ON ASYMPTOTIC STABILITY OF CONNECTIVE GROUPS

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Abstract. In their study of almost group representations, Manuilov and Mishchenko introduced and investigated the notion of asymptotic stability of a finitely presented discrete group. In this paper we establish connections between connectivity of amenable groups and asymptotic stability and exhibit new classes of asymptotically stable groups. In particular, we show that if G is an amenable and connective discrete group whose classifying space BG is homotopic to a finite simplicial complex, then G is asymptotically stable.

1. Introduction

Manuilov and Mishchenko [12] investigated the problem of when an almost representation of a finitely presented group can be extended to an asymptotic representation with control on approximate multiplicativity. They called the groups with this property asymptotically stable and they proved that the finitely generated abelian groups and the fundamental groups of orientable closed surfaces are asymptotically stable.

In joint work with Pennig [6], [7], we studied a property of discrete amenable groups called connectivity. Connectivity is interesting since it allows for a more geometric realization of the K-homology of the group C*-algebras. The torsion free discrete nilpotent groups are connective by a result of [7]. Other classes of connective groups were exhibited in [6] and [8].

In this paper we relate connectivity to asymptotic stability, see Theorem 2.7. This enables us to exhibit new classes of asymptotically stable groups. In particular, the finitely generated torsion free discrete nilpotent groups and the Bieberbach groups with cyclic holonomy are asymptotically stable, see Corollary 2.9.

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2. Definitions and results

Let G be a finitely presented discrete group, with presentation

$$G = \langle S|\, R\rangle. \tag{1}$$

We may assume that the generating set $S = \{s_1, \dots, s_k\} \subset G$ is symmetric and contains the identity element. R consists of a set $\{r_1, \dots, r_\ell\}$ of relations $r_j = r_j(s_1, \dots, s_k)$. Following [12], we assume that R contains the relations of the form $s_i s_i^{-1}$ as well. Recall that the group $U(\infty)$ is defined as the union of the groups $U(m)$, $m \geq 1$, where one embeds $U(m) \hookrightarrow U(m+1)$, by $u \mapsto u \oplus 1$.

Definition 2.1. ([12]) A map $\sigma : S \to U(m) \subset U(\infty)$, $\sigma(s_i) = u_i$, $i = 1, ..., k$, is called an *ε-almost representation* of the group G if

$$\|r_j(u_1, \dots, u_k) - 1\| \leq \varepsilon$$

for all relations $r_j \in R$. The set of all such σ is denoted by $\mathcal{R}_\varepsilon(G; U(m))$.

It was explained in [12] that this definition depends only in nonessential ways on a choice of a presentation of the group G. The set of all ε-almost representations of the group G is denoted by

$$\mathcal{R}_\varepsilon(G) = \mathcal{R}_\varepsilon(G; U(\infty)) = \bigcup_{m\geq 1} \mathcal{R}_\varepsilon(G; U(m)),$$

where we regard $\mathcal{R}_\varepsilon(G; U(m))$ as a subset of $\mathcal{R}_\varepsilon(G; U(\infty))$ via the inclusion $U(m) \subset U(\infty)$.

Define the *curvature* of an almost representation σ by

$$\|\!|\sigma|\!\| = \max_{r_j \in R} \|r_j(u_1, \dots, u_k) - 1\|. \tag{2}$$

For a C*-algebra B, we denote by B^+ the C*-algebra obtained by adding a unit to B. We denote by $U(B)$ the subgroup of $U(B^+)$ consisting of unitaries of the form $u = 1 + x$ where $x \in B$ and 1 is the unit of B^+. Note that we can identify $U(B)$ with the subset of normal elements $x \in B$ such that $x + x^* + xx^* = 0$.

Let $\mathbb{K}$ denote the C*-algebra of compact operators on $\ell^2(\mathbb{N})$. There is a canonical embedding of $U(\infty)$ in $U(\mathbb{K})$, which maps $u \in U(m)$ to $1_{\mathbb{K}^+} + (u - 1_m) \in U(\mathbb{K})$. We endow $U(\infty)$ with the topology induced from $U(\mathbb{K})$ via the embedding $U(\infty) \subset U(\mathbb{K})$.

Definition 2.2. ([12]) A k-tuple of continuous paths of unitaries

$$\pi_t = \{u_1(t), \dots, u_k(t)\} \subset U(\infty), \quad t \in [0, \infty),$$

is called an asymptotic representation of the group G if the almost representations defined by $\pi_t(s_i) = u_i(t)$, $i = 1, \dots, k$ satisfy the condition

$$\lim_{t\to\infty} \|\!|\pi_t|\!\| = 0.$$

The set of asymptotic representations of G is denoted by

$$\mathcal{R}_{asym}(G) = \mathcal{R}_{asym}(G; U(\infty)).$$

Notation 2.3. Throughout the paper we denote by $\mathcal{F}$ the set of all functions $f : [0,\infty) \to [0,\infty)$ such that $\lim_{\varepsilon\to 0} f(\varepsilon) = f(0) = 0$. Note that if $\alpha, \beta \in \mathcal{F}$ and $\lambda \in [0,\infty)$, then $\lambda\alpha, \alpha+\beta, \alpha\beta, \alpha\circ\beta \in \mathcal{F}$.

The following property was introduced by Manuilov and Mishchenko.

Definition 2.4. ([12]) Let G be a finitely presented discrete group as in (1). We say that G is asymptotically stable if there exists $\delta \in \mathcal{F}$ such that for any $\varepsilon > 0$ and any ε-almost representation $\sigma \in \mathcal{R}_\varepsilon(G)$ there exists an asymptotic representation $\{\pi_t\}_{t\in[0,\infty)} \in \mathcal{R}_{asym}(G)$ such that $\pi_0 = \sigma$ and $\|\!|\pi_t|\!\| \le \delta(\varepsilon)$ for all $t \in [0,\infty)$.

Lemma 2.5. *One obtains an equivalent definition of asymptotic stability for G if one allows the continuous family $\{\pi_t(s)\}_{t\in[0,\infty)}$, $s \in S$, in Definition 2.4 to consist of elements of the larger group $U(\mathbb{K})$ instead of just $U(\infty)$. In this case we write $\{\pi_t\}_{t\in[0,\infty)} \in \mathcal{R}_{asym}(G; U(\mathbb{K}))$.*

Proof. (sketch) Since $U(\infty)$ is dense in $U(\mathbb{K})$, by using functional calculus and polar decomposition one can approximate continuous paths in $U(\mathbb{K})$ by continuous paths in $U(\infty)$. More precisely, if $\alpha : [0,\infty) \to (0,1)$ is a continuous function such that $\lim_{t\to\infty} \alpha(t) = 0$, then for any continuous path $t \mapsto u(t) \in U(\mathbb{K})$, $t \in [0,\infty)$, with $u(0) \in U(\infty)$, there is a continuous map $t \mapsto v(t) \in U(\infty)$, $t \in [0,\infty)$, such that $v(0) = u(0)$ and $\|u(t) - v(t)\| \le \alpha(t)$ for all $t \ge 0$. Once this is verified, one completes the proof using the fact that R involves finitely many relations with finitely many generators.

Let us elaborate on the first part of the proof. This can be accomplished by applying inductively the following approximation procedure. Suppose that $u(t) = 1 + x(t)$, $t \in [n, n+1]$ is a continuous path in $U(\mathbb{K})$ such that $u(n) \in U(\infty)$, so that $x(n) \in M_\infty(\mathbb{C})$. Then we claim that there exists a continuous path $v(t)$, $t \in [n, n+1]$ in $U(\infty)$ such that

$v(n) = u(n)$ and $\|u(t) - v(t)\| \leq \alpha(t)$ for all $t \in [n, n+1]$. To verify the claim, we start with the observation that if w, u are elements of a unital C*-algebra such that u is a unitary and $\|u - w\| < 1/4$, then w is invertible and $\|u - w|w|^{-1}\| \leq 4\|u - w\|$. Then we choose an orthogonal projection $e \in M_\infty(\mathbb{C})$ such that $ex(n) = x(n)e = x(n)$ and $\sup_{t\in[n,n+1]} \|ex(t)e - x(t)\| \leq \frac{1}{5}\inf\{\alpha(r) : r \in [n, n+1]\}$. If we set $w(t) = 1 + ex(t)e$, and $v(t) = w(t)|w(t)|^{-1}$, then we have $v(n) = w(n) = u(n)$ and

$$\|u(t) - v(t)\| = \|u(t) - w(t)|w(t)|^{-1}\| \leq 4\|u(t) - w(t)\| \leq \alpha(t),$$

for all $t \in [n, n+1]$. □

Lemma 2.6. *([12]) The asymptotic stability property of a group does not depend on the choice of a presentation.*

The notion of connective discrete group was introduced in [6]. We will review this concept shortly, see Definition 2.10 below. All torsion free discrete nilpotent groups are connective by a result of [7]. More classes of connective groups were exhibited in [6] and [8].

Our main result is the following.

Theorem 2.7. *Let G be a finitely presented discrete group. Suppose that G is amenable, connective and that $K_*(C^*(G))$ is finitely generated. Then G is asymptotically stable.*

We prove this theorem in Section 5.

Corollary 2.8. *If G is an amenable and connective discrete group whose classifying space BG is homotopic to a finite simplicial complex, then G is asymptotically stable.*

Proof. If BG is a finite simplicial complex, then G is finitely presented since $G \cong \pi_1(BG)$ and the K-homology $K_*(BG)$ is finitely generated. It follows that $K_*(C^*(G))$ is finitely generated since by a result of Higson and Kasparov on the Baum-Connes conjecture [10] $K_*(C^*(G)) \cong K_*(BG)$ if G is amenable. We conclude that G is asymptotically stable by Theorem 2.7. □

From this result we obtain the following class of examples.

Corollary 2.9. *All finitely generated torsion free discrete nilpotent groups are asymptotically stable. All Bieberbach groups with cyclic holonomy are asymptotically stable.*

The idea of our approach is to consider a variant of the asymptotic stability property for C*-algebras, see Definition 5.4 and to relate this property to connectivity of C*-algebras.

Denote by $CB = C_0[0,1) \otimes B$ the cone over a C*-algebra B. Let H be a separable infinite-dimensional Hilbert space.

Definition 2.10. ([6]) A separable C*-algebra A is *connective* if there is a $*$-monomorphism

$$A \hookrightarrow \frac{\prod_n CL(H)}{\bigoplus_n CL(H)}$$

which is liftable to a completely positive contractive (cpc) map

$$\varphi : A \to \prod_n CL(H).$$

Let G be a countable discrete group and let $I(G)$ be the *augmentation* ideal defined as the kernel of the trivial representation $\iota : C^*(G) \to \mathbb{C}$ of the C*-algebra of G. The group G is *connective* if the ideal $I(G)$ is a connective C*-algebra.

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3. K-theory with coefficients

We begin by recalling the definition of K-theory with mod-m coefficients [14]. Let $\mathbb{T}$ denote the unit circle and let W_m denote the Moore space obtained by attaching the disk to the circle by a degree $m \geq 1$ map. The space W_m constructed in this way is unique up to homotopy equivalence. If X is a compact connected space we let $C_0(X)$ denote the subalgebra of $C(X)$ consisting of functions vanishing at a fixed base point. It is well known that $K_0(C_0(W_m)) = \mathbb{Z}/m$ and $K_1(C_0(W_m)) = 0$. One defines

$$K_0(A;\mathbb{Z}/m) = K_0(A \otimes C_0(W_m)), \quad K_1(A;\mathbb{Z}/m) = K_1(A \otimes C_0(W_m)).$$

It is convenient to work with the following natural realization of K-theory:

$$K_*(A) \cong K_*(A;\mathbb{Z}) \cong K_0(A \otimes C(\mathbb{T})), \text{ and}$$
$$K_*(A;\mathbb{Z}/m) \cong K_0(A \otimes C(\mathbb{T}) \otimes C_0(\mathbb{W}_m)), \; m \geq 1$$

The entire K-theory group is defined by

$$\underline{K}(A) = \bigoplus_{m=0}^{\infty} K_*(A; \mathbb{Z}/m).$$

The group $\underline{K}(A)$ is acted on by the ring Λ of Bockstein operations, [14].

Consider the category with objects separable C*-algebras and set of morphisms from A to B given by the Kasparov group $KK(A, B)$. Two C*-algebras that are isomorphic in this category are called KK-equivalent. It was shown by Rosenberg and Schochet [13] that the separable C*-algebras A that are KK-equivalent to abelian C*-algebras are exactly those satisfying the following universal coefficient exact sequence

$$0 \to \mathrm{Ext}(K_*(A), K_{*-1}(B)) \xrightarrow{\delta} KK(A, B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

for any separable C*-algebra B. If A has this property we say that A satisfies the UCT.

Theorem 3.1 ([4])**.** *Let A be a separable C*-algebra. Then A satisfies the UCT if and only if for any separable C*-algebra B there is a short exact sequence*

$$0 \to \mathrm{Pext}(K_*(A), K_{*-1}(B)) \xrightarrow{\delta'} KK(A, B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \to 0$$

which is natural in each variable.

Here $Pext$ denotes the subgroup of $\mathrm{Ext} = \mathrm{Ext}^1_{\mathbb{Z}}$ consisting of classes of pure extensions. The map δ' is the restriction of the map

$$\delta : \mathrm{Ext}(K_*(A), K_{*-1}(B))) \to KK(A, B).$$

$\mathrm{Pext}(K_*(A), K_{*-1}(B)) = 0$ if the group $K_*(A)$ is finitely generated.

If A is a separable C*-algebra and B is a general C*-algebra one defines

$$KK_{sep}(A, B) = \varinjlim_{B' \subset B} KK(A, B'),$$

where B' runs through the separable C*-subalgebras of B. If the group $K_*(A)$ is finitely generated, then it was shown in [4] and [5] that $\underline{K}(A)$ is a finitely generated Λ-module. Since

$$\underline{K}(B) = \varinjlim_{B' \subset B} \underline{K}(B'),$$

it follows from Theorem 3.1 that

$$KK_{sep}(A,B) \cong \mathrm{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(B)), \tag{3}$$

for all C^*-algebras A in the UCT class such that $K_*(A)$ is finitely generated.

The set of all projections in the union of the algebras $(A \otimes C(\mathbb{T}) \otimes C_0(\mathbb{W}_m))^+ \otimes \mathbb{K}$, $m \geq 0$, is denoted by $\underline{\mathcal{P}roj}(A)$.

4. K-theory classes associated to almost morphisms

Definition 4.1. Let A, $(B_n)_{n=1}^{\infty}$ be C*-algebras. A cpc discrete asymptotic morphism from A to $(B_n)_n$ is a sequence of completely positive linear contractions $\{\varphi_n : A \to B_n\}_n$ such that

$$\lim_{n\to\infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0$$

for all $a, b \in A$. If in addition each φ_n is unital, we say that $(\varphi_n)_n$ is a unital completely positive (ucp) discrete asymptotic morphism.

Any cpc discrete asymptotic morphism $\{\varphi_n : A \to B_n\}_n$ induces a $*$-homomorphism

$$\mathbf{\Phi} : A \to \prod_n B_n / \sum_n B_n \tag{4}$$

and hence a morphism of K-theory groups

$$\mathbf{\Phi}_\sharp : \underline{K}(A) \to \prod_n \underline{K}(B_n) / \sum_n \underline{K}(B_n)$$

obtained as the composition of the canonical maps

$$\underline{K}(A) \xrightarrow{\mathbf{\Phi}_*} \underline{K}(\prod_n B_n / \sum_n B_n) \xrightarrow{j} \prod_n \underline{K}(B_n) / \sum_n \underline{K}(B_n).$$

Two sequences (x_n) and (y_n) are called congruent, written $x_n \equiv y_n$, if they are tail equivalent, i.e. there is m such that $x_n = y_n$ for all $n \geq m$. For each $x \in \underline{K}(A)$ we denote by $(\varphi_{n\sharp}(x))_n$ an arbitrary fixed lifting of $\mathbf{\Phi}_\sharp(x)$ to $\prod_n \underline{K}(B_n)$. It is unique up to congruence. Homotopic discrete asymptotic morphisms $\{\varphi_n\}_n, \{\psi_n\}_n$ yield congruent sequences $\varphi_{n\sharp}(x) \equiv \psi_{n\sharp}(x)$. If each φ_n is a $*$-homomorphism, then $\varphi_{n\sharp}(x) \equiv \varphi_{n*}(x)$ for all $x \in \underline{K}(A)$.

Definition 4.2. If $\varphi : A \to B$ is a completely positive contraction and p is a projection in $A \otimes \mathbb{K}$ we define $\varphi_\sharp(p) \in K_0(B)$ by

$$\varphi_\sharp(p) = \begin{cases} [\chi(x)] & \text{if} \quad \|x^2 - x\| < 1/4, \\ 0 & \text{otherwise,} \end{cases}$$

where χ is the characteristic function of the interval $(1/2, 1]$ and $x = (\varphi \otimes id_{\mathbb{K}})(p) \in B \otimes \mathbb{K}$. In a similar manner we define $\varphi_\sharp(p) \in \underline{K}(B)$ for projections $p \in \underline{\mathcal{P}roj}(A)$.

One verifies immediately that for a cpc discrete asymptotic morphism $\{\varphi_n : A \to B_n\}_n$, the sequence $\left(\varphi_{n\,\sharp}([p])\right)_n$ is congruent to the sequence $\left((\varphi_n)_\sharp(p)\right)_n$.

If we set $B_n = \mathbb{K}$ for all $n \geq 1$, it follows by [9, Lemma 2.9] that the natural maps

$$\underline{K}\left(\prod_n B_n\right) \longrightarrow \prod_n \underline{K}(B_n)$$
$$\underline{K}\left(\frac{\prod_n B_n}{\bigoplus_n B_n}\right) \longrightarrow \prod_n \underline{K}(B_n) / \sum_n \underline{K}(B_n)$$

are isomorphisms. Set $B_\infty = \frac{\prod_n \mathbb{K}}{\bigoplus_n \mathbb{K}}$. If $K_*(A)$ is finitely generated, then $\mathrm{Pext}(K_*(A), K_*(B)) = 0$. If in addition A satisfies the UCT, then it follows from (3) that

$$KK_{sep}(A, B_\infty) \cong \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B_\infty)) \hookrightarrow \mathrm{Hom}(\underline{K}(A), \underline{K}(B_\infty)).$$

Moreover, since $K_*(A)$ is finitely generated, there is a finite set of projections $P \subset \underline{\mathcal{P}roj}(A)$ such that the set $\{[p] - [s(p)] : p \in P\}$, where $s(p)$ denotes the scalar component of p, generates $\underline{K}(A)$ as a Λ-module, see [4], [5].

Consequently, we obtain the following:

Proposition 4.3. *Let A be a separable C^*-algebra that satisfies the UCT and such that $K_*(A)$ is finitely generated. Let $P \subset \underline{\mathcal{P}roj}(A)$ be a finite set of projections such that the set $\{[p] : p \in P\}$ generates $\underline{K}(A)$ as a Λ-module. If $\{\varphi_n, \psi_n : A \to \mathbb{K}\}_n$ are two discrete cpc asymptotic morphisms such that $(\varphi_n)_\sharp(p) - (\varphi_n)_\sharp(s(p)) \equiv (\psi_n)_\sharp(p) - (\psi_n)_\sharp(s(p))$ for all $p \in P$, then the induced $*$-homomorphisms $\Phi, \Psi : A \to B_\infty$ satisfy $[\Phi] = [\Psi] \in KK_{sep}(A, B_\infty)$.*

5. Almost representations of C*-algebras

Let A, B be C*-algebras and let $F \subset A$ be a finite set. A cpc map $\theta : A \to B$ is called (F, ε)-multiplicative if $\|\theta(a)\theta(b) - \theta(ab)\| \leq \varepsilon$ for all $a, b \in F$.

The full C*-algebra of a discrete group G is denoted by $C^*(G)$. We identify G with canonical unitaries in $C^*(G)$. The following lemma provides approximations of almost representations of G by almost multiplicative ucp maps on $C^*(G)$.

Lemma 5.1. *Let $G = \langle S | R \rangle$ be a finitely presented discrete group. If G is amenable, then, for any finite set $F \subset G$, there exists a function $\alpha \in \mathcal{F}$ such that for any $\varepsilon > 0$ and any $\sigma \in \mathcal{R}_\varepsilon(G; U(\mathbb{K}))$, there is an $(F, \alpha(\varepsilon))$-multiplicative ucp map $\varphi : C^*(G) \to \mathbb{K}^+$ with $\varphi(I(G)) \subset \mathbb{K}$ and such that $\|\varphi(s) - \sigma(s)\| \leq \alpha(\varepsilon)$ for all $s \in S$.*

Proof. Let $ucp(G)$ denote the set of all ucp maps $\varphi : C^*(G) \to \mathbb{K}^+$ with $\varphi(I(G)) \subset \mathbb{K}$. For $\sigma : S \to U(\mathbb{K})$ and $\varphi \in ucp(G)$, let

$$d(\sigma, \varphi) = \max_{s \in S} \|\sigma(s) - \varphi(s)\| + \max_{a,b \in F} \|\varphi(ab) - \varphi(a)\varphi(b)\|$$

Define α for $\varepsilon > 0$ by

$$\alpha(\varepsilon) = \sup_{\sigma \in \mathcal{R}_\varepsilon(G, U(\mathbb{K}))} \inf\{d(\sigma, \varphi) \,:\, \varphi \in ucp(G)\}$$

It suffices to prove that $lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0$, since it is obvious that $\bar{\alpha}(\varepsilon) := \alpha(\varepsilon) + \varepsilon$ will satisfy the conclusion of the lemma. Seeking a contradiction, suppose that $\limsup_{\varepsilon \to 0} \alpha(\varepsilon) > 0$. Therefore there exist a decreasing sequence $\varepsilon_n \searrow 0$ and $\alpha_0 > 0$ such that $\alpha(\varepsilon_n) > \alpha_0$ for all $n \geq 1$. It follows that for each n there is $\sigma_n \in \mathcal{R}_{\varepsilon_n}(G, U(\mathbb{K}))$ such that

$$\inf\{d(\sigma_n, \varphi) \,:\, \varphi \in ucp(G)\} > \alpha_0. \tag{5}$$

Write $\sigma_n(s) = 1 + x_n(s)$, $s \in S$, with $x_n(s) \in \mathbb{K}$.

The sequence $(\sigma_n)_n$ induces a homomorphism of groups Ψ from G to the unitary group of the C*-algebra

$$B = \left(\prod_{n=1}^{\infty} \mathbb{K}\right)^+ / \bigoplus_{n=1}^{\infty} \mathbb{K},$$

in such a way that

$$(\sigma_n(s))_n = (1 + x_n(s))_n = 1 + (x_n(s))_n \in \left(\prod_{n=1}^{\infty} \mathbb{K}\right)^+$$

is a lifting of $\Psi(s)$, for each $s \in S$. By the universal property of group C*-algebras, Ψ extends to a unital $*$-homomorphism $\Psi : C^*(G) \to B$. Since G is amenable, $C^*(G)$ is nuclear by a result of Lance [11]. By the Choi-Effros theorem [3], the restriction of Ψ to $I(G)$ lifts to a cpc map $I(G) \to \prod_{n=1}^{\infty} \mathbb{K}$ whose unitalization gives a ucp map

$$\varphi : C^*(G) \to \left(\prod_{n=1}^{\infty} \mathbb{K}\right)^+ \subset \prod_{n=1}^{\infty} \mathbb{K}^+,$$

whose components are denoted by φ_n. Recall that if $(b_n)_n \in (\prod_{n=1}^{\infty} \mathbb{K})^+$ is a lifting of an element $b \in B$, then

$$\|b\| = \limsup_n \|b_n\|. \tag{6}$$

Since both sequences $(\sigma_n(s))_n$ and $(\varphi_n(s))_n$ are liftings of $\Psi(s)$, it follows immediately from (6) that

$$\lim_{n\to\infty} \|\sigma_n(s) - \varphi_n(s)\| = 0$$

for all $s \in S$. Arguing similarly, since Ψ is a unital $*$-homomorphism, we see that

$$\lim_n \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = \|\Psi(ab) - \Psi(a)\Psi(b)\| = 0,$$

for all $a, b \in C^*(G)$.

It follows that $\lim_{n\to\infty} d(\sigma_n, \varphi_n) = 0$, which contradicts (5). $\square$

The following lemma constructs almost representations of G from almost multiplicative ucp maps on $C^*(G)$.

Lemma 5.2. *Let $G = \langle S|\, R\rangle$ be a finitely presented discrete group. There is a function $\eta \in \mathcal{F}$ such that for any $\varepsilon > 0$ and any (S, ε)-multiplicative ucp map $\varphi : C^*(G) \to \mathbb{K}^+$, with $\varphi(I(G)) \subset \mathbb{K}$, there is map $\pi : S \to U(\mathbb{K})$ such that*

(i) $\pi \in \mathcal{R}_{\eta(\varepsilon)}(G, U(\mathbb{K}))$,
(ii) $\|\pi(s) - \varphi(s)\| \leq \eta(\varepsilon)$ for all $s \in S$.
(iii) If $\varepsilon \in (0, 1)$, π is defined by $\pi(s) = \varphi(s)|\varphi(s)|^{-1}$, $s \in S$.

Proof. By assumption, $\|1 - \varphi(s^{-1})\varphi(s)\| = \|\varphi(s^{-1}s) - \varphi(s^{-1})\varphi(s)\| < \varepsilon$ for all $s \in S = S^{-1}$. It follows that if $\varepsilon \in (0, 1)$, then $\varphi(s)$ is invertible for all (S,ε)-multiplicative ucp maps $\varphi : C^*(G) \to \mathbb{K}^+$. For $\varepsilon \geq 1$ we let π be the trivial representation, $\pi(s) = 1_{\mathbb{K}^+}$, $s \in S$ and $\eta(\varepsilon) = 2$. For $\varepsilon \in (0,1)$, let $ucp(S,\varepsilon)$ be the set of ucp maps $\varphi : C^*(G) \to \mathbb{K}^+$ with $\varphi(I(G)) \subset \mathbb{K}$, which are (S,ε)-multiplicative, and define

$$\eta(\varepsilon) = \sup\{|\!|\!|\pi|\!|\!| + \max_{s\in S} \|\varphi(s) - \pi(s)\| : \varphi \in ucp(S,\varepsilon)\},$$

where $\pi(s) = \varphi(s)|\varphi(s)|^{-1}$. Note that $\pi(s) \in U(\mathbb{K})$ since $\varphi(s) = 1 + \varphi(s-1) \in 1 + \mathbb{K}$. By its very definition, η satisfies the conditions (i)–(iii) from the statement. It remains to show that $\eta \in \mathcal{F}$. Seeking a contradiction, suppose that $\limsup_{\varepsilon\to 0} \eta(\varepsilon) > 0$. Therefore there exist a sequence $\varepsilon_n \searrow 0$, with $\varepsilon_n \in (0,1)$, and $\eta_0 > 0$ such that $\eta(\varepsilon_n) > \eta_0$ for all $n \geq 1$. It follows that there is a sequence $\{\varphi_n : C^*(G) \to \mathbb{K}^+\}_n$, $\varphi_n(I(G)) \subset \mathbb{K}$, of (S,ε_n)-multiplicative ucp maps and such that if we set $\pi_n(s) = \varphi_n(s)|\varphi_n(s)|^{-1} \in U(\mathbb{K})$, $s \in S$, then

$$|\!|\!|\pi_n|\!|\!| + \max_{s\in S} \|\varphi_n(s) - \pi_n(s)\| > \eta_0, \text{ for all } n \in \mathbb{N}, \tag{7}$$

The sequence $(\varphi_n)_n$ can be viewed as the components of a ucp map $C^*(G) \to \prod_n \mathbb{K}^+$. Moreover, this induces a ucp map

$$\Phi : C^*(G) \to B_\infty := \frac{\prod_n \mathbb{K}^+}{\bigoplus_n \mathbb{K}^+}$$

such that $\Phi(a^*a) = \Phi(a)^*\Phi(a)$ and for all $a \in S = S^*$. Since S generates $C^*(G)$, it follows by Proposition 1.5.7 from [1] (on multiplicative domains) that Φ is a $*$-homomorphism. It follows then that $\{\varphi_n : C^*(G) \to \mathbb{K}^+\}_n$ is a ucp discrete asymptotic morphism.

We have

$$\lim_{n\to\infty} \|\varphi_n(s_i^{-1})\varphi_n(s_i) - 1\| = \lim_{n\to\infty} \|\varphi_n(s_i)^*\varphi_n(s_i) - 1\| = 0,$$

so that $\varphi_n(s_i)$ becomes close to unitary elements for sufficiently large n, for all $s_i \in S$. Therefore

$$\lim_{n\to\infty} (\varphi_n(s_i) - \pi_n(s_i)) = \lim_{n\to\infty} \left(\varphi_n(s_i) - \varphi_n(s_i)|\varphi_n(s_i)|^{-1}\right) = 0.$$

Moreover, we observe that since $\{\varphi_n : C^*(G) \to \mathbb{K}^+\}_n$ is a ucp discrete asymptotic morphism, we must have for all relations $r_j \in R$:

$$\lim_{n\to\infty} \|r_j(\varphi_n(s_1), ..., \varphi_n(s_k)) - 1\| = \lim_{n\to\infty} \|\varphi_n(r_j(s_1, ..., s_k) - 1)\| = 0,$$

from which we deduce that

$$\lim_{n\to\infty} \|r_j(\pi_n(s_1), ..., \pi_n(s_k)) - 1\| = 0.$$

since
$$\lim_{n\to\infty} (\varphi_n(s_i) - \pi_n(s_i)) = 0, \ i = 1, .., k.$$
Combining the two lines above, we obtain a contradiction with (7) and conclude the proof. □

Lemma 5.3. *Let $G = \langle S | R\rangle$ be a finitely presented discrete group. Suppose that there exist $\delta \in \mathcal{F}$ and $\varepsilon_0 > 0$ such that for every almost representation $\sigma \in \mathcal{R}_\varepsilon(G; U(\infty))$ with $\varepsilon \in (0, \varepsilon_0)$, there exists an asymptotic representation $\{\pi_t\}_{t\in[0,\infty)} \in \mathcal{R}_{asym}(G; U(\mathbb{K}))$ such that $\|\pi_0(s) - \sigma(s)\| \leq \delta(\varepsilon)$ for all $s \in S$ and $\|\pi_t\| \leq \delta(\varepsilon)$ for all $t \in [0, \infty)$. Then G is asymptotically stable.*

Proof. This is a routine argument based on functional calculus and polar decomposition. As observed in the proof of Lemma 2.5, if x, u are contractions in a unital C*-algebra such that u is a unitary and $\|u - x\| < 1/4$, then x is invertible and $\|u - x|x|^{-1}\| \leq 4\|u - x\|$. Let δ, σ and $\{\pi_t\}_{t\in[0,\infty)}$ be as in the statement. After passing to a smaller ε_0 if necessary, we may arrange that $\delta(\varepsilon) < 1/4$ for all $\varepsilon \in (0, \varepsilon_0)$. We attach the path of unitaries $\pi'_r(s) = \rho_r(s)|\rho_r(s)|^{-1}$ arising from the polar decomposition of the path $r \mapsto \rho_r(s) := (1 - r)\sigma(s) + r\pi_0(s)$, $r \in [0, 1]$, to the path $\{\pi_{t-1}(s)\}_{t\in[1,\infty)}$, $s \in S$. It follows from the observation above that $\|\pi'_r(s) - \sigma(s)\| \leq 4\delta(\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_0)$ since $\delta(\varepsilon) < 1/4$. Since R involves finitely many relations and finitely many generators and since $\delta \in \mathcal{F}$, we see that $\|\pi'_r\| \leq \delta'(\varepsilon)$ for some $\delta' \in \mathcal{F}$ that depends only on δ, G, S and R and hence only on G, S and R.
□

We work with the following version of asymptotic stability for finitely generated C*-algebras.

Definition 5.4. Let A be a C*-algebra generated by a finite selfadjoint set S. We say that A is *asymptotically stable* if for every finite set $F \subset A$, there exists $\gamma \in \mathcal{F}$ with the property that for every $\varepsilon > 0$ and (S, ε)-multiplicative cpc map $\varphi : A \to \mathbb{K}$ there exists a cpc asymptotic morphism $\{\Phi_t : A \to \mathbb{K}\}_{t\in[0,\infty)}$ such that $\Phi_0 = \varphi$ and Φ_t is $(F, \gamma(\varepsilon))$-multiplicative for all $t \in [0, \infty)$.

It follows that if a finite set $F^+ \subset A^+$ was given, one can also arrange that $\Phi_t^+ : A^+ \to \mathbb{K}^+$ is $(F, \gamma(\varepsilon))$-multiplicative for all $t \in [0, \infty)$.

The main result of this section is Theorem 5.6 which relates connectivity of C*-algebras to asymptotic stability.

Let A be a separable nuclear connective C*-algebra. Suppose that A is generated as a C*-algebra by a selfadjoint finite set S. Suppose that its K-theory groups $K_i(A)$, $i = 0, 1$, are finitely generated and that A satisfies the UCT. Let $P \subset \underline{\mathcal{P}roj}(A)$ be a finite set of projections such that the set $\{[p] - [s(p)] : p \in P\}$, where $s(p)$ denotes the scalar component of p, generates $\underline{K}(A)$ as a Λ-module, as in Proposition 4.3.

In the next proposition we regard $\mathbb{K}$ as the subalgebra $\mathbb{K}\otimes e$ of $\mathbb{K}\otimes\mathbb{K}$, where $e \in \mathbb{K}$ is a fixed one-dimensional projection.

Proposition 5.5. *Let A, S and P be as in the paragraph above. Fix a finite set $F \subset A$. Then there is $\gamma \in \mathcal{F}$ such that for any $\varepsilon > 0$ and any two (S, ε)-multiplicative cpc maps $\varphi, \psi : A \to \mathbb{K}$ such that $\varphi_\sharp(p) - \varphi_\sharp(s(p)) = \psi_\sharp(p) - \psi_\sharp(s(p))$ for all $p \in P$, there is an $(F, \gamma(\varepsilon))$-multiplicative cpc map $h : A \to C[0, 1] \otimes \mathbb{K}$, $h = (h^{(r)})_{r\in[0,1]}$, such that $h^{(0)} = \varphi$ and $h^{(1)} = \psi$.*

Proof. For a cpc map $\theta : A \to B$, and $F \subset A$, we shall use the notation

$$|\theta|_F = \max_{a,b\in F} \|\theta(ab) - \theta(a)\theta(b)\|. \tag{8}$$

For $\varepsilon > 0$, define

$$\gamma(\varepsilon) = \sup_{(\varphi,\psi)} \inf_{h\in L(\varphi,\psi)} |h|_F \tag{9}$$

where (φ, ψ) runs in the set of pairs of (S, ε)-multiplicative cpc maps $A \to \mathbb{K}$ such that $\varphi_\sharp(p) - \varphi_\sharp(s(p)) = \psi_\sharp(p) - \psi_\sharp(s(p))$ for all $p \in P$, and $h = (h^{(r)})_{r\in[0,1]}$ runs in the subset $L(\varphi, \psi)$ of cpc maps $A \to C[0, 1] \otimes \mathbb{K} \otimes \mathbb{K}$ with the property that $h^{(0)} = \varphi$ and $h^{(1)} = \psi$.

We shall prove that $\lim_{\varepsilon\to0} \gamma(\varepsilon) = 0$. Seeking a contradiction, suppose that there exist a decreasing sequence $\varepsilon_n \to 0$ and $\gamma_0 > 0$ such that $\gamma(\varepsilon_n) > \gamma_0$ for all $n \geq 1$. It follows that there exists a sequence $(\varphi_n, \psi_n)_n$ where $\varphi_n, \psi_n : A \to \mathbb{K}$ are (S, ε_n)-multiplicative cpc maps such that $(\varphi_n)_\sharp(p) - (\varphi_n)_\sharp(s(p)) = (\psi_n)_\sharp(p) - (\psi_n)_\sharp(s(p))$ for all $p \in P$ and such that for any sequence $(h_n)_n$ with $h_n \in L(\varphi_n, \psi_n)$, one must have $|h_n|_F > \gamma_0$ for all $n \geq 1$. Equivalently: $\sup_{r\in[0,1]} \quad |h_n^{(r)}|_F > \gamma_0$. The sequences $(\psi_n)_n$ and $(\varphi_n)_n$ can be viewed as the components of cpc maps $A \to \prod_n \mathbb{K}$. Moreover, they induce cpc maps

$$\Phi, \Psi : A \to B_\infty := \frac{\prod_n \mathbb{K}}{\bigoplus_n \mathbb{K}}$$

such that $\Phi(a^*a) = \Phi(a)^*\Phi(a)$ and $\Psi(a^*a) = \Psi(a)^*\Psi(a)$ for all $a \in S = S^*$. Since S generates A, it follows by Proposition 1.5.7 from [1] (on multiplicative domains) that both Φ and Ψ are $*$-homomorphisms. Since $(\varphi_n)_\sharp(p) - (\varphi_n)_\sharp(s(p)) = (\psi_n)_\sharp(p) - (\psi_n)_\sharp(s(p))$, it follows by Proposition 4.3 that Φ and Ψ have the same KK-class in $KK(A, B')$ for some separable C*-subalgebra B' of B_∞ that contains $\Phi(A) \cup \Psi(A)$.

Since A is nuclear and connective, it follows by [7] that there is a cpc asymptotic morphism $\{H_t : A \to C[0,1] \otimes B_\infty \otimes \mathbb{K}\}_{t\in[0,\infty)}$, $H_t = (H_t^{(r)})_{r\in[0,1]}$ such that $H_t^{(0)} = \Phi$ and $H_t^{(1)} = \Psi$ for all $t \in [0,\infty)$. We identify B_∞ with the $B_\infty \otimes e$ corner of $B_\infty \otimes \mathbb{K}$. Fix a sufficiently large t_0 such that $|H_{t_0}|_F < \gamma_0/3$. Since A is nuclear, by the Choi-Effros theorem, there is a cpc lifting

$$h : A \to C[0,1] \otimes \left(\prod_n \mathbb{K}\right) \otimes \mathbb{K}$$

of H_{t_0}, whose components are cpc maps $h_n : A \to C[0,1] \otimes \mathbb{K} \otimes \mathbb{K}$. Since h is a lifting of H_{t_0}, we have that $\limsup_n \|h_n(a)\| = \|H_{t_0}(a)\|$ for all $a \in A$. Since $|H_{t_0}|_F < \gamma_0/3$, it follows that there is n_0 such that $|h_n|_F < \gamma_0/2$ for all $n \geq n_0$. Since H_{t_0} is a homotopy from Φ to Ψ, it follows by a similar argument that

$$\lim_n \|h_n^{(0)}(a) - \varphi_n(a)\| = \lim_n \|h_n^{(1)}(a) - \psi_n(a)\| = 0,$$

for all $a \in A$. Consider the cpc maps $\alpha_n, \beta_n : A \to C[0,1] \otimes \mathbb{K} \otimes \mathbb{K}$ defined by $\alpha_n^{(r)} = (1-r)\varphi_n + r h_n^{(0)}$ and $\beta_n^{(r)} = (1-r)h_n^{(1)} + r\psi_n$. The concatenation of homotopies $\alpha_n^{(r)}$, $h_n^{(r)}$ and $\beta_n^{(r)}$ gives a cpc map $\bar{h}_n : A \to C[0,1] \otimes \mathbb{K} \otimes \mathbb{K}$. Since $|\varphi_n|_F, |\psi_n|_F \to 0$ and $|h_n|_F < \gamma_0/2$ for all $n \geq n_0$, we see that $|\alpha_n|_F, |\beta_n|_F < \gamma_0/2$ for all sufficiently large n. It follows that there is $n_1 > n_0$ such that $\bar{h}_n \in L(\varphi_n, \psi_n)$ and

$$|\bar{h}_n|_F = \max\{|\alpha_n|_F, |h_n|_F, |\beta_n|_F\} \leq \gamma_0/2$$

for all $n \geq n_1$. But this means that $\gamma(\varepsilon_n) \leq \gamma_0/2$ for all $n \geq n_1$ which is a contradiction, since we started with $\gamma(\varepsilon_n) > \gamma_0$ for all $n \geq 1$.

Let j be the embedding $j : \mathbb{K} \cong \mathbb{K} \otimes e \subset \mathbb{K} \otimes \mathbb{K}$. So far we have obtained a homotopy $h_t : A \to \mathbb{K} \otimes \mathbb{K}$ from $j \circ \varphi$ to $j \circ \psi$. Let $\omega : \mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$ be an isomorphism. Then $\omega \circ h_t : A \to \mathbb{K}$ is a homotopy from $\omega \circ j \circ \varphi$ to $\omega \circ j \circ \psi$. There is a continuous path $\{g_t\}_{t\in[0,1]}$ of $*$-endomorphisms of $\mathbb{K}$ from $\mathrm{id}_{\mathbb{K}}$ to $\omega \circ j$, since both these endomorphisms preserve the rank of projections. We obtained a homotopy as in the statement by concatenation of the homotopies $g_t \circ \varphi$, $\omega \circ h_t$ and $g_{1-t} \circ \psi$. □

Theorem 5.6. *Let A be a nuclear C^*-algebra. Suppose that A and its K-theory groups are finitely generated and that A satisfies the UCT. If A is connective, then A is asymptotically stable.*

Proof. Let S and P be as in Proposition 5.5. Let $F \subset A$ be a finite set. Let $\gamma \in \mathcal{F}$ be given by the conclusion of Proposition 5.5. Let $\varphi : A \to \mathbb{K}$ be an (S, ε)-multiplicative cpc map. Since $K_*(A)$ is finitely generated, there is $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, then the map $p \mapsto \varphi_\sharp(p)$, $p \in P$, induces an element of α of $KK(A, \mathbb{C})$, as explained in Section 4. Since A is separable nuclear and connective, $KK(A, \mathbb{C}) \cong [[A, \mathbb{K}]]$ by the main result of [6]. It follows that there is a cpc asymptotic morphism $\{\Psi_t : A \to \mathbb{K}\}_{t\in[0,\infty)}$ such that $|\Psi_t|_{S\cup F} \leq \varepsilon$ for all $t \in [0, \infty)$ (see (8) for this notation) and $[[\Psi_t]] = \alpha$. In particular $\varphi_\sharp(p) = (\Psi_1)_\sharp(p)$ for all $p \in P$. By Proposition 5.5, there is an $(F, \gamma(\varepsilon))$-multiplicative cpc map $h : A \to C[0, 1] \otimes \mathbb{K}$, $h = (h^{(r)})_{r\in[0,1]}$, such that $h^{(0)} = \varphi$ and $h^{(1)} = \Psi_1$. Define a new cpc asymptotic morphism $\{\Phi_t : A \to \mathbb{K}\}_{t\in[0,\infty)}$ by concatenation of the families of maps $t \mapsto h^{(t)}$, $t \in [0, 1]$ and $t \mapsto \Psi_t$, $t \in [1, \infty)$. Define $\bar{\gamma} \in \mathcal{F}$ by $\bar{\gamma}(\varepsilon) = \max\{\varepsilon, \gamma(\varepsilon)\}$. Then $\bar{\gamma} \in \mathcal{F}$ and $\{\Phi_t : A \to \mathbb{K}\}_{t\in[0,\infty)}$ satisfy the conditions from the Definition 5.4 of asymptotic stability since $|\Psi_t|_F \leq \varepsilon$ and $|h^{(t)}|_F \leq \varepsilon$. The version $F^+ \subset A^+$ is proved by a minor modification of the argument above. □

Let us recall that for a groups G the augmentation ideal $I(G)$ is the kernel of the trivial representation $C^*(G) \to \mathbb{C}$. $I(G)$ is generated by the finite set $\{s - 1 : s \in S\}$.

Proposition 5.7. *Let G be a finitely presented amenable group as in (1). If the C^*-algebra $I(G)$ is asymptotically stable in the sense of Def. 5.4, then G is asymptotically stable in the sense of Def. 2.4.*

Proof. Let $\eta \in \mathcal{F}$ be given by Lemma 5.2. Let $\alpha \in \mathcal{F}$ be given by Lemma 5.1 applied to G and $F = S$. Suppose that $I(G)$ is asymptotically stable and let $\gamma \in \mathcal{F}$ be given by Definition 5.4 applied for the set $F = \{s - 1 : s \in S\} \subset I(G)$ or equivalently to $F^+ = S \subset G \subset I(G)^+$.

We are going to show that $\delta := \alpha + \eta \circ \gamma \circ \alpha$ has the properties from the equivalent definition of asymptotic stability of G as specified in Lemma 5.3.

Indeed, suppose now that $\sigma \in \mathcal{R}_\varepsilon(G, U(\infty))$ is given. By Lemma 5.1, it follows that there is an $(S, \alpha(\varepsilon))$-multiplicative ucp map $\varphi : C^*(G) \to \mathbb{K}^+$ with $\varphi(I(G)) \subset \mathbb{K}$ such that $\|\varphi(s) - \sigma(s)\| \leq \alpha(\varepsilon)$ for all $s \in S$.

Since $I(G)$ is asymptotically stable, by our choice of γ and Definition 5.4 applied for the input φ, there exists a cpc asymptotic morphism $\{\Phi_t : I(G) \to \mathbb{K}\}_{t\in[0,\infty)}$ such that $\{\Phi_t^+ : I(G)^+ = C^*(G) \to \mathbb{K}^+\}_{t\in[0,\infty)}$ satisfies $\Phi_0^+ = \varphi$ and ${\Phi_t}^+$ is $(S, \gamma(\alpha(\varepsilon)))$-multiplicative for all $t \in [0, \infty)$. Fix $\varepsilon_0 > 0$ small enough such that for any $\varepsilon \in (0, \varepsilon_0)$ it follows that $\gamma(\alpha(\varepsilon)) < 1$. Then by Lemma 5.2, the formula $\pi_t(s) = \Phi_t^+(s)|\Phi_t^+(s)|^{-1}$ defines a unitary-valued map such that $\|\pi_t\| \leq \eta(\gamma(\alpha(\varepsilon)))$ and $\|\pi_0(s) - \Phi_0^+(s)\| \leq \eta(\gamma(\alpha(\varepsilon)))$ for all $\varepsilon \in (0, \varepsilon_0)$. It follows that

$$\|\pi_0(s) - \sigma(s)\| \leq \|\pi_0(s) - \Phi_0^+(s)\| + \|\varphi(s) - \sigma(s)\| \leq \eta(\gamma(\alpha(\varepsilon))) + \alpha(\varepsilon)$$

for all $s \in S$. Note that $\Phi_t^+(s) \in U(\mathbb{K})$ since $\Phi_t(I(G)) \subset \mathbb{K}$. It follows that $\pi_t(s) \in U(\mathbb{K})$ for all $s \in S$ as required. □

Proof of Theorem 2.7

Let G be a finitely presented amenable discrete group such that $K_*(C^*(G))$ is finitely generated. Let $I(G)$ be the kernel of the trivial representation $C^*(G) \to \mathbb{C}$. By definition, a discrete group is connective if the C*-algebra $A = I(G)$ is connective. By Theorem 5.6, it follows that A is asymptotically stable in the sense of Definition 5.4. By applying Proposition 5.7, we obtain that G is asymptotically stable.

Proof of Corollary 2.9

A Bieberbach group is a torsion free group G given by a short exact sequence

$$1 \to \mathbb{Z}^n \to G \to H \to 1,$$

where H is a finite group, called the holonomy group of G. It is known that G acts by rigid motions on the Euclidean space $\mathbb{R}^n$ and that the classifying space of G is a closed flat manifold $BG = \mathbb{R}^n/G$, [2]. The classifying spaces of finitely generated torsion free nilpotent groups are finite simplicial complexes. It was proved in [6] and [7] that the groups from the statement of the corollary are connective. Since these groups are also (elementary) amenable, we conclude the proof by applying Corollary 2.8.

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