The Oshima-Sekiguchi and Liouville theorems on Heintze groups

Richard C. Penney
Purdue University

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1 Introduction

Let $\mathcal{L}$ be an elliptic operator on a manifold M. A function F is said to be $\mathcal{L}$-harmonic if $\mathcal{L}F = 0$. It was shown by Guivarc'h [11] and Raugi [19] around 1977, that if M is a homogeneous space of a Lie group S and $\mathcal{L}$ commutes with the S-action, then, under some additional technical hypotheses, the *bounded* $\mathcal{L}$-harmonic functions can be characterized as the Poisson integrals of the L^∞ functions on a certain homogeneous space of S against a "Poisson measure" on S. Similar results were developed independently in the context of general (non-homogeneous) manifolds of pinched negative curvature by D. Sullivan [21] and M. Anderson [2] in the early 1980's. In [7], Ewa Damek proved an explicit (and beautiful) version of these results in the case where S is a solvable Lie group. Our current manuscript also relies heavily on the paper [9] of E. Damek, A. Hulanicki, and J. Zienkiewicz as well as the work of R. Urban [22]. (For a (somewhat random) selection related work, see A. Ancona [1], E. Damek and A. Hulanicki [8], A. Korányi and E. Stein [14], R. Azencott and E. Wilson [3] and [4].)

Since the publication of the work of Guivarc'h and Raugi, a truly vast body of literature studying the $\mathcal{L}$-harmonic functions has appeared. The vast majority of these works have studied *bounded* harmonic functions. In fact, to our knowledge, there is no general description of a space of unbounded harmonic functions out side of the case of a rank one symmetric space. (The unbounded *positive* harmonic have also been studied.)

In this work, we provide a complete description of a space of harmonic functions that contains all harmonic functions of "moderate growth" on a very broad class of homogeneous Riemannian manifolds M of negative curvature. Specifically, we say that a function F on M has *moderate growth* if there are positive constants A and r_o (depending on F) such that

$$|F(x)| \leq Ae^{r_o\kappa(x)} \tag{1}$$

for all $x \in M$ where $\kappa(x)$ is the Riemannian distance in M from x to the base point x_o.

The class of manifolds we study is a subclass of the class of "Heintze Manifolds." E. Heintze[13] proved that the complete homogeneous Riemannian manifolds with negative curvature may be characterized as the set of connected, simply connected, solvable Lie groups S satisfying

1. $S = NA$ where N is a nilpotent, normal subgroup and A is isomorphic with $\mathbb{R}^+$.

2. There is an element A_o in the Lie algebra $\mathcal{A}$ of A such that the real part of all of the eigenvalues of $\mathrm{ad}\ A_o$ on the Lie algebra $\mathcal{N}$ of N are positive.

We assume in addition that

$$\delta(t) = \mathrm{Ad}\ (\exp((\log t)A_o)\Big|_N$$

acts diagonally on the Lie algebra $\mathcal{N}$ of N. Hence, there is a basis $X_1, X_2, \ldots, X_n$ of $\mathcal{N}$ such that for all i

$$\delta(t)X_i = t^{d_i}X_i \tag{2}$$

where $d_i > 0$ and

$$d_1 \leq d_2 \leq \cdots \leq d_n.$$

By an appropriate choice of A_0, we may (an will) also assume that $d_n = 1$. We set

$$d = \sum_1^n d_i. \tag{3}$$

We note that any rank one Riemannian symmetric space will satisfy these hypotheses. Of course, the vast majority of examples are non-symmetric.

We identify S with $N \times_s \mathbb{R}^+$ by means of the map

$$(n, t) \to n \exp_S((\log t)A_0).$$

Let R and L denote, respectively, the right and left regular representations of S and R_N and L_N the corresponding regular representations of N. We consider the differential operator $\mathcal{L}$ on S defined by

$$\begin{aligned}\mathcal{L} &= R\left(A_o^2 - \alpha A_o + \sum_1^n (X_i^2 + c_i X_i)\right) \\ &= \Theta^2 - \alpha\Theta + \sum_1^n (t^{2d_i}X_i^2 + c_i t^{d_i} X_i)\end{aligned} \tag{4}$$

where $\alpha > 0$, and, in the second equation, the X_i are considered as left-invariant vector fields on N and where

$$\Theta = t\partial_t.$$

According to a result of Hebisch and Sikora [12], there is a subadditive, smooth, homogeneous norm on N–i.e. there is a function $|\cdot|$ on N such that

$$\begin{aligned} |x| &> 0 \quad x \neq e \\ |\delta(t)x| &= t|x| \\ |xy| &\leq |x| + |y| \\ |x| &= |x^{-1}| \end{aligned} \tag{5}$$

Instead of condition (1) on page 2, we consider the following condition which we call "metric growth". It follows from Lemma 3 on page 20 below that moderate growth implies metric growth. We suspect that the converse is also valid, although we do not require this result.

Definition 1. *A function F on S has metric growth if*

$$|F(x,t)| \leq C(t^a + t^b)(1 + |x|)^k \tag{6}$$

where $k \geq 0$ and $a > b$.

We let

$$\mathcal{H}(a, b, k)$$

denote the set of all $\mathcal{L}$-harmonic functions on S satisfying (6). From the ellipticity of $\mathcal{L}$, it is clear that $\mathcal{H}(a, b, k)$ is a Banach space under the obvious norm.

Our main results are

1. A complete description of all functions which are $\mathcal{L}$-harmonic and "polynomial-like" . (See Definition 3 below.)

2. A "Liouville theorem" that states that an $\mathcal{L}$-harmonic function satisfying a certain growth condition must be a polynomial-like function which vanishes on N.

3. Asymptotic expansions of the Poisson kernel both at $t = 0$ and at $x = \infty$.

4. A representation Theorem (the Oshima-Sekeguci theorem) that states that a $\mathcal{L}$-harmonic function F has metric growth if and only if F is the Poisson integral (suitably defined) of a distribution over N plus a harmonic function of polynomial type which vanishes on N.

Remark. **The majority of our results are valid (with the same proofs) under the weaker assumption that the X_i generate $\mathcal{N}$ as a Lie algebra.** Specifically, Theorems 1, 3, the existence of the limit in Theorem 4 as well as the statement in this theorem beginning with "Conversely," and the first statement in Corollary 1, all hold in this more general context. Also, we use the Hebisch-Sikora norm only for sake of convenience; any homogeneous gauge would suffice. The subadditivity does, however, simplify some proofs.

To discuss these results further, we need a definition. Let A_N be the automorphism group of N. Then A_N is a real algebraic group and S is a subgroup of the real algebraic group $N \times_s A_N$. Let $\overline{S}$ be the algebraic closure of S in $N \times_s A_N$.

Definition 2. *We say that a function $p(x,t)$ on S is a polynomial if it is a restriction of a polynomial on $\overline{S}$ to S. More concretely, then, a function $p(x,t)$ on S will be a polynomial on S if and only if*

$$p(x,t) = \sum_{\beta \in \mathcal{I}} p_\beta(x) t^\beta \tag{7}$$

where $\mathcal{I}$ is a finite subset of

$$\tilde{\mathbb{N}} = \{\sum_1^n m_i d_i \mid m_i \in \mathbb{Z}, m_i \geq 0\} \tag{8}$$

and each $p_\beta(x)$ is a polynomial in the usual sense on N. We refer to $p_\beta(x)$ as the t^β-coefficient of p. The set of all polynomial functions on a particular space X is denoted $\mathcal{P}(X)$.

Definition 3. *Let α be as in (4) on page 3. We say that a function F on S is* polynomial-like *relative to $\mathcal{L}$ if there are polynomial functions (in the sense just defined) p, q, and h on S such that*

$$F(x,t) = \begin{cases} p(x,t) + t^\alpha q(x,t) & \alpha \notin \tilde{\mathbb{N}} \\ p(x,t) + t^\alpha q(x,t) + t^\alpha (\ln t) h(x,t) & \alpha \in \tilde{\mathbb{N}}. \end{cases} \tag{9}$$

Finally, if $p \equiv h \equiv 0$, we say that F is vanishes on N.

Our first main result is the following. (See Definition 5 below for more details.)

Theorem 1. *There is a linear isomorphism between the space of $\mathcal{L}$-harmonic, polynomial-like functions on S and $\mathcal{P}(N) \times \mathcal{P}(N)$. Thus the space of $\mathcal{L}$-harmonic, polynomial-like functions is infinite dimensional.*

In the theory of partial differential equations, a Liouville theorem for a linear differential operator D is a theorem that states that, under the correct interpretation of the term "polynomial," if $DF = 0$, then F is a polynomial if and only if F grows like a polynomial.

A simple, but enlightening, example is the case where $N = \mathbb{R}$, $\delta(t)x = tx$, and $\alpha = 1$. In this case S is identifiable with the upper-half plane H^+ in $\mathbb{C}$ and

$$\mathcal{L} = t^2 \left(\partial_t^2 + \partial_x^2\right)$$

which is just the invariant Laplacian on H^+. There are, of course, many non-constant bounded harmonic functions on H^+. (The Poisson integral of any L^1 function is an example.) Hence, we cannot expect a Liouville theorem in which "polynomial" has its standard meaning. However, it can be shown that the Liouville Theorem holds for H^+ *provided we only consider polynomials* $p(x,t)$ *where* $p(x,0) \equiv 0$. Specifically, we are saying that if F is harmonic on H^+ and satisfies an estimate of the form

$$|F(x,t)| \leq C(t^a + t^b)(1 + |x|)^k \tag{10}$$

where $a, b, k \in \mathbb{R}^+$, then

$$F(x,t) = tp(x,t)$$

for some polynomial $p(x,t)$. An example of such a harmonic function is $F(z) = \Im(z^n)$.

We prove that virtually the same result is true in our more general case. Specifically, we prove the following strong "Liouville Theorem" which does not even require harmonicity. Here $|\cdot|$ any sub-additive homogeneous norm on N, although the same result holds for any homogeneous gauge.

Theorem 2. *Suppose that F is a function on S that satisfies an estimate of the form (10) such that $\mathcal{L}F(x,t)$ depends polynomially on x. Then F depends polynomially on x. If in addition $\mathcal{L}F = 0$, then F is a polynomial-like harmonic function that vanishes on N in the sense of Definition 3 on page 5.*

Remark. It can be shown that the assumption of harmonicity in the final statement in the preceding theorem is unnecessary; if $\mathcal{L}F$ is polynomial in

x, then F is of polynomial type. We do not present the proof since it requires dealing with non-homogeneous equations in Section 2 which would complicate the exposition unnecessarily.

The starting point of the proof of Theorem 2 is an idea due to Geller [10]. Geller's idea, however, only provides a beginning. Our proof is somewhat lengthy and complicated. In particular it makes essential use of R. Urban's deep results on the growth of the derivatives of the Green kernel [22], our theory of asymptotic expansions, as well as a new asymptotic expansion of the Euclidean Fourier transformation of a harmonic function. Our exposition of the latter two results is self contained, although the work is motivated by works of N. Wallach [23] and E. van den Ban and H. Schlichtkrull [5]. Similar (but less precise) results are found in R. Penney and R. Urban [18] and R. Penney [17].

As mentioned previously, all rank one Riemannian symmetric spaces satisfy our hypotheses. If M is such a space, and $\mathcal{L}$ is the corresponding Laplace-Beltrami operator, then the Oshima-Sekiguchi Theorem [16] states that a harmonic function F on M has moderate growth if and only if F is the Poisson integral of a distribution over the Furstenberg boundary. It is key to the Oshima-Sekiguchi Theorem that the Furstenberg boundary is a compact manifold since this allows one to "integrate" an arbitrary distribution against the Poisson kernel.

For example, consider once again the case of the Laplace-Beltrami operator on the upper-half plane $\mathbb{R}\times\mathbb{R}^+$. In this case the Poisson kernel is

$$P(n,z) = -\Im\left(\frac{1}{z-n}\right)$$

where $n\in\mathbb{R}$ and $z = x+it\in\mathbb{R}\times\mathbb{R}^+$. The boundary value of the harmonic function $F(z) = \Re(z^k)$ is $f(n) = n^k$ which will not be integrable against $P(n,z)$ for $k\geq 1$.

We can avoid this difficulty as follows. Note that for $|n|>|z|$

$$-\frac{1}{z-n} = \sum_0^\infty \frac{z^k}{n^{k+1}}.$$

For $n\neq 0$, let

$$Q_m(n,z) = \Im\left(\sum_0^m \frac{z^k}{n^{k+1}}\right). \tag{11}$$

Then, for eacn $n \neq 0$, $Q_m(n,z)$ is a harmonic polynomial in z which is zero on the real axis. Let

$$P_m(n,z) = P(n,z) - Q_m(n,z).$$

From Taylor's Theorem, for all $m \in \mathbb{N}$ there are positive scalars a and C such that

$$|\partial_n^i P_m(n,z)| \leq C(1+|z|)^a |n|^{-(i+m+1)}.$$

It is easily seen that if ϕ is an Schwartz distribution on $\mathbb{R}$ whose support does not contain 0, then there is an $m \in \mathbb{N} \cup \{0\}$ such that

$$P_m(\phi)(z) =< \phi(\cdot), P_m(\cdot, z) >$$

is defined and harmonic in z. If ϕ has arbitrary support, then $\phi = \phi_1 + \phi_2$ where 0 is not in the support of ϕ_1 and ϕ_2 has compact support containing 0. We define the "mollified" Poisson integral of ϕ to be

$$P_m^{mol}(\phi)(z) = P_m(\phi_1)(z) + P(\phi_2)(z)$$

where m and $P_m(\phi_1)$ are as described above and $P(\phi_2)$ is the usual Poisson integral. It is clear that $P_m^{mol}(\phi)$ is a harmonic function.

Of course, $P_m^{mol}(\phi)$ is not uniquely defined; it depends on both the choice of m and the choice of ϕ_1 and ϕ_2. However, it is easily seen (see Proposition 1 on page 11 below) that $P_m^{mol}(\phi)(z)$ is uniquely determined modulo harmonic polynomials with null boundary value, i.e. if $\tilde{P}_{m'}^{mol}(\phi)$ is another mollified Poisson integral of ϕ resulting from possibly different choices of the ϕ_i and of m, then

$$\tilde{P}_{m'}^{mol}(\phi)(z) = P_m^{mol}(\phi)(z) + h(z)$$

where h is a harmonic polynomial on H^+ which is identically 0 on $\mathbb{R}$. The Oshima-Sekiguchi Theorem on the unit disk is equivalent with the statement that a harmonic function F on the upper-half plane has metric growth if and only if $F = P_m^{mol}(\phi) + h$ for some Schwartz distribution ϕ on $\mathbb{R}$ and a harmonic polynomial h on H^+ where h is zero on the real axis.

In this work we prove that the analogous result holds in our context. Specifically, it follows from work of E. Damek [7] that the space of bounded $\mathcal{L}$-harmonic functions is in one-to-one correspondence with $L^\infty(N)$. Explicitly, there is a C^∞ function P on $S/(\mathbb{R}^+) = N$ (the Poisson kernel function for $\mathcal{L}$)

such that every bounded harmonic function F may be written as a Poisson integral as

$$\begin{aligned} F(x,t) &= \int_{S/(\mathbb{R}^+)} f((x,t)n)P(n)\,dn \\ &= \int_N f(n)P(\delta(t^{-1})(x^{-1}n))t^{-d}\,dn \\ &= f * \tilde{P}_t(x) \\ &\equiv P(f)(x,t) \end{aligned} \tag{12}$$

where dn is Haar measure on N and

$$\begin{aligned} \tilde{P}_t(n) &= P(t^{-1}n^{-1}t)t^{-d} = \tilde{P}(n,t) \\ f(n) &= \lim_{t\to 0^+} F(n,t) \quad a.e. \end{aligned} \tag{13}$$

(Here d is as in (3) on page 3.)

Conversely the Poisson integral of any $f \in L^\infty(N)$ is a bounded $\mathcal{L}$-harmonic function on S such that for a.e. $n \in N$,

$$f(n) = \lim_{t\to 0^+} P(f)(n,t).$$

One of our main results generalizes the expansion (11). Recall that a function p on $N - \{0\}$ is said to be $\delta(t)$-homogeneous of homogeneous degree β if for all $t \in \mathbb{R}^+$

$$p(\delta(t)x) = t^\beta p(x).$$

Note that then for all $x \in N - \{0\}$

$$|p(x)| = |x|^\beta p(\delta(|x|^{-1})x) \leq C|x|^\beta$$

where $C = \sup_{|u|\leq 1} |p(u)|$ and $|\cdot|$ is any homogeneous gauge–e.g. the Hebisch and Sikora norm from (5).

In the following result, $\tilde{\mathbb{N}}$ is as defined in (8) on page 5. If $I = (i_1, i_2, \dots, i_n)$ is a multi-index we define

$$\begin{aligned} \|I\| &= \sum_{j=1}^n d_j i_j \\ X^I &= X_1^{i_1} X_1^{i_1} \dots X_1^{i_1}. \end{aligned} \tag{14}$$

Theorem 3. *There is a sequence $H_\beta(n,x,t) \in C^\infty((N - \{0\}) \times N \times \mathbb{R}^+)$ indexed by $\tilde{\mathbb{N}}$, which is uniquely defined by the following properties:*

1. *For each $\beta \in \tilde{\mathbb{N}}$ and each $n \in N - \{0\}$, $H_\beta(n,x,t)$ is a polynomial in (x,t) (in the sense of Definition 7 on page 5) which satisfies*

$$\begin{aligned} H_\beta(n,\delta(s)x,st) &= s^\beta H_\beta(n,x,t) \\ H_\beta(\delta(s)n,x,t) &= s^{-\beta-d-\alpha} H_\beta(n,x,t) \end{aligned}$$

2. *For each $\beta \in \tilde{\mathbb{N}}$ and each $n \in N - \{0\}$, $t^\alpha H_\beta(n,x,t)$ is $\mathcal{L}$-harmonic in (x,t).*

3. *For $\mu \in \tilde{\mathbb{N}}$, let*

$$P_\mu(n,x,t) = \tilde{P}(nx,t) - t^\alpha Q_\mu(n,x,t)$$

where

$$Q_\mu(n,x,t) = \sum_{\beta<\mu} H_\beta(n,x,t).$$

Then, for all $\beta \in \tilde{\mathbb{N}}$ and each milti-index I

$$\begin{aligned} &|R(X^I)P_\mu(n,x,t)| \\ &\qquad \le C(t^a+t^b)(1+|x|)^\tau |n|^{-d-\alpha-\mu-\|I\|} \end{aligned} \tag{15}$$

for $|n| \ge 1$ where $a,b,\tau \in \mathbb{R}$.

Theorem 3 allows us to define a "mollified Poisson integral" for any Schwartz distribution f on N. It follows from Lemma 7 on page 31 below that the following semi-norms for $p,k \in \mathbb{N} \cup \{0\}$, are finite on $\mathcal{S}(N)$ and define the Schwartz topology on $\mathcal{S}(N)$:

$$\|\eta\|_{p,k} = \sup\{(1+|x|)^{-k} |R_N(X^I)\eta(x)| \mid |I| \le p\}. \tag{16}$$

Let $f \in \mathcal{S}'(N)$. Then there are p and k such that

$$| < f(\cdot), \eta(\cdot) > | \le C\|\eta\|_{p,k}$$

for all $\eta \in \mathcal{S}(N)$. Hence, f extends continuously to the space $\mathcal{S}_{p,k}(N)$ consisting of all functions η such that $\|\eta\|_{p,k} < \infty$.

Let $\phi \in C_c^\infty(N)$ be supported in $\{|n| \le 2\}$ and satisfy

$$\begin{aligned} 0 \le &\phi(n) \le 1 \\ \phi(n) &= 1, \quad |n| \le 1 \end{aligned}$$

Let

$$\tilde{\phi} = 1 - \phi.$$

From Theorem 3, for $\mu \geq k$ and $(x,t) \in S$,

$$n \to \tilde{\phi}(n) P_\mu(n^{-1}, x, t) \in \mathcal{S}_{p,k}(N).$$

Definition 4. Let ϕ and $\tilde{\phi}$ be described above. Let $f \in \mathcal{S}'(N)$ and let p and k be such that $f \in \mathcal{S}'_{p,k}(N)$. Let μ be chosen as above. We define the mollified Poisson integral of f by

$$P^{mol}_{\mu,\phi}(f)(x,t) =< f(\cdot), \tilde{\phi}(\cdot) P_\mu((\cdot)^{-1}, x, t) > + < f(\cdot), \phi(\cdot)\tilde{P}((\cdot)^{-1}x, t) > .$$

It is clear from Theorem 3 that for $f \in \mathcal{S}'_{p,k}(N)$ and $\mu > k$, $P^{mol}_{\mu,\phi}(f)$ is a harmonic function of metric growth. The mollified Poisson integral, of course, depends on μ and ϕ. However, the following simple result shows that it is is well defined, modulo harmonic, polynomial-like functions that vanish on N.

Proposition 1. *Let $f \in \mathcal{S}'(N)$. Then for all (β, ψ) chosen in the same manner as (μ, ϕ) above, there is a polynomial (in the sense of Definition 2 on page 5) $h(x,t)$ where $t^\alpha h(x,t)$ is $\mathcal{L}$-harmonic, such that*

$$P^{mol}_{\beta,\psi}(f)(x,t) = P^{mol}_{\mu,\phi}(f)(x,t) + t^\alpha h(x,t).$$

Proof Note that $\tilde{\phi} - \tilde{\psi} = -(\phi - \psi)$ which has compact support. Hence

$$\begin{aligned}
&P^{mol}_{\mu,\phi}(f)(x,t) - P^{mol}_{\mu,\psi}(f)(x,t) \\
&=< (\tilde{\phi} - \tilde{\psi})(\cdot)(\tilde{P}((\cdot)^{-1}x, t) - t^\alpha Q_\mu((\cdot)^{-1}, x, t)), f > \\
&\quad + < (\phi - \psi)(\cdot)\tilde{P}((\cdot)^{-1}x, t), f > \\
&= t^\alpha < (\phi - \psi)(\cdot) Q_\mu((\cdot)^{-1}, x, t), f >
\end{aligned}$$

which is a harmonic function of the form of $t^\alpha h(x,t)$ where h is a polynomial.

Also, for $\beta > \mu$, let

$$Q_{\beta,\mu}(n, x, t) = Q_\beta(n, x, t) - Q_\mu(n, x, t).$$

Then

$$P^{mol}_{\beta,\phi}(f)(x,t) = P^{mol}_{\mu,\phi}(f)(x,t) - t^\alpha < f(\cdot), \phi(\cdot) Q_{\beta,\mu}((\cdot)^{-1}, x, t) > .$$

The term on the right is again a harmonic function of the form of $t^\alpha h(x,t)$ where h is a polynomial. This proves the proposition. $\square$

Let F be a harmonic function of metric growth. It follows from Lemma 7 on page 31 below that for all $t \in \mathbb{R}^+$, $F(\cdot,t)$ may be considered as a Schwartz distribution on N. We prove the following Oshima-Sekiguchi type theorem. See Definition 3 for the terminology.

Theorem 4. *Let F be an $\mathcal{L}$-harmonic function of metric growth. Then for each $t \in \mathbb{R}^+$,*

$$\lim_{t\to 0^+} F(\cdot,t) = f(\cdot) \tag{17}$$

exists in $\mathcal{S}'(N)$.

Furthermore, for any choice of (μ,ϕ) as described above, there is a polynomial-like harmonic function $t^\alpha h(x,t)$ on S that vanishes on N such that

$$F(x,t) = P^{mol}_{\mu,\phi}(f)(x,t) + t^\alpha h(x,t). \tag{18}$$

Conversely, for any distribution $f \in \mathcal{S}'(N)$ and any polynomial-like harmonic function $t^\alpha h(x,t)$ which vanishes on N, the above formula defines an $\mathcal{L}$-harmonic function of metric growth for which

$$\lim_{t\to 0^+} F(\cdot,t) = f(\cdot)$$

in $\mathcal{S}'(N)$.

Remark. We commented that in the case of the upper half-plane, the preceding theorem is equivalent with the Oshima-Sekiguchi Theorem on the unit disk. The same remains true in the case of a rank-one symmetric space, although we do not present a proof. We conjecture that there is a compactification of S for which the above theorem implies that the $\mathcal{L}$-harmonic functions of metric growth are precisely the Poisson integrals of distributions over an appropriate boundary. We have, in fact, proved such a result in the case that S is meta-abelian and algebraic: i.e. all of the d_i are rational numbers and N is abelian. Such a result is desirable in that it eliminates the ambiguity in defining the Poisson integral expressed in Proposition 1. We will discuss these ideas in depth at a later time.

We note the following interesting corollary of the preceding theorem.

Corollary 1. *Every $f \in \mathcal{S}'(N)$ is the boundary value of an $\mathcal{L}$-harmonic function $F(x,t)$ of metric growth in the sense that in $\mathcal{S}'(N)$*

$$\lim_{t\to 0^+} F(\cdot,t) = f(\cdot).$$

F is uniquely determined by f modulo harmonic polynomial-like functions which vanish on N.

A sketch of the proof of Theorem 4 is as follows. We provide the details in Section 5.

1. We use the theory of asymptotic expansions to prove the existence of the limit (17). (Corollary 8 on page 39.)

2. Let $\tilde{F} = F - P^{mol}_{\mu,\phi}(f)$. We show that $\lim_{t\to 0^+} \tilde{F}(\cdot,t) = 0$ in $\mathcal{S}'(N)$.

3. We show that in fact, $\tilde{F}$ satisfies the hypotheses of our Liouville Theorem (Theorem 2 on page 6); hence $\tilde{F}$ is a polynomial-like function which vanishes on N, proving Theorem 4.

1.1 Notational Conventions:

1. We use upper case Roman letters to denote Lie groups and the corresponding upper case script letter to denote the Lie algebra.

2. We use the exponential mapping to identify $\mathcal{N}$ and N–i.e. we assume that $N = \mathcal{N}$ where $\mathcal{N}$ is given the Campbell-Hausdorff product.

3. R_N and L_N denote, respectively, the right and left regular representations of N.

4. Throughout this work symbols such as "C", "C_i", "C'", etc. represent generic constants whose values can change from line to line.

5. The meaning of "$\|\cdot\|$" is "local" and will vary depending on context. We will attempt to clarify the meaning in any situation where confusion might arise.

6. The symbol $|\cdot|$ always denotes a subadditive semi-norm on N–i.e. a function satisfying (5) on page 4.

7. The set $\tilde{\mathbb{N}}$ defined by (8) on page 5.

8. Any summation in which the summation index is a lowercase Greek letter is always a summation over some subset of $\tilde{\mathbb{N}}$.

9. The symbols d_i, $1 \leq i \leq n$, represent the scalars defined in (2) on page 3 and $d = \sum d_i$. *We assume (with out loss of generality) that* $d_n = 1$.

10. The symbol "α" always refers to the number from equation (4) on page 3.

11. If $I = (i_1, i_2, \ldots, i_n)$ is a multi-index, then $\|I\|$ and X^I are defined in (14) on page 9 while $|I| = i_1 + i_2 + \cdots + i_n$.

The significance of d is that for integrable functions f on N

$$\int_N f(x)\,dx = \int_N f(\delta(t)x)t^d\,dx.$$

It is an important consequence of this equality that there is a finite measure ν on $\{|x| = 1\}$ such that

$$\int_N f(x)\,dx = \int_0^\infty \int_{|u|=1} f(\delta(t)u)t^{d-1}\,d\nu(u)dt. \tag{19}$$

2 Formal Harmonic Functions

In this section we study formal series of the form

$$F(\cdot, t) = \sum_{i=0}^{\infty} t^{\alpha_i} A_{\alpha_i}(\cdot) \tag{20}$$

where $\alpha_0 < \alpha_1 < \ldots, \alpha_i < \ldots$ and $A_{\alpha_i} \in \mathcal{S}'(N)$, $A_{\alpha_i} \neq 0$. We wish to find conditions under which such a series satisfies $\mathcal{L}F = 0$ where $\mathcal{L}$ is as in (4) on page 3. Actually, instead of $\mathcal{S}'(N)$ we may use any $\mathfrak{A}(\mathcal{N})$ module $\mathcal{V}$, in which case we suppress x, writing $F(t)$ instead of $F(\cdot, t)$ and A_{α_i} in place of $A_{\alpha_i}(\cdot)$. As the reader will certainly note, our discussion is modeled on the classical Frobenius theory of ODEs with regular singularities at 0.

Substituting (20) into $\mathcal{L}F = 0$ and equating like powers of t yields the recursion relation

$$\alpha_k(\alpha_k - \alpha)A_{\alpha_k} = -\sum_{i=1}^{n} X_i^2 A_{\alpha_k - 2d_i} - \sum_{i=1}^{n} c_i X_i A_{\alpha_k - d_i}$$

(If $\beta \neq \alpha_j$ for some j, we set $A_\beta = 0$.)

We say that A_{α_j} *is used in the computation of* A_{α_k} if $\alpha_k \in \alpha_j + \tilde{\mathbb{N}}$. Intuitively, this means that in using the recursion relation to compute A_{α_k}, we will eventually need to know A_{α_j} . For α_k given, let j be the smallest index for which A_{α_j} is used in the computation of A_{α_k}. From the recursion relation, $\alpha_j(\alpha_j - \alpha) = 0$ so either $\alpha_j = 0$ or $\alpha_j = \alpha$. Hence

$$\alpha_k \in \tilde{\mathbb{N}} \cup (\alpha + \tilde{\mathbb{N}})$$

showing that we can express the expansion (20) in the more explicit form

$$F(t) = \sum_{\beta \in \tilde{N}} t^\beta F_\beta + t^\alpha \sum_{\beta \in \tilde{N}} t^\beta G_\beta. \tag{21}$$

Our first result is:

Proposition 2. *Suppose that the formal series (20) satisfies $\mathcal{L}F = 0$. Then F is expandable as a series of the form of (21) where each of the two summations on the right is $\mathcal{L}$-harmonic. In this case the coefficients satisfy (22) and (23) below.*

Proof If $(\alpha + \tilde{\mathbb{N}}) \cap \tilde{\mathbb{N}} = \emptyset$, then the coefficients in (21) are unique. Substitution into $\mathcal{L}F = 0$ and equating coefficients of like powers of t yields the recursion relations

$$(\beta + \alpha)\beta G_\beta = -\sum_{i=1}^{n} X_i^2 G_{\beta - 2d_i} - \sum_{i=1}^{n} c_i X_i G_{\beta - d_i} \tag{22}$$

and

$$\beta(\beta - \alpha)F_\beta = -\sum_{i=1}^{n} X_i^2 \quad F_{\beta - 2d_i} - \sum_{i=1}^{n} c_i X_i F_{\beta - d_i} \tag{23}$$

from which the result follows.

If $(\alpha + \tilde{\mathbb{N}}) \cap \tilde{\mathbb{N}} \neq \emptyset$, then the coefficients in (21) will not be unique and the above recursion relations might not hold. Notice, however, that if none of the α_j in (20) equal α, then for each j, A_0 is used in computing A_{α_j}. Hence,

$\alpha_j \in \tilde{\mathbb{N}}$ for all k, showing that we may combine the second sum in (21) with the first, in which case our result follows trivially.

If there is a value j_o such that $\alpha_{j_o} = \alpha$, we set

$$G_0 = A_\alpha.$$

We then define G_β for $\beta > 0$, $\beta \in \tilde{\mathbb{N}}$, inductively by (22). This is possible since for $\beta \neq 0$, $(\beta + \alpha)\beta \neq 0$.

It is clear that as a formal series, under this definition, the term on the right in (21) is harmonic. In this case

$$\tilde{F}(t) = F(t) - t^\alpha \sum_{\beta \in \tilde{N}} t^\beta G_\beta$$

is a harmonic series where the coefficient of t^α in (20) on page 14 vanishes. From the comments immediately following (23), $\tilde{F}$ may be expressed as a sum of the form of (21) where the terms involving G_β vanish. The F_β are uniquely determined by F_0 from (23). Our result follows. □

Conversely, given an element $G_0 \in \mathcal{V}$, we may use (22) to define coefficients G_β which may be summed as on the right in (21) on page 15 to yield a harmonic series $G(t)$. *It follows that for any given $G_0 \in \mathcal{V}$, there is a harmonic series $G(t)$ of the form of the sum on the right in (21) on page 15.* Similarly, if $\alpha \notin \tilde{\mathbb{N}}$, the F_β may also be defined for all β and are uniquely determined. Hence we have the following result:

Proposition 3. *If $\alpha \notin \tilde{\mathbb{N}}$ then for any given elements F_0 and G_0 in $\mathcal{V}$, the relations (22) and (23) may be solved to yield a unique harmonic series (21) on page 15.*

If $\alpha \in \tilde{\mathbb{N}}$, the situation is somewhat more complicated. At $\beta = \alpha$, there will be no solution to (23) unless the term on the right is zero. The simplest way for this to happen is to have $F_\sigma = 0$ for all $\sigma < \alpha$, which produces a series having the form of the sum on the right in (21). Hence, we obtain nothing new in this fashion. In order to find a second class of solutions, we are forced to introduce logarithmic terms. Hence, we assume a solution of the form

$$F(t) = \sum_{\beta \in \tilde{\mathbb{N}}} t^\beta F_\beta + (\ln t) t^\alpha \sum_{\beta \in \tilde{\mathbb{N}}} t^\beta H_\beta \tag{24}$$

where $F_\alpha = 0$.

We substitute (24) into $\mathcal{L}F = 0$ and equate coefficients of both the logarithmic and the power terms, finding

$$\begin{aligned}(\beta+\alpha)\beta H_\beta &= -\sum_{i=1}^{n} X_i^2 H_{\beta-2d_i} - \sum_{i=1}^{n} c_i X_i H_{\beta-d_i}\\ \beta(\beta-\alpha)F_\beta &= -(2\beta-\alpha)H_{\beta-\alpha}\\ &\quad -\sum_{i=1}^{n} X_i^2 F_{\beta-2d_i} - \sum_{i=1}^{n} c_i X_i F_{\beta-d_i}.\end{aligned} \tag{25}$$

We claim that for F_0 given, these relations have a unique solution satisfying $F_\alpha = 0$. In fact, for $\beta < \alpha$, $H_{\beta-\alpha} = 0$ so the second equation determines F_β for all $\beta < \alpha$. For $\beta = \alpha$, the second equation uniquely determines H_0. The first equation then determines H_β for all $\beta > 0$, from which the remaining F_β may be determined using the second equation again. We have proved the following result.

Proposition 4. *Suppose that $\alpha \in \tilde{\mathbb{N}}$ and that F_0 and G_0 are given elements of $\mathcal{V}$. Then there is unique harmonic series of the form*

$$F(t) = \sum_{\beta\in\tilde{\mathbb{N}}} t^\beta (F_\beta + t^\alpha(\ln t)H_\beta) + t^\alpha \sum_{\beta\in\tilde{\mathbb{N}}} t^\beta G_\beta \tag{26}$$

where $F_\alpha = 0$, the F_β and H_β are determined by (25), and the G_β are determined by (22). Furthermore, each of the above sums is, by itself, a harmonic series.

The following definition uses the preceding propositions.

Definition 5. For $(F_0, G_0) \in \mathcal{V}\times\mathcal{V}$ let $P(F_0, G_0)(t) = F(t)$ where $F(t)$ is the formal harmonic series defined in Proposition 3 on page 16 ($\alpha \notin \tilde{\mathbb{N}}$) or in Proposition 4 on page 17 ($\alpha \in \tilde{\mathbb{N}}$). We call P the "Poisson transformation."

Corollary 2. *Suppose that $F(t)$ is a formal harmonic series for which $F_0 = 0$. Then*

$$F(t) = t^\alpha \sum_{\beta\in\tilde{\mathbb{N}}} t^\beta G_\beta.$$

Proof This follows from $P(F_0, G_0)(t) = F(t)$. □

We say that an element $q \in \mathcal{V}$ is nilpotent if there is an m such that

$$Y_1 Y_2 \dots Y_m q = 0$$

for all sequences $Y_1, \dots, Y_m$ of m elements of $\mathcal{N}$. The space of nilpotent elements is denoted $\mathcal{P}(\mathcal{N})$ due to the following lemma.

Lemma 1. *Let $\mathcal{V} = \mathcal{S}'(N)$ with the $\mathfrak{A}(N)$ action defined by either the right or the left regular representations. Then $\mathcal{P}(\mathcal{V})$ is the space $\mathcal{P}(N)$ of all polynomials on N.*

Proof

This result is true for any nilpotent group, homogeneous or not. However, the proof is particularly simple in our homogeneous case since any polynomial function on N may be written as a finite sum of homogeneous polynomials and the X_i from the basis (2) on page 3 decrease the homogeneous degree of a homogeneous polynomial. Hence, the polynomials are nilpotent elements.

Conversely, suppose that $f \in \mathcal{S}'(N)$ is nilpotent. Let X_n be the last basis element form the basis (2) on page 3. Then $X_n^m f = 0$ implies that f is a polynomial in x_n. Hence, we may write

$$f(x, x_n) = \sum_{k=0}^{p} x_n^k f_k(x)$$

where $x = (x_1, x_2, \dots, x_{n-1})$ and the f_i are distributions on $\mathbb{R}^{n-1}$. It is easily seen that f_p may be identified with a nilpotent distribution on $N/(\mathbb{R}X_n)$. By induction, f_p may be assumed to be a polynomial in x. We replace f by $f - x_n^p f_p$ and repeat this argument as many times as necessary to prove that f is a polynomial. □

We say that a $\mathcal{V}$-valued function on $\mathbb{R}^+$ is a polynomial if it is described by a formula such as (7) on page 5 where the p_β are elements of $\mathcal{P}(\mathcal{V})$. We then define the notion of polynomial-like just as in Definition 3 on page 5. The following result, which is a simple consequence of the above discussion, provides a complete characterization of the polynomial-like harmonic functions.

Theorem 5. *The Poisson transformation P from Definition 5 on page 17 defines a bijection between $\mathcal{P}(\mathcal{V}) \times \mathcal{P}(\mathcal{V})$ and the space of all $\mathcal{V}$-valued polynomial-like harmonic functions.*

There is a sense in which the formal series (21) on page 15 and (26) on page 17 converge to F asymptotically under appropriate assumptions. See Theorem 7 on page 36 below.

3 Riemannian Metric.

Let ω be a scalar product on $\mathcal{S}$. For $g \in S$, we identify the tangent space $T_g(S)$ with $\mathcal{S}$ using the *left*-invariant vector fields. Then ω may be considered as a left invariant Riemannian structure on S. Conversely, any left-invariant Riemannian structure arises in this manner.

We wish to estimate

$$\tau(g) = e^{\kappa(g)}.$$

where $\kappa(g) = d(g, e)$ is the Riemannian distance to e.

It follows from the left invariance of the metric that

$$\begin{aligned} \tau(x) &= \tau(x^{-1}) \\ \tau(xy) &\leq \tau(x)\tau(y). \end{aligned} \tag{27}$$

Lemma 2. *There is an $a > 0$ such that for all $t \in \mathbb{R}^+$, $\tau((0,t)) \leq t^a + t^{-a}$.*

Proof It suffices to assume $t \geq 1$. Let $\gamma : (1, t) \to S$ be defined by

$$\gamma(r) = (0, r) \in N \times \mathbb{R}^+.$$

For $g \in S$, let $\lambda_g : S \to S$ be left translation by g. Then

$$\begin{aligned} \gamma'(r) &= \frac{1}{r}\frac{d}{ds}\Big|_{s=1}\gamma(rs) \\ \gamma'(r) &= \frac{1}{r}\frac{d}{ds}\Big|_{s=1}(\lambda_{(0,r)}(\gamma(s)) \\ &= \frac{1}{r}\lambda^*_{(0,r)}\frac{d}{ds}\Big|_{s=1}\gamma(s) \\ &= \frac{1}{r}A_o. \end{aligned}$$

Hence, from the left invariance of the metric,

$$\begin{aligned} l(\gamma) &= \int_1^t \omega_{\gamma(r)}(\gamma'(r), \gamma'(r))^{\frac{1}{2}}\, dr \\ &= \|A_o\| \int_1^t r^{-1}\, dr \\ &= \|A_o\| \log t. \end{aligned}$$

where $\|\cdot\|$ is the ω-norm. Hence

$$d((0,t),(0,1)) \leq \|A_o\| \log t$$

which implies our lemma. $\square$

Lemma 3. *There are positive scalars C, a, and b such that*

$$\tau(n,t) \leq C(1+|n|)^b(t^a + t^{-a}).$$

Proof If $|n| \leq 1$, from (27),

$$\tau(n,t) \leq \tau(n,0)\tau(0,t)$$

and our lemma follows from Lemma 2 and the compactness of $\{|n| \leq 1\}$.

For $n \geq 1$, write

$$n = \delta(|n|^{-1})u$$

where $|u| = 1$. Then

$$(n,t) = (0,|n|^{-1})(u,t)(0,|n|).$$

Hence, from Lemma 2 and (27),

$$\tau(n,t) \leq (|n|^a + |n|^{-a})^2\tau(u,t).$$

Our lemma now follows from the $|n| \leq 1$ case. $\square$

4 The Poisson Kernel

4.1 Asymptotic Expansion of $\tilde{P}(x,t)$

The proof of Theorem 3 on page 9 involves combining an asymptotic expansion with a homogeneous Taylor expansion. In this section we present the asymptotic expansion.

According to Theorem (5.1), p. 144 of [9], for each multi-index I there is a constant C such that

$$|R_N(X^I)P(x)| \leq C(1+|x|)^{-d-\alpha-\|I\|}. \tag{28}$$

Proposition 5. *In formula (28) we may replace $L_N(X^I)$ by $R_N(X^I)$.*

Proof From (28),

$$\begin{aligned} |R_N(X^I)(P\circ\delta(t^{-1}))(x)| &= t^{-\|I\|}|(R_N(X^I)P)(\delta(t^{-1})(x))| \\ &\le Ct^{-\|I\|}(1+t^{-1}|x|)^{-d-\alpha-\|I\|} \\ &= Ct^{d+\alpha}(t+|x|)^{-d-\alpha-\|I\|} \\ &\le Ct^{d+\alpha}|x|^{-d-\alpha-\|I\|} \end{aligned} \tag{29}$$

Hence, for $|x|\ge 1$,

$$|R_N(X^I)(P\circ\delta(t^{-1}))(x)| \le Ct^{d+\alpha} \tag{30}$$

Conversely, suppose that we know that the above inequality holds for all $t\in\mathbb{R}^+$, but only for $1\le|x|\le 2$. We claim that inequality (28) follows. To see this note that it suffices to prove (28) for $|x|\ge 1$. Then there is an $n\in\mathbb{N}\cup\{0\}$ such that $2^n\le|x|\le 2^{n+1}$. Hence $1\le|\delta(2^{-n})x|\le 2$ and

$$\begin{aligned} |R_N(X^I)P(x)| &= |(R_N(X^I)P)\circ\delta(2^n)(\delta(2^{-n})(x))| \\ &= 2^{-n\|I\|}|R_N(X^I)(P\circ\delta(2^n))(\delta(2^{-n})(x))| \\ &\le 2^{-n\|I\|}C2^{-n(d+\alpha)} \\ &= C2^{d+\alpha+\|I\|}(2^{n+1})^{-d-\alpha-\|I\|} \\ &\le C'|x|^{-d-\alpha-\|I\|} \end{aligned}$$

proving (28).

By similar reasoning, *it suffices to prove that for* $1\le|x|\le 2$

$$|L_N(X^I)(P\circ\delta(t^{-1}))(x)| \le Ct^{d+\alpha}.$$

for all $t\in\mathbb{R}^+$. This, however, is simple. For each i there are polynomials $p^i_j(x)$ such that

$$L_N(X_i) = \sum_{j\ge i} p^i_j(x)R_N(X_j).$$

More generally, we may write

$$L_N(X^I) = \sum_{|J|=|I|} p_J(x)R_N(X^J). \tag{31}$$

Our inequality for $1 \leq |x| \leq 2$ follows from (29) and the fact that the p_J are bounded on $1 \leq |x| \leq 2$. $\square$

It follows from Proposition 9 and formula (13) on page 9 that

$$\begin{aligned} |R_N(X^I)\tilde{P}(x,t)| &= t^{-d-\|I\|}|(L_N(X^I)P)(\delta(t^{-1})x^{-1})| \\ &\leq Ct^{\alpha}(1+|\delta(t^{-1})x^{-1}|)^{-(d+\alpha+\|I\|)}t^{-d-\alpha-\|I\|} \\ &= Ct^{\alpha}(t+|x|)^{-(d+\alpha+\|I\|)} \end{aligned} \tag{32}$$

Corollary 3. *Suppose that I is a multi-index such that $m \leq |I| \leq 2m$ where $m \in \mathbb{N}$. Then if either $t^{-1}|x| \leq 1$ or $|x| \geq 1$,*

$$|t^{\|I\|}R_N(X^I)\tilde{P}(x,t)| \leq t^{\alpha}\xi(t)^{md_1}|x|^{-(d+\alpha+md_1)}$$

where

$$\xi(t) = t + t^{2/d_1}.$$

Proof Note that since for all i, $d_1 \leq d_i \leq d_n = 1$,

$$d_1|I| \leq \|I\| \leq |I|. \tag{33}$$

Hence $\|I\| \in [d_1 m, 2m]$. For $|x| \geq 1$, our corollary follows from (31). If $t^{-1}|x| \leq 1$, we note that

$$\begin{aligned} (t+|x|)^{-(d+\alpha+\|I\|)} &\leq |x|^{-(d+\alpha+\|I\|)} \\ &= t^{-(d+\alpha+\|I\|)}(t^{-1}|x|)^{-(d+\alpha+\|I\|)} \\ &\leq t^{-(d+\alpha+\|I\|)}(t^{-1}|x|)^{-(d+\alpha+d_1 m)} \\ &= t^{d_1 m-\|I\|}|x|^{-(d+\alpha+d_1 m)} \end{aligned}$$

from which our lemma follows. $\square$

The main result of this section is the following:

Theorem 6. *There is a a sequence $P_\beta \in C^\infty(N - \{0\})$ indexed by $\tilde{\mathbb{N}}$, where each P_β is $\delta(t)$-homogeneous of degree $-\beta - d - \alpha$, such that for all $\mu \in \tilde{\mathbb{N}}$*

$$\tilde{P}(x,t) = t^{\alpha}\sum_{\beta<\mu} t^{\beta}P_\beta(x) + R_\mu^{Asy}(x,t) \tag{34}$$

where for all multi-indecies I

$$|R_N(X^I)R_\mu^{Asy}(x,t)| \leq Ct^{\alpha}\xi(t)^{\mu}|x|^{-(d+\alpha+\mu+\|I\|)} \tag{35}$$

if either $t|x|^{-1} \leq 1$ or $|x| \geq 1$. Furthermore, the P_β are unique.

Proof

Throughout this proof, we assume, without further comment, that either $t|x|^{-1} \leq 1$ or $|x| \geq 1$. In particular, $x \neq 0$.

For $\beta \geq 0$, let Λ_β be the operator on functions on S defined by

$$\Lambda_\beta(f)(t,x) = t^\beta \int_0^t s^{-\beta-1} f(s,x)\, ds. \tag{36}$$

This integral exists provided $|f(x,t)| \leq K(x)t^\gamma$ where $\gamma - \beta > 0$, in which case Λ_β is a right inverse for $\Theta - \beta$. In this case

$$|\Lambda_\beta(f)(x,t)| \leq C\, K(x)t^\gamma \tag{37}$$

for the same $K(x)$. It is important to note also that for $\gamma > \beta$, t^γ is an eigenfunction for Λ_β. We note the following simple proposition.

Proposition 6. *If $\Theta(\Theta - \alpha)f$ is in the domain of $\Lambda_0\Lambda_\alpha$ then there are $f_1, f_2 \in \mathbb{R}$ such that*

$$\Lambda_0\Lambda_\alpha(\Theta(\Theta - \alpha))f(t) = f(t) + f_o + t^\alpha f_1$$

Proof This follows from

$$\Theta(\Theta - \alpha)\left(f - \Lambda_0\Lambda_\alpha\Theta(\Theta - \alpha)f\right) = 0.$$

□

Let

$$N_0 = \sum_1^n (t^{2d_i}X_i^2 + c_i t^{d_i} X_i). \tag{38}$$

From Corollary 3 on page 22, with $m = 1$,

$$|N_0\tilde{P}(x,t)| \leq t^\alpha C\xi(t)^{d_1}|x|^{-d_1-d-\alpha}. \tag{39}$$

Hence $G = \Lambda_0\Lambda_\alpha N_0\tilde{P}$ is defined. Furthermore, from (37), estimate (39) holds with $G(x,t)$ in place of $N_0\tilde{P}(x,t)$.

We apply $\Lambda_0\Lambda_\alpha$ to both sides of $\mathcal{L}F = 0$ and use Proposition 6, concluding that

$$(I + \Lambda_0\Lambda_\alpha N_0)\tilde{P}(x,t) = a(x) + P_0(x)t^\alpha. \tag{40}$$

Furthermore, from inequality (39) (applied to G) and (32) on page 22,

$$0 = a(x) + \lim_{t\to 0^+} \tilde{P}(x,t) = a(x).$$

Hence $a \equiv 0$.

Lemma 4. *For all $s \in \mathbb{R}^+$,*

$$P_0(\delta(s)x) = s^{-\alpha-d} P_0(x).$$

Proof

From formula (13) on page 9

$$\tilde{P}(\delta(s)x, st) = s^{-d} \tilde{P}(x,t).$$

From (40) and (39),

$$\begin{aligned} P_0(\delta(s)x) &= \lim_{t\to 0^+} t^{-\alpha} \tilde{P}(\delta(s)x, t) \\ &= \lim_{t\to 0^+} (st)^{-\alpha} \tilde{P}(\delta(s)x, st) \\ &= s^{-d-\alpha} P_0(x) \end{aligned}$$

proving the lemma. □

Let

$$\begin{aligned} Q_m(x,t) &= t^{-\alpha} \sum_{k=0}^{m} (-1)^k (\Lambda_0 \Lambda_\alpha N_0)^k [P_0(x) t^\alpha] \\ &= t^{-\alpha} \sum_{k=0}^{m} (-1)^k (\Lambda_0 \Lambda_\alpha N_0)^k (I + \Lambda_0 \Lambda_\alpha N_0) \tilde{P}(x,t) \\ &= t^{-\alpha} \left(I + (-1)^m (\Lambda_0 \Lambda_\alpha N_0)^{m+1} \right) \tilde{P}(x,t) \end{aligned} \tag{41}$$

Note that

$$N_0^k = \sum_{k \le |I| \le 2k} C(k,I) t^{\|I\|} X^I. \tag{42}$$

Since $t^{\|I\|}$ is an eigenfunction for $\Lambda_0 \Lambda_\alpha$ for each I, we see that

$$(\Lambda_0 \Lambda_\alpha N_0)^k [P_0(x) t^\alpha] = \sum_{k \le |I| \le 2k} \tilde{C}(k,I) t^{\alpha + \|I\|} X^I P_0(x).$$

From Lemma 4, $X^I P_0$ is $\delta(t)$-homogeneous of homogeneous degree $-d-\alpha-\|I\|$. We set

$$P_\beta^m(x) = \sum_{k=0}^{m} \sum_{\|I\|=\beta} (-1)^k \tilde{C}(k,I) X^I P_0(x).$$

The first equality in (41) together with (33) on page 22 shows that

$$Q_m(t,x) = \sum_{\beta \le 2m} t^\beta P_\beta^m(x). \tag{43}$$

Let

$$\tilde{R}_m^{Asy}(x,t) = \tilde{P}(x,t) - t^\alpha Q_m(x,t).$$

The third equality in (41) shows

$$\tilde{R}_m^{Asy}(x,t) = \pm(\Lambda_0 \Lambda_\alpha N_0)^{m+1} \tilde{P}(x,t) \tag{44}$$

From Corollary 3 on page 22 , formula (42) (with $k = m+1$), and the fact that X_i commutes with both t^{d_i} and $\Lambda_0\Lambda_\alpha$, we find that for all I

$$|R_N(X^I)\tilde{R}_m^{Asy}(x,t)| \le Ct^\alpha \xi(t)^{d_1(m+1)}|x|^{-(d+\alpha+d_1(m+1)+\|I\|)} \tag{45}$$

If $\beta \ge (m+1)d_1$ then, from the homogeneity of P_β^m, we may include $t^{\alpha+\beta}P_\beta^m(x)$ in the remainder $R_m^{Asy}(x,t)$ in formula (34) on page 22, showing that in the sum (43) we may may restrict the sum to $\beta < d_1(m+1)$. This proves formula (35) on page 22 in the case that $\mu = d_1(m+1)$. The general case follows by choosing an $m \in \mathbb{N}$ for which $d_1(m+1) \ge \mu$ and applying what was just proved, transferring appropriate terms to the remainder. We temporarily denote the coefficients in (34) by P_β^μ. (It will follow from the uniqueness that they are independent of μ.)

To prove the uniqueness of the P_β^μ, it suffices to note the following lemma that follows from a simple induction argument. (We include the logarithmic terms for later purposes.) A similar argument shows that the P_β^μ are, in fact, independent of μ. ☐

Lemma 5. *Suppose that b_β and c_β are sequences of scalars indexed by $\tilde{\mathbb{N}}$ for which an estimate of the form*

$$|\sum_{\beta<\mu} t^\beta(b_\beta + c_\beta \ln t)| < Ct^\mu |\ln t|$$

holds for $t \in (0,1)$. Then $b_\beta = c_\beta = 0$ for all $\beta < \mu$.

For later purposes, we note the following corollary, which is an immediate consequence of (44).

Corollary 4. *Let notation be as in Theorem 6 on page 22. Then*

$$|R_N(X^I)\Theta(\Theta-\alpha)R_\mu^{Asy}(x,t)| \le Ct^\alpha\xi(t)^\mu|x|^{-(d+\alpha+\mu+\|I\|)}$$

4.2 Taylor Expansion of $\tilde{P}(nx, t)$

In this section, we discuss the homogeneous Taylor expansion of $\tilde{P}(nx, t)$ in x. First we note the following fundamental result:

Lemma 6. *For all multi-indecies I there are $a, b \in \mathbb{R}$ such that*

$$|R_N(X^I)\tilde{P}(nx,t)| \leq C(t^\alpha + t^{-(d+\|I\|)})(1+|x|)^{d+\alpha+\|I\|}(1+|n|)^{-(d+\alpha+\|I\|)}.$$

Proof

From from (32) on page 22, it suffices to show that

$$\begin{aligned} &t^\alpha(1+|n|)^{d+\alpha+\|I\|}(t+|nx|)^{-(d+\alpha+\|I\|)} \\ &\qquad \leq C(t^\alpha + t^{-(d+\|I\|)})(1+|x|)^{d+\alpha+\|I\|}. \end{aligned}$$

If either $|n| \leq 2|x|$ or $|n| \leq 1$, this follows from $t + |nx| \geq t$ and, in the $|n| \leq 2|x|$ case, also $1 + |n| \leq 1 + 2|x|$.

If $|n| \geq 2|x|$ and $|n| \geq 1$, the reverse triangle inequality implies $t + |nx| \geq |n|/2$. Our lemma follows from

$$(1+|n|)^{d+\alpha+\|I\|}\left(\frac{|n|}{2}\right)^{-(d+\alpha+\|I\|)} \leq 4^{d+\alpha+\|I\|}$$

□

The following result describes the homogeneous Taylor expansion of $\tilde{P}$.

Proposition 7. *There is a sequence functions $p_\beta(n,x,t)$ which are C^∞ in (n,t) and polynomial in x, indexed by $\tilde{\mathbb{N}}$, such that for all $\mu \in \tilde{\mathbb{N}}$,*

$$\tilde{P}(nx,t) = \sum_{\beta<\mu} p_\beta(n,x,t) + R_\mu^{Tay}(n,x,t) \tag{46}$$

where for all multi-indecies I, there positive scalars b, c, and $\tau > \mu$ such that

$$\begin{aligned} &|R_N(X^I)_n R_\mu^{Tay}(n,x,t)| \\ &\qquad \leq C(t^\alpha + t^{-c})(|x|^\mu + |x|^\tau)(1+|n|)^{-(\mu+d+\alpha+\|I\|)}. \end{aligned} \tag{47}$$

(The subscript on $R_N(X^I)_n$ indicates that this operator acts in the n-variable.)

Furthermore, the p_β satisfy

$$\begin{aligned} p_\beta(n,\delta(s)x,st) &= s^\beta p_\beta(n,x,t) \\ p_\beta(\delta(s)n,x,st) &= t^{-d-\alpha-\beta} p_\beta(n,x,t). \end{aligned} \tag{48}$$

Proof

Assume for the moment that $\mu = d_1 m$ where $m \in \mathbb{N}$.

For f an integrable function on $\mathbb{R}^+$, let

$$\mathrm{Int}_r f(t) = \int_0^t f(r)\, dr.$$

(The subscript denotes the variable of integration.)

It is easily seen that if $h \in C^\infty(\mathbb{R})$ then, for all $m \in \mathbb{N}$,

$$h(r) - \sum_{k=0}^{m} \frac{h^{(k)}(0)}{k!} r^k = (\mathrm{Int}_r)^m (\partial_r^m h)(r).$$

Let $f \in C^\infty(\mathbb{R}^m)$. By applying the preceding equality at $r = 1$ to the function $g(r) = f(rx)$, we obtain the multi-variable Taylor expansion

$$f(x) = \sum_{|I|<m}^{m} \frac{\partial^I f(0)}{|I|!} x^I + R_m^{Tay}(x) \tag{49}$$

where

$$R_m^{Tay}(x) = (\mathrm{Int}_r)^m (\partial_r^m f(rx))|_{r=1}. \tag{50}$$

We apply this with $f(x) = \tilde{P}(nx, t)$. Note that

$$\partial_r|_{r=0} h(n(rx)) = \sum x_i R_N(X_i) h(n).$$

Hence, from (49),

$$\tilde{P}(nx, t) = \sum_{|I|<m} \frac{R_N(X^I)\tilde{P}(n,t)}{|I|!} x^I + \tilde{R}_m^{Tay}(n, x, t). \tag{51}$$

We divide the sum (51) into two sums, the first containing those terms where $\|I\| < \mu = d_1 m$. For each $\beta \in \tilde{\mathbb{N}}$, $\beta < \mu$, we define the summands in (46) by

$$p_\beta(n, x, t) = \sum_{\|I\|=\beta} \frac{R_N(X^I)\tilde{P}(n,t)}{|I|!} x^I. \tag{52}$$

It is clear that $p_\beta(n, x, t)$ is a polynomial in x which satisfies the homogeneity conditions (48).

From (33) on page 22, $|I| = m$ implies that $\|I\| \in [\mu, m]$. Suppose that $\|I\| \geq \mu$ and $|I| < m$. Then $\|I\| < m$ and from Lemma 6 on page 26

$$|R_N(X^I)\tilde{P}(n,t)| \leq C(t^\alpha + t^{-(d+m)})(1+|n|)^{-(\mu+d+\alpha)}.$$

Also, since x^I is $\delta(t)$-homogeneous of degree $\|I\|$,

$$|x^I| \leq C|x|^{\|I\|} \leq C(|x|^\mu + |x|^m).$$

Hence, the terms in the sum (51) corresponding to $\beta \geq \mu$ may be incorporated into the remainder.

To estimate the remainder we write

$$\begin{aligned}(\partial_r)^m \tilde{P}(n(rx),t) &= \big(\sum_{|I|=m} x^I R_N(X^I)[R_N(rx)\tilde{P}]\big)(n,t) \\ &= \sum_{|I|=m} x^I [R_N(\text{ Ad }((rx)^{-1})(X^I))\tilde{P}](n(rx),t)\end{aligned} \tag{53}$$

Since

$$\text{Ad }(\delta(s)x)(\delta(s)(X_i)) = \delta(s)(\text{ Ad }(x)(X_i))$$

we see that

$$\text{Ad }(x)(X_i) = \sum_{j\geq i} p_{i,j}(x)X_j$$

where the $p_{i,j}(x)$ are $\delta(s)$-homogeneous polynomials of homogeneous degree $d_j - d_i$. More generally

$$\text{Ad }(x)(X^I) = \sum_{\substack{\|J\| \geq \|I\| \\ |J| = |I|}} p_{I,J}(x)X^J$$

where the $p_{I,J}(x)$ are $\delta(s)$-homogeneous polynomials of homogeneous degree $\|J\| - \|I\|$.

We replace x by $(rx)^{-1} = -rx$, then substitute the above expression into (53). We find that

$$(\partial_r)^m \tilde{P}(n(rx),t) = \sum_{|J|=m} q_J(rx)(R_N(X^J)\tilde{P})(n(rx),t) \tag{54}$$

where $q_J(x)$ is $\delta(s)$-homogeneous of degree $\|J\| \in [\mu, m]$ in x. Hence, from Lemma 6 on page 26 and the homogeneity of the q_J, for $r \in [0,1]$,

$$\begin{aligned}|(\partial_r)^m \tilde{P}(n(rx),t)|& \\ &\leq C(|x|^\mu + |x|^m)(t^\alpha + t^{-(d+m)})(1+|x|)^{m+d+\alpha}(1+|n|)^{-(\mu+d+\alpha)} \\ &\leq C(t^\alpha + t^{-(d+m)})(|x|^\mu + |x|^\tau)(1+|n|)^{-(\mu+d+\alpha)}\end{aligned}$$

where $\tau > \mu$.

More generally, if we replace $\tilde{P}(n(rx),t)$ in (54) with $R_N(X^K)_n\tilde{P}(n(rx),t)$ and repeat the same reasoning, we find that a similar estimate holds with μ in the exponent of $(1+|n|)$ replaced by $\mu + \|K\|$. Estimate (47) on page 26 then follows from (50). □

For later purposes, we note the following corollary which follows from the observation that since $\tilde{P}$ is harmonic

$$\Theta(\Theta - \alpha)\tilde{P}(x,t) = -N_0\tilde{P}(x,t).$$

together with formulas (50) and (54). We leave the details to the reader.

Corollary 5. *Let notation be as in Proposition 7. Then there are positive scalars a, b, c, and $\tau > \mu$ such that*

$$\begin{aligned}|\Theta(\Theta - \alpha)R_N(X^I)_n R_\mu^{Tay}(n,x,t)|& \\ &\leq C(t^a + t^{-c})(|x|^\mu + |x|^\tau)(1+|n|)^{-(\mu+d+\alpha+\|I\|)}.\end{aligned}$$

4.3 Proof of Theorem 3 on page 9.

From formula (52) on page 27 the coefficients in the homogeneous Taylor expansion of $P(nx,t)$ are

$$p_\beta(n,x,t) = \sum_{\|I\|=\beta} \frac{R_N(X^I)\tilde{P}(n,t)}{|I|!}x^I.$$

We substitute the asymptotic expansion (34) for $\tilde{P}(n,t)$, obtaining a sum over I and γ of terms the form

$$\frac{R_N(X^I)P_\gamma(n)}{|I|!}x^I t^{\alpha+\gamma}.$$

Note that by hypothesis $|n| \geq 1$ so the hypotheses of Theorem 6 are fulfilled.

We set

$$H_\beta(n,x,t) = \sum_{\|I\|+\gamma=\beta} \frac{R_N(X^I)P_\gamma(n)}{|I|!} x^I t^\gamma$$

so that H_β satisfies the first homogeneity condition in Theorem 3. Note also that from Theorem 6 on page 22, $R_N(X^I)P_\gamma(n)$ is $\delta(t)$-homogeneous of degree $-\|I\| - \gamma - d - \alpha$ in n, showing that H_β also satisfies the second homogeneity condition.

We need to estimate

$$\begin{aligned} P_\mu(n,x,t) &= \tilde{P}(nx,t) - t^\alpha \sum_{\beta<\mu} H_\beta(n,x,t) \\ &= \tilde{P}(nx,t) - \sum_{\beta<\mu} p_\beta(n,x,t) \\ &+ \sum_{\|I\|<\mu} \frac{R_N(X^I)\tilde{P}(n,t)}{|I|!} x^I - t^\alpha \sum_{\|I\|+\gamma<\mu} \frac{R_N(X^I)P_\gamma(n)}{|I|!} \end{aligned}$$

But

$$\begin{aligned} \frac{R_N(X^I)\tilde{P}(n,t)}{|I|!} x^I - t^\alpha \sum_{\gamma<\mu-\|I\|} \frac{R_N(X^I)P_\gamma(n)}{|I|!} x^I t^\gamma \\ = \frac{x^I}{|I|!} R_N(X^I) R^{Asy}_{\mu-\|I\|}(n,t) \end{aligned}$$

Hence,

$$P_\mu(n,x,t) = R^{Tay}_\mu(n,x,t) + \sum_{\|I\|<\mu} \frac{x^I}{|I|!} R_N(X^I) R^{Asy}_{\mu-\|I\|}(n,t) \tag{55}$$

where R^{Tay}_μ is as in Proposition 7 on page 26. The estimate (15) on page 10 follows from (47) on page 26 and (35) on page 22.

To prove the uniqueness of the H_β, suppose that $\tilde{H}_\beta$ is a second sequence satisfying the conclusions of Theorem 3 on page 9. Then from Theorem 3, for fixed x, t and, $\mu \in \tilde{\mathbb{N}}$,

$$|\sum_{\beta<\mu} (H_\beta(n,x,t) - \tilde{H}_\beta(n,x,t))| \leq C(1+|n|)^{-d-\alpha-\mu}$$

We replace n by $\delta(s^{-1})n$ where $s \in (0,1)$. It then follows from the homogeneity of the H_β in n together with Lemma 5 on page 25, with t replaced by s, that $H_\beta - \tilde{H}_\beta \equiv 0$, proving the uniqueness.

Finally, we must prove the harmonicity of each the $H_\beta(n,x,t)$ in (x,t). For this, we note that

$$\begin{aligned} |\mathcal{L}Q_\mu(n,x,t)| &= |\mathcal{L}(\tilde{P}(nx,t) - Q_\mu(n,x,t))| \\ &= |\mathcal{L}P_\mu(n,x,t)| \\ &\le C(x,t)(1+|n|)^{-d-\alpha-\mu} \end{aligned}$$

where we used (55) and Corollaries 5 on page 29 and 4 on page 25 in the last estimate. It follows similarly to the proof of the uniqueness of the H_β that for each $\beta < \mu$, $\mathcal{L}H_\beta(n,x,t) \equiv 0$, finishing our proof. □

5 Boundary Values.

5.1 C^∞ asymptotic expansions

Suppose F is a $\mathcal{L}$-harmonic function on S which satisfies condition (6) on page 4. We initially replace F with a function with better regularity properties.

Lemma 7. *Let $\|\cdot\|$ be the Euclidean norm on $N = \mathcal{N}$. Then*

$$\begin{aligned} \|x\| &\le C(|x|^{d_1} + |x|) \\ |x| &\le C'(\|x\|^{1/d_1} + \|x\|) \end{aligned}$$

Proof Since all homogeneous gauges are comparable, we may replace $|\cdot|$ with $|\cdot|_\infty$ where

$$|x|_\infty = \sup_i |x_i|^{1/d_i}.$$

Similarly, we may replace $\|\cdot\|$ with $\|\cdot\|_\infty$ where

$$\|x\|_\infty = \sup_i |x_i|.$$

There is a j such that

$$|x_j|^{1/d_j} = |x|_\infty.$$

For this j,

$$|x_j|^{d_i/d_j} \geq |x_i|$$

for all i. Since $d_1 \leq d_i \leq 1$, it follows that

$$\|x\|_\infty \leq C(|x|_\infty^{d_1} + |x|_\infty)$$

proving the first inequality. The proof of the second is similar and left to the reader. $\square$

It follows that $\|\cdot\|_{p,k}$ in (16) on page 10 defines the Schwartz class on N. For $\eta \in \mathcal{S}(N)$ let

$$F_\eta(x,t) = \int \eta(y)F(y^{-1}x,t). \tag{56}$$

Lemma 8. *Let F be a $\mathcal{L}$-harmonic function on S which satisfies condition (6) on page 4. For each multi-index I, there is a scalar C such that*

$$|L_N(X^I)F_\eta(x,t)| \leq C\|\eta\|_{|I|,k+d+1}(t^a + t^b)(1+|x|)^k$$

where k, a, and b are as in condition (6).

Proof Since

$$L_N(X^I)F_\eta = F_{L_N(X^I)\eta}$$

it suffices to assume that $X^I = 1$. It follows from condition (6) that

$$|F_\eta(x,t)| \leq A(t^a + t^b)\int |\eta(y)|(|y^{-1}x|+1)^k\,dy.$$

We estimate the integrand in several cases:

Case 1: $|y| \geq 1$

Suppose first that $|x| \geq 1$. In this case

$$\begin{aligned}\frac{(|y^{-1}x|+1)^k}{|x|^k|y|^k} &\leq \frac{(|y|+|x|+1)^k}{|x|^k|y|^k}\\ &\leq 3^k.\end{aligned}$$

Hence, for $|x| \geq 1$,

$$\begin{aligned}&\int_{|y|\geq 1} |\eta(y)|(|y^{-1}x|+1)^k\,dy\\ &\qquad \leq C|x|^k\|\eta\|_{0,k+d+1}\int_{|y|\geq 1} |y|^k|(1+|y|)^{-k-d-1}\,dy.\end{aligned}$$

The integral is finite due to (19) on page 14.

On the other hand, for $|x| \leq 1$,

$$\int_{|y|\geq 1} |\eta(y)|(|y^{-1}x| + 1)^k \, dy \leq \int_N (|y| + 2)^k |\eta(y)| \, dy.$$

which is estimated similarly.

Case 2: $|y| \leq 1$.

In this case,

$$\int_{|y|\leq 1} |\eta(y)|(|y^{-1}x| + 1)^k \, dy \leq (|x| + 2)^k \int_{|y|\leq 1} |\eta(y)| \, dy.$$

which we estimate as before. Our lemma follows by summing the results of Case 1 and Case 2. □

We (temporarily) replace F with F_η. Hence we assume

$$|L_N(X^I)F(x,t)| \leq C\kappa_{|I|}(t^a + t^b)(1 + |x|)^k$$

where $\kappa_{|I|} = \|\eta\|_{|I|,k+d+1}$ for some $\eta \in \mathcal{S}(N)$. Our arguments will involve replacing the above inequality with sharper inequalities of a similar form. We will eventually need to understand the continuity properties with respect to η of our constructions. We say that a given scalar is " independent of κ" if it is independent of the *function* κ. Thus, any scalar that requires a value of κ_I in its definition will not be independent of κ. *In the discussion below, unless stated otherwise, we are stating that all of the scalars are independent of* κ.

For all I,

$$R_N(X^I) = \sum_{\substack{\|J\| \geq \|I\| \\ |J| = |I|}} p_J(x) L_N(X^J)$$

where $p_J(x)$ are $\delta(t)$-homogeneous polynomials of homogeneous degree $\|J\| - \|I\|$. From (33) on page 22, $\|J\| \leq |I|$. It follows that

$$|R_N(X^I)F(x,t)| \leq C\kappa_{|I|}(t^a + t^b)(1 + |x|)^{k'} \tag{57}$$

where a and b are as in condition (6) and $k' = k + |I| - \|I\|$.

We make use of the operators Λ_β which were defined in (36) on page 23 as well as the following operators

$$\Lambda^1_\beta(f)(t,x) = t^\beta \int_1^t s^{-\beta-1} f(s,x)\, ds.$$

Then Λ^1_β is a right inverse for $\Theta - \beta$. Furthermore, for all $\epsilon > 0$,

$$\Lambda^1_\beta(t^\gamma) = \begin{cases} C_1 t^\beta + C_2 t^\gamma & \gamma \neq \beta \\ C t^\beta \ln t & \gamma = \beta \end{cases} \tag{58}$$

Hence, if $|f(x,t)| \leq C(x) t^\gamma$ then, for all $\epsilon > 0$,

$$|\Lambda^1_\beta(f(x,t)| \leq C(x) C_\epsilon (t^\beta + t^{\gamma-\epsilon} + t^{\gamma+\epsilon}) \tag{59}$$

for the same $C(x)$, where ϵ can be chosen to be 0 if $\beta \neq \gamma$.

Throughout the following discussion, k represents a generic constant which, like C, may change from like to line. Typically, k will depend on at least I.

Proposition 8. *Suppose that F is an $\mathcal{L}$-harmonic function for which inequality (57) holds. Then an inequality of the form of (57) holds for some $0 \leq a < b$ where $\kappa_{|I|}$ is replaced by $\kappa_{|I|+p}$ for some p which is independent of I and k' is replaced by a value k which may depend on I.*

Proof

We assume with out loss of generality that $b > \alpha$.

Applying $\Lambda^1_0 \Lambda^1_\alpha$ to both sides of $\mathcal{L}F = 0$ and using the appropriate analogue of Proposition 6 on page 23 shows that

$$(I + \Lambda^1_0 \Lambda^1_\alpha N_0) F = \tilde{F}_0(x) + \tilde{F}_1(x) t^\alpha. \tag{60}$$

where N_0 is defined in (38) on page 23.

It follows from (57) and (59) with $\beta = \alpha$, together with $b > \alpha$, that all sufficiently small $\epsilon > 0$

$$|\Lambda^1_\alpha(t^{d_i} X_i F)(x,t)| \leq C_\epsilon \kappa_1 (t^\alpha + t^{a+d_i-\epsilon} + t^{b+d_i})(1+|x|)^k.$$

Hence, choosing $\epsilon < d_1/4$, using (59) with $\beta = 0$ in the second inequality,

$$\begin{aligned} |(\Lambda^1_\alpha N_0) F(x,t)| &\leq C\kappa_2 (t^\alpha + t^{a+\frac{3}{4}d_1} + t^{b+2})(1+|x|)^k \\ |(\Lambda^1_0 \Lambda^1_\alpha N_0) F(x,t)| &\leq C\kappa_2 (1 + t^\alpha + t^{a+d_1/2} + t^{b+2})(1+|x|)^k \\ &\leq C\kappa_2 (1 + t^{a+d_1/2} + t^{b+2})(1+|x|)^k. \end{aligned} \tag{61}$$

Thus from (60) and (57)

$$|\tilde{F}_0(x) + t^\alpha \tilde{F}_1(x)| \leq C\kappa_2(1 + t^a + t^{b+2})(1 + |x|)^k.$$

By choosing specific values of t, it follows that for $i = 1, 2$

$$|\tilde{F}_i(x)| \leq C\kappa_2(1 + |x|)^k. \tag{62}$$

More generally, inequality (57) with I general, together with (60), says that

$$|R_N(X^I)\tilde{F}_i(x)| \leq C\kappa_{|I|+2}(1 + |x|)^k. \tag{63}$$

We multiply both sides of (60) by $R_N(X^I)$, solve for $R_N(X^I)F$, and repeat the argument leading to inequality (61) finding

$$|R_N(X^I)F(x,t)| \leq C\kappa_{|I|+2}(1 + t^{a+d_1/2} + t^{b+2})(1 + |x|)^k.$$

Hence (57) holds with a replaced by $a' = \min\{a + d_1/2, 0\}$, b replaced by $b+2$, and $\kappa_{|I|}$ replaced by $\kappa_{|I|+2}$. Repeating this argument as many times as necessary, shows that (57) holds for some $a \geq 0$ and some k, and $\kappa_{|I|}$ replaced by $\kappa_{|I|+p}$ proving the lemma. $\square$

Corollary 6. *Under the hypotheses of Proposition 8*

$$\lim_{t \to 0^+} F(x,t) = F_0(x)$$

exists. Furthermore, for all multi-indecies I

$$|R_N(X^I)(F(x,t) - F_0(x))| \leq C\kappa_{|I|+p}(t^a + t^b)(1 + |x|)^k$$

for some k and some $b > a > 0$ where a, b, and p are independent of I.

Proof

Let Λ_β be as in (36) on page 23. From Proposition 8, $(\Lambda_0\Lambda^1_\alpha N_0)F$ is defined and, from (59) and (37) on page 23, satisfies

$$\begin{aligned} &|R_N(X^I)(\Lambda_0\Lambda^1_\alpha N_0)F(x,t)| \\ &\qquad \leq C\kappa_{p+|I|+2}(t^\alpha + t^{a+\frac{3}{4}d_1} + t^{b+2})(1 + |x|)^k. \end{aligned} \tag{64}$$

Applying $\Lambda_0\Lambda^1_\alpha$ to both sides of $\mathcal{L}F = 0$ and using the appropriate analogue of Proposition 6 on page 23

$$(I + \Lambda_0\Lambda^1_\alpha N_0)F = F_0(x) + \tilde{G}_0(x)t^\alpha. \tag{65}$$

where
$$|R_N(X^I)F_0(x)| \leq C\kappa_{p+|I|+2}(1+|x|)^k$$
$$|R_N(X^I)\tilde{G}_0(x)| \leq C\kappa_{p+|I|+2}(1+|x|)^k.$$

Thus
$$\begin{aligned}|R_N(X^I)(F(x,t)-F_0(x))| &= |R_N(X^I)(\tilde{G}_0(x)t^\alpha - \Lambda_0\Lambda^1_\alpha N_0 F)| \\ &\leq C\kappa_{p+2}(t^\alpha + t^{a+\frac{3}{4}d_1} + t^{b+2})(1+|x|)^k.\end{aligned}$$

This proves the corollary. □

The following is the main result of this subsection.

Theorem 7. *Suppose that F is an $\mathcal{L}$-harmonic satisfying (57) on page 33. Then for all $\beta \in \tilde{\mathbb{N}}$, there are uniquely defined functions F_β, G_β and H_β in $C^\infty(N)$ such that for all $\mu \in \tilde{\mathbb{N}}$ and all multi-indecies I*
$$\begin{aligned}&|R_N(X^I)(F(x,t)-Q_\mu(x,t))| \\ &\qquad \leq C\kappa_{|I|+p}(t^\mu \ln t + t^b)(1+|x|)^k\end{aligned} \tag{66}$$
where
$$Q_\mu(x,t) = \begin{cases}\sum_{\beta<\mu} t^\beta F_\beta(x) + t^{\alpha+\beta}G_\beta(x) & \alpha \notin \tilde{\mathbb{N}} \\ \sum_{\beta<\mu} t^\beta F_\beta(x) + t^{\alpha+\beta}((\ln t)H_\beta(x) + G_\beta(x)) & \alpha \in \tilde{\mathbb{N}}.\end{cases}$$
and p, and $b > \mu$ are independent of I (but not necessarily of μ). Furthermore, if T_β is either F_β, G_β ot H_β then
$$|R_N(X^I)T_\beta(x)| \leq C\kappa_{|I|+p}(1+|x|)^k. \tag{67}$$
Also, the relations (22) and (23) on page 15 and (25) on page 17 are satisfied. Finally, F is uniquely determined by F_0 and G_0.

Proof From Proposition 8 we may assume that in (57) on page 33, $a \geq 0$, provided we replace $\kappa_{|I|}$ with $\kappa_{p+|I|}$.

For $m \in \mathbb{N}$, we define
$$\tilde{Q}_m = \sum_{i=0}^{m}(-1)^i(\Lambda_0\Lambda^1_\alpha N_0)^i(F_0 + \tilde{G}_0 t^\alpha)$$
where F_0 and $\tilde{G}_0$ are as in (65). It is clear from (58) on page 34 that $\tilde{Q}_m(x,t)$ has an expansion of the form (9) on page 5 where $p(x,t)$, $q(x,t)$, and $h(x,t)$

depend polynomially on t (in the sense of Definition 2) and the coefficients $p_\beta(x)$, $q_\beta(x)$, and $h_\beta(x)$ of t^β in $p(x,t)$, $q(x,t)$ and $h(x,t)$ satisfy an estimate of the form of (67). (We are not claiming, however, that these terms depend polynomially on x.)

The reasoning that resulted in equation (45) on page 25 shows that

$$F(x,t) - \tilde{Q}_m(x,t) = (-1)^m (\Lambda_0 \Lambda^1_\alpha N_0)^{m+1} F(x,t) \tag{68}$$

We apply (59) on page 34 with $\beta = \alpha$ and (37) on page 23 with $\beta = 0$, finding that for some $b > \alpha$ and all sufficiently small $\epsilon > 0$

$$|F(x,t) - \tilde{Q}_m(x,t)| \leq C_\epsilon \kappa_{p+2m+2}(t^\alpha + t^{a+(m+1)d_1-\epsilon} + t^b)(1+|x|)^k.$$

Hence, by choosing m so that $d_1(m+1) > \alpha$ and $\epsilon < a + (m+1)d_1 - \alpha$, we see that

$$|F(x,t) - \tilde{Q}_m(x,t)| \leq C\kappa_{p'}(t^\alpha + t^b)(1+|x|)^k$$

where $b > \alpha$ and $p' = p + 2m + 2$. More generally, the same argument shows

$$|R_N(X^I)\left(F(x,t) - \tilde{Q}_m(x,t)\right)| \leq C\kappa_{p'+|I|}(t^\alpha + t^b)(1+|x|)^k \tag{69}$$

Let $H = F - \tilde{Q}_m$. Then

$$\mathcal{L}H = -\mathcal{L}\tilde{Q}_m \tag{70}$$

which is polynomial-like in t.

Note that (69) implies that $N_0 H$ is in the domain of $\Lambda_0\Lambda_\alpha$. Applying $\Lambda_0\Lambda_\alpha$ to both sides of (70) and using Proposition 6 on page 23 shows that

$$(I + \Lambda_0\Lambda_\alpha N_0)H(x,t) = Z(x,t) \tag{71}$$

where $Z(x,t)$ is again polynomial-like in t where Λ_0 and Λ_α are defined by (36) on page 23.

It also follows from (69) that

$$|R_N(X^I)Z(x,t)| \leq C\kappa_{p'+2+|I|}(t^\alpha + t^b)(1+|x|)^k. \tag{72}$$

Hence $N_0 Z$ is also in the domain of $\Lambda_0\Lambda_\alpha$. Let

$$H_q = \sum_{i=0}^{q} (-1)^i (\Lambda_0\Lambda_\alpha N_0)^i Z.$$

Clearly, H_q is also polynomial-like in t.

Reasoning as in formula (41) on page 24, we see

$$H - H_q = (-1)^q (\Lambda_0 \Lambda_\alpha N_0)^{q+1} Z.$$

Reasoning as in the proof of (45) on page 25, and using (57) on page 33, we see that for some p and some $b > d_1(q+1)$

$$|H(x,t) - H_q(x,t)| \le C\kappa_p (t^{d_1(q+1)} + t^b)(1+|x|)^k.$$

Choose q so that $d(q+1) > \mu$. Write H_q in the form of (9) on page 5 where $p(x,t)$, $q(x,t)$, and $h(x,t)$ depend polynomially on t and then expand each of these terms in a finite sum of the form in Definition 3 on page 5. We may omit any terms where $\beta \ge \mu$ without effecting the validity of (66) on page 36. The existence of the desired expansion follows. The uniqueness of the coefficients follows from Lemma 5 on page 25.

Next we show that the relations (22) and (23) on page 15 and (25) on page 17 are satisfied. Notice that

$$(\Theta - \alpha)\Theta \Lambda_0 \Lambda_\alpha N_0 = N_0$$

Hence, from (68),

$$\Theta(\Theta - \alpha)(H - H_q) = (-1)^q N_0 (\Lambda_0 \Lambda_\alpha N_0)^q Z$$

where Z is as in (71). It follows from this and (72) that for some $b > d_1(q+1)$ all sufficiently small $\epsilon > 0$

$$|\mathcal{L}(F - \tilde{Q}_m - H_q)(x,t)| \le C_\epsilon (t^{d_1(q+1)-\epsilon} + t^b)(1+|x|)^k.$$

Choose q and ϵ so that $d_1(q+1) - \epsilon > \mu$. Since $\mathcal{L}F = 0$, this implies that the coefficients of t^β in the expansion of $\mathcal{L}(Q_\mu + H_q)$ in powers of t must vanish for $\beta < \mu$. This implies that the desired recursion relations hold.

Finally, from Propositions 3 and 4 on page 17, the coefficients in the asymptotic expansion are uniquely determined by F_0 and G_0. If both of these terms are zero, then F vanishes to infinite order at 0, showing that F extends to a C^∞ function on $N \times \mathbb{R}$ which is zero on $N \times \mathbb{R}^-$. Theorem 2 of [6] shows that then $F \equiv 0$, finishing our theorem. □

Corollary 7. *Suppose that F satisfies the hypotheses of Theorem 7 and that $F(t,x)$ is a polynomial in x of degree at most m for all $t \in \mathbb{R}^+$. Then $F(t,x)$ is polynomial-like in the sense of Definition 2.*

Proof It follows from Corollary 6 on page 35 that $F_0(x)$ is a polynomial of degree at most m. Then, from Theorem 5 on page 18, $P(F_0, 0)$ is a polynomial-like harmonic function. Let

$$G(t, x) = F(t, x) - P(F_0, 0)(t, x).$$

Then G satisfies the same assumptions as F as well as $G(0, x) = 0$. It follows that $G = P(0, G_0)$ where

$$G_0(x) = \lim_{t \to 0} t^{-\alpha} G(t, x).$$

Hence, $G_0(x)$ is a polynomial as well. It then follows that $G(t, x)$ is polynomial-like, proving the result.

5.2 Distributional asymptotic expansions

Finally, we drop our assumption that F satisfies (57) on page 33, assuming only that F is a $\mathcal{L}$-harmonic function on S which satisfies condition (6) on page 4. It follows from Lemma 7 on page 31 that for each $t \in \mathbb{R}^+$, $F(\cdot, t)$ defines an element of $\mathcal{S}'(N)$. In fact, for $\eta \in \mathcal{S}(N)$,

$$< F(\cdot, t), \eta(\cdot) >= F_{\eta^*}(0, t)$$

where F_η is defined in (56) on page 32 and

$$\eta^*(x) = \eta(x^{-1}).$$

The following is immediate from Corollary 6 on page 35 applied to $F_\eta(0, t)$.

Corollary 8. *Suppose that F is a $\mathcal{L}$-harmonic function on S which satisfies condition (6) on page 4. Then*

$$\lim_{t \to 0^+} F(\cdot, t) = F_0(\cdot)$$

exists in $\mathcal{S}'(N)$, where $F_0 \in \mathcal{S}'(N)$.

A similar argument proves the following distributional version of Theorem 7. Specifically, $< F_\beta(\cdot), \eta(\cdot) >$, $< G_\beta(\cdot), \eta(\cdot) >$, and $< H_\beta(\cdot), \eta(\cdot) >$ are, respectively, $(F_\eta^*)_\beta(0)$, $(G_\eta^*)_\beta(0)$, and $(H_\eta^*)_\beta(0)$.

Theorem 8. *Suppose that F is a $\mathcal{L}$-harmonic function on S which satisfies condition (6) on page 4. Then for all $\beta \in \tilde{\mathbb{N}}$, there are uniquely defined Schwartz distributions F_β, G_β, and H_β on N such that for all $\mu \in \tilde{\mathbb{N}}$ and all multi-indecies I and all $\eta \in \mathcal{S}(N)$*

$$| < F(\cdot,t) - Q_\mu(\cdot,t), \eta(\cdot) > | \leq C(t^\mu \ln t + t^b)\|\eta\|_{p,k+d+1} \tag{73}$$

where

$$Q_\mu(\cdot,t) = \begin{cases} \sum_{\beta<\mu} t^\beta F_\beta(\cdot) + t^{\alpha+\beta} G_\beta(\cdot) & \alpha \notin \tilde{\mathbb{N}} \\ \sum_{\beta<\mu} t^\beta F_\beta(\cdot) + t^{\alpha+\beta}((\ln t)H_\beta(\cdot) + G_\beta(\cdot)) & \alpha \in \tilde{\mathbb{N}}. \end{cases}$$

p, and $b > \mu$ are independent of I and $\|\cdot\|_{p,k+d+1}$ is as in (56) on page 32.(But p will typically depend on μ.) Furthermore, the relations (23) and (22) on page 15 and (25) on page 17 are satisfied. Finally, F is uniquely determined by F_0 and G_0.

6 Liouville Theorem

6.1 Liouville Theorem: $b < -\alpha - 4$

In this section we prove the following "Liouville Theorem". Here $\chi_L = \chi_{(0,1]}$ and $\chi_R = \chi_{[1,\infty)}$.

Theorem 9. *Assume that F is $\mathcal{L}$-harmonic and satisfies*

$$|F(x,t)| \leq C(t^\alpha \chi_L(t) + t^b \chi_R(t))(1 + |x|)^k \tag{74}$$

where $b < -\alpha - 4$. Then $F \equiv 0$.

We prove that F is a polynomial-like function in the sense of Definition 3 on page 5. It then follows trivially from the above inequality that $F \equiv 0$. An approximate identity argument implies that we may assume that $F = F_\eta$ where F_η is defined in (56) on page 32. *Hence, from Lemma 8 on page 32, we may assume that inequality (57) on page 33 holds with $a = \alpha$ and $b < -\alpha - 4$.*

Lemma 9. *For $i = 1, 2$,*

$$|\Theta^i F(x,t)| \leq C(t^\alpha \chi_L(t) + t^{b+2} \chi_R(t))(1 + |x|)^k$$

for some k.

Proof

Let Λ_β be as in (36) on page 23. Applying $\Lambda_0\Lambda_\alpha$ to both sides of $\mathcal{L}F = 0$ and using Proposition 6 on page 23 we see

$$(I + \Lambda_0\Lambda_\alpha N_0)F = F_0(x) + \tilde{G}_0(x)t^\alpha.$$

where N_0 is defined in (38) on page 23. Hence

$$\Theta F = -\Lambda_\alpha N_0 F + \alpha\tilde{G}_0(x)t^\alpha$$

from which the desired estimate for ΘF on $(0, 1]$ follows. The estimate for $\Theta^2 F$ on $(0, 1]$ follows from

$$\Theta^2 F = \alpha\Theta F - N_0 F.$$

To prove the desired estimate on $[1, \infty)$ we repeat the preceding argument with Λ_β replaced by

$$\Lambda_\beta^\infty = -t^\beta \int_t^\infty s^{-\beta-1} f(s, x)\, ds.$$

In this case, the term corresponding to $\tilde{G}_0$ above vanishes due to (57). Our lemma follows from (57) and the fact that the highest power of t occurring in the definition of N_0 is t^2. □

It follows from Corollary 7 on page 38, that to prove that $F(x, t)$ is polynomial-like, it suffices to prove that $F(x, t)$ depends polynomially on x. For the proof, we recall an argument of Geller [Gel]. From Lemma 1 on page 18, we wish to show that

$$R_N(X^I)F = 0$$

whenever $|I|$ is sufficiently large.

This in turn is equivalent with showing that

$$< F, R_N(X^I)\phi >= 0$$

for all sufficiently large I and all $\phi \in C_c^\infty(S)$ where $< \cdot, \cdot >$ is defined by integration agains *right*-invariant Haar measure on S–i.e.

$$< f, h >= \int_{N\times\mathbb{R}^+} f(x, t)\overline{h}(x, t)\, t^{-1} dxdt$$

and dx is Haar measure on N.

We show that for $|I|$ sufficiently large, there is a C^∞ function ψ_I such that

$$\mathcal{L}^t \psi_I = R_N(X^I)\phi$$

where

$$\mathcal{L}^t = \Theta^2 + \alpha\Theta + \sum_1^n (t^{2d_i} X_i^2 - c_i t^{d_i} X_i).$$

With luck, it should follow that

$$\begin{aligned} < F, R_N(X^I)\phi > &=< F, \mathcal{L}^t \psi_I > \\ &=< \mathcal{L}F, \psi_I >= 0. \end{aligned} \tag{75}$$

In light of inequality (74) (or rather the version (57) on page 33 that contains $R_N(X^I)$) and Lemma 9, this computation will be valid provided for $p = 0, 1, 2,$

$$\begin{aligned} |t^{p\,d_i} R_N(X_i^p)\psi_I(x,t)| &\le C(\chi_L(t) + t^{\alpha+2}\chi_R(t))(1+|x|)^{-k-d-1} \\ |\Theta^p \psi_I(x,t)| &\le C(\chi_L(t) + t^{\alpha}\chi_R(t))(1+|x|)^{-k-d-1}. \end{aligned} \tag{76}$$

We prove the existence of ψ_I in Corollary 10 on page 45 below using some recent results of R. Urban [22] concerning the Green's function for a class of operators that includes $\mathcal{L}^t$. A Green's function for $\mathcal{L}^t$ is a function $G \in C^\infty(S \setminus \{(0,1)\}) \cap L^1_{loc}(S)$ for which

$$\mathcal{L}^t G = \delta_{(0,1)}$$

as a distribution, where functions are identified with distributions using right invariant Haar measure on S. The modular function for right-invariant Haar measure on S is

$$\chi(n,t) = t^{-d}.$$

We define

$$f * h(x) = \int_S f(g)h(g^{-1}x)\chi(g)\, dg$$

where dg is right-invariant Haar measure. (Note that $\chi(g)dg$ is left-invariant Haar measure.) Since $\mathcal{L}^t$ is left invariant, it is easily seen that for all $\phi \in C_c^\infty(S)$,

$$\mathcal{L}^t(\phi * G) = \phi.$$

In [22], Urban proved the following result.

Theorem 10 (R. Urban). *There exists a Green's function G for $\mathcal{L}^t$ which satisfies the following conditions:*

For every compact neighborhood $\mathcal{U}$ of $(0,1)$ in S and for all multi-indecies I and all $k \in \mathbb{N} \cup \{0\}$ there is a constant C such that

$$|\Theta^k R_N(X^I)G(x,t)| \le \begin{cases} C(|x|+t)^{-\|I\|-d-\alpha}t^\alpha & (x,t) \in (\mathcal{Q}_1 \cup \mathcal{U})^c, \\ C & (x,t) \in \mathcal{Q}_1 \setminus \mathcal{U}. \end{cases} \tag{77}$$

where $\mathcal{Q}_\epsilon = \{|x| \le \epsilon\} \times \{0 < t \le \epsilon\}$ and $d = \sum_1^n d_j$.

Remark. Theorem 10 implies that for all $\epsilon > 0$, $R_N(X^I)G(x,t)$ is uniformly bounded on $Q_\epsilon \backslash \mathcal{U}$. Hence, we may replace Q_1 by Q_ϵ in the second inequality.

We require a slight variant on Urban's Theorem.

Proposition 9. *Theorem 10 holds with $L_N(X^I)$ in place of $R_N(X^I)$.*

Proof The proof is very similar to that of Proposition 5 on page 21 so we omit many details. We may assume, without loss of generality, that $\mathcal{U} \subset \{|x| \le 1\} \times \mathbb{R}^+$. On $\{|x| \le 1\} \times \mathbb{R}^+$, the result follows from Theorem 10 together with equation (31) on page 21. Hence we may assume $|x| \ge 1$, in which case we are attempting to prove the first inequality.

Let $\sigma(s)(x,t) = (\delta(s)x, st)$. From the reasoning in the proof of Proposition 5 on page 21 it suffices to show that on $\{1 \le |x| \le 2\} \times \mathbb{R}^+$, for all $s \ge 1$,

$$|L_N(X^I)\Theta^k(G \circ \sigma(s))(x,t)| \le C_I s^{-d}(|x|+t)^{-d-\alpha-\|I\|}t^\alpha.$$

This, however, follows from Theorem 10 together with equation (31) on page 21. $\square$

Corollary 9. *Let $\phi \in C_c^\infty(S)$ and let $\psi_I = (R_N(X^I)\phi) * G$. Then there is a $\tau > 0$ such that*

$$|\psi_I(x,t)| \le \begin{cases} C(|x|+t)^{-\|I\|-d-\alpha}t^\alpha & (x,t) \in (\mathcal{Q}_\tau \cup \mathcal{U})^c, \\ C & (x,t) \in \mathcal{Q}_\tau \setminus \mathcal{U}. \end{cases} \tag{78}$$

Proof

We may assume with loss of generality that $(0,1) \in \text{supp } \phi$. Let $\tilde{\phi}(x,t) = t^{-d}\phi(x,t)$. Then

$$\begin{aligned}\psi_I(x,t) &= \int R_N(X^I)\tilde{\phi}(y,u)G((y,u)^{-1}(x,t))\,\frac{dydu}{u}\\ &= \int R_N(X^I)\tilde{\phi}(y,u)G(\delta(u^{-1})(y^{-1}x),u^{-1}t)\,\frac{dydu}{u}\\ &= \int \tilde{\phi}(y,u)u^{-\|I\|}L_N(X^I)G(\delta(u^{-1})(y^{-1}x),u^{-1}t)\,\frac{dydu}{u}\\ &= \int \tilde{\phi}(y,u)u^{-\|I\|}L_N(X^I)G((y,u)^{-1}(x,t))\,\frac{dydu}{u}\end{aligned}$$

There are $\epsilon > 0$ and $\tau > 0$ such that

$$\begin{aligned}(\text{supp } \phi)\mathcal{Q}_1 &\subset \mathcal{Q}_\tau\\ (\text{supp } \phi)^{-1}\mathcal{Q}_\tau &\subset \mathcal{Q}_\epsilon.\end{aligned}$$

We set

$$\mathcal{U}_1 = (\text{supp } \phi)\mathcal{U}.$$

Then

$$\begin{aligned}(\text{supp } \phi)^{-1}\mathcal{U}_1^c &\subset \mathcal{U}^c\\ (\text{supp } \phi)^{-1}\mathcal{Q}_\tau^c &\subset \mathcal{Q}_1^c.\end{aligned}$$

Hence

$$\begin{aligned}(\text{supp } \phi)^{-1}\left(\mathcal{Q}_\tau \setminus \mathcal{U}_1\right) &\subset \mathcal{Q}_\epsilon \setminus \mathcal{U}\\ (\text{supp } \phi)^{-1}\left(\mathcal{Q}_\tau \cup \mathcal{U}_1\right)^c &\subset \left(\mathcal{Q}_1 \cup \mathcal{U}\right)^c\end{aligned}$$

Hence, from Corollary 9, along with the remark following Theorem 10, for $(x,t) \in \mathcal{Q}_\tau \setminus \mathcal{U}_1$,

$$\begin{aligned}|\psi_I(x,t)| \leq& C\int_{\text{supp } \phi} |\tilde{\phi}(y,u)|u^{-\|I\|}\frac{dydu}{u}\\ &= C'\end{aligned}$$

while for $(x,t) \in \left(\mathcal{Q}_\tau \cup \mathcal{U}_1\right)^c$,

$$\begin{aligned}&|\psi_I(x,t)|\\ &\leq C\int_{\text{supp } \phi} |\tilde{\phi}(y,u)|u^{-\|I\|}(u^{-1}|y^{-1}x| + u^{-1}t)^{-\|I\|-d-\alpha}t^\alpha u^{-\alpha}\,\frac{dydu}{u}\\ &= C\int_{\text{supp } \phi} |\tilde{\phi}(y,u)|u^{d-1}(|y^{-1}x| + t)^{-\|I\|-d-\alpha}t^\alpha\,dydu\\ &\leq C'(|x| + t)^{-\|I\|-d-\alpha}t^\alpha\end{aligned}$$

Our proposition follows. $\square$

Corollary 10. *For $\|I\| > k+1$, the function $\psi_I(x,t)$ defined above satisfies the inequalities (76) on page 42.*

Proof On $\mathcal{U}$, the desired inequalities are automatic from the compactness of $\mathcal{U}$ and the fact that ψ_I is C^∞. If $p = 0$ in the inequalities (76), then the desired inequalities off of $\mathcal{U}$ are trivial consequences of Corollary 9. Finally, note that $t^{\|J\|}R_N(X^J)$ is left invariant on S. Hence for $\phi \in C_c^\infty(S)$,

$$t^{\|J\|}R_N(X^J)(\phi * G) = \phi * (t^{\|J\|}R_N(X^J)G)$$
$$\Theta^k(\phi * G) = \phi * (\Theta^k G)$$

Our result follows by repeating the reasoning from Corollary 9, replacing G with either $t^{\|J\|}R_N(X^J)G$ or $\Theta^k G$ as needed and using Theorem 10 on page 43. $\square$

This finishes the proof of Theorem 9.

6.2 The general Liouville Theorem.

In this section we prove Theorem 2 on page 6. Specifically, we assume

$$\mathcal{L}F(x,t) = H(x,t)$$

where $H(x,t)$ depends polynomially on x and F satisfies the estimate (10) on page 6. By replacing F with F_η where F_η is as in (56) on page 32, we may assume that for all multi-indecies I,

$$|R_N(X^I)F(x,t)| \leq C(t^\alpha \chi_L(t) + t^b \chi_R(t))(1+|x|)^k \tag{79}$$

where b is independent of I but k may depend on I. (See the discussion of (57) on page 33.)

There is a "trick" that allows us to assume that $H = 0$. Let

$$Z = \text{span}\,\{X_i \mid d_i = 1\}.$$

Then Z is a $\delta(t)$-invariant central subgroup of N. By $F(x,y,t)$ we mean the function on $N \times Z \times \mathbb{R}^+$ defined by

$$F(x,y,t) = F(xy,t).$$

For $(x,t) \in S$, let

$$F^{\wedge}(x,\cdot,t) = F(x,\cdot,t)^{\wedge} \tag{80}$$

where " $\wedge$ " denotes the (Euclidean) distributional Fourier transformation on Z which is an element of $\mathcal{S}'(Z^*)$. Let ξ be the variable in Z^*. From (79), $(\mathcal{L}F)^{\wedge}$ is a tempered distribution in ξ supported at $\xi = 0$.

Let $\phi \in C_c^{\infty}(Z^*)$ be supported in $I(c,d)$ where $c > 0$ and

$$I(c,d) = \{\xi \in Z^* \mid c < \|\xi\| < d\} \tag{81}$$

and $\|\cdot\|$ is the Euclidean norm on Z^*. Let

$$G(x,t) = \int_Z \phi^{\vee}(y) F(xy^{-1},t)\,dy$$

so that

$$G^{\wedge}(x,\xi,t) = \phi(\xi)F^{\wedge}(x,\xi,t).$$

Then $(\mathcal{L}G)^{\wedge} = 0$. Hence, G is $\mathcal{L}$-harmonic.

Using Theorem 9 on page 40, we will prove the following:

Proposition 10. *Suppose that F is an $\mathcal{L}$-harmonic function such that $R_N(X^I)F$ satisfies the estimate (79) where C and k may depend on I. Suppose also that $F^{\wedge}$ is supported in $I(c,d)$ where $c > 0$. Then $F \equiv 0$.*

We claim that Proposition 10 implies Theorem 2. To see this, note that it follows that the function G above is zero for all choices of ϕ. Hence $F^{\wedge}$ is supported at $\xi = 0$, showing that $F(x,y,t)$ depends polynomially on y. Let

$$N' = \text{span}\,\{X_i \mid d_i < 1\}.$$

For $(x,y,t) \in N' \times Z \times \mathbb{R}^+$, write

$$F(x,y,t) = \sum_{|I| \leq m} F_I(x,t) y^I.$$

where I ranges over the set of multi-indecies of length equal to the dimension n_z of Z and the coordinates in Z are defined by the X_i basis. We identify N/Z with N'. The F_I define functions on $S_1 = S/Z = N' \times \mathbb{R}^+$.

From the Campbell-Hausdorff Theorem, together with the centrality of Z, we may express the product on N in terms of the product on $N/Z = N'$ as

$$(x,y)(x_1,y_1) = (xx_1, y + y_1 + \pi_Z\left(p(\text{ ad } x)(x_1)\right) + R(x,x_1))$$

where $p(\cdot)$ is a polynomial function on $\text{ad}\,(N)$, π_Z is the projection onto Z in the decompositions $N = N' + Z$, and

$$\|R(x, x_1)\| \leq C(x)\|x_1\|^2.$$

It follows that for $X_i \in N'$ and $f \in C^\infty(N)$,

$$\begin{aligned} R_N(X_i)f(x,y) &= R_{N_1}(X_i)f(x,y) + df(\pi_z(p(\text{ ad } x)(X_i)) \\ &= R_{N_1}(X_i)f(x,y) + \sum_{j=1}^{n_z} q_i^j(x_1, \ldots, x_{i-1})\partial_{y_j} \end{aligned}$$

where the q_i^j are polynomials. In particular, for $(x, y, t) \in N' \times Z \times \mathbb{R}^+$,

$$(\mathcal{L}F)(xy, t) = \mathcal{L}_1 F_m(x, y, t) \quad \text{mod } (C^\infty(N_1) \otimes (\text{span }_{|J|<m}\{y^J\}).$$

where

$$F_m(x, y, t) = \sum_{|I|=m} y^I F_I(x, t)$$

and $\mathcal{L}_1$ is the operator on S_1 defined by equation (4) on page 3 where now the summation is only over the indecies for which $d_i < 1$.

Since $\mathcal{L}F$, by hypothesis, is polynomial in (x, y), the same must be true for $\mathcal{L}_1 F_m$. It follows by induction on the dimension of S that we may assume that F_m is polynomial on N_1. We replace F by $F' = F - F_m$. Then F' satisfies the same assumptions as F and is of total degree at most $m-1$ in y. We may assume by induction on the degree of F in y that F' also depends polynomially on x, proving the polynomial dependence of F. Thus, our proof of Theorem 2 will be complete, once we have proved Proposition 10 on page 46. (Note that the last statement in Theorem 2 follows from Corollary 7 on page 38.)

6.3 Asymptotics as $t \to \infty$.

In this section we assume that F is an $\mathcal{L}$-harmonic function that satisfies the estimate (79).

We may eliminate the first order ∂_t terms in $\mathcal{L}$ with a change of dependent variable. Let $\tilde{F}(x, t) = t^{1-s}F(x, t)$ where

$$s = \frac{1 + \alpha}{2} > \frac{1}{2}.$$

Then, as the reader can check, $\tilde{\mathcal{L}}\tilde{F} = 0$ where

$$\tilde{\mathcal{L}} = t^2\partial_t^2 + s(1-s) + \sum_1^n (t^{2d_i}X_i^2 + c_i t^{d_i} X_i). \tag{82}$$

Division by t^2 shows

$$(\mathcal{L}_0 + Q)\tilde{F} = 0$$

where

$$\begin{aligned} \mathcal{L}_0 &= \partial_t^2 + s(1-s)t^{-2} + \sum_{d_i=1} X_i^2, \\ Q &= \sum_{d_i<1} t^{-\epsilon_i} X_i^2 + \sum_1^n t^{-\tau_i} c_i X_i, \\ \epsilon_i &= 2 - 2d_i > 0, \tau_i = 2 - d_i > 0. \end{aligned} \tag{83}$$

Then $\tilde{F}$ satisfies an inequality of the form of (79) where α is replaced by s and $b > s$.

Notice that

1. For large t the coefficients of X_i and X_i^2 in Q become small.
2. $t^2\mathcal{L}_0$ is essentially the Laplace-Beltrami operator on the upper-half hyperspace in $\mathbb{R}^{n_z+1}$.

This suggests analyzing $\tilde{\mathcal{L}}$ as a perturbation of $\mathcal{L}_0$ for large t.

We work with the Fourier transform $\tilde{F}^\wedge$ which is defined via formula (80) on page 46. Then $\tilde{F}^\wedge$ is $(\mathcal{L}_0^\wedge + Q)$-harmonic where

$$\mathcal{L}_0^\wedge = \partial_t^2 + s(1-s)t^{-2} - \|\xi\|^2.$$

and $\|\cdot\|$ is the Euclidean norm on Z^*. Replacing t by $(2\|\xi\|)^{-1}t$ transforms $\mathcal{L}_0^\wedge$ into

$$(2\|\xi\|)^2 \left(\partial_t^2 - \frac{1}{4} + \frac{s(1-s)}{t^2}\right).$$

The operator in parentheses is the Whittaker operator with the parameter $k = 0$ and $m = s - 1/2 = \alpha/2$. (Equation (1.6.2), p. 9 of [20]). From p. 50-51 of [20], equations (3.5.3) and (3.5.10), the null space of the Whittaker operator is spanned by the following two functions

$$\begin{aligned} M_s(t) &= \frac{\Gamma(2s)}{\Gamma(s)^2} t^s e^{-t/2} \int_0^1 e^{-ut}[u(1-u)]^{s-1}\,du \\ W_s(t) &= \frac{1}{\Gamma(s)} t^s e^{-t/2} \int_0^\infty e^{-ut}[u(1+u)]^{s-1}\,du. \end{aligned} \tag{84}$$

where the integrals exist for $s > 0$.

Following Lang [15], p. 291-293, we let

$$\begin{aligned} K_s(\xi,t) &= \frac{\Gamma(s)}{\Gamma(2s)} M_s(2\|\xi\|t) \\ J_s(\xi,t) &= W_s(2\|\xi\|t) \end{aligned} \tag{85}$$

Then $K_s(\xi,t)$ and $J_s(\xi,t)$ are $\mathcal{L}_0^\wedge$-harmonic functions such that

$$\begin{aligned} \lim_{t\to 0^+} t^{-s} K_s(\xi,t) &= \frac{\Gamma(2s)}{\Gamma(s)} (2\|\xi\|)^s \\ \lim_{t\to 0^+} t^{s-1} J_s(\xi,t) &= \frac{\Gamma(2s-1)}{\Gamma(s)} (2\|\xi\|)^{1-s}. \end{aligned} \tag{86}$$

(See (89) and (91) below for the computation of the limits.)

For $f \in C_c^\infty((Z^* \setminus \{0\}) \times \mathbb{R}^+)$, let $\Lambda^\pm f$ be defined by

$$\begin{aligned} \Lambda^+ f(\xi,t) &= (2\|\xi\|)^{-1} J_s(\xi,t) \int_0^t K_s(\xi,u) f(\xi,u)\,du \\ \Lambda^- f(\xi,t) &= (2\|\xi\|)^{-1} K_s(\xi,t) \int_t^\infty J_s(\xi,u) f(\xi,u)\,du \end{aligned} \tag{87}$$

According to the first boxed equation on p. 291 of Lang [15],

$$\Lambda = \Lambda^+ + \Lambda^-$$

is a right inverse for $\mathcal{L}_0^\wedge$ on $C_c^\infty((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^+)$.

In order to define $\Lambda^\pm$ on distributions, we need asymptotics for the derivatives of M_s and W_s at both $t = 0$ and $t = \infty$. The following results are known. We provide proofs for the convenience of the reader.

Lemma 10. *For $s > 0$, the following limits exist.*

$$\begin{aligned} W^n_{s,0} &= \lim_{t\to 0} t^{s+n-1}\partial_t^n W_s(t) \\ W^n_{s,\infty} &= \lim_{t\to\infty} e^{t/2}\partial_t^n W_s(t) \\ M^n_{s,0} &= \lim_{t\to 0} t^{n-s}\partial_t^n M_s(t) \\ M^n_{s,\infty} &= \lim_{t\to\infty} e^{-t/2}\partial_t^n M_s(t). \end{aligned}$$

Proof

From (84) on page 49, with $v = ut$, $du = t^{-1}dv$,

$$\begin{aligned} &\partial_t^n \left(t^{-s}e^{t/2}W_s(t)\right) \\ &= \frac{(-1)^n}{\Gamma(s)}\int_0^\infty e^{-ut}u^{s+n-1}(1+u)^{s-1}\,du \\ &= \frac{(-1)^n}{\Gamma(s)}t^{-s-n}\int_0^\infty e^{-v}v^{s+n-1}(1+\frac{v}{t})^{s-1}\,dv \\ &= \frac{(-1)^n}{\Gamma(s)}t^{-2s+1-n}\int_0^\infty e^{-v}v^{s+n-1}(t+v)^{s-1}\,dv \end{aligned} \tag{88}$$

Hence

$$\begin{aligned} \lim_{t\to 0} t^{2s-1+n}\partial_t^n \left(t^{-s}e^{t/2}W_s(t)\right) &= \frac{(-1)^n\Gamma(2s-n-1)}{\Gamma(s)} \\ \lim_{t\to\infty} t^{s+n}\partial_t^n \left(t^{-s}e^{t/2}W_s(t)\right) &= \frac{(-1)^n\Gamma(s+n)}{\Gamma(s)} \end{aligned} \tag{89}$$

The first equation implies, in particular, the existence the first limit in Lemma 10.

From Leibnitz's rule

$$\partial_t^n \left(t^{-s}e^{t/2}W_s(t)\right) = \sum_{k=0}^n \binom{n}{k} (C_k t^{-s-(n-k)} + \dots)e^{t/2}\partial_t^k W_s(t) \tag{90}$$

where "…" indicates a finite linear combination of $\{t^j\}$ where $j > -s-(n-k)$. We multiply (90) by t^{2s-1+n} and let $t \to 0^+$, finding from (89) that

$$\frac{(-1)^n\Gamma(2s-n-1)}{\Gamma(s)} = \sum_{k=0}^n \binom{n}{k} C_k \lim_{t\to 0}\left(t^{s+k-1}\partial_t^k W_s(t)\right).$$

The nth limit in the above sum exists provided all of the preceding $n-1$ limits exist. Thus, the existence of all of the limits follows by induction on n.

For the limit as $t \to \infty$, we reason similarly, writing (90) as

$$\partial_t^n \left(t^{-s}e^{t/2}W_s(t)\right) = \sum_{k=0}^{n} \binom{n}{k} (C_k t^{-s} + \dots) e^{t/2} \partial_t^k W_s(t)$$

where now "…" indicates a finite linear of the form $C_j t^j$ where $j < -s$. Our result follows as before by multiplying the above equation by t^s and applying the $n = 0$ case.

For M_s, we note that from (84) on page 49

$$\begin{aligned}\partial_t^n \left(t^{-s}e^{t/2}M_s(t)\right) &= (-1)^n \frac{\Gamma(2s)}{\Gamma(s)^2} \int_0^1 e^{-ut} u^{s+n-1}(1-u)^{s-1}\, du \\ &= (-1)^n \frac{\Gamma(2s)}{\Gamma(s)^2} t^{-n-s} \int_0^t e^{-v} v^{s+n-1}(1-\frac{v}{t})^{s-1}\, dv\end{aligned}$$

These two equations imply, respectively, that

$$\begin{aligned}\lim_{t\to 0} \partial_t^n \left(t^{-s}e^{-t/2}M_s(t)\right) &= \frac{\Gamma(2s)(-1)^n}{\Gamma(s)^2} B(s+n, s) \\ &= (-1)^n \frac{\Gamma(2s)\Gamma(s+n)}{\Gamma(s)\Gamma(2s+n)} \\ \lim_{t\to\infty} t^{n+s} \partial_t^n \left(t^{-s}e^{-t/2}M_s(t)\right) &= (-1)^n \frac{\Gamma(2s)\Gamma(s+n)}{\Gamma(s)^2}\end{aligned} \tag{91}$$

where B is the beta function. The latter equality implies in particular the existence of

$$\lim_{t\to\infty} t^s \partial_t^n \left(t^{-s}e^{-t/2}M_s(t)\right).$$

Our result follows using Leibnitz's rule and induction just as before. □

Corollary 11. *For all compact subsets K of $Z^* \setminus \{0\}$ there is a $C > 0$ such that for $\xi \in K$,*

$$\begin{aligned}|\partial_\xi^n(K_s(\xi, t))| &\le C \left(t^s \chi_L(t) + e^{\|\xi\| t} \chi_R(t)\right) \\ |\partial_\xi^n(J_s(\xi, t))| &\le C \left(t^{1-s} \chi_L(t) + e^{-\|\xi\| t} \chi_R(t)\right).\end{aligned}$$

Suppose that $t \to f(\cdot,t)$ is a mapping of $\mathbb{R}^+$ into the space of distributions on $Z^* \setminus \{0\}$. We define

$$\begin{aligned}
&<\Lambda^+ f(\cdot,t), \phi(\cdot)> \\
&\quad = \int_0^t < f(\cdot,u), (2\|\cdot\|)^{-1} J_s(\cdot,t) K_s(\cdot,u)\phi(\cdot) > \, du \\
&<\Lambda^- f(\cdot,t), \phi(\cdot)> \\
&\quad = \int_t^\infty < f(\cdot,u), (2\|\cdot\|)^{-1} K_s(\cdot,t) J_s(\cdot,u)\phi(\cdot) > \, du.
\end{aligned} \tag{92}$$

provided the integrals exist and define distributions on $Z^* \setminus \{0\}$.

For $\phi \in C_c^\infty(Z^*)$ and $0 < c < d$ let

$$\|\phi\|_{m,c,d} = \sup\{|\partial^J \phi(\xi)| \xi \in I(c,d), 0 \le |J| \le m\} \tag{93}$$

where $I(c,d)$ is as in (81) on page 46. A distribution ϕ that is continuous in $\|\cdot\|_{m,c,d}$ for some m is supported in $\overline{I(c,d)}$. Conversely, a distribution which is supported in a compact subset of $I(c,d)$ is continuous in $\|\cdot\|_{m,c,d}$ for some m.

Definition 6. Let $t \to f(x,\cdot,t)$ be a set of mappings of $\mathbb{R}^+$ into the set of distributions on $Z^* \setminus \{0\}$ indexed by a parameter $x \in \Lambda$ and let $M : \Lambda \times \mathbb{R}^+ \to \mathbb{R}^+$. We say that $f(x,\cdot,t)$ is uniformly bounded by $M(x,t)$ on $I(c,d)$ where $c > 0$ if there exist $c < c' < d' < d$ and a constant C such that for all $\phi \in C_c^\infty(Z^* \setminus \{0\})$

$$| < f(x,\cdot,t), \phi(\cdot) > | \le C M(x,t) \|\phi\|_{m,c',d'} \tag{94}$$

where C, c', d' and k are all independent of ϕ and t.

Lemma 11. *Suppose $t \to f(\cdot,t)$ be a mapping of $\mathbb{R}^+$ into the set of distributions on $Z^* \setminus \{0\}$ that is uniformly bounded by $M(t) = t^a \chi_L(t) + t^b \chi_R(t)$ on $I(c,d)$ where $c > 0$ where $a > -s-1$. Then, the integrals in (92) exist and define elements of $\mathcal{D}'(Z^*)$.*

Proof

From Corollary 11, and inequality (94), the first integrand in (92) is bounded by a function of the form

$$\begin{aligned}
&C(t^{1-s}\chi_L(t) + e^{-ct}\chi_R(t)) \\
&\qquad \cdot (u^{s+a}\chi_L(u) + u^b e^{du}\chi_R(u)) \|\phi\|_{m,c,d}
\end{aligned} \tag{95}$$

while the second integrand is bounded by

$$C(t^s\chi_L(t)+e^{dt}\chi_R(t)) \cdot(u^{1-s+a}\chi_L(u)+u^be^{-cu}\chi_R(u))\|\phi\|_{m,c,d}. \tag{96}$$

□

Proposition 11. *Suppose that* $t \to f(\cdot,t)$ *satisfies the hypotheses of Lemma 11 where* $a > s-2$*. Then* $\Lambda^{\pm}f$ *also satisfies the hypotheses of Lemma 11 with* a *replaced by* s*.*

Proof

We first note that $a+s > 2s-2 \geq -1$ since $s \geq \frac{1}{2}$. Hence, the hypotheses of Lemma 11 are fulfilled.

We need to show that for $\phi \in C_c^\infty(Z^*)$ there is an $m \in \mathbb{N}$, and $0 < c < c'' < d'' < d$ such that

$$| < \Lambda^{\pm}f(\cdot,t),\phi(\cdot) > | \leq C\left(t^s\chi_L(t)+t^b\chi_R(t)\right)\|\phi\|_{m,c'',d''}. \tag{97}$$

By hypothesis there are $c < c' < d' < d$ for which inequality (94) on page 52 holds. In particular, $f(\cdot,t)$ is supported in $\overline{I(c',d')}$ for all t.

We let c'' and d'' be any numbers satisfying $c < c'' < c'$ and $d' < d'' < d$. Let $\phi_o \in C_c^\infty(I(c'',d''))$ equal 1 on $I(c',d')$. Then,

$$< \Lambda^{\pm}f(\cdot,t),\phi(\cdot) >=< \Lambda^{\pm}f(\cdot,t),\phi_o\phi(\cdot) > .$$

Hence, in establishing (97), we may assume that $\phi \in C_c^\infty(I(c'',d''))$.

Case 1: $0 < t \leq 1$**.**

From (95),

$$\begin{aligned} < \Lambda^+f(\cdot,t),\phi(\cdot) > &\leq Ct^{1-s}\|\phi\|_{m,c',d'}\int_0^t u^{s+a}\,du \\ &= C\frac{t^{2+a}}{s+a-1}\|\phi\|_{m,c',d'} \\ &\leq C't^s\|\phi\|_{m,c'',d''} \end{aligned}$$

as desired.

We also note that from (96)

$$\begin{aligned}
&< \Lambda^- f(\cdot, t), \phi(\cdot) > \\
&\quad \leq Ct^s \|\phi\|_{m,c',d'} \left(\int_t^1 u^{1-s+a}\, du + \int_1^\infty u^b e^{-cu}\, du \right) \\
&\quad = Ct^s \|\phi\|_{m,c',d'} (t^{2-s+a} + K) \\
&\quad \leq C' t^s \|\phi\|_{m,c'',d''}.
\end{aligned}$$

Case 2: $1 < t < \infty$.

This case is somewhat subtler than Case 1 since in formulas (87) on page 49, we must show that the exponential growth of $K_s(\xi, t)$ is precisely canceled by the exponential decay of $J_s(\xi, t)$. If $f(\xi, t)$ is a function, this is accomplished using a point wise estimate on the integrand. In the case of a distribution, we make use of the fact that every distribution is obtained by differentiating a function. We note also that it is sufficient to prove the boundedness condition for all sufficiently large t.

Let $f^\wedge(x, t)$ be the Fourier transformation of $f(\xi, t)$ in ξ. From the boundedness condition (94) on page 52 applied to $\phi(\xi) = e^{-ix\cdot\xi}$, $f^\wedge(x, t)$ is C^∞ in (x, t) and satisfies

$$|f^\wedge(x, t)| \leq C(1 + \|x\|^2)^{m/2} (t^a \chi_L(t) + t^b \chi_R(t)).$$

For $n > \frac{1}{2}(n_z + m)$, $n \in \mathbb{N}$, where $n_z = \dim Z$, let

$$\begin{aligned}
g^\wedge(x, t) &= (1 + \|x\|^2)^{-n} f^\wedge(x, t) \\
g(\xi, t) &= (g^\wedge)^\vee(\xi, t)
\end{aligned}$$

Then g is continuous in ξ and satisfies

$$\begin{aligned}
\|g(\cdot, t)\|_\infty &\leq C(t^a \chi_L(t) + t^b \chi_R(t)) \\
(1 - \sum_i \partial_{\xi_i}^2)^n g(\xi, t) &= f(\xi, t)
\end{aligned} \tag{98}$$

where the last equality is in the sense of distributions.

From (96), Corollary 11 on page 51, the Fubini theorem, and the assumption on supp ϕ,

$$
\begin{aligned}
&| <\Lambda^- f(\cdot,t), \phi(\cdot)> | \\
&\quad = | \int_t^\infty \int_{-\infty}^\infty g(\xi,u)(1-\sum_i \partial_{\xi_i}^2)^n \left((2\|\xi\|)^{-1} K_s(\xi,t) J_s(\xi,u)\phi(\xi)\right) \, d\xi du| \\
&\quad \le C\|\phi\|_{m,c'',d''} \int_{c''<\|\xi\|<d''} e^{\|\xi\| t} \int_t^\infty u^b e^{-\|\xi\| u} \, du d\xi.
\end{aligned}
$$

Our result follows from Lemma 12 below.

The reasoning for Λ^+ is similar. Let $t_o \ge \max\{-2c^{-1}b, 1\}$. We split the first integral in (92) on page 52 into the sum of an integral over $[0, t_o]$ and an integral over $[t_o, t]$. We express f as in (98) and estimate the integrals as before using (95) on page 52. From Corollary 11 on page 51, the first integral decays exponentially as $t \to \infty$. The analysis of the second integral follows from Lemma 13 below. We leave the details to the reader. $\square$

Lemma 12. *For $t \in \mathbb{R}^+$, $t \ge 2b\,\|\xi\|^{-1}$,*

$$
\int_t^\infty e^{-\|\xi\| u} u^b \, du < 2\|\xi\|^{-1} e^{-\|\xi\| t} t^b.
$$

Proof

Let $t_o > t$ be given. From the Cauchy Mean Value Theorem applied in the t variable, there is a $C \in (t, t_o)$ such that

$$
\begin{aligned}
\frac{\int_t^{t_o} e^{-\|\xi\| u} u^b \, du}{\left(e^{-\|\xi\| u} u^b \big|_t^{t_o}\right)} &= \frac{-e^{-\|\xi\| C} C^b}{\|\xi\| e^{-\|\xi\| C} C^b - b e^{-\|\xi\| C} C^{b-1}} \\
&= \frac{C}{-\|\xi\| C + b}
\end{aligned}
$$

Note that for $t > b\|\xi\|^{-1}$, the denominator in the last fraction is negative and $e^{-t\|\xi\|} t^b$ is decreasing. Hence the above formula implies

$$
\int_t^{t_o} e^{-\|\xi\| u} u^b \, du \le \frac{C}{\|\xi\| C - b} e^{-\|\xi\| t} t^b < 2\|\xi\|^{-1} t^b e^{-t/2}
$$

proving the lemma. $\square$

Lemma 13. *Let $t > t_o \ge -2\|\xi\|^{-1} b$ where $t_o \in \mathbb{R}^+$. Then*

$$
\int_{t_o}^t e^{\|\xi\| u} u^b \, du < 2\|\xi\|^{-1} e^{\|\xi\| t} t^b.
$$

Proof As in the proof of Lemma 12

$$\int_{t_o}^{t} e^{\|\xi\|u} u^b \, du = \frac{C}{\|\xi\|C + b} \left(e^{\|\xi\|t} t^b - e^{\|\xi\|t_o} t_o^b \right)$$

for some $C \in (t_o, t)$. Furthermore, our assumptions on t_o imply that $e^{\|\xi\|t} t^b$ is increasing on (t_o, t) and that the fraction on the right of the above equality is less than $2\|\xi\|^{-1}$, proving our lemma. □

We require one additional lemma before proving Proposition 10:

Lemma 14. *Suppose that $t \to f(\cdot, t)$ is a C^∞ mapping into the space of distributions on $Z^* \setminus \{0\}$ which satisfies the hypotheses of Lemma 11 on page 52 where $a = s$. Suppose also that*

$$\mathcal{L}_0^\wedge f(\cdot, t) = 0.$$

Then there are unique distributions $a(\cdot)$ and $b(\cdot)$ supported in $I(c, d)$ such that

$$f(\cdot, t) = a(\cdot) K_s(\cdot, t) + b(\cdot) J_s(\cdot, t).$$

Proof Let

$$\tilde{F}(x, t) = < f(\cdot, t), e^{i(\cdot)x} >$$

be the inverse Fourier transform of $f(\cdot, t)$. Then $\mathcal{L}_0 \tilde{F} = 0$. Let $F = t^{s-1} \tilde{F}$. Then

$$(\Theta^2 - \alpha\Theta + \sum_{i=1}^{n_z} t^2 \partial_i^2) F(x, t) = 0. \tag{99}$$

Furthermore, from our hypotheses,

$$|F(x, t)| \le C \left(t^s \chi_L(t) + t^b \chi_R(t) \right) (1 + |x|)^k.$$

Hence F satisfies condition (6) on page 4 relative to the group $Z \times_s \mathbb{R}^+$.

It follows from Corollary 8 on page 39 that

$$\lim_{t \to 0^+} F(\cdot, t) = a^\vee(\cdot)$$

exists in $\mathcal{S}'(Z^*)$. Let

$$a(\cdot) = (a^\vee)^\wedge(\cdot)$$

be the distributional Fourier transform of $a(\cdot)$.

Since the Fourier transformation is continuous on $\mathcal{S}'(Z^*)$,

$$\lim_{t\to 0^+} t^{s-1} f(\cdot,t) = a(\cdot).$$

Hence for $\phi \in C_c^\infty(Z^*)$,

$$| < a(\cdot), \phi(\cdot) > | \leq C\|\phi\|_{m,c,d}$$

for some $C > 0$.

Let

$$g(\cdot,t) = f(\cdot,t) - \frac{\Gamma(s)}{\Gamma(2s-1)}(2\|\cdot\|)^{s-1} J_s(\cdot,t)a(\cdot)$$

where J_s is as in (85) on page 49. Then g is $\mathcal{L}_0^\wedge$ harmonic and from (86) on page 49

$$\lim_{t\to 0^+} t^{s-1} g(\cdot,t) = 0$$

Let G be constructed from g in precisely the same manner as F was constructed from f. Then G satisfies the differential equation (99) as well as

$$\lim_{t\to 0^+} G(\cdot,t) = 0.$$

It now follows from Corollary 8 on page 40 and Corollary 2 on page 17 that

$$\lim_{t\to 0^+} t^{-\alpha} G(\cdot,t) = b^\vee(\cdot)$$

exists in $\mathcal{S}'(Z^*)$. Then, reasoning as above, we see that $b(\cdot) = (b^\vee)^\wedge(\cdot)$ is supported in $I(c,d)$.

Finally, from (85) on page 49 and the first equation in (91) on page 51,

$$h(\cdot,t) = g(\cdot,t) - \frac{\Gamma(s)}{\Gamma(2s-1)}(2\|\cdot\|)^{s-1} K_s(\cdot,t)b(\cdot)$$

is an $\mathcal{L}_0^\wedge$ harmonic function. It is easily seen by forming the Fourier transformation as above and applying Theorem 8 on page 40 and formula (86) on page 49, that $h \equiv 0$, proving our lemma. □

6.4 Proof of Proposition 10 on page 46.

Let notation be as in Section 6.3 and let F satisfy the hypotheses of Proposition 10 on page 46. Thus F is an $\mathcal{L}$-harmonic function satisfying the estimate (79) on page 45 such that $\tilde{F}^\wedge(x,\cdot,t)$ is a distribution which is supported in $I(c,d)$. By decreasing c and enlarging d if necessary, we may assume that $\tilde{F}^\wedge(x,\cdot,t)$ is supported in $I(c',d')$ where $c < c' < d' < d$ and c' and d' are independent of (x,t). We will prove that under these conditions, F satisfies the hypotheses of Proposition 9 on page 40, proving Proposition 10.

Lemma 15. *For all multi-indecies I, $R_N(X^I)\tilde{F}^\wedge(x,\cdot,t)$ is uniformly bounded on $I(c,d)$ by $M(t)$ where*

$$M(t) = (t^s\chi_L(t) + t^{b'}\chi_R(t))(1+|x|)^k. \tag{100}$$

where $b' = b+1-s$.

Proof There are $c < c' < d' < d$ such that $\tilde{F}^\wedge(x,\cdot,t)$ is supported in $I(c',d')$. Let $c < c'' < c'$ and $d < d'' < d'$ be arbitrary. In proving that an estimate of the form of (94) on page 52 holds, we may assume that supp $\phi \subset I(c'',d'')$. (See the proof of Proposition 11 on page 53.) From the definition of the Fourier transform, (79) on page 45, and the subadditive property of $|\cdot|$, we see

$$\begin{aligned}
&| < R_N(X^I)\tilde{F}^\wedge(x,\cdot,t), \phi(\cdot) > | \\
&= |\int_Z t^{1-s} R_N(X^I)F(xy,t)\overline{(\overline{\phi})^\vee}(y)\,dy| \\
&\le Ct^{1-s}(t^\alpha\chi_L(t) + t^b\chi_R(t))(1+|x|)^k \int_Z (1+|y|)^k|\overline{(\overline{\phi})^\vee}(y)|\,dy.
\end{aligned}$$

Since $1-s+\alpha = s$, our lemma follows. □

Lemma 15, together with Lemma 11 on page 52, implies that for each x, $\tilde{F}^\wedge(x,\cdot,\cdot)$ is in the domain of Λ.

Lemma 16.

$$\tilde{F}^\wedge(x,\cdot,t) = \Lambda\mathcal{L}_0^\wedge\tilde{F}^\wedge(x,\cdot,t).$$

Proof

Since Λ is a right inverse for $\mathcal{L}_0^\wedge$, $\tilde{F}^\wedge - \Lambda\mathcal{L}_0^\wedge\tilde{F}^\wedge$ is $\mathcal{L}_0^\wedge$ harmonic. Hence, from Lemma 14 on page 56,

$$\tilde{F}^\wedge(x,\cdot,t) = \Lambda\mathcal{L}_0^\wedge\tilde{F}^\wedge(x,\cdot,t) + a(x,\cdot)K_s(\cdot,t) + b(x,\cdot)J_s(\cdot,t). \tag{101}$$

where J_s and K_s are as in (85) on page 49 and, $a(x,\cdot)$ and $b(x,\cdot)$ are distributions supported in $I(c,d)$.

It follows from the above equality, Lemma 15, Proposition 11 on page 53 and (86) on page 49 that

$$\begin{aligned} 0 &= \lim_{t\to 0^+} t^{s-1}(a(x,\cdot)K_s(\cdot,t) + b(x,\cdot)J_s(\cdot,t)) \\ &= b(x,\cdot)\frac{\Gamma(2s-1)}{\Gamma(s)}(2\|\cdot\|)^{1-s} \end{aligned}$$

showing that $b \equiv 0$.

Next we show that $a \equiv 0$ as well, which will finish the proof of the lemma. From 83 on page 48,

$$\begin{aligned} &\Lambda\mathcal{L}_0^\wedge\tilde{F}^\wedge(x,\cdot,t) = \\ &\quad -\sum_{d_i<1}\Lambda\left(t^{-\epsilon_i}X_i^2\tilde{F}^\wedge(x,\cdot,t)\right) - \sum_1^n\Lambda\left(t^{-\tau_i}c_iX_i\tilde{F}^\wedge(x,\cdot,t)\right). \end{aligned} \tag{102}$$

The smallest possible exponent in the above sum is $-\epsilon_1 = 2d_1 - 2 > -2$ and the largest is $-\tau_n = -1$. It follows from Proposition 11 on page 53 that the above expression is uniformly bounded on $I(c,d)$ by

$$(t^s\chi_R(t) + t^{b'-1}\chi_R(t))(1+|x|)^k. \tag{103}$$

Hence, from (101), $a(x,\cdot)K_s(\cdot,t)$ is uniformly bounded on $I(c,d)$ for $t \geq 1$ by $t^{b'}(1+|x|)^k$.

Let $\phi(\xi,t) \in C_c^\infty(I(c,d))$ satisfy

$$t \to \|\phi(\cdot,t)\|_{m,c,d} \leq M \tag{104}$$

for $t \geq 1$ where M is some scalar and m is as in Definition 6 on page 52 relative to $f(x,\cdot,t) = a(x,\cdot)K_s(\cdot,t)$. It follows that for $t \geq 1$,

$$| < a(x,\cdot), K_s(\cdot,t)\phi(\cdot,t) > | \leq C(1+|x|)^k t^{b'}.$$

We will show that if $a \not\equiv 0$, there is a choice of $\phi(\cdot, t)$ such that the expression on the left grows at least exponentially in t, which is a contradiction. In fact, if $a(x, \cdot) \not\equiv 0$, then $< a(x, \cdot), \psi_o(\cdot) > \neq 0$ for some $\psi_o \in C_c^\infty(I(c, d))$. Let

$$\phi(\xi, t) = e^{ct} K_s(\xi, t)^{-1} \psi_o(\xi).$$

We claim that

$$\lim_{t \to \infty} \|\phi(\cdot, t)\|_{m,c,d} = 0$$

so that $\phi(\xi, t)$ satisfies (104). In fact, it is easily seen by mathematical induction that for any differentiable function f on an open subset of $\mathbb{R}$,

$$\partial_t^k (\frac{1}{f}) = \frac{p_k(f, \partial_t f, \ldots, \partial_t^k f)}{f^{k+1}}$$

where $p_k(x_o, x_1, \ldots, x_k)$ is a polynomial of total degree at most k. From the second equation in (91) on page 51 with $n = 0$,

$$M_s(t) \geq C e^{t/2}$$

for $t \geq 1$ and some $C \neq 0$, while from Lemma 10 on page 49

$$\partial_t^k M_s(t) \leq C' e^{t/2}$$

on the same set. It follows that

$$e^{t/2} \partial_t^k (M_s(t))^{-1}$$

is bounded on $[1, \infty)$. Our claim follows from this and the definition of $K(\xi, t)$ (formula (85) on page 49).

Also

$$< a(x, \cdot), K_s(\cdot, t)\phi(\cdot, t) > = e^{ct} < a(x, \cdot), \psi_o(\cdot) >$$

which grows exponentially in t. Our lemma follows. □

It follows from the preceding lemma and (102), that $\tilde{F}^\wedge(x, \cdot, t)$ is uniformly bounded on $I(c, d)$ by the function $M(t)$ from (103). Repeating this argument as many times as necessary shows that $\tilde{F}^\wedge(x, \cdot, t)$ is uniformly bounded on $I(c, d)$ by an expression such as (100) where $b' < -s - 4$. By taking the inverse Fourier transform, it follows that F satisfies the hypotheses of Theorem 9 on page 40, proving Proposition 10. □

7 Proof of Theorem 4

We complete the proof sketch given on page 13. The first step is Corollary 8 on page 39. For the second, it suffices to show the following proposition.

Proposition 12. *Let $f \in \mathcal{S}'(N)$ and let μ and ϕ be chosen as described in Definition 4 on page 11. Then in $\mathcal{S}'(N)$*

$$\lim_{t \to 0^+} P^{mol}_{\mu,\phi}(f)(\cdot, t) = f(\cdot).$$

Proof

Let $F(x,t) = P^{mol}_{\mu,\phi}(f)(x,t)$. From Corollary 8 on page 39 and Theorem 8 on page 40, it suffices to show that the first term F_0 of the asymptotic expansion of F is f. If $f \in C^\infty_c(N)$ this is clear since, in this case, Definition 4 on page 11 implies that

$$P^{mol}_{\mu,\phi}(f)(x,t) = P(f)(x,t) + t^\alpha < Q_\mu((\cdot)^{-1}, x, t), \phi(\cdot) f(\cdot) >$$

where $P(f)$ is the standard Poisson integral of f. Hence

$$\lim_{t \to 0^+} P^{mol}_{\mu,\phi}(f)(x,t) = \lim_{t \to 0^+} P(f)(x,t) = f(x).$$

The general case follows from a density argument together with the observations that

1. For each $\mu \geq k$, are real numbers (a,b,m) such that $f \to P^{mol}_{\mu,\phi}(f)(x,t)$ is continuous as a mapping of $\mathcal{S}'_{p,k}(N)$ into the space $\mathcal{H}(a,b,m)$ which is defined below (6) on page 4.

2. For $\beta \in \tilde{\mathbb{N}}$, the mapping of $\mathcal{H}(a,b,m)$ into $\mathcal{S}'(N)$ defined by $F \to F_\beta$, where F_β is as in Theorem 8, is continuous as a map into $\mathcal{S}'(N)$.

The continuity of the above maps is easily seen either from the closed graph theorem or directly from the proofs of their existence. □

For the third step of our proof, recall that we set $\tilde{F} = F - P^{mol}_{\mu,\phi}(f)$. From step 2 in the proof, the $\tilde{F}_0$ term in $\tilde{F}$'s asymptotic expansion in Theorem 8 vanishes. It then follows from Corollary 2 on page 17 that this expansion contains only the $\tilde{G}_\beta$ terms. Let $\eta \in C^\infty_c(N)$ and let $\tilde{F}_\eta$ is defined in (56) on page 32. The asymptotic expansion of $\tilde{F}_\eta$ will also contain only $(\tilde{G}_\eta)_\beta$ terms. It then follows from Theorem 7 on page 36 that

$$|\tilde{F}_\eta(x,t) - t^\alpha (\tilde{G}_0)_\eta(x)| \leq C(t^\mu + t^b)(1+|x|)^k \tag{105}$$

for some $b > \mu > \alpha$ where k, b, and μ are independent of η. Thus

$$|\tilde{F}_\eta(x,t)| \leq C(t^\alpha + t^b)(1+|x|)^k.$$

Then, from Theorem 2 on page 6, $\tilde{F}_\eta(x,t)$ is a polynomial-like function that vanishes on N–i.e.

$$\tilde{F}_\eta(x,t) = t^\alpha \sum_{I,\beta} C^\eta_{I,\beta} x^I t^\beta$$

where the sum is finite. Furthermore, from (105), in this sum $\|I\| \leq k$ and $\beta \leq b$. Taking a limit where η ranges over an approximate identity shows that $\tilde{F}(x,t)$ is a polynomial-like harmonic function that vanishes on N, proving our theorem. □

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