MA 301 Test 4, Spring 2007

2 hours, calculator allowed, no notes. Provide paper for the students to do work on. Students should not write answers on test sheet.

TA Grades 1, 5, 6, 7

All answers must be justified. Simply stating an answer is not worth any credit.

(1) State the "official" definition of "$\lim_{x\to a} f(x) = L$." 8 pts

0, 4, or 8 pts.

DEFINITION 1. *We say that*

$$\lim_{x\to a} f(x) = L$$

provided that for all numbers $\epsilon > 0$ *there is a number* $\delta > 0$ *such that*

$$|f(x) - L| < \epsilon$$

for all x *satisfying* $0 < |x - a| < \delta$.

(2) Show that it is impossible to list all numbers in the interval $(0, 1)$. How does this prove that the set in question is uncountable. 12 pts

Solution:

Imagine that we have somehow managed to list all real numbers in this interval. Our list might look something like:

$$b_1 = .31415027\ldots$$
$$b_2 = .14936815\ldots$$
$$b_3 = .22719664\ldots$$
$$b_4 = .97652234\ldots$$
$$b_5 = .62718891\ldots$$
$$\vdots$$

We imagine the decimal expansions extending out to infinity and the list extending down the page to infinity. Some numbers have both a finite decimal expansion and an infinite expansion. In such cases we use the infinite expansion. Thus, for example, we use $.4999\ldots$ insteead of $.5$. We claim that no matter what the specific numbers in the list, there will always be some real number r which is not in the list. To see this, look at the first digit of the first number in the list. In our case, it is 3. We choose some number between 1 and 9, other than 3, and make it be the first digit of r. Lets choose 4, so $r = .4+$. This insures that $r \neq b_1$. Next, we look at the second digit of the second number on the list: 4. We change it to something between 1 and 9, declaring, say, $r = .47+$. This guarantees that r is also not equal to b_2. We continue this way, at each step choosing the n^{th} digit of r to be some some number between 1 and 9 which differs from the n^{th} digit of b_n. For the list above, r might look like $r = .47647+$. Since we never choose 0, r will not be a finite decimal. It is clear that in this manner we produce a number r which appears nowhere on our list.

Since we cannot list the real numbers in $(0, 1)$, they cannot be put into a one-to-one correspondence with the natural numbers; hence they are uncountable.

(3) Let A be the set of all rational numbers in the open interval $(0, 1)$ and let B be the set of all rational numbers in the closed interval $[0, 1]$. Describe an explicit one-to-one correspondence between A and B.

10 pts

Solution: We define the correspondence by the following table:

$$\begin{array}{ccccccccccc} \frac{1}{2} & \frac{1}{3} & \frac{2}{3} & \frac{1}{4} & \frac{3}{4} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{1}{6} & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\ 0 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} & \frac{1}{4} & \frac{3}{4} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \cdots \end{array}$$

The pattern is that in each list we list the fractions in order of increasing denominators, and then by increasing numerators, omitting non-reduced fractions, except that in the second list, we first list 0 and 1.

(4) For the following function, find a value of a for which $f(x)$ is continuous at $x = 1$

10 pts

$$f(x) = \begin{cases} (x+a)^2 & x < 1 \\ 3 + x & x \geq 1 \end{cases}$$

Solution:

$$f(1) = 4 \quad 2 \text{ pts.}$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3 + x = 4 \quad 2 \text{ pts.}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+a)^2 = (1+a)^2 \quad 2 \text{ pts.}$$

Hence, $f(x)$ will be continuous at $x = 1$ if and only if $(1+a)^2 = 4$; hence $a = \pm 2 - 1$ *2 pts.* ($a = -3$ or $a = 1$ are both acceptable.)

(5) Use a δ-ϵ argument to prove that 12 pts

$$\lim_{x \to 2} \frac{2x+1}{x+1} = \frac{5}{3}.$$

Scratch Work: *Let $\epsilon > 0$ be given. We want*

$$\left|\frac{2x+1}{x+1} - \frac{5}{3}\right| < \epsilon \quad \textit{2 pts.}$$

$$\left|\frac{x-2}{3(x+1)}\right| = |x-2|\,\frac{1}{3|x+1|} < \epsilon$$

2 pts. for simplification

Assume that $x = 2 \pm 1$ (1 pt. for $2 \pm \delta$ (any δ)) so that $1 < x < 3$ and $2 < x+1 < 4$. Then

$$|x-2|\,\frac{1}{3|x+1|} < \frac{1}{6}|x-2| \quad \textit{3 pts.}$$

This will be $< \epsilon$ if $|x-2| < 6\epsilon$.

Proof: Let $\epsilon > 0$ be given *1 pt.* and let $\delta = \min\{1, 6\epsilon\}$ *1 pt.*. Assume that $0 < |x-2| < \delta$.*1 pt.* Then from the scratch work

$$\left|\frac{2x+1}{x+1} - \frac{5}{3}\right| < \epsilon, \quad \text{1 pt.}$$

proving the limit statement.

(6) Use a δ-ϵ argument to prove that 12 pts

$$\lim_{x \to 1} \sqrt{3x+1} = 2.$$

Scratch Work: *Let $\epsilon > 0$ be given. We want*

$$\left|\sqrt{3x+1} - 2\right| < \epsilon \quad \textit{2 pt.}$$

$$\frac{|(\sqrt{3x+1}-2)(\sqrt{3x+1}+2)|}{|\sqrt{3x+1}+2|} < \epsilon \quad \textit{2 pt.}$$

$$\frac{|3(x-1)|}{|\sqrt{3x+1}+2|} < \epsilon \quad \textit{1 pt. for simplification}$$

Assume that $x = 1 \pm 1$ *(1 pt. for* $2 \pm \delta$*, any* δ*) so that*

$$0 < x < 2$$
$$1 < 3x + 1 < 7$$
$$1 < \sqrt{3x+1} < \sqrt{7}$$
$$3 < 2 + \sqrt{3x+1} < 2 + \sqrt{7} \quad \textit{2 pt.}$$

Then

$$|x-1| \frac{1}{|\sqrt{3x+1}+2|} < |x-1| \frac{1}{3} \quad \textit{1 pt.}$$

This will be $< \epsilon$ *if* $|x - 1| < 3\epsilon$.

Proof: *Grade same as Problem (5)* Let $\epsilon > 0$ be given and let $\delta = \min\{1, 3\epsilon\}$. Assume that $0 < |x - 1| < \delta$. Then from the scratch work

$$\left|\sqrt{3x+1} - 2\right| < \epsilon,$$

proving the limit statement.

12 pts (7) Use a δ-ϵ argument to prove that

$$\lim_{x \to 2} \frac{1}{3-x} = 1.$$

Scratch Work: *Let* $\epsilon > 0$ *be given. We want*

$$\left|\frac{1}{3-x} - 1\right| < \epsilon \quad \textit{1 pt.}$$
$$\left|\frac{x-2}{3-x}\right| < \epsilon$$
$$|x-2| \frac{1}{|3-x|} < \epsilon$$

2 pt. for simplification

Assume that $x = 2 \pm \frac{1}{2}$ *2 pt. for* $2 \pm \delta$*, any* δ *. Then*

$$1.5 < x < 2.5$$
$$-1.5 > -x > -2.5$$
$$1.5 > 3 - x > .5$$

1 pt.: but interval cannot contain 0

Hence

$$|x-2| \frac{1}{|3-x|} < |x-2| \frac{1}{.5}$$

This will be $< \epsilon$ *if* $|x-2| < \frac{1}{2}\epsilon$.
2 pt.

Proof: *Grade as in Exercise (5)* Let $\epsilon > 0$ be given and let $\delta = \min\{.5, \frac{1}{2}\epsilon\}$. Assume that $0 < |x-2| < \delta$. Then from the scratch work

$$\left|\frac{1}{3-x} - 1\right| < \epsilon,$$

proving the limit statement.

(8) Assume that $\lim_{x\to a} f(x) = 1$. Use a δ-ϵ argument to prove that 12 pts

$$\lim_{x\to a} \frac{f(x)}{f(x)+2} = \frac{1}{3}.$$

Scratch work: *Let* $\epsilon > 0$ *be given. We want*

$$\left|\frac{f(x)}{f(x)+2} - \frac{1}{3}\right| < \epsilon \quad \textit{2 pts.}$$

$$\frac{2}{3}\left|\frac{f(x)-1}{f(x)+2}\right| < \epsilon$$

$$|f(x)-1| \frac{2}{3|f(x)+2|} < \epsilon \quad \textit{2 pts.}$$

The term on the left is our "gold" since it becomes small as x *approaches* a. *The other term is our "trash" which we will bound. Specifically, we reason that for all* x *sufficiently close to* a, $f(x) = 1 \pm 1$. *Thus, for such* x,

$$0 < f(x) < 2 \quad \textit{2 pts.}$$

$$2 < f(x) + 2 < 4 \quad \textit{1 pts.}$$

Hence

$$|f(x)-1| \frac{2}{3|f(x)+2|} < \frac{1}{3}|f(x)-1| \quad \textit{1 pts.}$$

This is $< \epsilon$ *if* $|f(x)-1| < 3\epsilon$, *which is true for all* x *sufficiently close to* a.

Proof: Let $\epsilon > 0$ 1 pts. be given and choose $\delta_1 > 0$ 1 pts. so that

$$|f(x) - 1| < 1$$

for $0 < |x-a| < \delta_1$.

Choose $\delta_2 > 0$ such that

$$|f(x) - 1| < 3\epsilon \quad 1 \text{ pts.}$$

for $0 < |x - a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$ 1 pts. . From the scratch work, $0 < |x - a| < \delta$ implies that

$$\left|\frac{f(x)}{f(x)+2} - \frac{1}{3}\right| < \epsilon$$

proving the limit statement.

(9) Use a δ-ϵ argument to prove Theorem 3 on p. 164 of the notes:

12 pts

THEOREM 3 (Sequence). *Let $f(x)$ be continuous at a and let x_n be a sequence such that $\lim_{n\to\infty} x_n = a$. Then*

$$\lim_{n\to\infty} f(x_n) = f(a).$$

Proof Let $\epsilon > 0$ 1 pt. be given. Since $\lim_{x\to a} f(x) = f(a)$, there is a $\delta > 0$ such that

(1) $$|f(x) - f(a)| < \epsilon. \quad 4 \text{ pt.}$$

for $|x - a| < \delta$, $x \neq a$. This inequality holds even if $x = a$ since in this case the left hand quantity is zero. 1 pt.

But, since $\lim_{n\to\infty} x_n = a$, there is an N such that

$$|x_n - a| < \delta \quad 4 \text{ pt.}$$

for all $n > N$. Replacing x with x_n in (1) shows that

$$|f(x_n) - f(a)| < \epsilon \quad 3 \text{ pt.}$$

for $n > N$, which proves our theorem.

PROPOSITION 1 (2, p. 138). *Let Z be an irrational number and let x and y be rational numbers with $x \neq 0$. Then $W = xZ + y$ is irrational.*

Proof

Solution: There are integers p, q, r, and s with p, q and s non-zero such that

$$x = \frac{p}{q},\ y = \frac{r}{s}.$$

We work by contradiction, showing that assuming W rational leads to nonsense. Specifically, suppose that $W = u/v$ where u and v are integers. Then

$$\begin{aligned}\frac{u}{v} &= \frac{p}{q} + \frac{r}{s}Z \\ \frac{s}{r}\left(\frac{u}{v} - \frac{p}{q}\right) &= Z \\ \frac{s(uq - vp)}{rpq} &= Z\end{aligned}$$

showing that Z is rational, which is a contradiction.