MA 301 Test 3, Spring 2005

TA grades: 1,3,4,5,7

Prof. Grades 2,6,8,9,10

(1) Question 1: Study for Test 4..
State the "official" definition of "$\lim_{x\to a} f(x) = L$."
0, 5, 9 or 10 pts. The 9 pts. is if they omit $0 < |x - a|$.

10 pts

DEFINITION 1. *We say that*

$$\lim_{x\to a} f(x) = L$$

provided that for all numbers $\epsilon > 0$ *there is a number* $\delta > 0$ *such that*

$$|f(x) - L| < \epsilon$$

for all x *satisfying* $0 < |x - a| < \delta$.

(2) Assume that it is given that $y = f(x)$ is increasing on $[0, 5]$ and decreasing on $[5, \infty)$. (See the figure below for a possible graph of f.) Let $a_n = f(n)$ and $s = \sum_1^\infty a_n$.

10 pts

(a) Find a specific value of n, a and b such that the following inequality is guaranteed to hold. Choose both n and a as large as possible and b as small as possible, consistent with the information provided. Justify your answer with a diagram. You may either use the figure below or draw your own.

$$a_1 + a_2 + \cdots + a_n < \int_a^b f(x)\,dx$$

Solution:

$$a_1 + a_2 + a_3 + a_4 \le \int_1^5 f(x)\,dx$$

For the figure draw 5 rectangles of width 1 with their left edges beginning at $x = 1, 2, 3, 4$ respectively and extending up to the curve. The above inequality is true because the rectangles have their top edges below the curve since f is increasing over $[0, 5]$.

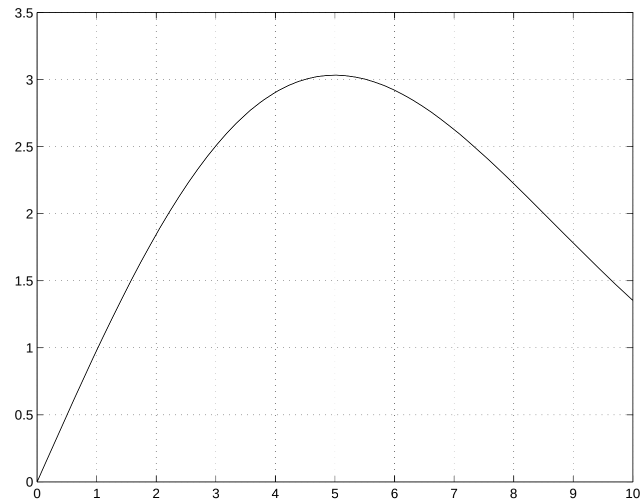

FIGURE 1

(b) Find a specific value of n and a such that the following inequality is guaranteed to hold. Choose a and n as small as possible, consistent with the information provided. Justify your answer with a diagram. You may either use the figure below or draw your own.

$$s - s_n < \int_a^\infty f(x)\,dx$$

Solution:

$$s - s_5 \leq \int_5^\infty f(x)\,dx$$

For the figure draw rectangles of width 1 with their right edges beginning at $x = 6, 7, 8, 9, 10, \ldots$ and extending up to the curve. The sum of the areas of these rectangles is $s - s_5$. The above inequality is true because the rectangles have their top edges below the curve since f is decreasing over $[5, \infty)$.

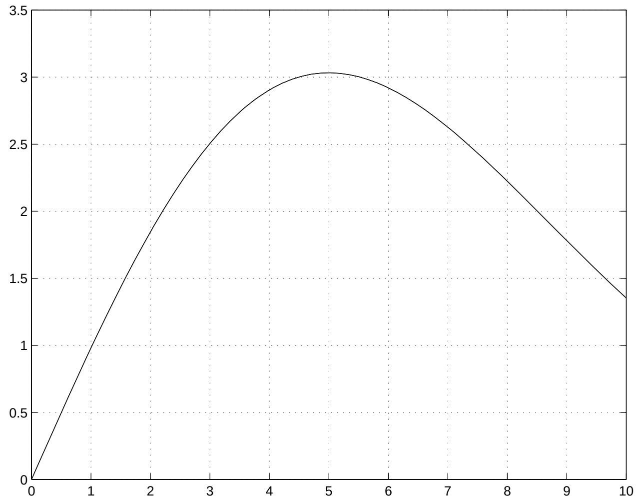

FIGURE 2

(3) Question 3: Study for Test 4..
Prove that $Z = \frac{1}{\sqrt{1+\sqrt{2}}}$ is irrational. You may assume that $\sqrt{2}$ is irrational. You MAY NOT use Proposition 1 from Chapter 9. 10 pts

Solution: Assume that Z is rational. *2 pt*

Then $Z = \frac{p}{q}$ where p and q are integers *2 pt. It must be stated somewhere in the solution that p and q are integers.* with $q \neq 0$. Then

$$\frac{p}{q} = \frac{1}{\sqrt{1+\sqrt{2}}} \qquad \textit{2 pts.}$$

$$\frac{q}{p} = \sqrt{1+\sqrt{2}}$$

$$\frac{q^2}{p^2} = 1+\sqrt{2}$$

$$\frac{q^2-p^2}{p^2} = \sqrt{2} \ \textit{2 pts.}$$

Since $q^2 - p^2$ and p^2 are both integers*2 pts.*, we conclude that $\sqrt{2}$ is rational, which is nonsense. Hence Z must be irrational.

10 pts (4) Write a sum that expresses s to within $\pm 10^{-3}$ where

$$s = \sum_{1}^{\infty} \frac{2}{(2n+1)^3}.$$

Solution:

According to Theorem 1,

$$\begin{aligned} s - s_n &< \int_n^{\infty} \frac{2}{(2x+1)^3}\, dx \\ &= \frac{1}{2(2n+1)^2} \end{aligned}$$

4 pts., -2 if they miss evaluate integral

This will be less than 10^{-3} provided

$$\begin{aligned} \frac{1}{2(2n+1)^2} &< 10^{-3} \\ 2(2n+1)^2 &> 10^3 \\ 2n+1 &> \left(\frac{10^3}{2}\right)^{1/2} \\ n &> \frac{1}{2}\left(\frac{10^3}{2}\right)^{1/2} - \frac{1}{2} = 10.68 \end{aligned}$$

4 pts. They may have the wrong answer for the integral. As long as they attempt to solve for n they get these points.

Hence

$$s = \sum_{1}^{11} \frac{2}{(2n+1)^3} \pm 10^{-3}.$$

2 pts.

5 pts (5) Write a sum that expresses s within $\pm 10^{-3}$ where

$$s = \sum_{1}^{\infty} \frac{2}{(2n+1)^3 + 17\ln(n+2) + 5}.$$

Solution:

$$\frac{2}{(2n+1)^3 + 17\ln(n+2) + 5} \le \frac{2}{(2n+1)^3}$$

3 pts.

From Problem 4, 11 terms suffice. Hence

$$s = \sum_1^{11} \frac{2}{(2n+1)^3 + 17\ln(n+2) + 5} \pm 10^{-3}.$$

2 pts.

(6) Prove, using M, that the following series diverges. 10 pts

$$\sum_1^{\infty} \frac{1}{\sqrt{n+3}}$$

Scratch work: According to Theorem 4

$$\begin{aligned} s_n &\ge \int_1^{n+1} (x+3)^{-\frac{1}{2}}\,dx \\ &= 2\left((4+n)^{\frac{1}{2}} - 2\right) \\ &= 2\sqrt{4+n} - 4 \end{aligned}$$

2 pts+1 pts

Then s_n is greater than M if:

$$\begin{aligned} 2\sqrt{4+n} - 4 &> M \quad \textit{2 pts} \\ \sqrt{4+n} &> \frac{1}{2}M + 2 \\ n &> \left(\frac{1}{2}M + 2\right)^2 - 4 \quad \textit{2 pts} \end{aligned}$$

Proof: Let $M > 0$ be given.*1 pt* Let $N = \left(\frac{1}{2}M + 2\right)^2 - 4$.*1 pt* From the scratch work, for $n > N$, $s_n > M$,*1 pt* proving that $\lim_{n\to\infty} s_n = \infty$.

(7) Is the following series convergent or divergent? Prove your answer. 10 pts

$$\sum_1^{\infty} \frac{\ln n}{n^{1.1} + 1}$$

Solution: Convergent. There is an $N > 0$ such that for all $n > N$,

$$\ln n < n^{.05} \quad (\text{or } n^a \text{ for any } 0 < a < .1.) \quad .$$

3 pts

Then (*3 pts* for dealing with N)

$$\sum_{N+1}^{\infty} \frac{\ln n}{n^{1.1}+1} < \sum_{N+1}^{\infty} \frac{n^{.05}}{n^{1.1}}$$
$$= \sum_{N+1}^{\infty} \frac{1}{n^{1.05}} < \infty$$

3 pts

since $\sum \frac{1}{n^p} < \infty$ for $p > 1$.*1 pts* It follows that

$$\sum_{1}^{\infty} \frac{\ln n}{n^{1.1}+1} < \infty$$

proving convergence.

15 pts (8) For which values of p, $p \geq 0$, is the following series:
(a) Divergent?
(b) Conditionally convergent?
(c) Absolutely convergent?
You must justify all of your answers.

$$\sum_{1}^{\infty} (-1)^n \frac{\sqrt{n^5+1}}{n^p+2}$$

Solution:

$$\frac{\sqrt{n^5+1}}{n^p+2} \sim \frac{n^{2.5}}{n^p} \sim \frac{1}{n^{p-2.5}}.$$

(a) If $2.5 \geq p \geq 0$, the series diverges since $\lim_{n\to\infty} \frac{\sqrt{n^5+1}}{n^p+2} \neq 0$ in this case.
(b) If $3.5 \geq p > 2.5$ the series converges conditionally because it is an alternating series and $\lim_{n\to\infty} \frac{\sqrt{n^5+1}}{n^p+2} = 0$ in this case.
(c) If $p > 3.5$ the series will converge absolutely since $\sum \frac{1}{n^q}$ converges for $q > 1$.

(9) What is the set of x for which the following series converges? You need not prove your answer. However, you should explain your reasoning. 10 pts

$$\sum_{1}^{\infty} \frac{\ln n}{2^n(n+1)} x^n$$

Solution: $-2 \leq x < 2$. *3 pts*

This is the same as

$$\sum_{1}^{\infty} \frac{\ln n}{n+1} \left(\frac{x}{2}\right)^n.$$

If $\frac{|x|}{2} > 1$, this diverges because $\left(\frac{x}{2}\right)^n$ grows exponentially while $\frac{\ln n}{n+1}$ decays slowly. (Or one can say that in this case $\lim_{n\to\infty} |a_n| = \infty$.) *2 pts*

Similarly, if $\frac{|x|}{2} < 1$ it converges because $\left(\frac{x}{2}\right)^n$ decays exponentially while $\frac{\ln n}{n+1}$ decays. *3 pts*

If $x = 2$, it diverges because $\frac{\ln n}{n+1} > \frac{1}{n+1}$ for $n > N$. If $x = -2$, it converges since it is an alternating series and $\lim_{n\to\infty} \frac{\ln n}{n+1} = 0$. *2 pts*

(10) Question 10: Study for Test 4..

Find an explicit one-to-one correspondence between the set of odd integers and the integers that are multiples of 3. 10 pts

Solution:

$$f(n) = 3\frac{n+1}{2}.$$

The following is worth 5 pts. (It is not really "explicit".)

...	-3	-1	1	3	5	7	...
...	-3	0	3	6	9	12	...

Theorem (2'). *Suppose $a_n > 0$ for all n and $f(x)$ is an integrable, decreasing function on $[0, \infty)$ such that $a_n = f(n)$ for all $n \in \mathbb{N}$. Then $s = \sum_1^\infty a_n$ exists if*

$$\int_0^\infty f(x)\, dx < \infty$$

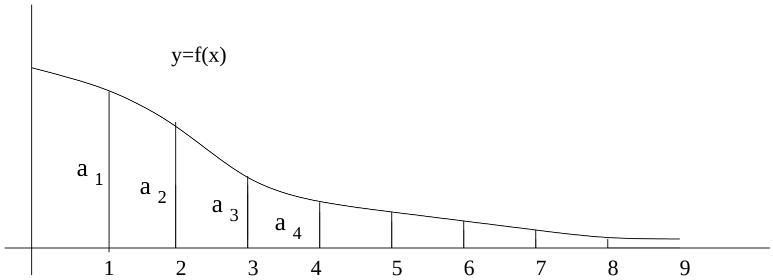

FIGURE 3. Theorems 2' and 4'

Proof Each a_n is the length of a line segment drawn from the point $(n, 0)$ on the x-axis to the graph of $y = f(x)$ as in Figure 3.

The area of a rectangle of width one having this line segment as its right edge is a_n. (See Figure 4). This rectangle also lies entirely below the graph of $y = f(x)$ *since this graph is decreasing.*

Since the left side of the first rectangle extends to $x = 0$,

$$s_n = a_1 + a_2 + \cdots + a_n \leq \int_0^n f(x)\, dx \leq \int_0^\infty f(x)\, dx. \tag{1}$$

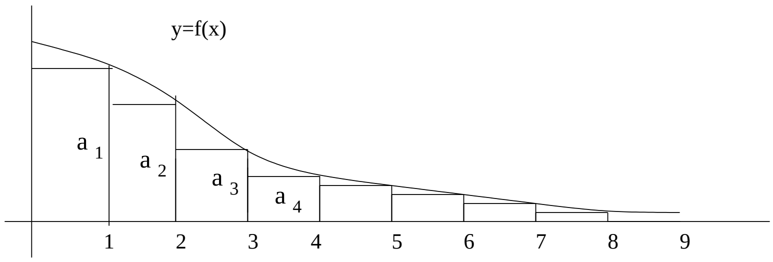

FIGURE 4. Theorem 2'

Finally, since the a_n are all positive, s_n *is an increasing sequence.* From the Bounded Increasing Theorem, $\lim s_n$ either exists or equals ∞. Formula (1) proves that the limit is not ∞. Hence the limit exists, proving the convergence of the sum. □

Various Results From The Text

PROPOSITION (1, p.89). *If* $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_1^\infty a_n$ *cannot converge.*

THEOREM (1, p.89). *Suppose $a_n > 0$ for all n and $f(x)$ is an integrable, decreasing function on $[0, \infty)$ such that $a_n = f(n)$ for all $n \in \mathbb{N}$. Then*

$$s - s_n \leq \int_n^\infty f(x)\,dx$$

THEOREM (2, p.89). *Suppose $a_n > 0$ for all n and $f(x)$ is an integrable, decreasing function on $[0, \infty)$ such that $a_n = f(n)$ for all $n \in \mathbb{N}$. Then $s = \sum_1^\infty a_n$ exists if there is a k such that*

$$\int_k^\infty f(x)\,dx < \infty$$

THEOREM (3, p.91). *The following series converges for all $p > 1$.*

$$\sum_1^\infty \frac{1}{n^p} \tag{2}$$

Remark 1: The series in Theorem 3 above diverges if $p \leq 1$.

THEOREM (4, p.94). *Suppose $a_n > 0$ for all n and $f(x)$ is an integrable, decreasing function on $[0, \infty)$ such that $a_n = f(n)$ for all $n \in \mathbb{N}$. Then*

$$s_n \geq \int_1^{n+1} f(x)\,dx$$

THEOREM (5, p. 95). *Suppose that $0 \leq a_n \leq b_n$ for all n. Then $\sum_1^\infty a_n$ will converge if $\sum_1^\infty b_n$ converges.*

THEOREM (6, p. 96). *Suppose that in Theorem 5 above, the sum of the first N b_n approximates $\sum_1^\infty b_n$ to within $\pm\epsilon$. Then the same will be true for a_n: i.e. the sum of the first N a_n will approximate $\sum_1^\infty a_n$ to within $\pm\epsilon$.*

THEOREM (7, p. 98). *Let x be a real number. Then the series on the right side of the following equality converges if, and only if,*

$|x| < 1$. *Furthermore, when it converges, it converges to the stated value.*

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \ldots$$

Remark 2: $\sum_1^\infty a_n$ converges if and only if there is an N such that $\sum_N^\infty a_n$ converges.

THEOREM (1, p. 111). *Let a_n be a sequence of real numbers. Then $\sum_1^\infty a_n$ will converge if $\sum_1^\infty |a_n|$ converges.*

THEOREM (2, p. 114). *Suppose that a_n is a positive, decreasing sequence where $\lim_{n\to\infty} a_n = 0$. Then*

$$s = \sum_1^\infty (-1)^n a_n$$

converges. Furthermore

$$|s - s_n| < a_{n+1}$$