MA 301 Practice Test 4, Spring 2006

(1) State the "official" definition of "$\lim_{x\to a} f(x) = L$."

(2) Suppose that $f(x)$ and $g(x)$ are both continuous at $x = a$. Prove that $h(x) = f(x)g(x)$ is also continuous at $x = a$. You may use the product theorem for limits of functions from Chapter 10.

Solution:

$$\begin{aligned}\lim_{x\to a} h(x) &= \lim_{x\to a}(f(x)g(x)) \\ &= (\lim_{x\to a} f(x))(\lim_{x\to a} g(x)) \quad \text{Product Theorem} \\ &= f(a)g(a) = h(a) \quad \text{Continuity of } f \text{ and } g\end{aligned}$$

Since $\lim_{x\to a} h(x) = h(a)$, h is continuous at $x = a$.

(3) Use a δ-ϵ argument to prove the following limit statements:

(a) $$\lim_{x\to 2} \frac{3x}{x+1} = 2$$

(b) $$\lim_{x\to 2} \frac{1}{x^2} = \frac{1}{4}$$

(c) $$\lim_{x\to 1} (x^2+3) = 4$$

(d) $$\lim_{x\to 1} \frac{1}{x^2+3} = \frac{1}{4}$$

(e) $$\lim_{x\to 0} \frac{1}{1-x} = 1$$

(f) $$\lim_{x\to 1} \frac{1}{\sqrt{3+x}} = \frac{1}{2}$$

Solutions:

(a):

Scratch Work: *Let $\epsilon > 0$ be given. We want*

$$\left|\frac{3x}{x+1} - 2\right| < \epsilon$$

$$\left|\frac{x-2}{x+1}\right| = |x-2|\,\frac{1}{|x+1|} < \epsilon$$

Assume that $x = 2 \pm 1$ *so that* $1 < x < 3$ *and* $2 < x+1 < 3$. *Then*

$$|x-2|\,\Big|\frac{1}{x+1}\Big| < \frac{1}{2}|x-2|$$

This will be $< \epsilon$ *if* $|x-2| < 2\epsilon$.

Our Proof: Let $\epsilon > 0$ be given and let $\delta = \min\{1, 2\epsilon\}$. Assume that $0 < |x-2| < \delta$. Then from the scratch work

$$\Big|\frac{3x}{x+1} - 2\Big| < \epsilon,$$

proving the limit statement.

(b):

Scratch Work: *Let* $\epsilon > 0$ *be given. We want*

$$\Big|\frac{1}{x^2} - \frac{1}{4}\Big| < \epsilon$$

$$\Big|\frac{x^2-4}{4x^2}\Big| = |x-2|\,\frac{|x+2|}{4x^2} < \epsilon$$

Assume that $x = 2 \pm 1$. *Then*

$$1 < x < 3$$

$$1 < x^2 < 9$$

$$4 < 4x^2 < 36$$

$$3 < x+2 < 5$$

Then

$$|x-2|\,\frac{|x+2|}{4x^2} < \frac{5}{4}|x-2|$$

This will be $< \epsilon$ *if* $|x-2| < \frac{4}{5}\epsilon$.

Our Proof: Let $\epsilon > 0$ be given and let $\delta = \min\{1, \frac{4}{5}\epsilon\}$. Assume that $0 < |x-2| < \delta$. Then from the scratch work

$$\Big|\frac{1}{x^2} - \frac{1}{4}\Big| < \epsilon,$$

proving the limit statement.

(c):

Scratch Work: *Let* $\epsilon > 0$ *be given. We want*

$$\left|(x^2+3)-4\right| < \epsilon$$

$$\left|(x-1)(x+1)\right| = |x-1|\,|x+1| < \epsilon$$

Assume that $x = 1 \pm 1$ *so that* $0 < x < 2$. *Then*

$$1 < x+1 < 3$$

Then

$$|x-1|\,|x+1| < 3|x-1|$$

This will be $< \epsilon$ *if* $|x-1| < \frac{1}{3}\epsilon$.

Our Proof: Let $\epsilon > 0$ be given and let $\delta = \min\{1, \frac{1}{3}\epsilon\}$. Assume that $0 < |x-1| < \delta$. Then from the scratch work

$$|(x^2+3)-4| < \epsilon,$$

proving the limit statement.

(d):

Scratch Work: *Let* $\epsilon > 0$ *be given. We want*

$$\left|\frac{1}{x^2+3} - \frac{1}{4}\right| < \epsilon$$

$$\left|\frac{1-x^2}{4(x^2+3)}\right| = |x-1|\,\frac{|1+x|}{4(x^2+3)} < \epsilon$$

Assume that $x = 1 \pm 1$ *so that* $0 < x < 2$. *Then*

$$0 < x < 2$$

$$1 < 1+x < 3$$

$$0 < x^2 < 4$$

$$3 < x^2+3 < 7$$

$$12 < 4(x^2+3) < 28$$

Hence

$$|x-1|\,\frac{|1+x|}{4(x^2+3)} < \frac{3}{12}\,|x-1|$$

This will be $< \epsilon$ *if* $|x-1| < 4\epsilon$.

Our Proof: Let $\epsilon > 0$ be given and let $\delta = \min\{1, 4\epsilon\}$. Assume that $0 < |x - 1| < \delta$. Then from the scratch work

$$\left|\frac{1}{x^2+3} - \frac{1}{4}\right| < \epsilon,$$

proving the limit statement.

(e):

Scratch Work: *Let $\epsilon > 0$ be given. We want*

$$\left|\frac{1}{1-x} - 1\right| < \epsilon$$

$$\left|\frac{x}{1-x}\right| = |x|\,\frac{1}{|1-x|} < \epsilon$$

If we assume that $x = 0 \pm 1$ we get

$$-1 < x < 1$$

$$1 > -x > -1$$

$$2 > 1 - x > 0$$

We cannot have 0 in the denominator. Hence, we assume instead that $x = 0 \pm .5$. Then

$$-.5 < x < .5$$

$$.5 > x > -.5$$

$$-.5 < -x > .5$$

$$.5 > 1 - x > 1.5$$

Hence

$$|x|\,\frac{1}{|1-x|} < \frac{1}{.5}|x|$$

This will be $< \epsilon$ if $|x| < .5\epsilon$.

Our Proof: Let $\epsilon > 0$ be given and let $\delta = \min\{.5, .5\epsilon\}$. Assume that $0 < |x - 0| < \delta$. Then from the scratch work

$$\left|\frac{1}{1-x} - 1\right| < \epsilon,$$

proving the limit statement.

(f):

Scratch Work: *Let $\epsilon > 0$ be given. We want*

$$\left|\frac{1}{\sqrt{3+x}} - \frac{1}{2}\right| < \epsilon$$

$$\left|\frac{2-\sqrt{3+x}}{2\sqrt{3+x}}\right| < \epsilon$$

$$\frac{|2-\sqrt{3+x}|\,|2+\sqrt{3+x}|}{(2\sqrt{3+x})\,|2+\sqrt{3+x}|} < \epsilon$$

$$|1-x|\,\frac{1}{(2\sqrt{3+x})\,|2+\sqrt{3+x}|} < \epsilon$$

Assume that $x = 1 \pm 1$ so that

$$0 < x < 2$$

$$3 < 3+x < 5$$

$$\sqrt{3} < \sqrt{3+x} < \sqrt{5}$$

$$2+\sqrt{3} < 2+\sqrt{3+x} < 2+\sqrt{5}$$

$$2\sqrt{3} < 2\sqrt{3+x} < 2\sqrt{5}$$

Then

$$|1-x|\,\frac{1}{(2\sqrt{3+x})\,|2+\sqrt{3+x}|} < |1-x|\,\frac{1}{(2\sqrt{3})(2+\sqrt{3})}$$

This will be $< \epsilon$ if $|x-1| < (2\sqrt{3})(2+\sqrt{3})\epsilon$.

Our Proof: Let $\epsilon > 0$ be given and let $\delta = \min\{1, (2\sqrt{3})(2+\sqrt{3})\epsilon\}$. Assume that $0 < |x-1| < \delta$. Then from the scratch work

$$\left|\frac{1}{\sqrt{3+x}} - \frac{1}{2}\right| < \epsilon,$$

proving the limit statement.

(4) Assume that $\lim_{x\to a} f(x) = 2$. Use a δ-ϵ argument to prove:

(a) $$\lim_{x\to a} \frac{2}{f(x)+2} = \frac{1}{2}$$

(b) $$\lim_{x\to a} \sqrt{f(x)+2} = 2$$

(c) $$\lim_{x\to a} f(x)^2 = 4$$

(d) $$\lim_{x\to a} \frac{3f(x)}{f(x)+1} = 2$$

Solutions:

(a):

Scratch work: *Let* $\epsilon > 0$ *be given. We want*

$$\left|\frac{2}{f(x)+2} - \frac{1}{2}\right| < \epsilon$$

$$\frac{|2-f(x)|}{2|f(x)+2|} < \epsilon$$

$$|f(x)-2|\,\frac{1}{2|f(x)+2|} < \epsilon$$

The term on the left is our "gold" since it becomes small as x *approaches* a*. The other term is our "trash" which we will bound. Specifically, we reason that for all* x *sufficiently close to* a*,* $f(x) = 2 \pm 1$*. Thus, for such* x*,*

$$1 < f(x) < 3$$

$$3 < f(x)+2 < 5$$

$$6 < 2|f(x)+2| < 10$$

Hence

$$|f(x)-2|\,\frac{1}{2|f(x)+2|} < \frac{1}{6}\,|f(x)-2|$$

This is $< \epsilon$ *if* $|f(x)-2| < 6\epsilon$*, which is true for all* x *sufficiently close to* a*.*

Proof: Let $\epsilon > 0$ be given and choose $\delta_1 > 0$ so that

$$|f(x)-2| < 1$$

for $0 < |x-a| < \delta_1$.

Choose $\delta_2 > 0$ such that

$$|f(x)-2| < 2\epsilon$$

for $0 < |x-a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. From the scratch work, $0 < |x-a| < \delta$ implies that

$$\left|\frac{2}{f(x)+2} - \frac{1}{2}\right| < \epsilon$$

proving the limit statement.

(b):

Scratch work: *Let $\epsilon > 0$ be given. We want*

$$\begin{aligned} \left|\sqrt{f(x)+2}-2\right| &< \epsilon \\ \frac{|(\sqrt{f(x)+2}-2)(\sqrt{f(x)+2}+2)|}{\sqrt{f(x)+2}+2} &< \epsilon \\ \frac{|f(x)-2|}{\sqrt{f(x)+2}+2} &< \epsilon \end{aligned}$$

The numerator is our "gold" since it becomes small as x approaches a. The other term is our "trash" which we will bound. Specifically, we reason that for all x sufficiently close to a, $f(x) = 2 \pm 1$. Thus, for such x,

$$\begin{gathered} 1 < f(x) < 3 \\ 3 < f(x)+2 < 5 \\ \sqrt{3} < \sqrt{f(x)+2} < \sqrt{5} \\ 2+\sqrt{3} < \sqrt{f(x)+2}+2 < 2+\sqrt{5} \end{gathered}$$

Hence

$$\frac{|f(x)-2|}{\sqrt{f(x)+2}+2} < \frac{1}{2+\sqrt{3}}\,|f(x)-2|$$

This is $< \epsilon$ if $|f(x)-2| < (2+\sqrt{3})\epsilon$, which is true for all x sufficiently close to a.

Proof: Let $\epsilon > 0$ be given and choose $\delta_1 > 0$ so that

$$|f(x)-2| < 1$$

for $0 < |x-a| < \delta_1$.

Choose $\delta_2 > 0$ such that

$$|f(x)-2| < (2+\sqrt{3})\epsilon$$

for $0 < |x-a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. From the scratch work, $0 < |x-a| < \delta$ implies that

$$\left|\sqrt{f(x)+2}-2\right| < \epsilon$$

proving the limit statement.

(c):

Scratch work: *Let $\epsilon > 0$ be given. We want*

$$\begin{aligned} \left|f(x)^2 - 4\right| &< \epsilon \\ |f(x) - 2|\,|f(x) + 2| &< \epsilon \end{aligned}$$

The term on the left is our "gold" since it becomes small as x approaches a. The other term is our "trash" which we will bound. Specifically, we reason that for all x sufficiently close to a, $f(x) = 2 \pm 1$. Thus, for such x,

$$\begin{aligned} 1 < f(x) &< 3 \\ 3 < f(x) + 2 &< 5 \end{aligned}$$

Hence

$$|f(x) - 2|\,|f(x) + 2| < 5\,|f(x) - 2|$$

This is $< \epsilon$ if $|f(x) - 2| < \epsilon/5$, which is true for all x sufficiently close to a.

Proof: Let $\epsilon > 0$ be given and choose $\delta_1 > 0$ so that

$$|f(x) - 2| < 1$$

for $0 < |x - a| < \delta_1$.

Choose $\delta_2 > 0$ such that

$$|f(x) - 2| < \epsilon/5$$

for $0 < |x - a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. From the scratch work, $0 < |x - a| < \delta$ implies that

$$\left|f(x)^2 - 4\right| < \epsilon$$

proving the limit statement.

(d):

Scratch work: *Let $\epsilon > 0$ be given. We want*

$$\begin{aligned} \left|\frac{3f(x)}{f(x) + 1} - 2\right| &< \epsilon \\ \frac{|f(x) - 2|}{|f(x) + 1|} &< \epsilon \\ |f(x) - 2|\,\frac{1}{|f(x) + 1|} &< \epsilon \end{aligned}$$

The term on the left is our "gold" since it becomes small as x approaches a. The other term is our "trash" which we will bound. Specifically, we reason that for all x sufficiently close to a, $f(x) = 2 \pm 1$. Thus, for such x,

$$\begin{aligned} 1 < f(x) < 3 \\ 2 < f(x) + 1 < 4 \end{aligned}$$

Hence

$$|f(x) - 2| \frac{1}{|f(x) + 1|} < \frac{1}{2} |f(x) - 2|$$

This is $< \epsilon$ if $|f(x) - 2| < 2\epsilon$, which is true for all x sufficiently close to a.

Proof: Let $\epsilon > 0$ be given and choose $\delta_1 > 0$ so that

$$|f(x) - 2| < 1$$

for $0 < |x - a| < \delta_1$.

Choose $\delta_2 > 0$ such that

$$|f(x) - 2| < 2\epsilon$$

for $0 < |x - a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. From the scratch work, $0 < |x - a| < \delta$ implies that

$$\left| \frac{3f(x)}{f(x) + 1} - 2 \right| < \epsilon$$

proving the limit statement.

(5) Find a value of a for which the following function is continuous at $x = 2$.

$$f(x) = \begin{cases} x + a & x < 2 \\ x^2 & x \geq 2 \end{cases}$$

Solution:

$$\begin{aligned} f(2) &= 4 \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} x^2 = 4 \\ \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} x + a = 2 + a \end{aligned}$$

Hence, $f(x)$ will be continuous at $x = 2$ if and only if $2 + a = 4$; hence $a = 2$.

(6) Prove that there is a point (x, y) at which the graphs of $y = e^x$ and $y = 2\cos x$ cross. What theorem are you using?

Solution:

The graphs cross at (x, y) if and only if $e^x = 2\cos x$ which is the same as $e^x - 2\cos x = 0$. Let

$$g(x) = e^x - 2\cos x.$$

Then $g(0) = e^0 - 2\cos 0 = -1$ while $g(\pi/2) = e^{\pi/2} - 2\cos \pi/2 = e^{\pi/2}$. Since $g(0) < 0$ and $g(\pi/2) > 0$, it follows *from the Intermediate Value Theorem* that there is an x between 0 and $\pi/2$ such that $g(x) = 0$, proving that the graphs cross.

(7) Suppose that f is continuous at every x in $[0, 1]$ and that for all x in this interval, $0 \leq f(x) \leq 1$. Prove that there is an $x \in [0, 1]$ such that $f(x) = x^2$.

Solution: Let

$$g(x) = f(x) - x^2.$$

Then

$$g(0) = f(0) - 0^2 = f(0) \geq 0.$$

If $f(0) = 0$ then $f(0) = 0^2$ so there is an x such that $f(x) = x^2$. Hence we may assume that $g(0) > 0$.

Also

$$g(1) = f(1) - 1^2 = f(1) - 1 \leq 0.$$

If $f(1) = 1$ then $f(1) = 1^2$ so there is an x such that $f(x) = x^2$. Hence we may assume that $g(1) < 0$.

Since $g(0) > 0$ and $g(1) < 0$, it follows *from the Intermediate Value Theorem* that there is an x between 0 and 1 such that $g(x) = 0$; hence there is an x such that $f(x) = x^2$.

(8) Prove Theorem 3 on p. 180 of the notes:

THEOREM 3 (Sequence). *Let $f(x)$ be continuous at a and let x_n be a sequence such that $\lim_{n\to\infty} x_n = a$. Then*

$$\lim_{n\to\infty} f(x_n) = f(a).$$

Proof: See the proof of Theorem 3, on p. 180.