Fomenko index theorem in order to extend an index theorem of Connes Gromov and Moscovici from [14].

Fix $F \subset \pi_1(M,p)$ and m as in Notation 2.5 and the map $s \mapsto \gamma_s$ as in Notation 3.6.

Definition 3.1 ([14])**.** A unitary connection ∇ on a bundle $E \in \mathrm{Bdl}_A^V(M)$ is called ε-flat, $\varepsilon > 0$, if its norm

$$\|R^\nabla\| = \sup_{p\in M}\{\|R_p^\nabla(X,Y)\| \colon \|X\wedge Y\| \le 1, X, Y \in TM_p\},$$

satisfies $\|R^\nabla\| < \varepsilon$. In this case, the couple (E,∇) is called ε-flat.

The topological counterpart of ε-flatness is the following.

Definition 3.2. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of M. A bundle $E \in \mathrm{Bdl}_A^V(M)$ is called $(\mathcal{U},\varepsilon)$-flat if is represented by a cocycle $v_{ij} : U_i \cap U_j \to U(V)$ such that $\|v_{ij}(p) - v_{ij}(q)\| < \varepsilon$ for all $p, q \in U_i \cap U_j$ and all $i, j \in I$.

In the sequel we will occasionally identify a bundle E in $\mathrm{Bdl}_A^V(M)$ with the corresponding bundle in $\mathrm{Bdl}_{\mathcal{L}(V)}^{\mathcal{L}(V)}(M)$ constructed from the same cocycle $v_{ij} : U_i \cap U_j \to U(V) = U(\mathcal{L}(V))$. A and $\mathcal{L}(V)$ are Morita equivalent as V is a finitely generated Hilbert A-module. In particular, for $A = M_r(\mathbb{C})$, we can identify and go back-and-forth between rank-one bundles of Hilbert $M_r(\mathbb{C})$-modules and rank-r hermitian complex vector bundles constructed from the same transition functions.

We are going to explain how Atiyah's map $\pi \mapsto L_\pi$ can be extended to approximate group representations. Let M, $\Gamma = \pi_1(M,p)$, $(\chi_i)_{i\in I}$ with $|I| = m$ and $F = \{s_{ij}\}$ be as in Notation 2.5.

For $\varepsilon > 0$ and V a projective Hilbert A-module, we define

$$\mathrm{Rep}^V_{(F,\varepsilon)}(\Gamma) = \{\rho : \Gamma \to U(V) : \|\rho(st) - \rho(s)\rho(t)\| < \varepsilon, \rho(s^{-1}) = \rho(s)^*,\ s,t \in F,\ \rho(1) = 1\}.$$

For the next definition, we use the following elementary spectral theory argument. Suppose that x is a selfadjoint element of a unital C^*-algebra such that $\|x^2 - x\| < \varepsilon$ where $\varepsilon < 1/\lambda - 1/\lambda^2$ for some $1 < \lambda < 2$. Then the spectrum of x is contained in $(-\varepsilon, \lambda\varepsilon) \cup (1 - \lambda\varepsilon, \varepsilon)$. Since $\lambda\varepsilon < \lambda(1/\lambda - 1/\lambda^2) < \lambda/4 < 1/2$, the selfadjoint projection $\ell = \chi_{(\frac{1}{2},\infty)}(x)$ is an element of A and $\|x - \ell\| < \lambda\varepsilon$. In particular, if $\|x^2 - x\| < \varepsilon < 1/5$, then $\|x - \ell\| < 5\varepsilon/3 < 1/3$, (take $\lambda = 5/3$).

Definition 3.3. (a) The map $\mathrm{Rep}^V_{(F,\frac{1}{5m^2})}(\Gamma) \to \mathrm{Bdl}^V_A(M)$,

$$\rho \mapsto L_\rho$$

is defined as follows. Consider the selfadjoint element $x_\rho = \sum_{i,j\in I} e_{ij} \otimes \chi_i\chi_j \otimes \rho(s_{ij})$ of the C^*-algebra $M_m(\mathbb{C}) \otimes C(M) \otimes \mathcal{L}(V)$. Since

$$x_\rho^2 - x_\rho = \sum_{i,k} \left(\sum_j e_{ik} \otimes \chi_i\chi_k\chi_j^2 \otimes (\rho(s_{ij})\rho(s_{jk}) - \rho(s_{ik})) \right)$$

we see that $\|x_\rho^2 - x_\rho\| < 1/5$ and hence the spectrum of x_ρ is contained in $(-1/5, 1/3) \cup (2/3, 6/5)$. It follows by functional calculus that

$$\ell_\rho = \chi_{(\frac{1}{2},\infty)}(x_\rho)$$

is a selfadjoint projection in the same C^*-algebra such that $\|x_\rho - \ell_\rho\| < 1/3$. The Hilbert A-module bundle corresponding to ℓ_ρ is denoted L_ρ.

(b) Let us note that if $\varphi : C^*(\Gamma) \to A$ is a unital completely positive map such that $\|\varphi(st) - \varphi(s)\varphi(t)\| < \frac{1}{5m^2}$ for all $s, t \in F$, then the selfadjoint element $x_\varphi = \sum_{i,j\in I} e_{ij} \otimes \chi_i\chi_j \otimes \varphi(s_{ij})$ satisfies $\|x_\varphi^2 - x_\varphi\| < 1/5$ so that we can define the projection ℓ_φ and the corresponding bundles L_φ as above.

The following Lemma shows that approximate representations in close proximity yield isomorphic bundles.

Lemma 3.4. *If $\rho, \rho' \in \mathrm{Rep}^V_{(F,\frac{1}{5m^2})}(\Gamma)$ and $\sup_{s\in F} \|\rho(s) - \rho'(s)\| < \frac{1}{3m^2}$ then $\|\ell_\rho - \ell_{\rho'}\| < 1$ and hence $L_\rho \cong L_\rho$.*

Proof. $\|\ell_\rho - \ell_{\rho'}\| \le \|\ell_\rho - x_\rho\| + \|x_\rho - x_{\rho'}\| + \|x_{\rho'} - \ell_{\rho'}\| < \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$ □

Remark 3.5. The following version of Lemma 3.4 holds. Depending only on the data from Notation 2.5, there is $\varepsilon > 0$ such that for $\rho \in \mathrm{Rep}^V_{(F,\frac{1}{5m^2})}(\Gamma)$ and any homomorphism $\pi : \Gamma \to GL(V)$ satisfying

$$\sup_{s\in F} \|\rho(s) - \pi(s)\| < \varepsilon,$$

the idempotent ℓ_π defined by the equation (3) is sufficiently closed to the selfadjoint projection ℓ_ρ so that they are conjugated by an invertible element. In particular, they define the same class in $K^0(M)$. The idempotent ℓ_π is not necessarily selfadjoint since π takes values in $GL(V)$ rather than $U(V)$.

It is now well understood that the monodromy correspondence described in Proposition 2.4 extends to an approximate monodromy correspondence for almost flat bundles. This idea which was introduced in [14] and was explored in detail in [11], [32] and [42], it is central for our paper. In the sequel we will use a version of the approximate monodromy correspondence in a smooth setting.

We will assume that $\Gamma = \pi_1(M,p)$ does not have elements of order 2. This restriction is not really necessary, but we make it in order to streamline some of the arguments. In any case, our main application involves only torsion free groups.

Notation 3.6. For each $s \in \Gamma = \pi_1(M,p)$ choose a piecewise smooth loop γ_s that represents s with the provision that $\gamma_{s^{-1}}(\tau) = \gamma_s(1-\tau)$, $\tau \in [0,1]$. This defines a map $\gamma : \Gamma \to \Omega_1(M,p)$. With γ fixed as above, for each a smooth hermitian vector bundle E on M endowed with a unitary connection ∇, we consider the map

$$\rho = \rho_{(E,\nabla)} : \Gamma = \pi_1(M,p) \to U(E_p)$$

$\rho(s) = h(\gamma_s) = P_{\gamma_s}$, $s \in \Gamma$, where $P = P^{\nabla}$ is the parallel transport in E defined by ∇.

Consider two smooth curves $f_0, f_1 : [0,1] \to M$ from p to q and let (f_t), $0 \le t \le 1$, be a smooth homotopy between f_0 and f_1 with fixed endpoints. Let $P_{f_i} : E_p \to E_q$ be the parallel transport along f_i. Assume for the area of the homotopy (f_t) that:

$$\iint \|\partial_t f_t(s) \wedge \partial_s f_t(s)\| \, ds \, dt \le C.$$

Proposition 3.7 (Buser-Harcher)**.**

$$\|P_{f_0} - P_{f_1}\| \le \iint \|R^{\nabla}(\partial_t f_t(s), \partial_s f_t(s))\| \, ds \, dt \le \|R^{\nabla}\| \iint \|\partial_t f_t(s) \wedge \partial_s f_t(s)\| \, ds \, dt \le \|R^{\nabla}\| C.$$

Proof. This is proved in [10, 6.2.1], see also [32, Prop.2.7]. □

It is routine to extend Proposition 3.7 to piecewise smooth curves.

Lemma 3.8 ([14])**.** *For M a compact connected Riemannian manifold and F a finite subset of $\Gamma = \pi_1(M,p)$, there is a constant C that depends only on M and F such that for any couple (E,∇),*

$$\|\rho(st) - \rho(s)\rho(t)\| \le C\|R^{\nabla}\|, \quad \textit{for all} \quad s,t \in F.$$

Proof. For $s,t \in F$, fix a homotopy between $\gamma_t * \gamma_s$ and γ_{st}. Since F is finite, there is a constant C larger than the areas of all these homotopies. It follows then by Proposition 3.7 that $\|P_{\gamma_t * \gamma_s} - P_{\gamma_{st}}\| \le C\|R^{\nabla}\|$, for all $s,t \in F$. Since $\rho(s)\rho(t) = P_{\gamma_s} \circ P_{\gamma_t} = P_{\gamma_t * \gamma_s}$, this completes the proof. □

The following Proposition collects several facts from papers of Hanke and Schick [31] and Hanke [30].

Proposition 3.9 ([30])**.** *Let M be a compact connected Riemannian manifold and let $\varepsilon > 0$. Let (E_n, ∇_n) be a sequence with $E_n \in \mathrm{Bdl}_{\mathbb{C}}^{r_n}(M)$ and $\|R^{\nabla_n}\| \le \varepsilon$ for all n. Let $A_n = M_{r_n}(\mathbb{C})$, and set $A = \prod_n A_n$ and $B = \prod_n A_n / \bigoplus_n A_n$. Then*

(i) *There is $E_A \in \mathrm{Bdl}_A(M)$ with transition functions in diagonal form and such that the $n^{th}-$component of E_A is isomorphic to E_n as Hilbert A_n-modules bundles.*

(ii) *The holonomy representations* $h_n : \mathcal{P}_1(M) \to \mathcal{T}(E_n)$ *defined via parallel transport for* (E_n, ∇_n) *assemble to a holonomy representation* $h_A : \mathcal{P}_1(M) \to \mathcal{T}(E_A)$.

(iii) *If* $\lim_n \|R^{\nabla_n}\| = 0$, *then the composition* $h_B : \mathcal{P}_1(M) \xrightarrow{h_A} \mathcal{T}(E_A) \longrightarrow \mathcal{T}(E_A \otimes_A B)$ *descends to a morphism of groupoids* $\Pi_1(M) \to \mathcal{T}(E_A \otimes_A B)$ *and hence induces a group homomorphism* $\rho_B : \pi_1(M,p) \to U(B)$. *The Hilbert* B*-module bundle* $E_A \otimes_A B$ *is isomorphic to* L_{ρ_B}.

Proof. B is viewed as a left A module via the quotient map $q : A \to B$. In the notation of Kasparov, $E_A \otimes_A B = E_A \otimes_q B$. The statement of the proposition is the combination of Propositions 3.4, 3.12 and 3.13 from [30]. The proof of the these properties is based on the following fact established in Proposition 3.4 of [30]. There are constants $C, \lambda > 0$ depending only on M such that for any couple (E, ∇) one has

$$\|P_\gamma - \mathrm{id}_{E_p}\| \leq C\|R^\nabla\| \cdot \mathrm{length}(\gamma)$$

for each closed loop $\gamma \in \Omega_1(M,p)$ with $\mathrm{length}(\gamma) \leq \lambda$. The above estimate or Lemma 3.8 also explain why the map h_B descends to $\pi_1(M,p)$, when $\lim_n \|R^{\nabla_n}\| = 0$. □

Fix $F \subset \pi_1(M,p)$ and m as in Notation 2.5 and the map $s \mapsto \gamma_s$ as in Notation 3.6.

Theorem 3.10. *There is* $\varepsilon = \varepsilon_M > 0$ *such that for any smooth hermitian vector bundle* E *on* M *which admits a unitary connection* ∇ *with* $\|R^\nabla\| < \varepsilon$, *the map* $\rho(s) = P^\nabla_{\gamma_s}$, $s \in \Gamma = \pi_1(M,p)$, *defined by parallel transport is an approximate representation* $\rho \in \mathrm{Rep}^r_{(F,\frac{1}{5m^2})}(\Gamma)$, $r = \mathrm{rank}(E)$, *with the property that* $L_\rho \cong E$.

Proof. By Lemma 3.8, there is ε_0 such that if $\|R^\nabla\| < \varepsilon_0$, then $\rho \in \mathrm{Rep}^r_{(F,\frac{1}{5m^2})}(\Gamma)$. Seeking a contradiction, suppose that the statement is false for all $\varepsilon > 0$. Consequently, if we fix a sequence (ε_n) convergent to 0 with $\varepsilon_n < \varepsilon_0$, then there is a sequence (E_n, ∇_n) such that $E_n \in \mathrm{Bdl}^{r_n}_{\mathbb{C}}(M)$, $\|R^{\nabla_n}\| < \varepsilon_n$ but $E_n \ncong L_{\rho_n}$ for all $n \in N$.

Let E_A, h_A, h_B and ρ_B be as in Proposition 3.9 so that in particular $E_A \otimes_A B \cong L_{\rho_B}$. Define $\rho_A : \Gamma \to U((E_A)_p) \cong U(A)$ by $\rho_A(s) = h_A(\gamma_s)$. Thus the components of ρ_A are (ρ_n) and $q \circ \rho_A = \rho_B$, where $q : A \to B$ is the quotient map. Consider the element

$$x_{\rho_A} = \sum_{i,j \in I} e_{ij} \otimes \chi_i \chi_j \otimes \rho_A(s_{ij}) \in M_m(\mathbb{C}) \otimes C(M) \otimes A,$$

and let $\ell_{\rho_A} = \chi_{(2/3,1]}(x_{\rho_A})$ be the projection representing L_{ρ_A} as Definition 3.3. Then $(\mathrm{id}_{M_m(\mathbb{C}) \otimes C(M)} \otimes q)(\ell_{\rho_A}) = \ell_{\rho_B}$ is a projection representing L_{ρ_B}. Using Proposition 3.9(iii), it follows that

$$L_{\rho_A} \otimes_A B \cong L_{\rho_B} \cong E_A \otimes_A B.$$

The components of ℓ_{ρ_A} are (ℓ_{ρ_n}) by naturality of functional calculus.

This implies that $L_{\rho_n} \cong (L_{\rho_A})_n \cong (E_A)_n \cong E_n$, for all sufficiently large n, and this contradicts our assumption. □

4. Uniform-to-local non-stable groups

Let M be a closed connected Riemannian manifold with nonpositive sectional curvature, $K(M) \leq 0$. Fix a base point $p \in M$ and consider the fundamental group $\Gamma = \pi_1(M,p)$. It is a classic result that Γ is torsion free and that each homotopy class $s \in \pi_1(M,p)$ contains a unique constant speed geodesic $\gamma_s : [0,1] \to M$, see [7, Thm.4.13, p.200]. In particular $\gamma_{s^{-1}}$ is γ_s with the opposite parametrization, $\gamma_{s^{-1}}(\tau) = \gamma_s(1-\tau)$. Thus, for manifolds with nonpositive sectional curvature we have a preferred choice for the map $\Gamma = \pi_1(M,p) \to \Omega_1(M,p)$, $s \mapsto \gamma_s$, considered in Notation 3.6. For an ε-flat couple (E,∇) as in Definition 3.1, parallel transport along these geodesics define a map $\rho : \Gamma \to U(E_p)$,

$$\rho(s) = P_{\gamma_s},\ \ s \in \Gamma. \tag{4}$$

with $\rho(s^{-1}) = \rho(s)^{-1}$ for all $s \in \Gamma$. The map ρ is an approximate unitary representation by Lemma 3.8. Morever, as indicated by Gromov on page 166 of [28], ρ is a uniform approximate representation if M has strictly negative sectional curvature.

Proposition 4.1 (Gromov)**.** *Let M be a closed connected Riemannian manifold with sectional curvature $K(M) \le -\kappa < 0$. Then $\|\rho(st) - \rho(s)\rho(t)\| \le \kappa\pi\|R^{\nabla}\|$ for all $s,t \in \Gamma$.*

Proof. Consider the universal cover $\Phi : \widetilde{M} \to M$. We endowed $\widetilde{M}$ with the pullback of the Riemannian metric from M so that Φ becomes a local isometry. Fix a base point $\tilde{p} \in \widetilde{M}$ with $\phi(\tilde{p}) = p$. Γ acts by isometries on $\widetilde{M}$: $\Gamma \times \widetilde{M} \to \widetilde{M}$, $(t,\tilde{x}) \mapsto t\cdot\tilde{x}$. For each $t \in G$, let $\tilde{\gamma}_t : [0,1] \to \widetilde{M}$ be the unique geodesic joining $\tilde{p}$ with $t\cdot\tilde{p}$. Then $\Phi \circ \tilde{\gamma}_t = \gamma_t$ by the uniqueness result mentioned earlier, see [7, p.201]. Fix s and t in Γ. Let $\bar{\gamma}_s$ be the unique lift of s to a path in $\widetilde{M}$ with $\bar{\gamma}_s(0) = t\cdot\tilde{p}$. Then $\gamma_t * \bar{\gamma}_s$ lifts st. Consider the geodesic triangle in $\widetilde{M}$ with vertices $\tilde{p}$, $t\cdot\tilde{p}$, and $(st)\cdot\tilde{p}$ and edges $\tilde{\gamma}_t$, $\bar{\gamma}_s$ and $\tilde{\gamma}_{st}$. Following [10, p.106], we span a ruled surface S into this geodesic triangle by joining $\tilde{\gamma}_t(\tau)$ to $\tilde{\gamma}_{st}(\tau)$ by the unique minimizing geodesic for each $\tau \in [0,1]$ (geodesics vary continuously and smoothly on their endpoints). From Gauss' equation the intrinsic curvature of S is $\le K(M)$ and by A.D. Aleksandrov's area comparison theorem [10, Prop.6.7], the area of S is less than equal to the area of the model triangle T with the same edge length. Since the model space has constant curvature $-\kappa < 0$, it follows that $\text{area}(T) \le \kappa\pi$ by Gauss-Bonnet, and hence $\text{area}(S) \le \kappa\pi$. This shows that there is a homotopy $(\tilde{f}_\tau)$ of area $\le \kappa\pi$ between the paths and $\tilde{\gamma}_t * (\bar{\gamma}_s)$ and $\tilde{\gamma}_{st}$. Since Φ is a local isometry, it follows that $f_\tau := \Phi \circ \tilde{f}_\tau$ is a homotopy of area $\le \kappa\pi$ between $f_0 = \gamma_t * \gamma_s$ and $f_1 = \gamma_{st}$. It follows by Proposition 3.7 that

$$\|P_{\gamma_s} \circ P_{\gamma_t} - P_{\gamma_{st}}\| = \|P_{\gamma_t * \gamma_s} - P_{\gamma_{st}}\| \le \|R^{\nabla}\|\kappa\pi.$$

□

The dual assembly map of Kasparov $\nu : K^0(C^*(\Gamma)) \to RK^0(B\Gamma)$ is a generalization of the Atiyah's map $\text{Rep}(\Gamma) \to RK^0(B\Gamma)$, $\pi \mapsto [L_\pi]$. Kasparov has shown that if Γ has a γ-element, then ν is surjective, [35]. For the proof of Theorem 4.2 we only use the surjectivity of ν in the case when Γ is the fundamental group of a closed Riemannian manifold with nonpositive sectional curvature. This was proven by Kasparov in [35, Thm.6.7] by showing that Γ has a γ-element. By the Hadamard-Cartan theorem the universal cover of M is contractible and hence M is a model for $B\Gamma$. Moreover, it turns out that for quasidiagonal groups (e.g. residually finite groups), the map ν remains surjective even if one restricts its domain to quasidiagonal K-homology classes $K^0(C^*(\Gamma))_{qd}$, [16], [43], [17]. We will elaborate on this point in the proof of Theorem 4.2 below.

Theorem 4.2. *Let M be a closed connected Riemannian manifold with sectional curvature $K(M) \leq -\kappa < 0$. Assume that $\Gamma = \pi_1(M)$ is residually finite, and $b_{2i}(M) \neq 0$ for some $i > 0$. Then there exist a finite subset $F \subset \Gamma$ and $C > 0$ with the following property. For any $\varepsilon > 0$ there is a unital map $\rho : \Gamma \to U(n)$, satisfying $\sup_{s,t\in\Gamma} \|\rho(st) - \rho(s)\rho(t)\| < \varepsilon$, such that for any representation $\pi : \Gamma \to GL_n(\mathbb{C})$, $\sup_{s\in F} \|\rho(s) - \pi(s)\| > C$.*

Proof. Let $F \subset \Gamma$ be the finite set from Notation 2.5. We construct a sequence of unital maps $\{\rho_n : \Gamma \to U(H_n)\}_n$ with H_n finite dimensional Hilbert spaces and

$$\lim_{n\to\infty} \sup_{s,t\in\Gamma} \|\rho(st) - \rho_n(s)\rho_n(t)\| = 0,$$

for which does not exist a sequence of homomorphisms $\{\pi_n : \Gamma \to GL(H_n)\}_n$ such that

$$\lim_{n\to\infty} \|\rho_n(s) - \pi_n(s)\| = 0, \text{ for all}\, s \in F. \tag{5}$$

Since $M = B\Gamma$ and $b_{2k}(\Gamma) \neq 0$, one has $H^{2k}(M, \mathbb{Q}) \neq 0$. Since the Chern character is a rational isomorphism, there is a nontorsion element $y \in \tilde{K}^0(M)$. Since the dual assembly map $\nu : K^0(C^*(\Gamma))_{qd} \to K^0(B\Gamma) = K^0(M)$ is surjective by [17, Thm.4.6], there is $x \in K^0(C^*(\Gamma))_{qd}$ such that $\nu(x) = y$. The following realization of ν on quasidiagonal KK-classes was introduced in [16]. The element x is represented by a pair of unital $*$-representations $\Phi, \Psi : C^*(\Gamma) \to \mathcal{L}(H)$, such that $\Phi(a) - \Psi(a) \in K(H)$, $a \in C^*(\Gamma)$, and with property that there is an increasing approximate unit $(p_n)_n$ of $K(H)$ consisting of projections such that $(p_n)_n$ commutes asymptotically with both $\Phi(a)$ and $\Psi(a)$, for all $a \in C^*(\Gamma)$. Thus:

$$\lim_n \|p_n\Phi(a) - \Phi(a)p_n\| = 0, \quad \lim_n \|p_n\Psi(a) - \Psi(a)p_n\| = 0, \quad a \in C^*(\Gamma). \tag{6}$$

Moreover, since the group Γ is residually finite, there is a residually finite dimensional C^*-algebra D which is intermediate between $C^*(\Gamma)$ and $C^*_r(\Gamma)$ so that x is in the image of the map $K^0(D) \to K^0(C^*(\Gamma))$. This is explained in [17] (Prop. 3.8 and Thm.4.6) where we used Kubota's idea [43] of considering quasidiagonal C^*-algebras which are intermediate between the full and the reduced group C^*-algebras. This fact allows us to assume that Ψ is given by a direct sum of finite dimensional unitary representations of Γ, so that we may arrange that each p_n commutes with Ψ.

If $r_n = \mathrm{rank}(p_n)$, set $A_n = M_{r_n}(\mathbb{C})$, $A = \prod_n A_n$, $B = \prod_n A_n / \bigoplus_n A_n$, and let $q : A \to B$ be the quotient map. Define unital maps $\varphi, \psi : C^*(\Gamma) \to A$ by

$$\varphi(a) = (\varphi_n(a))_n \quad \text{and} \quad \psi(a) = (\psi_n(a))_n,$$

where $\varphi_n(a) = p_n\Phi(a)p_n$ and $\psi_n(a) = p_n\Psi(a)p_n$. Then φ is a unital completely positive map with asymptotically multiplicative components and ψ is a unital $*$-homomorphism. Let L_φ and L_ψ be the corresponding Hilbert A-module bundles, with components L_{φ_n} and L_{ψ_n}, see Definition 3.3. As we noted in [16] based on work of Phillips and Stone [49, 3.18], [50, Thm.1], the bundles L_{φ_n} are $(\mathcal{U}, \varepsilon'_n)$-flat with $\lim \varepsilon'_n = 0$ and each L_{ψ_n} is flat since ψ_n is a true representation. Moreover, by [16]:

$$y = \nu(x) = [L_{\varphi_n}] - [L_{\psi_n}], \quad \text{for all } n \geq 1.$$

The definition of almost flat bundles based on smooth connections as defined in [14], see Definition 3.1, is equivalent with the definition based on flatness of transition functions, see Definition 3.2, as verified by Hunger [32]. It follows that one can replace the bundles L_{φ_n} by smooth bundles E_n isomorphic to L_{φ_n} and which are endowed with metric compatible connections (E_n, ∇_n) such that $\|R^{\nabla_n}\| \le \varepsilon_n$ where $\varepsilon_n \le C'\varepsilon_n'$ and hence $\lim \varepsilon_n = 0$. The constant C' depends only on M and the fixed cover $\mathcal{U}$. Let $h_n : \mathcal{P}_1(M) \to \mathcal{T}(E_n)$ be the holonomy representations defined via parallel transport corresponding to (E_n, ∇_n) and let $\rho_n(s) = h_n(s) = P^{\nabla_n}_{\gamma_s}$, $s \in \Gamma$. It follows from Theorem 3.10 that

$$L_{\rho_n} \cong E_n \cong L_{\varphi_n}$$

for all sufficiently large n. On the other hand, since we work with geodesic loops γ_s representing the elements of Γ, it follows by Proposition 4.1 that

$$\sup_{s,t\in\Gamma} \|\rho_n(st) - \rho_n(s)\rho_n(t)\| \le \kappa\pi\varepsilon_n.$$

We claim that the sequence (ρ_n) cannot be perturbed to a sequence of representations satisfying (5). Seeking a contradiction, assume that there exists a sequence of representations $\pi_n : \Gamma \to GL_{r_n}(\mathbb{C})$ such that $\lim_n \|\rho_n(s) - \pi_n(s)\| = 0$ for all $s \in F$. In this case, it follows from Remark 3.5 that $L_{\rho_n} \cong L_{\pi_n}$ for all sufficiently large n and hence $y = [L_{\rho_n}] - [L_{\psi_n}] = [L_{\pi_n}] - [L_{\psi_n}] \in \widetilde{K}^0(M)$. By [46], all Chern classes of a flat complex bundle vanish over $\mathbb{Q}$, so that the Chern character of y is zero as both π_n and ψ_n are representations. We obtained a contradiction, since y is a nontorsion element of $\widetilde{K}^0(M)$. ∎

Remark 4.3. There is a related approach to Theorem 1.2 based on the notion K-area of a Riemannian manifold introduced by Gromov [27, §4]. Indeed, if M is an orientable closed connected Riemannian manifold with non-positive sectional curvature of dimension $\dim(M) = 2m$ and residually finite fundamental group, then M has infinite K-area by [27, Ex.(v') p.25], see also [31]. This yields a sequence (E_n, ∇_n) with $\lim_n \|R^{\nabla_n}\| = 0$ and top Chern class $c_m(E_n) \neq 0$. Having this sequence at hand, one proceeds like in the second part of the proof of Theorem 1.2.

5. Obstructions to C^*-stability

In this section we establish Theorem 1.7 as a consequence of a stronger result which assumes weaker forms of stability, see Theorem 5.15 and Theorem 5.17.

We refer the reader to [35] for the definitions and the basic properties of the various KK-theory groups introduced by Kasparov. We will freely employ the same notation as there. Thus if X is a locally compact group and A, B are separable C^*-algebras we write $RKK^0(X; A, B)$ for $\mathcal{R}KK(X; C_0(X) \otimes A, C_0(X) \otimes B)$ and $RK^0(X; B)$ for $RKK^0(X; \mathbb{C}, B)$.

Let $\underline{E}\Gamma$ be the classifying space for proper actions of Γ, [5]. It is known that $\underline{E}\Gamma$ admits a locally compact model, [34]. Let us recall that Γ has a γ-element if there exists a $\Gamma - C_0(\underline{E}\Gamma)$-algebra A in the sense of [35] and two elements $d \in KK_G(A, \mathbb{C})$ and $\eta \in KK_\Gamma(\mathbb{C}, A)$ (called Dirac and dual-Dirac elements, respectively) such that the element $\gamma = \eta \otimes_A d \in KK_\Gamma(\mathbb{C}, \mathbb{C})$ has the property that $p^*(\gamma) = 1 \in \mathcal{R}KK_\Gamma(\underline{E}\Gamma; C_0(\underline{E}\Gamma), C_0(\underline{E}\Gamma))$ where $p : \underline{E}\Gamma \to \text{point}$, [56]. Let B be a separable C^*-algebra endowed with the trivial Γ-action. Consider the dual assembly map with coefficients in B:

$$\alpha : KK_\Gamma(\mathbb{C}, B) \to RKK^0_\Gamma(\underline{E}\Gamma; \mathbb{C}, B)$$

defined by $\alpha(y) = p^*(y)$ where $p : \underline{E}\Gamma \to$ point. As in [35], we write $RKK^0_\Gamma(\underline{E}\Gamma;\mathbb{C},B)$ for $\mathcal{R}KK_\Gamma(\underline{E}\Gamma; C_0(\underline{E}\Gamma), C_0(\underline{E}\Gamma)\otimes B)$. .

We need the following result which is essentially due to Kasparov, [35, Th.6.5]. It was discussed in [45, Thm.7.1], [22, Thm.23] and [23, Lem.10.1].

Theorem 5.1 (Kasparov)**.** *If Γ is a countable discrete group that admits a γ-element, then the dual assembly map $\alpha : KK_\Gamma(\mathbb{C},B) \to RKK^0_\Gamma(\underline{E}\Gamma;\mathbb{C},B)$ is split surjective with kernel $(1-\gamma)KK_\Gamma(\mathbb{C},B)$.*

By universality of $\underline{E}\Gamma$, there is a Γ-equivariant map (unique up to homotopy) $\sigma : E\Gamma \to \underline{E}\Gamma$. It induces a map $\sigma^* : RKK^0_\Gamma(\underline{E}\Gamma;\mathbb{C},B) \to RKK^0_\Gamma(E\Gamma;\mathbb{C},B)$. Recall that $\mathcal{Q} = \bigotimes_n M_n$ is universal UHF algebra and $K_0(\mathcal{Q}) = \mathbb{Q}$ and $K_1(\mathcal{Q}) = 0$. We can identify the representable rational K-theory group $RK^0(X;\mathbb{Q})$ with $RK^0(X;\mathcal{Q})$, as in [36]. We view $\mathcal{Q}$ as a trivial Γ-algebra.

Corollary 5.2. *Let Γ be a countable discrete group that admits a γ-element. Let B be a separable trivial Γ-algebra such that $B \cong B\otimes\mathcal{Q}$. Then the composition*

$$\gamma KK_\Gamma(\mathbb{C},B) \hookrightarrow KK_\Gamma(\mathbb{C},B) \xrightarrow{\alpha} RKK^0_\Gamma(\underline{E}\Gamma;\mathbb{C},B) \xrightarrow{\sigma^*} RKK^0_\Gamma(E\Gamma;\mathbb{C},B)$$

is a surjective map.

Proof. It was shown in [5, p.275-6] that σ induces a rationally injective homomorphism

$$\sigma_* : RK^\Gamma_0(E\Gamma) \to RK^\Gamma_0(\underline{E}\Gamma).$$

It follows that the map $(\sigma_*)^* : \mathrm{Hom}(RK^\Gamma_0(\underline{E}\Gamma), K_0(B)) \to \mathrm{Hom}(RK^\Gamma_0(E\Gamma), K_0(B))$ is surjective since $K_0(B) \cong K_0(B)\otimes\mathbb{Q}$. By the universal coefficient theorems stated as Lemma 2.3 of [33] and Lemma 3.4 of [38] applied for the coefficient algebra B, the horizontal maps in the commutative diagram

$$\begin{array}{ccc} RKK^0_\Gamma(\underline{E}\Gamma;\mathbb{C},B) & \longrightarrow & \mathrm{Hom}(RK^\Gamma_0(\underline{E}\Gamma), K_0(B)) \\ \downarrow{\scriptstyle \sigma^*} & & \downarrow{\scriptstyle (\sigma_*)^*} \\ RKK^0_\Gamma(E\Gamma;\mathbb{C},B) & \longrightarrow & \mathrm{Hom}(RK^\Gamma_0(E\Gamma), K_0(B)) \end{array}$$

are bijections. It follows that the restriction map $\sigma^* : RKK^0_\Gamma(\underline{E}\Gamma;\mathbb{C},B) \to RKK^0_\Gamma(E\Gamma;\mathbb{C},B)$ is surjective. The statement follows now from Theorem 5.1. ☐

Let us recall that a set of operators $S \subset \mathcal{L}(H)$ on a separable Hilbert space H is quasidiagonal if there exists an approximate unit of projections $(p_n)_n$ of $K(H)$ such that

$$\lim_{n\to\infty} \|[a,p_n]\| = 0, \quad \text{for all } a \in S.$$

A representation $\pi : A \to \mathcal{L}(H)$ of a separable C^*-algebra A is quasidiagonal if the set $\pi(A)$ is quasidiagonal. A C^*-algebras A is quasidiagonal if it admits a faithful quasidiagonal representation.

Let A, B be separable C^*-algebras. Any class $x \in KK(A,B)$ is represented by some Cuntz pair, i.e. a pair of $*$-homomorphisms $\varphi,\psi : A \to M(K(H)\otimes B)$, such that $\varphi(a)-\psi(a) \in K(H)\otimes B$, for all $a \in A$. Assume that B is unital.

Definition 5.3 ([16], [17])**.** An element $x \in KK(A,B)$ is quasidiagonal if it is represented by a Cuntz pair (φ,ψ) with the property that there exists an approximate unit of projections $(p_n)_n$ of

$K(H)$ such that $\lim_{n\to\infty} \|[\psi(a), p_n \otimes 1_B]\| = 0$, for all $a \in A$. The quasidiagonal elements form a subgroup of $KK(A,B)$, denoted by $KK(A,B)_{qd}$. For other contexts it will be useful to modify the definition by asking for the existence of an approximate unit of $K(H)\otimes B$ consisting of projections (q_n) such that $\lim_{n\to\infty} \|[\psi(a), q_n]\| = 0$, for all $a \in A$.

Remark 5.4. (a) If $\theta : A \to D$ is a $*$-homomorphism, then $\theta^*[\varphi, \psi] = [\varphi\circ\theta, \psi\circ\theta]$ and hence

$$\theta^*(KK(D,B)_{qd}) \subset KK(A,B)_{qd}.$$

(b) Let D, B be separable unital C^*-algebras with B nuclear. Fix a faithful unital representation $\psi_0 : D \to M(K(H))$ such that $\psi_0(D)\cap K(H) = \{0\}$. Then any element $x \in KK(D,B)$ is represented by a Cuntz pair (φ, ψ) where $\psi = \psi_0 \otimes 1_B : A \to M(K(H)\otimes B)$, [52]. Therefore, if D is quasidiagonal, then $\psi_0(D)$ is a quasidiagonal subset of $M(K(H))$ and hence $KK(D,B) = KK(D,B)_{qd}$.

Let us recall that a countable discrete group Γ is *quasidiagonal* if there is a faithful unitary representation $\pi : \Gamma \to U(H)$ such that the set $\pi(\Gamma)$ is quasidiagonal. Note that residually finite groups or more generally maximally periodic groups (abbreviated MAP) are quasidiagonal. Residually amenable groups are quasidiagonal by [55]. Quasidiagonality of a group Γ is at least formally weaker than quasidiagonality of the full group C^*-algebra $C^*(\Gamma)$. For example, the question of quasidiagonality of $C^*(\Gamma)$ is completely open for all residually finite, infinite, property T groups.

Proposition 5.5 ([17]). *For a countable discrete group Γ, the following assertions are equivalent:*

(i) Γ *is quasidiagonal.*
(ii) λ_Γ *is weakly contained in a quasidiagonal representation π of Γ.*
(iii) *The canonical map $q_\Gamma : C^*(\Gamma) \to C^*_r(\Gamma)$ factors through a unital quasidiagonal C^*-algebra.*

Let j_Γ and $j_{\Gamma,r}$ be the descent maps of Kasparov [35, Thm.3.11]. Thus $\gamma \in KK_\Gamma(\mathbb{C},\mathbb{C})$ gives an element $j_\Gamma(\gamma) \in KK(C^*(\Gamma), C^*(\Gamma))$ which induces a map

$$j_\Gamma(\gamma)^* = j_\Gamma(\gamma) \otimes_{C^*(\Gamma)} - : KK(C^*(\Gamma), B) \to KK(C^*(\Gamma), B).$$

The image of $j_\Gamma(\gamma)^*$ is usually denoted by $\gamma KK(C^*(G), B)$, while $\gamma_r KK(C^*_r(\Gamma), B)$ is defined similarly as the image of $j_{\Gamma,r}(\gamma)^*$. Since G is discrete and acts trivially on B, there is a canonical isomorphism (called dual Green-Julg isomorphism), [35],

$$\kappa : KK_\Gamma(\mathbb{C}, B) \xrightarrow{\cong} KK(C^*(\Gamma), B)$$

which is compatible with the module structure over the group ring of Γ. Moreover, by [48, Lemma 11], for every $x \in KK_\Gamma(\mathbb{C},\mathbb{C})$, the following diagram is commutative.

$$\begin{array}{ccc} KK_\Gamma(\mathbb{C}, B) & \xrightarrow{\kappa} & KK(C^*(\Gamma), B) \\ \downarrow{\scriptstyle x\otimes -} & & \downarrow{\scriptstyle j_\Gamma(x)\otimes -} \\ KK_\Gamma(\mathbb{C}, B) & \xrightarrow[\kappa]{} & KK(C^*(\Gamma), B) \end{array} \tag{7}$$

Kasparov [35, 3.12] has shown that the canonical surjection $q_\Gamma : C^*(\Gamma) \to C^*_r(\Gamma)$ induces an isomorphism of γ-parts $q_\Gamma^* : \gamma_r KK(C^*_r(\Gamma), B) \xrightarrow{\cong} \gamma KK(C^*(\Gamma), B)$. In particular:

Proposition 5.6 (Kasparov [35])**.** *If Γ is a discrete countable group that admits a γ-element, then $\gamma KK(C^*(\Gamma), B) \subset q_\Gamma^*(KK(C_r^*(\Gamma), B))$.*

We gave an exposition of Proposition 5.6 in [17]. By [35, Thm.3.4], see also [38, p.313], there is natural descent isomorphism

$$\lambda^\Gamma : RKK_\Gamma^0(E\Gamma; \mathbb{C}, B) \xrightarrow{\cong} RKK^0(B\Gamma; \mathbb{C}, B).$$

Let $\nu : KK^0(C^*(\Gamma), B) \to RK^0(B\Gamma; B)$ be the map

$$\nu = \lambda^\Gamma \circ \sigma^* \circ \alpha \circ \kappa^{-1} : KK(C^*(\Gamma), B) \cong KK_\Gamma(\mathbb{C}, B) \to RKK_\Gamma^0(E\Gamma; \mathbb{C}, B) \cong RKK^0(B\Gamma; \mathbb{C}, B).$$

Note that $\sigma^* \circ \alpha = p^*$ where $p : E\Gamma \to point$. Abusing terminology, we shall also refer to ν as the dual assembly map.

As in [17], we rely on Kubota's idea [43] of using a quasidiagonal C^*-algebra intermediate between $C_r^*(\Gamma)$ and $C^*(\Gamma)$ which strengthens significantly the construction of almost flat K-theory classes based on K-quasidiagonality of $C^*(\Gamma)$, introduced in [16].

In particular Thm. 4.6 from [17] admits a version with coefficients:

Theorem 5.7. *Let Γ be a countable discrete quasidiagonal group and let B be a separable nuclear unital C^*-algebra. If Γ admits a γ-element, then $\gamma KK(C^*(\Gamma), B)) \subset KK(C^*(\Gamma), B)_{qd}$. It follows that $\nu(KK^0(C^*(\Gamma), B)) = \nu(KK(C^*(\Gamma), B)_{qd})$ and hence $\nu(KK(C^*(\Gamma), B)_{qd}) = RK^0(B\Gamma; B)$ if $B \cong B \otimes \mathcal{Q}$.*

Proof. The factorization $C^*(\Gamma) \xrightarrow{q_D} D \to C_r^*(\Gamma)$ of q_Γ with D unital and quasidiagonal given by Proposition 5.5 in conjunction with Remark 5.4 implies that

$$q_\Gamma^*(KK(C_r^*(\Gamma), B)) \subset q_D^*(KK(D, B)) = q_D^*(KK(D, B)_{qd}) \subset KK(C^*(\Gamma), B)_{qd}.$$

From this and Proposition 5.6 we obtain that $\gamma KK(C^*(\Gamma), B)) \subset KK(C^*(\Gamma), B)_{qd}$.

By Theorem 5.1, α vanishes on $(1-\gamma)KK_\Gamma(\mathbb{C}, B))$. Since the diagram (7) is commutative, this group is mapped to $(1-\gamma)KK^0(C^*(\Gamma), B)$ by κ and hence

$$\nu(KK^0(C^*(\Gamma), B)) = \nu(\gamma KK^0(C^*(\Gamma), B)) = \nu(KK(C^*(\Gamma), B)_{qd}).$$

By Corollary 5.2, the map ν is surjective if $B \cong B \otimes \mathcal{Q}$. □

Definition 5.8. Let Γ be a countable discrete group and let $\mathcal{B}$ be a class of unital C^*-algebras.

(a) Γ is called locally $GL(\mathcal{B})$-stable if for any sequence of unital maps $\{\varphi_n : \Gamma \to U(B_n)\}$ with $B_n \in \mathcal{B}$ and

$$\lim_{n\to\infty} \|\varphi_n(st) - \varphi_n(s)\varphi_n(t)\| = 0, \quad \text{for all } s, t \in \Gamma, \tag{8}$$

there exists a sequence of homomorphisms $\{\pi_n : \Gamma \to GL(B_n)\}$ such that

$$\lim_{n\to\infty} \|\varphi_n(s) - \pi_n(s)\| = 0, \quad \text{for all } s \in \Gamma. \tag{9}$$

(b) Γ is called locally $U(\mathcal{B})$-stable if for any sequence of unital maps $\{\varphi_n : \Gamma \to U(B_n)\}$ that satisfies (8), there exists a sequence of homomorphisms $\{\pi_n : \Gamma \to U(B_n)\}$ satisfying (9).

(c) If $\mathcal{B} = \{B_n : n \geq 1\}$, property (a) will be also called local $\{GL(B_n) : n \geq 1\}$-stability and property (b) local $\{U(B_n) : n \geq 1\}$-stability

Remark 5.9. The following observations are immediate.

(i) If Γ is locally $U(\mathcal{B})$-stable, then Γ is locally $GL(\mathcal{B})$-stable.

(ii) If $\mathcal{B}$ is the class of all unital separable C^*-algebras, then Γ is locally $U(\mathcal{B})$-stable if and only if Γ is C^*-stable in the sense [21] or equivalently $C^*(\Gamma)$ is weakly semiprojective.

(iii) If $\mathcal{B} = \{M_n(\mathbb{C}) : n \in \mathbb{N}\}$, then Γ is locally $U(\mathcal{B})$-stable if and only if Γ is matricially stable in the sense of [21], [17], or locally stable in the sense of Definition 1.1.

(iv) Let X be a compact metrizable space. It is easily verified that if an MF group Γ is locally $\{GL_n(C(X)) : n \in \mathbb{N}\}$-stable, then Γ is also locally $\{GL_n(\mathbb{C}) : n \in \mathbb{N}\}$-stable.

Lemma 5.10. *Let Γ be a countable discrete MF group.*

(i) *If Γ is locally $\{U_n(\mathbb{C}) : n \in \mathbb{N}\}$-stable, then Γ is MAP and hence quasidiagonal.*

(ii) *If Γ is finitely generated and Γ is locally $\{GL_n(\mathbb{C}) : n \in \mathbb{N}\}$-stable, then Γ is residually finite and hence quasidiagonal.*

Proof. (i) Since Γ is MF, it embeds in $U(\prod_n M_n / \bigoplus_n M_n)$. By local $\{U_n(\mathbb{C}) : n \in \mathbb{N}\}$-stability we obtained an embedding of Γ in $\prod_n U(n)$. (ii) By local $\{GL_n(\mathbb{C}) : n \in \mathbb{N}\}$-stability we obtained an embedding of Γ in $\prod_n GL_n(\mathbb{C})$. If Γ is finitely generated, it follows by Malcev's theorem [8, 6.4.13] that Γ is residually finite and hence quasidiagonal. ☐

Notation 5.11. For a compact space Y, let $RK^0(Y;B)_{\mathrm{flat}}$ be the subgroup of $RK^0(Y;B)$ generated by locally trivial bundles with typical fiber projective B-modules fB^k, $f^2 = f = f^* \in M_k(B)$, constructed from finite open covers $\mathcal{U} = (U_i)_{i\in I}$ of Y and 1-Cech cocycles $v_{ij} : U_i \cap U_j \to GL(fM_k(B)f)$ with the property that each function v_{ij} is constant. Recall that we identify $RK^0(Y;B)$ with the operator K-theory group $K_0(C(Y) \otimes B)$. For a compact subspace Y of $B\Gamma$, we denote by ν_Y the composition of the dual assembly map $\nu : KK(C^*(\Gamma), B) \to RK^0(B\Gamma; B)$ with the restriction map $RK^0(B\Gamma; B) \to RK^0(Y;B)$. For $B = C(\mathbb{T}) \otimes \mathcal{Q}$, we will use the natural isomorphisms

$$RK^0(Y;B) \cong K_0(C(Y) \otimes C(\mathbb{T}) \otimes \mathcal{Q}) \cong RK^0(Y \times \mathbb{T}; \mathbb{Q}). \tag{10}$$

Proposition 5.12. *Let $\mathcal{B}$ be a class of unital separable C^*-algebras such that if $B \in \mathcal{B}$, and $f^2 = f = f^* \in M_k(B)$ for $k \in \mathbb{N}$, then $fM_k(B)f \in \mathcal{B}$. Let Γ be a discrete countable group and let $Y \subset B\Gamma$ be a finite connected CW-complex. If Γ is locally $GL(\mathcal{B})$-stable, then for any $B \in \mathcal{B}$,*

$$\nu_Y(KK(C^*(\Gamma), B)_{qd}) \subset RK^0(Y;B)_{\mathit{flat}} \tag{11}$$

More generally, if (B_k) in an increasing sequence of C^*-algebras in $\mathcal{B}$ sharing the same unit and B is the C^*-completion of $\bigcup_k B_k$, then

$$\nu_Y(KK(C^*(\Gamma), B)_{qd}) \subset \varinjlim_k RK^0(Y;B_k)_{\mathit{flat}} \subset RK^0(Y;B)_{\mathit{flat}}. \tag{12}$$

Proof. We prove (11) first. Let $\jmath : \Gamma \hookrightarrow U(C^*(\Gamma))$ be the natural inclusion. The orbit space of the left action of Γ on $\widetilde{E\Gamma} \times C^*(\Gamma)$, defined by $s \cdot (\tilde{p}, v) = (s \cdot \tilde{p}, \jmath(s)v)$, is the total space of a (flat) Hilbert $C^*(\Gamma)$-module bundle $E\Gamma \times_\jmath C^*(\Gamma) \to B\Gamma$ denoted by $L_\jmath$.

The restriction of $L_\jmath$ to Y, denoted L_Y, yields a self-adjoint projection $p = p_Y$ in matrices over the ring $C(Y) \otimes \mathbb{C}[\Gamma]$ constructed as follows. Let $(U_i)_{i\in I}$ be finite covering of Y by open sets such that $L_\jmath$ is trivial on each U_i and $U_i \cap U_j$ is connected. Using trivializations of $L_\jmath$ on U_i one obtains group elements $s_{ij} \in G$ which define a 1-cocycle that is constant on each nonempty set $U_i \cap U_j$ and which represents L_Y. Thus $s_{ij}^{-1} = s_{ji}$ and $s_{ij} \cdot s_{jk} = s_{ik}$ whenever $U_i \cap U_j \cap U_k \neq \emptyset$. Let $(\chi_i)_{i\in I}$ be positive continuous functions with χ_i supported in U_i and such that $\sum_{i\in I} \chi_i^2 = 1$. Set $m = |I|$ and let (e_{ij}) be the canonical matrix unit of $M_m(\mathbb{C})$. Then L_Y is represented by the selfadjoint projection

$$p = \sum_{i,j\in I} e_{ij} \otimes \chi_i\chi_j \otimes s_{ij} \in M_m(\mathbb{C}) \otimes C(Y) \otimes C^*(\Gamma). \tag{13}$$

It was shown by Kasparov [35, Lemma 6.2], [37], that the map

$$\nu_Y : KK(C^*(\Gamma), B) \xrightarrow{\nu} K^0(B\Gamma; B) \to K^0(Y; B) \cong K_0(C(Y) \otimes B)$$

is given by $\nu_Y(x) = [p] \otimes_{C^*(\Gamma)} x$.

The restriction of ν_Y to quasidiagonal KK-classes can be described as follows, see [16]. Each element $x \in KK(C^*(\Gamma), B)_{qd}$ is represented by a pair of nonzero $*$-representations $\Phi^{(r)} : C^*(\Gamma) \to M(K(H) \otimes B)$, $r = 0, 1$, such that $\Phi^{(0)}(a) - \Phi^{(1)}(a) \in K(H) \otimes B$, $a \in C^*(\Gamma)$, and with the property that there is an increasing approximate unit $(p_n)_n$ of $K(H)$ consisting of projections such that $(p_n \otimes 1_B)_n$ commutes asymptotically with both $\varphi^{(0)}(a)$ and $\varphi^{(1)}(a)$, for all $a \in C^*(\Gamma)$. It follows that the compressions $\varphi_n^{(r)}(\cdot) = (p_n \otimes 1_B)\Phi^{(r)}(\cdot)(p_n \otimes 1_B)$ are completely positive asymptotic homomorphisms $\varphi_n^{(r)} : C^*(\Gamma) \to K(H) \otimes B$, $r = 0, 1$. Let 1 denote the unit of $C^*(\Gamma)$. It is routine to further perturb these maps to completely positive asymptotic homomorphisms such that $f_n^{(r)} := \varphi_n^{(r)}(1)$ are selfadjoint projections in matrices over B. Hence we can view these maps as unital completely positive maps $\varphi_n^{(r)} : C^*(\Gamma) \to D_n^{(r)}$, where each $D_n^{(r)} = f_n^{(r)}(K(H) \otimes B)f_n^{(r)}$ is Morita equivalent to B. As argued in [16, Prop.2.5], if $\mathrm{id} = \mathrm{id}_{M_m(C(Y))}$,

$$\nu_Y(x) = [p] \otimes_{C^*(\Gamma)} x = (\mathrm{id} \otimes \varphi_n^{(0)})_\sharp(p) - (\mathrm{id} \otimes \varphi_n^{(1)})_\sharp(p), \tag{14}$$

for all sufficiently large n. Here $(\mathrm{id} \otimes \varphi_n^{(r)})_\sharp(p) \in K_0(C(Y) \otimes B)$ is the class of the projection $\chi_{(1/2,1]}(x_n^{(r)})$ obtained by continuous functional calculus from the approximate projection $x_n^{(r)} = \sum_{i,j\in I} e_{ij} \otimes \chi_i\chi_j \otimes \varphi_n^{(r)}(s_{ij})$. Since $\varphi_n^{(r)}(s)$ are almost unitary elements and since Γ is countable, there exist sequences of unital maps $\sigma_n^{(r)} : \Gamma \to U(D_n^{(r)})$ such that $\lim_n \|\varphi_n^{(r)}(s) - \sigma_n^{(r)}(s)\| = 0$ for all $s \in \Gamma$. The sequences $(\sigma_n^{(r)})$, $r = 0, 1$ are asymptotically multiplicative in the sense of (8). If Γ satisfies Definition 5.8 (b), then there exist sequences of group homomorphisms $\{\pi_n^{(r)} : \Gamma \to GL(D_n^{(r)})\}$, $r = 0, 1$, such that

$$\lim_n \|\sigma_n^{(r)}(s) - \pi_n^{(r)}(s)\| = 0 \tag{15}$$

for all $s \in \Gamma$.

Note that the projections $e_n^{(r)}$, $r = 0, 1$, defined by

$$e_n^{(r)} = (\mathrm{id} \otimes \pi_n^{(r)})(p) = \sum_{i,j\in I} e_{ij} \otimes \chi_i\chi_j \otimes \pi_n^{(r)}(s_{ij}) \in M_m(\mathbb{C}) \otimes C(Y) \otimes D_n^{(r)},$$

correspond to flat bundles since they are realized via the constant cocycles $\pi_n^{(r)}(s_{ij})$ and hence $[e_n^{(0)}] - [e_n^{(1)}] \in RK^0(Y, B)_{\mathrm{flat}}$. From (13), (14) and (15) we deduce that $\nu_Y(x) = [e_n^{(0)}] - [e_n^{(1)}]$ for all sufficiently large n, since $\|\sigma_n^{(r)}(s_{ij}) - \pi_n^{(r)}(s_{ij})\| \to 0$. We conclude that $\nu_Y(x) \in RK^0(Y, B)_{\mathrm{flat}}$, as desired.

The proof for (12) is similar. One makes a small adjustment to the previous argument. By assumption, $\bigcup_k B_k$ is dense in B. Enumerate $\Gamma = \{g_n\}_n$ and modify the maps $(\sigma_n^{(r)})$, $r = 0, 1$ as follows. Choose selfadjoint projections $h_n^{(r)}$ in $K(H) \otimes B_{k(n,r)}$ for sufficiently large $k(n,r) \in \mathbb{N}$ such that $h_n^{(r)}$ approximate $f_n^{(r)}$ and $\mathrm{dist}(\varphi_n^{(r)}(s), U(h_n^{(r)}(K(H) \otimes B_{k(n,r)})h_n^{(r)})) < 1/n$ for all $s \in \{g_1, ..., g_n\}$. Letting $D_n^{(r)} = h_n^{(r)}(K(H) \otimes B_{k(n,r)})h_n^{(r)}$, we can now construct $\sigma_n^{(r)} : \Gamma \to U(D_n^{(r)})$ with the property that $\lim_n \|\varphi_n^{(r)}(s) - \sigma_n^{(r)}(s)\| = 0$ for all $s \in \Gamma$. The rest of the proof is as in (i). Note that $[e_n^{(0)}] - [e_n^{(1)}] \in RK^0(Y, B_{k(n,r)})_{\text{flat}}$. □

Let X be a finite connected CW complex. Let Γ be a discrete countable group and let $\rho : \Gamma \to GL_r(C(X))$ be a group homomorphism. Consider the flat bundle L_ρ defined as $E\Gamma \times_\rho C(X)^r \to B\Gamma$ whose typical fiber is $C(X)^r$. Let $Y \subset B\Gamma$ be a finite CW complex. We can view the restriction of L_ρ to Y, denoted $L_\rho|_Y$, as a rank r complex vector bundle over $Y \times X$.

Theorem 5.13 (Baird-Ramras, [4]). *Suppose that $H^k(X; \mathbb{Q}) = 0$ for $k > d$ Then for all $m > 0$, the Chern classes $c_{d+m}(L_\rho|_Y) \in H^{2d+2m}(Y \times X; \mathbb{Z})$ map to zero in $H^{2d+2m}(Y \times X; \mathbb{Q})$.*

We shall apply Theorem 5.13 for $X = \{\text{point}\}$ with $d = 0$ and for $X = \mathbb{T}$ with $d = 1$.

Lemma 5.14. *Let X be a topological space with $H^2(X, \mathbb{Q}) = 0$ and such that $H^{even}(X \times \mathbb{T}, \mathbb{Q})$ is generated as a ring by $H^0(X \times \mathbb{T}, \mathbb{Q}) \oplus H^2(X \times \mathbb{T}, \mathbb{Q})$. Then $H^k(X, \mathbb{Q}) = 0$ for all $k > 1$.*

Proof. Let $\pi_1 : X \times \mathbb{T} \to X$ and $\pi_2 : X \times \mathbb{T} \to \mathbb{T}$ be the canonical projections. By [47, Thm.61.6], the cross product $(x, y) \mapsto x \times y = \pi_1^*(x) \cup \pi_2^*(y)$ induces an isomorphism of algebras

$$\theta : H^*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(\mathbb{T}, \mathbb{Q}) \to H^*(X \times \mathbb{T}, \mathbb{Q}). \tag{16}$$

Since $H^2(X, \mathbb{Q}) = 0$, we must have $H^1(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(\mathbb{T}, \mathbb{Q}) \cong H^2(X \times \mathbb{T}, \mathbb{Q})$.

By [47, Thm.61.5], in the cohomology ring $H^*(X \times \mathbb{T}, \mathbb{Q})$, we have

$$(\alpha \times \beta) \cup (\alpha' \times \beta') = (-1)^{(deg\beta)(deg\alpha')}(\alpha \cup \alpha') \times (\beta \cup \beta')$$

for $\alpha, \alpha' \in H^*(X, \mathbb{Q})$ and $\beta, \beta' \in H^*(\mathbb{T}, \mathbb{Q})$. It follows that in our situation $\gamma \cup \gamma' = 0$ for all $\gamma, \gamma' \in H^2(X \times \mathbb{T}, \mathbb{Q})$ as $\beta \cup \beta' = 0$ for $\beta, \beta' \in H^1(\mathbb{T}, \mathbb{Q})$. Since $H^{even}(X \times \mathbb{T}, \mathbb{Q})$ is generated as a ring by $H^0(X \times \mathbb{T}, \mathbb{Q}) \oplus H^2(X \times \mathbb{T}, \mathbb{Q})$ we deduce that $H^{2k}(X \times \mathbb{T}, \mathbb{Q}) = 0$ for all $k \geq 2$. Since

$$H^{2k}(X, \mathbb{Q}) \oplus H^{2k-1}(X, \mathbb{Q}) \cong H^{2k}(X \times \mathbb{T}, \mathbb{Q})$$

by (16), it follows that $H^j(X, \mathbb{Q}) = \{0\}$ for all $j \geq 2$. □

The following is the main result of the second part of our paper. Its first part (i) strengthens (at least formally) the main result of [17], since in principle, local $\{GL_n(\mathbb{C}) : n \in \mathbb{N}\}$-stability is weaker than local $\{U_n(\mathbb{C}) : n \in \mathbb{N}\}$-stability.

Theorem 5.15. *Let Γ be a countable discrete group that admits a γ-element. Suppose that Γ is either MF and finitely generated or that Γ is quasidiagonal.*

(i) *If Γ is locally $\{GL_n(\mathbb{C}) : n \in \mathbb{N}\}$-stable, then $H^{2k}(\Gamma, \mathbb{Q}) = 0$ for all $k > 0$.*
(ii) *If Γ is locally $\{GL_n(C(\mathbb{T})) : n \in \mathbb{N}\}$-stable, then $H^k(\Gamma, \mathbb{Q}) = 0$ for all $k > 1$.*

Remark 5.16. Theorem 5.15 applies to finitely generated linear groups and to residually finite hyperbolic groups, among others. Indeed, linear groups are MF by [17] and exact by [29] and hence they admit a γ-element by [56]. In particular, since it was shown in [3] that if Γ is a cocompact lattice in a real semisimple Lie group G which is not locally isomorphic to either $SO(n,1)$ for n odd or $SL_3(\mathbb{R})$, then $b_{2i}(\Gamma)>0$ for some $i>0$, and since Γ is finitely generated by cocompactness, we deduce that Γ is not $\{GL_n(\mathbb{C}):n\in\mathbb{N}\}$-stable. Hyperbolic groups are exact by [8]. It is not known whether there exists a hyperbolic group which is not residually finite.

Proof. (for Thm. 5.15) By Lemma 5.10 and Remark 5.9 (iv), MF finitely generated groups that satisfy the assumptions from (i) or from (ii) are quasidiagonal. Thus we may assume that Γ is quasidiagonal for the remainder of the proof.

Let $B=\mathcal{Q}$ or $B=C(\mathbb{T})\otimes\mathcal{Q}$. By Theorem 5.7 we have a surjective map

$$\nu: KK(C^*(\Gamma),B)_{qd}\to RK^0(B\Gamma;B).$$

If $B\Gamma$ is written as the union of an increasing sequence $(Y_i)_i$ of finite CW complexes, then as explained in the proof of Lemma 3.4 from [38], there is a Milnor $\varprojlim^1$ exact sequence which gives

$$RK^0(B\Gamma;B)\cong\varprojlim RK^0(Y_i;B) \tag{17}$$

since $K_0(B)$ is divisible. We denote by ν_i the composition of the map ν defined above with the restriction map $RK^0(B\Gamma;B)\to RK^0(Y_i;B)$.

(i) For $B=\mathcal{Q}$, using the naturality of the Chern character, we have a commutative diagram

$$\begin{array}{ccccc} KK(C^*(\Gamma),\mathcal{Q})_{qd} & \xrightarrow{\nu} & RK^0(B\Gamma;\mathbb{Q}) & \xrightarrow{ch} & H^{even}(B\Gamma;\mathbb{Q}) \\ & \searrow^{(\nu_i)} & \downarrow & & \downarrow \\ & & \varprojlim RK^0(Y_i;\mathbb{Q}) & \xrightarrow{ch} & \varprojlim H^{even}(Y_i;\mathbb{Q}) \end{array}$$

with bijective vertical maps. Recall that we identify $RK^0(Y;\mathbb{Q})$ with $RK^0(Y;\mathcal{Q})$, [36]. Write $\mathcal{Q}$ as inductive limit of matrix algebras $B_k\cong M_{k!}(\mathbb{C})$. By Proposition 5.12, the image of each ν_i is contained in $\varinjlim_k RK^0(Y_i;B_k)_{\mathrm{flat}}$ and hence by Theorem 5.13 applied for $X=\{\text{point}\}$, the image of the map $ch\circ\nu_i: RK^0(Y_i;\mathbb{Q})\to H^{even}(Y_i;\mathbb{Q})$ is contained in $H^0(Y_i;\mathbb{Q})$. Since ν is surjective, it follows that $ch(RK^0(B\Gamma;\mathbb{Q}))=H^{even}(B\Gamma;\mathbb{Q})$ and hence $H^{2k}(\Gamma,\mathbb{Q})=0$ for all $k\geq 1$.

(ii) Now let $B=C(\mathbb{T})\otimes\mathcal{Q}$ and write B as the inductive limit of $B_k=C(\mathbb{T})\otimes M_{k!}(\mathbb{C})$. Just as above, we denote by ν_i the composition of the map ν with the restriction map $RK^0(B\Gamma;B)\to RK^0(Y_i;B)\cong RK^0(Y_i\times\mathbb{T};\mathbb{Q})$. By (10) and (17), we have

$$\begin{aligned} &RK^0(B\Gamma;B)=RK^0(B\Gamma;C(\mathbb{T})\otimes\mathcal{Q})\cong\varprojlim RK^0(Y_i;C(\mathbb{T})\otimes\mathcal{Q})\cong\\ &\varprojlim RK^0(Y_i\times\mathbb{T};\mathcal{Q})\cong\varprojlim RK^0(Y_i\times\mathbb{T};\mathbb{Q})\cong RK^0(B\Gamma\times\mathbb{T};\mathbb{Q}).\end{aligned}$$

Abusing the notation, we denote the map $KK(C^*(\Gamma),B)_{qd}\to RK^0(B\Gamma\times\mathbb{T};\mathbb{Q})$ obtained by composing ν with the isomorphism $RK^0(B\Gamma;B)\cong RK^0(B\Gamma\times\mathbb{T};\mathbb{Q})$ from above, again by ν.

We are going to argue that each map $\nu_i: KK(C^*(\Gamma),B)_{qd}\to RK^0(Y_i\times\mathbb{T};\mathbb{Q})$ has the property that the rational Chern classes satisfy $c_m(\nu_i(y))=0$ for all $y\in KK(C^*(\Gamma),B)_{qd}$ and $m\geq 2$. Indeed, by Proposition 5.12, the image of each ν_i is contained in $\varinjlim_k RK^0(Y_i;B_k)_{\mathrm{flat}}$ and hence by Theorem 5.13 applied for $X = \mathbb{T}$, we deduce the desired property. It follows that, for all $y \in KK(C^*(\Gamma), B)_{qd}$, the Chern character of $\nu_i(y)$ can be computed as

$$ch(\nu_i(y)) = exp(c_1(\nu_i(y))).$$

Using the commutative diagram with bijective vertical arrows

$$\begin{array}{ccccc} KK(C^*(\Gamma), C(\mathbb{T}) \otimes \mathcal{Q})_{qd} & \xrightarrow{\nu} & RK^0(B\Gamma \times \mathbb{T}; \mathbb{Q}) & \xrightarrow{ch} & H^{even}(B\Gamma \times \mathbb{T}; \mathbb{Q}) \\ & \searrow^{(\nu_i)} & \downarrow & & \downarrow \\ & & \varprojlim RK^0(Y_i \times \mathbb{T}; \mathbb{Q}) & \xrightarrow{ch} & \varprojlim H^{even}(Y_i \times \mathbb{T}; \mathbb{Q}) \end{array}$$

we deduce that $ch(\nu(y)) = exp(c_1(\nu(y)))$ for all $y \in KK(C^*(\Gamma), B)_{qd}$ and hence $ch(z) = exp(c_1(z))$ for all $z \in RK^0(B\Gamma \times \mathbb{T}; \mathbb{Q})$, by surjectivity of ν. Since the Chern character is a rational isomorphism and the image of the first Chern class has degree two, it follows that $H^{even}(B\Gamma \times \mathbb{T}, \mathbb{Q})$ is generated as a ring by $H^0(B\Gamma \times \mathbb{T}, \mathbb{Q}) \oplus H^2(B\Gamma \times \mathbb{T}, \mathbb{Q})$. The conclusion of the theorem follows now from Lemma 5.14 since $H^*(B\Gamma, \mathbb{Q}) \cong H^*(\Gamma, \mathbb{Q})$ and we already know that $H^2(B\Gamma, \mathbb{Q}) = 0$ by part (i) above and Remark 5.9 (iv). □

The first part of the theorem below was already proved in our earlier paper [17]. We include the statement for the sake of completeness.

Theorem 5.17. *Let Γ be an MF countable discrete group that admits a γ-element.*

(i) *If Γ is locally $\{U_n(\mathbb{C}) : n \in \mathbb{N}\}$-stable, then $H^{2k}(\Gamma, \mathbb{Q}) = 0$ for all $k > 0$.*
(ii) *If Γ is locally $\{U_n(C(\mathbb{T})) : n \in \mathbb{N}\}$-stable, then $H^k(\Gamma, \mathbb{Q}) = 0$ for all $k > 1$.*

Proof. By Lemma 5.10(i), Γ is quasidiagonal. Since local $U(\mathcal{B})$-stability implies local $GL(\mathcal{B})$-stability, Theorem 5.17 follows from Theorem 5.15. □

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