

# QUASI-REPRESENTATIONS OF GROUPS AND TWO-HOMOLOGY

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ABSTRACT. The Exel-Loring formula asserts that two topological invariants associated to a pair of almost commuting unitary matrices coincide. Such a pair can be viewed as a quasi-representation of  $\mathbb{Z}^2$ . We give a generalization of this formula for countable discrete groups. We also show the nontriviality of the corresponding invariants for quasidiagonal groups which are coarsely embeddable in a Hilbert space and have nonvanishing second Betti number.

## 1. INTRODUCTION

Kazhdan [18] and Voiculescu [28] exhibited sequences of pairs of almost commuting unitaries without commuting approximants. In their proofs, Kazhdan used a winding number argument and Voiculescu used a Fredholm index argument. Another proof was given later by Loring using K-theory [21]. For two unitaries  $u, v \in U(n)$  such that  $\|uv - vu\|$  is smaller than a positive universal constant, Loring introduced a K-theory invariant  $k(u, v) \in \mathbb{Z}$  which can be described informally as follows. The pair  $u, v$  gives rise to a group quasi-representation  $\varphi : \mathbb{Z}^2 \rightarrow U(n)$  and hence to a contractive quasi-representation of  $*$ -algebras  $\varphi : \ell^1(\mathbb{Z}^2) \rightarrow M_n(\mathbb{C})$ . Then  $k(u, v)$  is defined as the pushforward  $\varphi_{\#}(\beta)$  of the Bott element, where  $K_0(\ell^1(\mathbb{Z}^2)) \cong K_0(C^*(\mathbb{Z}^2)) \cong \mathbb{Z} \oplus \mathbb{Z}\beta$ . The virtual rank of  $\beta$  is 0 and the first Chern class of  $\beta$  is 1. On the other hand, Exel and Loring [11] rediscovered Kazhdan's invariant  $\omega(u, v)$  defined as the winding number in  $\mathbb{C} \setminus \{0\}$  (abbreviated wn) of the loop  $t \mapsto \det((1-t)1_n + t[v, u])$  and proved the equality  $k(u, v) = \omega(u, v)$ , [12]. Exel gave another proof of this equality using the soft torus  $C^*$ -algebra, see [10]. We extended the Exel-Loring formula to quasi-representations  $\pi : \Gamma_g \rightarrow U(n)$  of surface groups of genus  $g \geq 1$  in [6] and in joint work with Carrión [4] to quasi-representations  $\rho : \Gamma_g \rightarrow U(A)$  for  $A$  a unital tracial  $C^*$ -algebra, see Theorems 2.5, 2.6 below. A key step in these generalizations was to realize that the Exel-Loring formula is related to an index theorem of Connes, Gromov and Moscovici [5] and to its extension studied in [6].

By Hopf's formula

$$H_2(\Gamma, \mathbb{Z}) = R \cap [F, F]/[R, F],$$

for the second homology of a discrete group  $\Gamma$  in terms of a free presentation

$$(1) \quad 0 \rightarrow R \rightarrow F \xrightarrow{q} \Gamma \rightarrow 0, \quad q(a) = \bar{a},$$

each element  $x \in H_2(\Gamma, \mathbb{Z})$  is represented by a product of commutators  $\prod_{i=1}^g [a_i, b_i]$  with  $a_i, b_i \in F$ , for some integer  $g \geq 1$ , such that  $\prod_{i=1}^g [\bar{a}_i, \bar{b}_i] = 1$ .

Consider the (rationally injective) homomorphism

$$\beta^\Gamma : H_2(\Gamma, \mathbb{Z}) \cong H_2(B\Gamma, \mathbb{Z}) \rightarrow RK_0(B\Gamma),$$

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studied in [2], [22], [23] and define the map  $\alpha^\Gamma : H_2(\Gamma, \mathbb{Z}) \rightarrow K_0(\ell^1(\Gamma))$  as the composition  $\alpha^\Gamma = \mu_1^\Gamma \circ \beta^\Gamma$  where  $\mu_1^\Gamma$  is the  $\ell^1$ -version of the assembly map of [19]:

$$\alpha^\Gamma : H_2(\Gamma, \mathbb{Z}) \xrightarrow{\beta^\Gamma} RK_0(B\Gamma) \xrightarrow{\mu_1^\Gamma} K_0(\ell^1(\Gamma)).$$

We generalize the Exel-Loring formula to arbitrary discrete countable groups  $\Gamma$  as follows.

**Theorem 1.1.** *Let  $\Gamma$  be a discrete countable group. Let  $x \in H_2(\Gamma, \mathbb{Z})$  be represented by a product of commutators  $\prod_{i=1}^g [a_i, b_i]$  with  $a_i, b_i \in F$  and  $\prod_{i=1}^g [\bar{a}_i, \bar{b}_i] = 1$ . Let  $p_0$  and  $p_1$  be projections in some matrix algebra over  $\ell^1(\Gamma)$  such that  $\alpha^\Gamma(x) = [p_0] - [p_1] \in K_0(\ell^1(\Gamma))$ . There exist a finite set  $S \subset G$  and  $\varepsilon > 0$  such that if  $\pi : \Gamma \rightarrow U(n)$  is unital map with  $\|\pi(st) - \pi(s)\pi(t)\| < \varepsilon$  for all  $s, t \in S$ , then*

$$(2) \quad \pi_\#(\alpha^\Gamma(x)) = \text{wn det} \left( (1-t)1_n + t \prod_{i=1}^g [\pi(\bar{a}_i), \pi(\bar{b}_i)] \right) = \frac{1}{2\pi i} \text{Tr} \left( \log \left( \prod_{i=1}^g [\pi(\bar{a}_i), \pi(\bar{b}_i)] \right) \right).$$

More generally if  $A$  is a unital  $C^*$ -algebra with a trace  $\tau$  and  $\pi : \Gamma \rightarrow U(A)$  is unital map with  $\|\pi(st) - \pi(s)\pi(t)\| < \varepsilon$  for all  $s, t \in S$ , then

$$(3) \quad \tau_*(\pi_\#(\alpha^\Gamma(x))) = \frac{1}{2\pi i} \tau \left( \log \left( \prod_{i=1}^g [\pi(\bar{a}_i), \pi(\bar{b}_i)] \right) \right).$$

Here  $\pi_\#(\alpha^\Gamma(x)) = \pi_\#(p_0) - \pi_\#(p_1)$  where  $\pi_\#(p_i)$  is the  $K$ -theory class of the perturbation of  $(\text{id} \otimes \pi)(p_i)$  to a projection via analytic functional calculus.

Moreover, we show in Theorem 3.2 that if  $\Gamma$  is a quasidiagonal group which admits a  $\gamma$ -element and  $x \in H_2(\Gamma, \mathbb{Z})$  is not of finite order, then there are finite dimensional unitary quasi-representations  $\pi : \Gamma \rightarrow U(n)$  for which the winding number of the closed loop

$$t \mapsto \det \left( (1-t)1_n + t \prod_{i=1}^g [\pi(\bar{a}_i), \pi(\bar{b}_i)] \right)$$

from Theorem 1.1 is nonzero. In particular these quasi-representations are not perturbable to genuine representations, see Corollary 3.3. The proof of Theorem 1.1 combines results from [6], [4] with results of Loday [20] and Matthey [22], [23]. For the proof of Theorem 3.2 we rely on our previous paper [8].

Eilers, Shulman and Sørensen [9] showed that certain concrete groups with homogeneous relations are not matricially stable by using winding number invariants of Kazhdan/Exel-Loring type and quasi-representations constructed ad-hoc. Theorem 1.1 explains how these invariants are connected to the two-homology of the groups and Theorem 3.2 gives general abstract criteria for their nonvanishing.

## 2. TWO-HOMOLOGY AND WINDING NUMBERS

If  $s, t$  are elements of a group  $\Gamma$ , we denote by  $[s, t]$  their commutator  $sts^{-1}t^{-1}$ . The commutator subgroup of  $\Gamma$ , denoted  $[\Gamma, \Gamma]$ , consists of finite products of commutators.

If  $\omega : [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$  is a loop,  $\omega(0) = \omega(1)$ , we denote by  $\text{wn } \omega(t)$  its winding number. Let  $\log$  be the principal branch of the logarithm defined on  $\mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}$ ,  $\log 1 = 0$ . Let  $\text{Tr} : M_n(\mathbb{C}) \rightarrow \mathbb{C}$  be the canonical trace with  $\text{Tr}(1_n) = n$ . Let  $w \in SU(n)$  with  $\|w - 1_n\| < 2$ . If  $w$  is written

as  $w = \exp(2\pi i h)$  with  $h = h^* = \frac{1}{2\pi i} \log(w)$ , then  $\text{Tr}(h) \in \mathbb{Z}$  since  $\det(w) = \exp(2\pi i \text{Tr}(h)) = 1$ . Define the map  $\kappa : \{w \in SU(n) : \|w - 1\| < 2\} \rightarrow \mathbb{Z}$ ,

$$(4) \quad \kappa(w) = \frac{1}{2\pi i} \text{Tr}(\log(w)).$$

The function  $\kappa$  is continuous and hence locally constant as it assumes only integral values.

If  $A$  is a unital  $C^*$ -algebra with a trace  $\tau$ , we define  $\kappa_\tau : \{w \in U(A) : \|w - 1\| < 1\} \rightarrow \mathbb{R}$ , by

$$(5) \quad \kappa_\tau(w) = \frac{1}{2\pi i} \tau(\log(w)).$$

It is clear that if  $A = M_n(\mathbb{C})$  and  $\tau = \text{Tr}$ , then  $\kappa_\tau = \kappa$ .

**Lemma 2.1** ([10]). *If  $w \in SU(n)$  and  $\|w - 1\| < 2$ , then  $\text{wn det}((1-t)1_n + tw) = \kappa(w)$ .*

*Proof.* This is proved in [10, Lemma 3.1] for a commutator  $w = [u, v]$  with  $u, v \in U(n)$ . Let us review the argument. One verifies that if  $h = h^* = \frac{1}{2\pi i} \log(w)$ , then for all  $0 \leq t \leq 1$ ,

$$\|((1-t)w^* + t1_n) - \exp(2\pi i t h)w^*\| = \|((1-t)1_n + t \exp(2\pi i h) - \exp(2\pi i t h))\| < 1.$$

Thus the two paths  $\omega_0(t) = (1-t)w^* + t1_n$  and  $\omega_1(t) = \exp(2\pi i t h)w^*$  are homotopic with endpoints fixed as maps into  $GL(n, \mathbb{C})$  via the linear homotopy  $\omega_s(t) = (1-s)\omega_0(t) + s\omega_1(t)$ . It follows that

$$\text{wn det}((1-t)1_n + tw) = \text{wn det}(\exp(2\pi i t h)) = \text{wn exp}(2\pi i t \text{Tr}(h)) = \text{Tr}(h). \quad \square$$

**Lemma 2.2** (Lemma 5, [18]). *Let  $(u_i)_{i=1}^g, (v_i)_{i=1}^g, (u'_i)_{i=1}^g, (v'_i)_{i=1}^g$ , be elements of  $U(n)$  such that  $\|\prod_{i=1}^g [u_i, v_i] - 1_n\| < 1/5g$ ,  $\|u_i - u'_i\| < 1/5g$  and  $\|v_i - v'_i\| < 1/5g$  for  $i = 1, \dots, g$ . Then*

$$\kappa\left(\prod_{i=1}^g [u_i, v_i]\right) = \kappa\left(\prod_{i=1}^g [u'_i, v'_i]\right).$$

*It follows that if  $\kappa(\prod_{i=1}^g [u_i, v_i]) \neq 0$ , then  $\prod_{i=1}^g [u'_i, v'_i] \neq 1_n$ .*

*Proof.* Kazdan considers the continuous paths in  $U(n)$

$$u_i(t) = u_i \exp(t \log(u_i^{-1} u'_i)), \quad v_i(t) = v_i \exp(t \log(v_i^{-1} v'_i)), \quad i = 1, \dots, g.$$

Then  $\|u_i(t) - 1_n\| < 1/5g$ ,  $\|v_i(t) - 1_n\| < 1/5g$ ,  $t \in [0, 1]$ . It follows that  $w(t) = \prod_{i=1}^g [u_i(t), v_i(t)]$  is a continuous path in  $SU(n)$  such that  $w(0) = \prod_{i=1}^g [u_i, v_i]$ ,  $w(1) = \prod_{i=1}^g [u'_i, v'_i]$  and  $\|w(t) - 1_n\| < 1$  for all  $t \in [0, 1]$ . One concludes that  $\kappa(w(0)) = \kappa(w(1))$  since  $t \mapsto \kappa(w(t)) \in \mathbb{Z}$  is continuous.  $\square$

**Example 2.3.** Kazdan's and Voiculescu's examples involve the sequence of pairs of unitaries

$$u_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad v_n = \begin{pmatrix} \lambda_n & 0 & 0 & 0 & 0 \\ 0 & \lambda_n^2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_n^3 & \cdot & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & \lambda_n^n \end{pmatrix}, \quad \lambda_n = e^{2\pi i/n}$$

$$[u_n, v_n] = \exp(-2\pi i/n) \cdot 1_n, \quad \|[u_n, v_n] - 1_n\| = |\exp(2\pi i/n) - 1| < 2\pi/n$$

$$\kappa([u_n, v_n]) = \kappa(\exp(-2\pi i/n)1_n) = \frac{1}{2\pi i} \text{Tr}(\log(\exp(-2\pi i/n)1_n)) = -1.$$

As noted in [18] and rediscovered in [11], Lemma 2.2 implies that the sequence of pairs of unitaries  $u_n$  and  $v_n$  does not admit commuting approximants.

**Remark 2.4.** Suppose that  $\{\pi_n : A \rightarrow D_n\}_n$  is a bounded asymptotic homomorphism of unital  $C^*$ -algebras. Thus  $\lim_{n \rightarrow \infty} \|\pi_n(aa') - \pi_n(a)\pi_n(a')\| = 0$  for all  $a, a' \in A$ . The sequence  $\{\pi_n\}_n$  induces a unital  $*$ -homomorphism  $A \rightarrow \prod_n D_n / \bigoplus_n D_n$  and hence a group homomorphism  $K_0(A) \rightarrow \prod_n K_0(D_n) / \bigoplus_n K_0(D_n)$ . This gives a canonical way to push forward an element  $x \in K_0(A)$  to a sequence  $(\pi_{n\#}(x))_n$  with components in  $K_0(D_n)$  which is well-defined up to tail equivalence: two sequences are tail equivalent,  $(y_n) \equiv (z_n)$ , if there is  $m$  such that  $x_n = y_n$  for all  $n \geq m$ . Note that  $\pi_{n\#}(x + x') \equiv \pi_{n\#}(x) + \pi_{n\#}(x')$ . Of course, if  $\pi_n$  are genuine  $*$ -homomorphisms then  $\pi_{n\#}(x) = \pi_{n*}(x)$ . One can extend these considerations to Banach algebras. Occasionally it is convenient to work with a local version of this construction. For instance, if  $\pi : A \rightarrow B$  is a unital linear contraction which is almost multiplicative in the sense that  $\|\pi(aa') - \pi(a)\pi(a')\| < \varepsilon$  for  $a, a'$  in a finite subset  $S$  of  $A$ , then one can pushforward specific projections  $p$  in matrices over  $A$  to projections in matrices over  $B$ . Assuming that  $S$  is sufficiently large and  $\varepsilon$  is sufficiently small,  $(\pi \otimes id)(p)$  is close to a projection (use analytic functional calculus) whose K-theory class is denoted by  $\pi_{\#}(p)$ . Moreover given  $p$  and  $q$  with  $[p] = [q] \in K_0(A)$ , it will follow that  $\pi_{\#}(p) = \pi_{\#}(q)$  provided that  $S$  is sufficiently large and  $\varepsilon$  is sufficiently small. Using this observation, we will sometimes abuse notation and write  $\pi_{\#}(x)$  for  $\pi_{\#}(p) - \pi_{\#}(p')$  where  $x = [p] - [p'] \in K_0(A)$  and the representatives  $p, p'$  are fixed.

Let  $\Gamma$  be a discrete countable group with classifying space  $B\Gamma$ . If  $B\Gamma$  is written as an increasing union of finite simplicial complexes  $Y_i$ , then the K-homology of  $B\Gamma$  is  $RK_0(B\Gamma) \cong \varinjlim_i K_0(Y_i)$ . Let  $\mu^\Gamma : RK_0(B\Gamma) \rightarrow K_0(C^*(\Gamma))$  denote the full assembly map [15]. Let  $j : \ell^1(\Gamma) \rightarrow C^*(\Gamma)$  be the canonical homomorphism. There is a factorization of  $\mu^\Gamma$  through its  $\ell^1$ -version [19]:

$$(6) \quad \begin{array}{ccc} RK_0(B\Gamma) & \xrightarrow{\mu_1^\Gamma} & K_0(\ell^1(\Gamma)) \\ & \searrow \mu^\Gamma & \downarrow j_* \\ & & K_0(C^*(\Gamma)) \end{array}$$

A unital map  $\pi : \Gamma \rightarrow U(n)$  is called a quasi-representation of  $\Gamma$ . It induces a linear contraction  $\pi : \ell^1(\Gamma) \rightarrow M_n$ . Let  $S \subset \Gamma$  be a symmetric finite subset and let  $\varepsilon > 0$ . We say that  $\pi$  is  $(S, \varepsilon)$ -multiplicative if  $\|\pi(st) - \pi(s)\pi(t)\| < \varepsilon$  for all  $s, t \in S$ . Since  $S$  is symmetric we see that  $\|\pi(s^{-1}) - \pi(s)^*\| < \varepsilon$  for all  $s \in S$ . One can use sufficiently multiplicative quasi-representations  $\pi$  to pushforward K-theory elements of  $K_0(\ell^1(\Gamma))$  via a partially defined map  $\pi_{\#} : K_0(\ell^1(\Gamma)) \rightarrow \mathbb{Z}$  as discussed in Remark 2.4. By Lemma 3.3. of [6], if  $x, y \in K_0(\ell^1(\Gamma))$  are such that  $j_*(x) = j_*(y) \in K_0(C^*(\Gamma))$ , then  $\pi_{\#}(x) = \pi_{\#}(y)$  provided that  $\pi$  is sufficiently multiplicative.

A one-relator group is a group with a presentation of the form  $\langle S; r \rangle$ , where  $r$  is single element in the free group  $F(S)$  on the countable generating set  $S$ . An important example is the surface group

$$\Gamma_g = \pi_1(\Sigma_g) = \langle s_1, t_1, \dots, s_g, t_g; \prod_{i=1}^g [s_i, t_i] \rangle,$$

where  $\Sigma_g = B\Gamma_g$  is a connected closed orientable surface of genus  $g \geq 1$ . We regard  $s_i, t_i$  as the generators of the free group  $\mathbb{F}_{2g}$ . Their images in  $\Gamma_g$  are denoted by  $\bar{s}_i, \bar{t}_i$ , so that  $\prod_{i=1}^g [\bar{s}_i, \bar{t}_i] = 1$ .

Let  $[\Sigma_g]_K$  denote the fundamental class of  $\Sigma_g$  in K-homology. It is independent of the choice of the spin structure of  $\Sigma_g$  and  $K_0(\Sigma_g) \cong \mathbb{Z} \oplus \tilde{K}_0(\Sigma_g) \cong \mathbb{Z} \oplus \mathbb{Z}[\Sigma_g]_K$ . In [6], we extended the Exel-Loring formula from  $\mathbb{Z}^2$  to all surface groups  $\Gamma_g, g \geq 1$  as follows:

**Theorem 2.5** (Thm.4.2, [6]). *There exist a finite set  $S \subset \Gamma_g$  and  $\varepsilon > 0$  such that if  $\rho : \Gamma_g \rightarrow U(n)$  is any  $(S, \varepsilon)$ -multiplicative quasi-representation, then*

$$(7) \quad \rho_{\sharp}(\mu^{\Gamma_g}[\Sigma_g]) = -\frac{1}{2\pi i} \text{Tr} \left( \log \left( \prod_{i=1}^g [\rho(\bar{s}_i), \rho(\bar{t}_i)] \right) \right)$$

The result above was extended to quasi-representations  $\rho : \Gamma_g \rightarrow U(A)$  for  $A$  a unital tracial  $C^*$ -algebra in [4].

**Theorem 2.6** (Thm.2.3, [4]). *There exist a finite set  $S \subset \Gamma_g$  and  $\varepsilon > 0$  such that if  $\rho : \Gamma_g \rightarrow U(A)$  is any  $(S, \varepsilon)$ -multiplicative quasi-representation, then*

$$(8) \quad \tau_*(\rho_{\sharp}(\mu^{\Gamma_g}[\Sigma_g])) = -\frac{1}{2\pi i} \tau \left( \log \left( \prod_{i=1}^g [\rho(\bar{s}_i), \rho(\bar{t}_i)] \right) \right)$$

Here  $\tau_* : K_0(A) \rightarrow \mathbb{R}$  is the homomorphism induced by  $\tau$ .

**Remark 2.7.** The formula (8) was stated in [4] without the negative sign. This was due to an inadvertent omission of the sign in the statement of Theorem 5.2 from [4], even though the correct sign was obtained in its proof.

We are going to show that the formulae (7), (8) can be generalized to arbitrary countable discrete groups, as stated in Theorem 1.1.

For a connected pointed CW complex  $X$  there is a natural homomorphism  $\beta^X : H_2(X, \mathbb{Z}) \rightarrow RK_0(X)$  which is a rational right inverse of the Chern character in the sense that:  $(ch_2 \otimes \text{id}_{\mathbb{Q}}) \circ (\beta^X \otimes \text{id}_{\mathbb{Q}}) = \text{id}_{H_2(X, \mathbb{Q})}$  and hence it is rationally injective, see [2] and [22]. The map  $\beta^X$  is defined by composing the isomorphisms  $H_2(X, \mathbb{Z}) \cong H_2(X^{(3)}, \mathbb{Z}) \cong RK_0(X^{(3)})$  with the map  $RK_0(X^{(3)}) \rightarrow RK_0(X)$  induced by the inclusion of the 3-skeleton  $X^{(3)} \hookrightarrow X$ .

Let  $\Gamma$  be a countable discrete group. For  $X = B\Gamma$ , we denote by  $\beta^{\Gamma}$  the corresponding (rationally injective) homomorphism, [22],

$$\beta^{\Gamma} : H_2(\Gamma, \mathbb{Z}) \cong H_2(B\Gamma, \mathbb{Z}) \rightarrow RK_0(B\Gamma).$$

Consider the map  $\alpha^{\Gamma} : H_2(\Gamma, \mathbb{Z}) \rightarrow K_0(\ell^1(\Gamma))$  defined by  $\alpha^{\Gamma} = \mu_1^{\Gamma} \circ \beta^{\Gamma}$ :

$$\alpha^{\Gamma} : H_2(\Gamma, \mathbb{Z}) \xrightarrow{\beta^{\Gamma}} RK_0(B\Gamma) \xrightarrow{\mu_1^{\Gamma}} K_0(\ell^1(\Gamma))$$

Chose a free resolution of  $\Gamma$ :

$$(9) \quad 0 \rightarrow R \rightarrow F \xrightarrow{q} \Gamma \rightarrow 0, \quad q(a) = \bar{a},$$

where  $F$  and  $R$  are free groups. By Hopf's formula [3],

$$H_2(\Gamma, \mathbb{Z}) = \frac{R \cap [F, F]}{[R, F]}.$$

Thus each element  $x \in H_2(\Gamma, \mathbb{Z})$  is represented by a product of commutators,  $\prod_{i=1}^g [a_i, b_i]$  with  $a_i, b_i \in F$  for some integer  $g \geq 1$  and such that  $\prod_{i=1}^g [\bar{a}_i, \bar{b}_i] = 1$ .

*Proof of Theorem 1.1*

*Proof.* We shall prove only (2). The proof of (3) is entirely similar except that one uses Theorem 2.6 instead of Theorem 2.5 and  $\kappa_{\tau}$  instead of  $\kappa$ .

Let  $x \in H_2(\Gamma, \mathbb{Z})$  be represented by a product of commutators  $\prod_{i=1}^g [a_i, b_i]$  with  $a_i, b_i \in F$  with  $F$  as in (9). Let us recall that in the case of surface groups  $\Gamma_g$ , with resolution

$$(10) \quad 0 \rightarrow R_{2g} \rightarrow F_{2g} \xrightarrow{q} \Gamma_g \rightarrow 0,$$

it was shown in [20, 2.2.4] that under the isomorphism

$$H_2(\Sigma_g, \mathbb{Z}) \cong H_2(\Gamma_g, \mathbb{Z}) = \frac{R_{2g} \cap [F_{2g}, F_{2g}]}{[R_{2g}, F_{2g}]},$$

the fundamental class  $[\Sigma_g]$  of  $H_2(\Sigma_g, \mathbb{Z})$  corresponds to the element  $-x_g \in H_2(\Gamma_g, \mathbb{Z})$  where  $x_g$  is the class of  $\prod_{i=1}^g [s_i, t_i]$ . Following Loday, we consider the homomorphism  $F_{2g} \rightarrow F$  which maps  $s_i$  to  $a_i$  and  $t_i$  to  $b_i$ . This induces an homomorphism  $f : \Gamma_g \rightarrow \Gamma$  such that  $f(\bar{s}_i) = \bar{a}_i$  and  $f(\bar{t}_i) = \bar{b}_i$ ,  $i = 1, \dots, g$  and the corresponding map  $Bf : B\Gamma_g \rightarrow B\Gamma$ . We make the identification  $\Sigma_g = B\Gamma_g$ . If  $[\Sigma_g]$  denotes the fundamental class of  $H_2(\Sigma_g; \mathbb{Z})$  then  $\beta^{\Sigma_g}([\Sigma_g]) = [\Sigma_g]_K$ , see [23, p.324]. From the previous discussion we then obtain  $\beta^{\Gamma_g}(x_g) = -[\Sigma_g]_K$  and hence we can rewrite equation (7) as

$$(11) \quad \rho_{\#}(\alpha^{\Gamma_g}(x_g)) = \kappa \left( \prod_{i=1}^g [\rho(\bar{s}_i), \rho(\bar{t}_i)] \right).$$

By naturality of  $\beta$ , [23] and  $\mu$ , [1], [19], the following diagram is commutative.

$$\begin{array}{ccc} H_2(\Gamma_g, \mathbb{Z}) & \xrightarrow{\alpha^{\Gamma_g}} & K_0(\ell^1(\Gamma_g)) \\ f_* \downarrow & & \downarrow f_* \quad \searrow (\pi \circ f)_{\#} \\ H_2(\Gamma, \mathbb{Z}) & \xrightarrow{\alpha^{\Gamma}} & K_0(\ell^1(\Gamma)) \xrightarrow{\pi_{\#}} \mathbb{Z} \end{array}$$

Since  $x_g$  is the generator of  $H_2(\Gamma_g, \mathbb{Z})$  given by the product  $\prod_{i=1}^g [s_i, t_i]$ , it follows that  $f_*(x_g) = x$ . By fixing representatives of the relevant  $K$ -theory classes and by choosing  $S$  sufficiently large and  $\varepsilon$  sufficiently small we may arrange that  $\pi_{\#}(f_*(y)) = (\pi \circ f)_{\#}(y)$  for finitely many elements  $y \in K_0(\ell^1(\Gamma_g))$  and in particular for  $y = \alpha^{\Gamma_g}(x_g)$ . Thus:

$$(12) \quad \pi_{\#}(\alpha^{\Gamma}(x)) = \pi_{\#}(\alpha^{\Gamma}(f_*(x_g))) = \pi_{\#}(f_*(\alpha^{\Gamma_g}(x_g))) = (\pi \circ f)_{\#}(\alpha^{\Gamma_g}(x_g)).$$

On the other hand, the formula (11) applied for the quasi-representation  $\rho = \pi \circ f : \Gamma_g \rightarrow U(n)$  implies that

$$(13) \quad (\pi \circ f)_{\#}(\alpha^{\Gamma_g}(x_g)) = \kappa \left( \prod_{i=1}^g [\pi(f(\bar{s}_i)), \pi(f(\bar{t}_i))] \right)$$

Since  $f(\bar{s}_i) = \bar{a}_i$  and  $f(\bar{t}_i) = \bar{b}_i$  we obtain from (12) and (13) that

$$\pi_{\#}(\alpha^{\Gamma}(x)) = \kappa \left( \prod_{i=1}^g [\pi(\bar{a}_i), \pi(\bar{b}_i)] \right).$$

□

**Remark 2.8.** The integer  $\text{wn det}((1-t)\mathbf{1}_n + t \prod_{i=1}^g [\pi(\bar{a}_i), \pi(\bar{b}_i)])$  depends only on the class  $x$  of  $\prod_{i=1}^g [a_i, b_i]$  in  $H_2(\Gamma, \mathbb{Z})$ . This means that if we represent  $x$  by a different product of commutators,

$\prod_{i=1}^{g'} [a'_i, b'_i]$ , then

$$\kappa \left( \prod_{i=1}^g [\pi(\bar{a}_i), \pi(\bar{b}_i)] \right) = \kappa \left( \prod_{i=1}^{g'} [\pi(\bar{a}'_i), \pi(\bar{b}'_i)] \right)$$

for all sufficiently multiplicative quasi-representations  $\pi$ , since both these integers are equal to  $\pi_{\#}(\alpha^{\Gamma}(x))$  by equation 2.

### 3. QUASI-REPRESENTATIONS WITH NONTRIVIAL INVARIANTS

Our next goal is to exhibit classes of groups that admit quasi-representations  $\pi$  for which the invariants from Theorem 1.1 do not vanish. This is addressed in Theorem 3.2.

Let  $\Gamma$  be a discrete countable group. Let  $\mathcal{Q}$  be the universal UHF-algebra,  $\mathcal{Q} \cong \bigotimes_{n \geq 1} M_n(\mathbb{C})$ . Consider the natural pairing

$$KK(\mathbb{C}, C^*(\Gamma)) \times KK(C^*(\Gamma), \mathcal{Q}) \rightarrow KK(\mathbb{C}, \mathcal{Q}) \cong \mathbb{Q},$$

given by  $(x, y) \mapsto x \otimes_{C^*(\Gamma)} y$ . Consider the full assembly map  $\mu : RK_0(B\Gamma) \rightarrow K_0(C^*(\Gamma))$  and the dual assembly map with rational coefficients  $\nu : KK(C^*(\Gamma), \mathcal{Q}) \rightarrow RK^0(B\Gamma, \mathbb{Q})$ , [15], [16]. For each finite CW complex  $Y \subset B\Gamma$ , let  $\nu_Y : KK(C^*(\Gamma), \mathcal{Q}) \rightarrow RK^0(B\Gamma, \mathbb{Q}) \rightarrow K^0(Y, \mathbb{Q})$  be the composition of  $\nu$  with the restriction map  $RK^0(B\Gamma, \mathbb{Q}) \rightarrow K^0(Y, \mathbb{Q})$ . Let  $\mu_Y : K_0(Y) \rightarrow K_0(C^*(\Gamma))$  be the composition of  $\mu$  with  $K_0(Y) \rightarrow K_0(B\Gamma)$ . By [15, 6.2] these maps satisfy the identity:

$$(14) \quad \nu_Y(y) \otimes_{C(Y)} z = \mu_Y(z) \otimes_{C^*(\Gamma)} y$$

for all  $z \in K_0(Y)$  and  $y \in KK(C^*(\Gamma), \mathcal{Q})$ .

If  $B\Gamma$  is written as the union of an increasing sequence  $(Y_i)_i$  of finite CW complexes, then as explained in the proof of Lemma 3.4 from [17], there is a Milnor  $\varprojlim^1$  exact sequence which implies that

$$(15) \quad RK^0(B\Gamma; \mathbb{Q}) \cong \varprojlim K^0(Y_i; \mathbb{Q}).$$

We denote  $\nu_{Y_i}$  by  $\nu_i$  and  $\mu_{Y_i}$  by  $\mu_i$ . On the other hand  $RK_0(B\Gamma) = \varinjlim K_0(Y_i)$  and  $\mu$  is just the limit of the compatible maps  $\mu_i : K_0(Y_i) \rightarrow K_0(C^*(\Gamma))$ . Using (14) we deduce that the following diagram is commutative

$$(16) \quad \begin{array}{ccc} KK(C^*(\Gamma), \mathcal{Q}) & \longrightarrow & \text{Hom}(K_0(C^*(\Gamma)), \mathbb{Q}) \\ \nu_i \downarrow & & \downarrow \mu_i^* \\ RK^0(Y_i; \mathbb{Q}) & \xrightarrow{\delta_i} & \text{Hom}(RK_0(Y_i), \mathbb{Q}) \end{array}$$

where the horizontal arrows correspond to natural pairings of K-theory with K-homology.

Let  $\underline{E}\Gamma$  be the classifying space for proper actions of  $\Gamma$ , [1]. It is known that  $\underline{E}\Gamma$  admits a locally compact model, [14]. Let us recall that  $\Gamma$  has a  $\gamma$ -element if there exists a  $\Gamma$ - $C_0(\underline{E}\Gamma)$ -algebra  $A$  in the sense of Kasparov [15] and two elements  $d \in KK_{\Gamma}(A, \mathbb{C})$  and  $\eta \in KK_G(\mathbb{C}, A)$  (called Dirac and dual-Dirac elements, respectively) such that the element  $\gamma = \eta \otimes_A d \in KK_{\Gamma}(\mathbb{C}, \mathbb{C})$  has the property that  $p^*(\gamma) = 1 \in \mathcal{R}KK_G(\underline{E}\Gamma; C_0(\underline{E}\Gamma), C_0(\underline{E}\Gamma))$  where  $p : \underline{E}\Gamma \rightarrow \text{point}$ , [27]. We refer the reader to [15] for the definitions and the basic properties of these groups. The groups which are coarsely embeddable in a Hilbert space admit a  $\gamma$ -element, [27]. The class of groups which are coarsely embeddable in a Hilbert space include the amenable groups, the exact (boundary amenable) groups, the linear groups and the hyperbolic groups.

**Proposition 3.1.** *Suppose that  $\Gamma$  has a  $\gamma$ -element. Then for any homomorphism  $h : K_0(C^*(\Gamma)) \rightarrow \mathbb{Q}$  there is  $y \in KK(C^*(\Gamma), \mathbb{Q})$  such that  $h(\mu(z)) = \mu(z) \otimes_{C^*(\Gamma)} y$  for all  $z \in RK_0(B\Gamma)$ .*

*Proof.* Since  $RK^0(B\Gamma; \mathbb{Q}) \cong \varprojlim K^0(Y_i; \mathbb{Q})$  and  $RK_0(B\Gamma) = \varinjlim K_0(Y_i)$ , after passing to limit in (16), we deduce that the following diagram is commutative

$$(17) \quad \begin{array}{ccc} KK(C^*(\Gamma), \mathbb{Q}) & \longrightarrow & \text{Hom}(K_0(C^*(\Gamma)), \mathbb{Q}) \\ \nu \downarrow & & \downarrow \mu^* \\ RK^0(B\Gamma; \mathbb{Q}) & \xrightarrow{\delta} & \text{Hom}(RK_0(B\Gamma), \mathbb{Q}) \end{array}$$

The horizontal arrows correspond to natural pairings of K-theory with K-homology.

The map  $\delta$  is surjective by Lemma 3.4 of [17]. If  $\Gamma$  has a  $\gamma$ -element it is known that the vertical maps are surjective as well. Indeed  $\mu$  is rationally injective by [24], [27] and hence  $\mu^*$  is surjective. For the surjectivity of  $\nu$  (due to Kasparov) see [8, Cor.4.2]. Let  $h \in \text{Hom}(K_0(C^*(\Gamma)), \mathbb{Q})$ . Then  $h \circ \mu \in \text{Hom}(RK_0(B\Gamma), \mathbb{Q})$ . Since both  $\nu$  and  $\delta$  are surjective, there is  $y \in KK(C^*(\Gamma), \mathbb{Q})$  such that  $\delta(\nu(y)) = h \circ \mu$ .

Thus  $\delta(\nu(y)) = h \circ \mu$  implies that  $\delta_i(\nu_i(y)) = h \circ \mu_i$  for some  $i_0$  and hence for all indices  $i \geq i_0$ . Every  $z \in RK_0(B\Gamma)$  is the image of some  $z_i \in K_0(Y_i)$  with  $i \geq i_0$ . It follows from (14) that

$$h \circ \mu_i(z_i) = \delta_i(\nu_i(y))(z_i) = \nu_i(y) \otimes_{C(Y_i)} z_i = \mu_i(z_i) \otimes_{C^*(\Gamma)} y,$$

and hence  $h(\mu(z)) = \mu(z) \otimes_{C^*(\Gamma)} y$ .  $\square$

A countable discrete group  $G$  is *quasidiagonal* if it is isomorphic to a subgroup of the unitary group of a quasidiagonal  $C^*$ -algebra [8]. Equivalently, there is a faithful representation  $\pi : \Gamma \rightarrow U(H)$  on a Hilbert space for which there is an increasing sequence  $(p_n)_n$  of finite dimensional projections which converges strongly to  $1_H$  and such that  $\lim_{n \rightarrow \infty} \|\pi(s) - p_n \pi(s)\| = 0$  for all  $s \in \Gamma$ . Thus, a maximally almost periodic group (MAP) is quasidiagonal. Amenable groups, or more generally, residually amenable groups are also quasidiagonal as a consequence of [25].

If  $\Gamma$  has a  $\gamma$ -element, then it is known that  $\mu^\Gamma$  is rationally injective [26] and therefore so is the map  $\bar{\alpha}^\Gamma : H_2(\Gamma, \mathbb{Z}) \rightarrow K_0(C^*(\Gamma))$  defined by  $\bar{\alpha}^\Gamma = \mu^\Gamma \circ \beta^\Gamma = j_* \circ \alpha^\Gamma$ , where  $j_* : K_0(\ell^1(\Gamma)) \rightarrow K_0(C^*(\Gamma))$ . We shall use notation as in (9).

**Theorem 3.2.** *Let  $\Gamma$  be a quasidiagonal group which admits a  $\gamma$ -element. Suppose that  $x$  is a non-torsion element of  $H_2(\Gamma, \mathbb{Z})$  represented by a product of commutators  $\prod_{i=1}^g [a_i, b_i]$  with  $a_i, b_i \in F$  and  $\prod_{i=1}^g [\bar{a}_i, \bar{b}_i] = 1$ . Then there is an asymptotic homomorphism  $\{\pi_n : \Gamma \rightarrow U(k_n)\}_n$  such that*

$$\text{wn det} \left( (1-t)1_{k_n} + t \prod_{i=1}^g [\pi_n(\bar{a}_i), \pi_n(\bar{b}_i)] \right) \neq 0$$

for all sufficiently large  $n$ .

*Proof.* Let us recall that  $\alpha^\Gamma = \mu_1^\Gamma \circ \beta^\Gamma$  and  $\bar{\alpha}^\Gamma = \mu^\Gamma \circ \beta^\Gamma$ . By Theorem 1.1 it suffices to find  $(\pi_n)_n$  such that  $(\pi_n)_\#(\alpha^\Gamma(x)) \neq 0$  for all sufficiently large  $n$ . We claim that it suffices to find a unital completely positive (ucp) asymptotic morphism  $\{\psi_n : C^*(\Gamma) \rightarrow M_{k_n}\}_n$  such that  $(\psi_n)_\#(\bar{\alpha}^\Gamma(x)) \neq 0$  for all sufficiently large  $n$ . Indeed, by functional calculus one can perturb the restriction to  $\Gamma$  of each  $\psi_n$  to a unital map  $\pi_n : \Gamma \rightarrow U(k_n)$  such that  $\lim_n \|\pi_n(s) - \psi_n(s)\| = 0$  for all  $s \in \Gamma$ . Then the asymptotic homomorphism  $\{\pi_n : \Gamma \rightarrow U(k_n)\}_n$  induces  $*$ -homomorphisms  $\pi : \ell^1(\Gamma) \rightarrow \prod_n M_{k_n} / \bigoplus_n M_{k_n}$  and



$\underline{\pi} : C^*(\Gamma) \rightarrow \prod_n M_{k_n} / \bigoplus_n M_{k_n}$  with  $j \circ \underline{\pi} = \pi$  such that  $\underline{\pi}$  is equal to the  $*$ -homomorphism induced by  $\{\psi_n\}_n$ . It follows that  $(\pi_n)_\#(\alpha^\Gamma(x)) = (\psi_n)_\#(\bar{\alpha}^\Gamma(x)) \neq 0$  for all sufficiently large  $n$ .

Since  $x$  is a non-torsion element and since  $\bar{\alpha}^\Gamma$  is a composition of rationally injective maps ( $\Gamma$  has a  $\gamma$ -element), there is  $h : K_0(C^*(\Gamma)) \rightarrow \mathbb{Q}$  such that  $h(\bar{\alpha}^\Gamma(x)) \neq 0$ . Since  $\Gamma$  has a  $\gamma$ -element and it is quasidiagonal, it follows by [8, Thm.4.6] that  $\nu(KK(C^*(\Gamma), \mathcal{Q})_{qd}) = \nu(KK(C^*(\Gamma), \mathcal{Q})) = RK^0(B\Gamma; \mathbb{Q})$ . Therefore in the proof of Proposition 3.1 we can choose  $y \in KK(C^*(\Gamma), \mathcal{Q})_{qd}$  such that  $h(\mu(z)) = \mu(z) \otimes_{C^*(\Gamma)} y$  for all  $z \in RK_0(B\Gamma)$ . In particular, we obtain that  $h(\bar{\alpha}^\Gamma(x)) = \bar{\alpha}^\Gamma(x) \otimes_{C^*(\Gamma)} y \neq 0$ . Since  $y \in KK(C^*(\Gamma), \mathcal{Q})_{qd}$ ,  $y$  is represented by a pair of nonzero  $*$ -representations  $\varphi, \psi : C^*(\Gamma) \rightarrow M(K(H) \otimes \mathcal{Q})$ , such that  $\varphi(a) - \psi(a) \in K(H) \otimes \mathcal{Q}$ ,  $a \in C^*(\Gamma)$ , and with property that there is an increasing approximate unit  $(p_n)_n$  of  $K(H)$  consisting of projections such that  $(p_n \otimes 1_{\mathcal{Q}})_n$  commutes asymptotically with both  $\varphi(a)$  and  $\psi(a)$ , for all  $a \in C^*(\Gamma)$ , see [8, Def.4.4]. It is then clear that  $\varphi_n^{(0)} = (p_n \otimes 1_{\mathcal{Q}})\varphi(p_n \otimes 1_{\mathcal{Q}})$  and  $\varphi_n^{(1)} = (p_n \otimes 1_{\mathcal{Q}})\psi(p_n \otimes 1_{\mathcal{Q}})$  are contractive completely positive asymptotic homomorphisms from  $C^*(\Gamma)$  to  $K(H) \otimes \mathcal{Q}$ . Let 1 denote the unit of  $C^*(\Gamma)$ . It is routine to further perturb these maps to completely positive asymptotic homomorphisms such that  $\varphi_n^{(r)}(1)$ ,  $r = 0, 1$ , are projections so that we can view this maps as ucp maps into matrix subalgebras of  $\mathcal{Q}$ . By [7, Prop.2.5] the Kasparov product  $\bar{\alpha}^\Gamma(x) \otimes_{C^*(\Gamma)} y$  can be computed as

$$(18) \quad (\varphi_n^{(0)})_\#(\bar{\alpha}^\Gamma(x)) - (\varphi_n^{(1)})_\#(\bar{\alpha}^\Gamma(x)) \equiv \bar{\alpha}^\Gamma(x) \otimes_{C^*(\Gamma)} y \neq 0.$$

It follows that there is  $n_0$  such that for each  $n \geq n_0$  there is  $r_n \in \{0, 1\}$  such that  $(\varphi_n^{(r_n)})_\#(\bar{\alpha}^\Gamma(x))$  is nonzero. Then  $\psi_n := \varphi_n^{(r_n)}$  has the desired properties.  $\square$

Any finitely generated linear group  $\Gamma$  is residually finite by Malcev's theorem and exact by [13] and so it satisfies the hypotheses of Theorem 3.2. In particular, this is the case for finitely generated torsion free nilpotent groups [8].

**Corollary 3.3.** *Let  $\Gamma$  be a quasidiagonal group which admits a  $\gamma$ -element and such that  $H_2(\Gamma, \mathbb{Q}) \neq 0$ . Then there is an asymptotic homomorphism  $\{\pi_n : \Gamma \rightarrow U(k_n)\}_n$  for which there exist no genuine representations  $\{\pi'_n : \Gamma \rightarrow U(k_n)\}_n$  such that  $\lim_{n \rightarrow \infty} \|\pi_n(s) - \pi'_n(s)\| = 0$  for all  $s \in \Gamma$ .*

*Proof.* This follows from Theorem 3.2 and Lemma 2.2 as

$$\kappa \left( \prod_{i=1}^g [\pi'_n(\bar{a}_i), \pi'_n(\bar{b}_i)] \right) = 0$$

for genuine representations  $\pi'_n$  of  $\Gamma$ . A more general result proved in [8] asserts that it suffices to assume the nonvanishing of some  $H_{2k}(\Gamma, \mathbb{Q})$ ,  $k \geq 1$ .  $\square$

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