MA 301 Test 3, Spring 2006

All answers must be justified. Simply stating an answer is not worth any credit.

(1) Question 1: Study for Test 4..
State the "official" definition of "$\lim_{x\to a} f(x) = L$." 10 pts

(2) Assume that it is given that $y = f(x)$ is increasing on $[0, 5]$ and decreasing on $[5, \infty)$. (See the figure below for a possible graph of f.) Let $a_n = f(n)$ and $s = \sum_1^\infty a_n$. 10 pts

(a) Find a specific value of n, a and b such that the following inequality is guaranteed to hold. Choose both n and a as large as possible and b as small as possible, consistent with the information provided. Justify your answer with a diagram. You may either use the figure below or draw your own.

$$a_1 + a_2 + \cdots + a_n < \int_a^b f(x)\,dx$$

FIGURE 1

(b) Find a specific value of n and a such that the following inequality is guaranteed to hold. Choose a and n as small as possible, consistent with the information provided. Justify your answer with a diagram. You may either use the figure below or draw your own.

$$s - s_n < \int_a^\infty f(x)\,dx$$

FIGURE 2

(3) Question 3: Study for Test 4..
Prove that $Z = \frac{1}{\sqrt{1+\sqrt{2}}}$ is irrational. You may assume that $\sqrt{2}$ is irrational. You MAY NOT use Proposition 1 from Chapter 9.

10 pts

10 pts

(4) Write a sum that expresses s to within $\pm 10^{-3}$ where

$$s = \sum_1^\infty \frac{2}{(2n+1)^3}.$$

5 pts

(5) Write a sum that expresses s within $\pm 10^{-3}$ where

$$s = \sum_1^\infty \frac{2}{(2n+1)^3 + 17\ln(n+2) + 5}.$$

10 pts

(6) Prove, using M, that the following series diverges.

$$\sum_1^\infty \frac{1}{\sqrt{n+3}}$$

10 pts

(7) Is the following series convergent or divergent? Prove your answer.

$$\sum_{1}^{\infty} \frac{\ln n}{n^{1.1}+1}$$

(8) For which values of p, $p \geq 0$, is the following series: 15 pts
 (a) Divergent?
 (b) Conditionally convergent?
 (c) Absolutely convergent?
 You must justify all of your answers.

$$\sum_{1}^{\infty} (-1)^n \frac{\sqrt{n^5+1}}{n^p+2}$$

(9) What is the set of x for which the following series converges? You need not prove your answer. However, you should explain your reasoning. 10 pts

$$\sum_{1}^{\infty} \frac{\ln n}{2^n(n+1)} x^n$$

(10) Question 10: Study for Test 4.
 Find an explicit one-to-one correspondence between the set of odd integers and the integers that are multiples of 3. 10 pts

Various Results From The Text

PROPOSITION (1, p.89). *If* $\lim_{n\to\infty} a_n \neq 0$*, then* $\sum_1^\infty a_n$ *cannot converge.*

THEOREM (1, p.89). *Suppose* $a_n > 0$ *for all* n *and* $f(x)$ *is an integrable, decreasing function on* $[0, \infty)$ *such that* $a_n = f(n)$ *for all* $n \in \mathbb{N}$*. Then*

$$s - s_n \leq \int_n^\infty f(x)\,dx$$

THEOREM (2, p.89). *Suppose* $a_n > 0$ *for all* n *and* $f(x)$ *is an integrable, decreasing function on* $[0, \infty)$ *such that* $a_n = f(n)$ *for all* $n \in \mathbb{N}$*. Then* $s = \sum_1^\infty a_n$ *exists if there is a* k *such that*

$$\int_k^\infty f(x)\,dx < \infty$$

THEOREM (3, p.91). *The following series converges for all $p > 1$.*

$$\sum_{1}^{\infty} \frac{1}{n^p} \tag{1}$$

Remark 1: The series in Theorem 3 above diverges if $p \leq 1$.

THEOREM (4, p.94). *Suppose $a_n > 0$ for all n and $f(x)$ is an integrable, decreasing function on $[0, \infty)$ such that $a_n = f(n)$ for all $n \in \mathbb{N}$. Then*

$$s_n \geq \int_1^{n+1} f(x)\, dx$$

THEOREM (5, p. 95). *Suppose that $0 \leq a_n \leq b_n$ for all n. Then $\sum_1^\infty a_n$ will converge if $\sum_1^\infty b_n$ converges.*

THEOREM (6, p. 96). *Suppose that in Theorem 5 above, the sum of the first N b_n approximates $\sum_1^\infty b_n$ to within $\pm\epsilon$. Then the same will be true for a_n: i.e. the sum of the first N a_n will approximate $\sum_1^\infty a_n$ to within $\pm\epsilon$.*

THEOREM (7, p. 98). *Let x be a real number. Then the series on the right side of the following equality converges if, and only if, $|x| < 1$. Furthermore, when it converges, it converges to the stated value.*

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \ldots$$

Remark 2: $\sum_1^\infty a_n$ converges if and only if there is an N such that $\sum_N^\infty a_n$ converges.

THEOREM (1, p. 111). *Let a_n be a sequence of real numbers. Then $\sum_1^\infty a_n$ will converge if $\sum_1^\infty |a_n|$ converges.*

THEOREM (2, p. 114). *Suppose that a_n is a positive, decreasing sequence where $\lim_{n\to\infty} a_n = 0$. Then*

$$s = \sum_1^\infty (-1)^n a_n$$

converges. Furthermore

$$|s - s_n| < a_{n+1}$$