MA 301 Test 4, Spring 2006

TA Grades 1-4

(1) State the "official" definition of "$\lim_{x\to a} f(x) = L$." 8 pts

0, 7, or 8 pts.

DEFINITION 1. *We say that*

$$\lim_{x\to a} f(x) = L$$

provided that for all numbers $\epsilon > 0$ *there is a number* $\delta > 0$ *such that*

$$|f(x) - L| < \epsilon$$

for all x *satisfying* $0 < |x - a| < \delta$.

(2) Find a value of a for which the following function is continuous at $x = 2$. Justify your answer. 8 pts

$$f(x) = \begin{cases} 2^{ax} & x > 2 \\ \sqrt{x} & 0 < x \le 2 \end{cases}$$

Solution:

$$f(2) = \sqrt{2} = 2^{1/2} \quad \text{2 pts.}$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2^{ax} = 2^{2a} \quad \text{2 pts.}$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \sqrt{x} = 2^{1/2} \quad \text{2 pts.}$$

Hence, $f(x)$ will be continuous at $x = 2$ if and only if $2^{2a} = 2^{1/2}$; hence $a = \frac{1}{4}$ *2 pts.*.

(3) Use a δ-ϵ argument to prove that 12 pts

$$\lim_{x\to 3} \frac{x-1}{x+1} = \frac{1}{2}.$$

Scratch Work: *Let* $\epsilon > 0$ *be given. We want*

$$\left|\frac{x-1}{x+1} - \frac{1}{2}\right| < \epsilon \quad \textit{2 pts.}$$

$$\left|\frac{x-3}{2(x+1)}\right| = |x-3|\,\frac{1}{2|x+1|} < \epsilon$$

2 pts. for simplification

Assume that $x = 3 \pm 1$ *(1 pt. for* $3 \pm \delta$ *(any* δ*)) so that* $2 < x < 4$ *and* $3 < x + 1 < 5$*. Then*

$$|x-3|\,\frac{1}{2|x+1|} < \frac{1}{6}|x-3| \quad \textit{3 pts.}$$

This will be $< \epsilon$ *if* $|x-3| < 6\epsilon$*.*

Proof: Let $\epsilon > 0$ be given *1 pt.* and let $\delta = \min\{1, 6\epsilon\}$ *1 pt.*. Assume that $0 < |x-3| < \delta$.*1 pt.* Then from the scratch work

$$\left|\frac{x-1}{x+1} - \frac{1}{2}\right| < \epsilon, \quad \text{1 pt.}$$

proving the limit statement.

12 pts (4) Use a δ-ϵ argument to prove that

$$\lim_{x \to 1} \frac{1}{\sqrt{2x+7}} = \frac{1}{3}.$$

Scratch Work: *Let* $\epsilon > 0$ *be given. We want*

$$\left|\frac{1}{\sqrt{2x+7}} - \frac{1}{3}\right| < \epsilon \quad \textit{2 pt.}$$

$$\frac{|3-\sqrt{2x+7}|}{3\sqrt{2x+7}} < \epsilon \quad \textit{1 pt.}$$

$$\frac{|(3-\sqrt{2x+7})(3+\sqrt{2x+7})|}{3\sqrt{2x+7}(3+\sqrt{2x+7})} < \epsilon \quad \textit{1 pt.}$$

$$\frac{|2-2x|}{3\sqrt{2x+7}(3+\sqrt{2x+7})} < \epsilon$$

$$|x-1|\,\frac{2}{3\sqrt{2x+7}(3+\sqrt{2x+7})} < \epsilon \quad \textit{1 pt. for simplification}$$

Assume that $x = 1 \pm 1$ *1 pt. for* $1 \pm \delta$*, any* δ *so that*

$$0 < x < 2$$

$$7 < 2x + 7 < 11$$

$$\sqrt{7} < \sqrt{2x+7} < \sqrt{11}$$

$$3 + \sqrt{7} < 3 + \sqrt{2x+7} < 3 + \sqrt{11} \quad \textit{2 pt.}$$

$$3\sqrt{7}(3+\sqrt{7}) < 3\sqrt{2x+7}(3+\sqrt{2x+7})$$

Then

$$|x-1|\frac{2}{3\sqrt{2x+7}(3+\sqrt{2x+7})} < |x-1|\frac{2}{3\sqrt{7}(3+\sqrt{7})} \quad \textit{1 pt.}$$

This will be $< \epsilon$ *if* $|x-1| < \frac{3\sqrt{7}(3+\sqrt{7})}{2}\epsilon$.

Proof: *Grade same as Problem (3)* Let $\epsilon > 0$ be given and let $\delta = \min\{1, \frac{3\sqrt{7}(3+\sqrt{7})}{2}\epsilon\}$. Assume that $0 < |x-1| < \delta$. Then from the scratch work

$$\left|\frac{1}{\sqrt{2x+7}} - \frac{1}{3}\right| < \epsilon,$$

proving the limit statement.

(5) Use a δ-ϵ argument to prove that 12 pts

$$\lim_{x\to .5}\frac{1}{x^2} = 4.$$

Scratch Work: *Let* $\epsilon > 0$ *be given. We want*

$$\left|\frac{1}{x^2} - 4\right| < \epsilon \quad \textit{1 pt.}$$

$$\left|\frac{1-4x^2}{x^2}\right| < \epsilon$$

$$\left|\frac{(1-2x)(1+2x)}{x^2}\right| < \epsilon$$

$$2\left|x-\frac{1}{2}\right|\frac{|1+2x|}{x^2} < \epsilon$$

2 pt. for simplification

Assume that $x = .5 \pm .25$ *2 pt. for* $.5 \pm \delta$*, any* δ *. Then*

$$.25 < x < .75$$

$$(.25)^2 < x^2 < (.75)^2$$

1 pt.: but interval cannot contain 0

Also

$$.25 < x < .75$$

$$.5 < 2x < 1.5$$

$$1.5 < 2x+1 < 2.5 \quad \textit{1 pt.}$$

Hence

$$2\left|x-\frac{1}{2}\right|\frac{|1+2x|}{x^2} < \frac{5}{(.25)^2}\left|x-\frac{1}{2}\right|$$

This will be $< \epsilon$ *if* $|x - \frac{1}{2}| < \frac{(.25)^2}{5}\epsilon$.

2 pt.

Proof: *Grade as in Exercise (3)* Let $\epsilon > 0$ be given and let $\delta = \min\{.25, \frac{(.25)^2}{5}\epsilon\}$. Assume that $0 < |x - \frac{1}{2}| < \delta$. Then from the scratch work

$$\left|\frac{1}{x^2} - 4\right| < \epsilon,$$

proving the limit statement.

12 pts

(6) Assume that $\lim_{x\to a} f(x) = 5$. Use a δ-ϵ argument to prove that

$$\frac{f(x)+3}{f(x)-1} = 2.$$

Scratch work: *Let* $\epsilon > 0$ *be given. We want*

$$\left|\frac{f(x)+3}{f(x)-1} - 2\right| < \epsilon \quad \textit{2 pts.}$$

$$\left|\frac{f(x)-5}{f(x)-1}\right| < \epsilon$$

$$|f(x)-5|\,\frac{1}{|f(x)-1|} < \epsilon \quad \textit{2 pts.}$$

The term on the left is our "gold" since it becomes small as x *approaches* a*. The other term is our "trash" which we will bound. Specifically, we reason that for all* x *sufficiently close to* a, $f(x) = 5 \pm 1$*. Thus, for such* x,

$$4 < f(x) < 6 \quad \textit{2 pts.}$$

$$3 < f(x) - 1 < 5 \quad \textit{1 pts.}$$

$$3 < |f(x) - 1| < 5$$

Hence

$$|f(x)-5|\,\frac{1}{|f(x)-1|} < \frac{1}{3}|f(x)-5| \quad \textit{1 pts.}$$

This is $< \epsilon$ *if* $|f(x) - 5| < 3\epsilon$*, which is true for all* x *sufficiently close to* a*.*

Proof: Let $\epsilon > 0$ 1 pts. be given and choose $\delta_1 > 0$ 1 pts. so that

$$|f(x) - 5| < 1$$

for $0 < |x - a| < \delta_1$.

Choose $\delta_2 > 0$ such that

$$|f(x) - 5| < 3\epsilon \quad \text{1 pts.}$$

for $0 < |x - a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$ 1 pts. . From the scratch work, $0 < |x - a| < \delta$ implies that

$$\left|\frac{f(x) + 3}{f(x) - 1} - 2\right| < \epsilon$$

proving the limit statement.

(7) Use a δ-ϵ argument to prove Theorem 3 on p. 180 of the notes:

12 pts

THEOREM 3 (Sequence). *Let $f(x)$ be continuous at a and let x_n be a sequence such that* $\lim_{n\to\infty} x_n = a$. *Then*

$$\lim_{n\to\infty} f(x_n) = f(a).$$

Proof Let $\epsilon > 0$ 1 pt. be given. Since $\lim_{x\to a} f(x) = f(a)$, there is a $\delta > 0$ such that

(1) $$|f(x) - f(a)| < \epsilon. \quad \text{4 pt.}$$

for $|x - a| < \delta$, $x \neq a$. This inequality holds even if $x = a$ since in this case the left hand quantity is zero. 1 pt.

But, since $\lim_{n\to\infty} x_n = a$, there is an N such that

$$|x_n - a| < \delta \quad \text{4 pt.}$$

for all $n > N$. Replacing x with x_n in (1) shows that

$$|f(x_n) - f(a)| < \epsilon \quad \text{3 pt.}$$

for $n > N$, which proves our theorem.