VAN DEN BAN-SCHLICHTKRULL-WALLACH ASYMPTOTIC EXPANSIONS ON NON-SYMMETRIC DOMAINS

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Introduction

Let $X = G/K$ be a homogeneous Riemannian manifold where G is the identity component of its isometry group. A C^∞ function F on X is *strongly harmonic* if it is annihilated by every element of $D_G(X)$, the algebra of all G-invariant differential operators without constant term. One of the most beautiful results in the harmonic analysis of symmetric spaces is the Helgason Conjecture, which states that on a Riemannian symmetric space of non-compact type, a function is strongly harmonic if and only if it is the Poisson integral of a hyperfunction over the Furstenberg boundary G/P_o where P_o is a minimal parabolic subgroup. (See [He], [KKMOOT].) One of the more remarkable aspects of this theorem is its generality; one obtains a complete description of all solutions to the system of invariant differential operators on X without imposing any boundary conditions or growth conditions.

If X is a Hermitian symmetric space, then one is typically interested in complex function theory, in which case one is interested in functions whose boundary values are supported on the Shilov boundary rather than the Furstenberg boundary. (The Shilov boundary is G/P where P is a certain maximal parabolic containing P_o.) In this case, it turns out that the algebra of G invariant differential operators is not necessarily the most appropriate one for defining harmonicity. Johnson and Korányi [JK], generalizing earlier work of Hua [Hu], Korányi-Stein [KS], and Korányi-Malliavin [KM], introduced an invariant system of second order differential operators (the HJK system) defined on any Hermitian symmetric space. In [DHP2], we noted that this system could be defined entirely in terms of the geometric structure of X as

$$\mathrm{HJK}\ (f) = -\sum \nabla^2 f(Z_i, \overline{Z}_j) R(\overline{Z}_i, Z_j)|T^{01}$$

where ∇ denotes covariant differentiation, R is the curvature operator, T^{01} is the bundle of anti-holomorphic tangent vectors, and Z_i is a local frame field for T^{10} that is orthonormal with respect to the canonical Hermitian scalar product H on T^{10}. (It is easily seen that HJK does not depend on the choice of the Z_i.) Thus, HJK maps $C^\infty(\mathcal{D})$ into sections of $\mathrm{Hom}_{\mathbb{C}}(T^{01}, T^{01})$. (See [DHP2] for more details.) A C^∞ function f is said to be *Hua-harmonic* if HJK $(f) = 0$.

In [JK] the following results were proved in the Hermitian symmetric case:

(a) All Hua-harmonic functions are harmonic.

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(b) The boundary hyperfunctions are constant on right cosets of P and hence project to hyperfunctions on the Shilov boundary.
(c) Every Hua-harmonic function on X is the Poisson integral of its boundary hyperfunction over the Shilov boundary.
(d) If X is tube-type then Poisson integrals of hyperfunctions are harmonic.[1]

Thus, in the tube case, these results yield a complete description of all solutions to the Hua system, while in the non-tube case, we lack only a characterization of those hyperfunctions on the Shilov boundary whose Poisson integrals are Hua-harmonic.

Since the Hua system is meaningful for any Kähler manifold X, it seems natural to ask to what extent these results are valid out side of the symmetric case. One might, for example, consider homogeneous Kähler manifolds. There is a structure theory for such manifolds that was proved in special cases by by Vindberg and Gindikin [VG] and in general by Dorfmeister and Nakajima [DN] that states that every such manifold admits a holomorphic fibration whose base is a bounded homogeneous domain in $\mathbb{C}^n$, and whose fiber is the product of a flat, homogeneous Kähler manifold and a compact, simply connected, homogeneous, Kähler manifold. It follows that one should first consider generalizations to the class of bounded homogeneous domains in $\mathbb{C}^n$.

This problem was considered in [DHP2] and [PK]. In both of these works, however, extremely restrictive growth conditions were imposed on the solutions: in [DHP2] the solutions were required to be bounded and in [PK] an $\mathcal{H}^2$ type condition was imposed.

The technical difficulties involved in eliminating these growth assumptions at first seem daunting. In the non-symmetric case, K can be quite small. Thus, arguments which are based on concepts such as K-finiteness and bi-K invariance tend not to generalize. Entirely new proofs must be discovered.

The most problematic issues, however, come from the the boundary. In general, G may have no non-trivial boundaries in the sense of Furstenberg. Hence, it is not at all clear how to even define the Furstenberg boundary. The Shilov boundary is, of course, meaningful. However, in the symmetric case, the Shilov boundary is a homogeneous space for K; hence a manifold. In the solvable case it is almost certainly false that the Shilov boundary is a manifold. All that is known is that there is a nilpotent subgroup N of G, of nilpotence degree at most 2, which acts on the Shilov boundary in such a way that there is a dense, open orbit which we call the *principal open subset*. The principal open subset is well understood and easily described. Its compliment in the Shilov boundary is, to our knowledge, completely unstudied outside of the symmetric case. This does not cause difficulties for bounded or $\mathcal{H}^2$ solutions since the corresponding boundary hyperfunctions are functions and we only need to know them a.e. Understanding general unbounded solutions seems to require being able to describe their boundary values on this potentially singular and poorly understood set. In fact, it is not at all clear how to define the notion of a hyperfunction (or even a distribution) on the Shilov boundary, much less the boundary hyperfunction for a solution.

[1]Statement (d) is false in the general Hermitian symmetric case ([BV]).

There are, however, two works of E. van den Ban and H. Schlichtkrull ([BS1] and [BS2]) and a work of N. Wallach [Wa] which provide some hope of at least understanding the solutions with distributional boundary values. To describe these results, let $\tau(x)$ be the Riemannian distance in X from x to the base point $x_o = eK$. A result of Oshima and Sekiguci [OS] says that the boundary hyperfunction of a harmonic function F is a distribution if and only if there are positive constants A and r (depending on F) such that

$$|F(x)| \leq Ae^{r\tau(x)} \tag{0.1}$$

for all $x \in X$. In [BS1], using ideas from [Wa], it was shown that any harmonic function satisfying (0.1) has an "asymptotic expansion" as x approaches the Furstenberg boundary where the coefficients are distributions on this boundary. The boundary distribution occurs as one of the coefficients in this expansion. Actually, in [BS1], a finite set of these coefficients were singled out as boundary distributions. It was then shown how to choose one particular boundary distribution whose Poisson integral is F, providing a new proof of the Oshima-Sekaguci theorem. (Wallach also obtained a new proof of the same theorem using his asymptotic expansions.)

In [BS2] it was shown that F is uniquely determined by the restrictions of its boundary distributions to *any open subset of the boundary.* In this case, however, one needs all of the boundary functions, not just the particular one mentioned above.

Thus, in the non-symmetric case, one might hope to

(1) Prove the existence of a distribution asymptotic expansion for Hua-harmonic functions satisfying (0.1) as x approaches the principal open subset of the Shilov boundary.
(2) Choose a particular finite subset of the coefficients to be the boundary distributions which uniquely determine the solution.
(3) Describe the inverse of the boundary map. (The "Poisson transformation.")
(4) Describe the image of the boundary map.

In this work we carry out the first three steps of above program and make progress on the fourth. Specifically, in the general case it is still possible to write $G = AN_LK$ where A is an $\mathbb{R}$ split algebraic torus, N_L is a unipotent aubgroup normalized by A, K is a maximal compact subgroup. (See Section 3 for details.) Then $L = AN_L$ acts simply-transitively on $\mathcal{D}$, allowing us to identify $\mathcal{D}$ with L. As an algebraic variety,

$$L = N_L \times (\mathbb{R}^+)^d \subset N_L \times \mathbb{R}^d$$

where d is the rank of X. Under this identification, N_L is contained in the topological boundary of AN_L. We use N_L as a substitute for the Furstenberg boundary. In the semi-simple case this amounts to restricting to a dense, open, subset of the Furstenberg boundary.

We prove that any Hua harmonic function that satisfies (0.1) has an asymptotic expansion as $a \to 0$ with coefficients from the space of Schwartz distributions on N_L. We then single out a set of at most 2^d of these coefficients which serve as the boundary values and show that the boundary values uniquely determine the solution. Finally, we give an inductive construction, based on our work [P1], of a

Poisson transformation that "reconstructs" F from its boundary values. (See the remark following the proof of Proposition (3.23)).

Actually, all of the above statements hold, with "Schwartz distribution" replaced by "distribution" under the weaker assumption that for all compact sets $K \subset N_L$, there is a constant C_K such that

$$\sup_{n\in K} |F(na)| \leq C_K e^{r\tau(a)} \tag{0.2}$$

for all $a \in A$, except that in this case our construction of the Poisson kernel does not work since there seems to be no way of definining the integrals we require.

We also prove a version of the Johnson-Korányi result relating to the projection of the boundary distribution to the Shilov boundary. The Johnson-Korányi result that in the semi-simple tube case, the Hua harmonic functions are Poisson integrals of hyperfunctions over the Shilov boundary follows. (Theorem (3.41).)

Concerning the fourth step, as mentioned above, the description of the space of boundary values for the Hua system is unknown, even for a Hermitian-symmetric domain of non-tube type. (The Johnson-Korányi result shows that in the tube case, the space of boundary values is just the space of all hyperfunctions on the Shilov boundary.) In [BV], Berline and Vergne conjectured that this space could be characterized as null space of a "tangential" Hua system, although, to our knowledge, this conjecture has never been resolved.

However, in the symmetric case, it is possible it describe the boundary values for the "$\mathcal{H}^2_{\mathrm{HJK}}$" functions–which are Hua harmonic functions satisfying an $\mathcal{H}^2$ like condition. (See Section 3 below.) In [BBDHPT], the current author, together with Bonami, Buraczewski, Damek, Hulanicki, and Trojan, showed that for a non-tube type Hermitian symmetric domain, the $\mathcal{H}^2_{\mathrm{HJK}}$ harmonic functions are pluriharmonic–i.e. they are complex linear combination of the real and imaginary parts of $\mathcal{H}^2$ functions. Theorem (5.4) states that this same result holds in the non-symmetric case, at least for domains that are sufficiently non-tube like (Definition (2.21)).

The ability to generalize this result to the non-symmetric case is, we feel, a significant accomplishment. The symmetric space proof utilized the symmetry of the domain in many ways, but most significantly in its use of the full force of Johnson-Korányi theorem for tube domains. Explicitly, it required knowing that Poisson integrals are Hua-harmonic. It is a result of [PK] that this result is equivalent with the symmetry of the domain. One seems to require entirely new techniques (such as asymptotic expansions) to avoid its use in the general case.

We should also mention that our section on asymptotic expansions is quite general. The proofs, while inspired by those in [BS] and [BS2], which were, in turn, inspired by those in [Wa], are in actuality, quite different (and somewhat less involved) since we do not have as much algebraic machinery at our disposal. It is our expectation that this theory will have far reaching implications in many other contexts. It has already found application in [PU]. We expect it to play a major role in understanding the Helgeson program for other systems of equartions and other boundaries as well.

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Remarks on notation *Throughout this work, we will usually denote Lie groups by upper case Roman letters, in which case the corresponding Lie algebra will automatically be denoted by the corresponding upper case script letter. The main exceptions to this rule will be abelian Lie groups which will be identified with their Lie algebras. We also use "C" to denote a generic constant which may change from line to line.*

Section 1: Asymptotic Expansions

Let $\mathcal{V}$ be a Fréchet space over $\mathbb{C}$ and let $\mathcal{C} = C^\infty((-\infty, 0], \mathcal{V})$, given the topology of uniform convergence on compact subsets of functions and their derivatives. For $r \in \mathbb{R}$, let $\mathcal{C}_r^o$ be the set of $F \in \mathcal{C}$ such that

$$\{e^{-rt}F(t) \mid t \in (-\infty, 0]\}$$

is bounded in $\mathcal{V}$. Let $\|\cdot\|_m$, $m = 1, 2, \dots$ be a family of continuous semi-norms on $\mathcal{V}$ that defines its topology. We equip $\mathcal{C}_r^o$ with the topology defined by the semi-norms

$$\begin{aligned} \|F\|_{r,m} &= \sup_{t \in (-\infty,0]} e^{-rt}\|F(t)\|_m \\ \|F\|_{k,n,m} &= \sup_{-k \le t \le 0} \|F^{(n)}(t)\|_m \end{aligned} \tag{1.1}$$

where $k \in \mathbb{N}$ and

$$n \in \mathbb{N}_o = \mathbb{N} \cup \{0\}$$

We let

$$\mathcal{C}_r = \cap_{s<r} \mathcal{C}_s^o$$

given the inverse limit topology. It is easily seen that $\mathcal{C}_r$ is a Fréchet space. The space $\mathcal{C}_r$ is used since, unlike $\mathcal{C}_r^o$, it is closed under multiplication by polynomials. Let F and G belong to $\mathcal{C}$.

We say that

$$F \sim_r G$$

if $F - G \in \mathcal{C}_r$. Note that $F \sim_r G$ implies that $F \sim_s G$ for all $s < r$.

Let $I \subset \mathbb{C}$ be finite. An *exponential polynomial with exponents from I* is a sum

$$F(t) = \sum_{\alpha \in I} \sum_{n=0}^{n_\alpha} e^{\alpha \cdot t} t^n F_{\alpha,n} \tag{1.2}$$

where $F_\alpha \in \mathcal{V}$ and $n_\alpha \in \mathbb{N}_o$. In this case, we set

$$F_\alpha(t) = \sum_{n=0}^{n_\alpha} t^n F_{\alpha,n}$$

which is (by definition) a $\mathcal{V}$ valued polynomial. We also consider the case where $I \subset \mathbb{C}$ is countably infinite, in which case (1.2) is considered as a formal sum which we refer to it as an *exponential series.*

Definition 1.3: *Let $F \in \mathcal{C}$ and let $\check{F}$ be an exponential series as in (1.2). We say that $G \sim \check{F}$ if*

(a) *for all $r \in \mathbb{R}$, there is a finite subset $I(r) \subset I$ such that $G \sim_r F_r$ where*

$$F_r(t) = \sum_{\alpha \in I(r)} e^{\alpha t} F_\alpha(t) \tag{1.4}$$

and

(b) $I = \cup_r I(r)$.

In this case, we say that $\check{F}$ is an asymptotic expansion for F.

Remark: In formula (1.4), any term corresponding to an index α with $\operatorname{re} \alpha \geq r$ belongs to $\mathcal{C}_r$ and may be omitted. Thus, we may, and will, take $I(r)$ to be contained in the set of $\alpha \in I$ where $\operatorname{re} \alpha < r$.

We note the following lemma, which is a simple consequence of Lemma 3.3 of [BS].

Lemma 1.5. *If the function from (1.2) belongs to $\mathcal{C}_r$, then $F_\alpha(t) = 0$ for all $\operatorname{re} \alpha < r$ and all $t \in \mathbb{R}$.*

Lemma 1.6. *Suppose $G \sim \tilde{F}$ as in Definition (1.3), where all of the $F_\alpha(t)$ for $\alpha \in I$, are non-zero. Then $I(r) = \{\alpha \in I \mid \operatorname{re} \alpha < r\}$. In particular, the set of such α is finite.*

Proof Let $r < s$. Then $F \sim_r \check{F}_r$ and $F \sim_r \check{F}_s$. Hence $D_r = \check{F}_r - \check{F}_s \in \mathcal{C}_r$. Then D_r is an exponential polynomial with index set

$$(I(r) \cup I(s)) \setminus (I(r) \cap I(s))$$

Lemma (1.5) shows that this set is disjoint from $\operatorname{re} \alpha < r$, implying that it is disjoint from $I(r)$. Hence $I(r) \subset I(s)$. It then follows that $I(s) \setminus I(r)$ is disjoint from $\{\operatorname{re} \alpha < r\}$. Hence $\{\alpha \in I \mid \operatorname{re} \alpha < r\} \cap I \subset I(r)$, which proves our lemma. $\square$

Corollary 1.7. *Let $F \in \mathcal{C}$. Suppose that for each $r \in \mathbb{R}$, there is an exponential* *polynomial S^r such that $F \sim_r S^r$. Then there is an exponential series $\check{F}$ such that $F \sim \check{F}$.*

Proof Each S^r may be written

$$S^r(t) = \sum_{\alpha \in I(r)} e^{\alpha t} S^r_\alpha(t)$$

where $I(r)$ is a finite subset of $\mathbb{C}$ such that $S^r_\alpha(t) \neq 0$ for all $\alpha \in I(r)$. As before, we may assume that for all $\alpha \in I(r)$, $\operatorname{re} \alpha \leq r$. Then from the proof of Lemma (1.6), for $r < s$, $I(r) \subset I(s)$. Lemma (1.5) then implies that $S^r_\alpha(t) = S^s_\alpha(t)$ for $\alpha \in I(r)$.

Our corollary now follows: we let I be the union of the $I(r)$ and let

$$F_\alpha(t) = S^r_\alpha(t)$$

where r is chosen so that $\alpha \in I(r)$. The previous remarks show that this is independent of the choice of r. $\square$

The following is left to the reader. The minimum exists due to Corollary (1.6).

Proposition 1.8. *Suppose that $F \in \mathcal{C}$ has an asymptotic expansion with exponents I. Then $F \in \mathcal{C}_r$ where*

$$r = \min\{\operatorname{re} \alpha \mid \alpha \in I, F_\alpha \neq 0\}$$

Furthermore, suppose that there is a unique $\alpha \in I$ with $\operatorname{re} \alpha = r$ and that for this α, F_α is independent of t. Then

$$\lim_{t \to -\infty} e^{-\alpha t} F(t) = F_\alpha.$$

We consider a differential equation on $\mathcal{C}$ of the form

$$F'(t) = (Q_0 + Q(t))F(t) + G(t) \tag{1.9}$$

where $G \in \mathcal{C}$,

$$Q(t) = \sum_{i=1}^{d} e^{\beta_i t} Q_i,$$

$$1 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_d, \tag{1.10}$$

and the Q_k are continuous linear operators on $\mathcal{V}$. We also assume that Q_0 is *finitely triangularizable*, meaning that

(a) There is a direct sum decomposition

$$\mathcal{V} = \sum_{i=1}^{q} \mathcal{V}^i \tag{1.11}$$

where the $\mathcal{V}^i$ are closed subspaces of $\mathcal{V}$ invariant under Q_0.

(b) For each i there is an $\alpha_i \in \mathbb{C}$ and an integer n_i such that

$$(Q_0 - \alpha_i I)^{n_i}\big|_{\mathcal{V}^i} = 0.$$

(c) $\alpha_i \neq \alpha_j$ for $i \neq j$.

For the set of exponents we use $I = \{\alpha_i\} + I_o$ where

$$I_o = \{\sum_j \beta_j k_j \mid k_j \in \mathbb{N}_o\}.$$

The first main result of this section is the following:

Theorem 1.12. *Let $F \in \mathcal{C}_r$ satisfy (1.9). Assume that G has an asymptotic expansion with exponents from I' . Then F has an asymptotic expansion with exponents from $I'' = (\{\alpha_i\} \cup I') + I_0$.*

Proof From Corollary (1.7) it suffices to prove that for all $n \in \mathbb{N}$, there is an exponential polynomial $S_n(t)$ with exponents from I'' such that

$$F(t) - S_n(t) \in \mathcal{C}_{r+n}.$$

We reason by induction on n. Let

$$P(t) = \sum_i e^{(\beta_i - 1)t} Q_i$$

so that $Q(t) = e^t P(t)$. Note $\beta_i - 1 \geq 0$ for all i.

We apply the method of Picard iteration to (1.9). Explicitly, (1.9) implies that

$$F(t) = e^{tQ_0}F(0) - \int_t^0 e^{(t-s)Q_0} e^s P(s) F(s)\, ds - \int_t^0 e^{(t-s)Q_0} G(s)\, ds. \tag{1.13}$$

We begin with the term on the far right. Let

$$G(t) = R_u^G(t) + G(t)_u$$

where $u > \max\{r+1,\ \text{re}\ \alpha_i\}$, $R_u^G \in \mathcal{C}_u$, and

$$G(t)_u = \sum_{\alpha \in I'(u)} G_\alpha(t) e^{\alpha t} \tag{1.14}$$

is an exponential polynomial.

Let $B_i = (Q_0 - \alpha_i I)\big|_{\mathcal{V}_i}$. On $\mathcal{V}^i$

$$e^{tQ_0} = e^{\alpha_i t} A_i(t) \tag{1.15}$$

where

$$A_i(t) = e^{tB_i} = \sum_{j=0}^{n_i} B_i^j \frac{t^j}{j!}.$$

It follows that the integrals in the following equality converge where the superscript indicates the i^{th} component in the decomposition (1.11).

$$\int_t^0 e^{(t-s)Q_0} (R_u^G)^i(s)\, ds = e^{\alpha_i t} A_i(t) G_o^i - \int_{-\infty}^t e^{\alpha_i (t-s)} A_i(s-t)(R_u^G)^i(s)\, ds \tag{1.16}$$

where

$$G_o^i = \int_{-\infty}^0 e^{-s\alpha_i} A_i(s)(R_u^G)^i(s)\, ds.$$

The second term on the right in (1.16) is easily seen to belong to $\mathcal{C}_u$ and the G_o^i term will become part of S_1. Note that its exponents belong to $I \subset I''$.

On the other hand, replacing $G(s)$ in (1.13) with $G_\alpha(s)^i e^{\alpha s}$ from (1.14) produces a term of the form

$$e^{\alpha_i t} H_i(s) e^{(-\alpha_i + \alpha)s}\Big|_{s=0}^{s=t}$$

where H_i is a $\mathcal{V}$ valued polynomial. Both terms are exponential polynomials with exponents from I'' which become part of S_1.

Next we consider the second term on the right in (1.13). Its i^{th} component is

$$\begin{aligned} &-\int_t^0 e^{(t-s)\alpha_i} e^s A_i(t-s)(P(s)F(s))^i\, ds \\ &= \sum_{k=0}^{n_i} \sum_{j=0}^{n_i} t^k e^{\alpha_i t} \int_t^0 s^j e^{(1-\alpha_i)s} C_{k,j}(P(s)F(s))^i\, ds \end{aligned} \tag{1.17}$$

where the $C_{k,j}$ are continuous operators on $\mathcal{V}^i$.

Since $s \to P(s)F(s)$ belongs to $\mathcal{C}_r$, it follows that for each $v < r$ and each $m \in \mathbb{N}_o$ there is a constant $M_{v,m}$ such that

$$\|C_{k,j}(P(s)F(s))^i\|_m \le M_{v,m} e^{vs} \tag{1.18}$$

for all $s < 0$. Hence, (1.17) is bounded in $\|\cdot\|_m$ by

$$C(|t|^N + 1)(e^{(v+1)t} + e^{t(\ \text{re}\ \alpha_i)})$$

where C and N are positive constants. It follows that the left side of (1.17) belongs to $\mathcal{C}_{r+1}$ if $\operatorname{re}\alpha_i \geq r+1$.

On the other hand, if $\operatorname{re}\alpha_i < r+1$, then we may express the right side of (1.17) as

$$e^{\alpha_i t} H_i(t) + \int_{-\infty}^{t} e^{(t-s)\alpha_i} e^s A_i(t-s)(P(s)F(s))^i \, ds$$

where

$$H_i(t) = -\int_{-\infty}^{0} e^{s(-\alpha_i+1)} A_i(t-s)(P(s)F(s))^i \, ds.$$

(Note that the integrals converge in the topology of $\mathcal{V}$ since we may choose $v > \operatorname{re}\alpha_i - 1$ in (1.18).) The H_i term is an exponential polynomial which becomes part of S_1 and the other term belongs to $\mathcal{C}_{r+1}$. It now follows that there does indeed exist an exponential polynomial $S_1(t)$ with exponents from I'' such that $F(t) - S_1(t) \in \mathcal{C}_{r+1}$.

Next suppose by induction that we have proved the existence of an exponential polynomial S_n such that $R_n = F - S_n \in \mathcal{C}_{r+n}$ for some n. We provisionally define

$$S_{n+1}(t) = e^{tQ_0} F(0) - \int_{t}^{0} e^{(t-s)Q_0} e^s P(s) S_n(s) \, ds - \int_{t}^{0} e^{(t-s)Q_0} G(s)_u \, ds \qquad (1.19)$$

where u is greater than both $r+n+1$ and $\operatorname{re}\alpha_i$ for all i. Then from (inteq) $F - S_{n+1} = R_{n+1}$ where

$$R_{n+1}(t) = -\int_{t}^{0} e^{(t-s)Q_0} e^s P(s) R_n(s) \, ds + \int_{t}^{0} e^{(t-s)Q_0} R_u^G(s) \, ds.$$

Now, we project onto $\mathcal{V}^i$ as before and split the argument into two cases, depending on whether or not $\operatorname{re}\alpha_i \geq r+n+1$. An argument virtually identical to that done above shows that in each case, R_{n+1} is the sum of an exponential polynomial, which becomes part of S_{n+1}, and an element of $\mathcal{C}_{r+1}$. We leave the details to the reader. □

From this point on, until we begin discussing multi-variable expansions, we assume that $F \in \mathcal{C}_r$ satisfies (1.9) where $G = 0$ so $I'' = \{\alpha_i\} + I_o$.

Proposition 1.20. *For all $n \in \mathbb{N}_o$ $F^{(n)} \in \mathcal{C}_r$ and*

$$F^{(n)} \sim \sum_{\alpha \in I} e^{\alpha t} F_\alpha^n(t)$$

where

$$F_\alpha^n(t) = e^{-\alpha t} \frac{d^n}{dt^n}(e^{\alpha t} F_\alpha)(t).$$

Proof

Let $\tilde{\mathcal{V}}_r$ be the space of all elements $F \in \mathcal{C}_r$ for which $F^{(n)} \in \mathcal{C}_r$ for all $n \in \mathbb{N}_o$, topologized via the semi-norms

$$F \to \|F^{(n)}\|_{s,m}$$

where $m \in \mathbb{N}$, $n \in \mathbb{N}_o$, $\|\cdot\|_{s,m}$ is as in (1.1), and $s < r$. It is easily seen that $\tilde{\mathcal{V}}_r$ is a Fréchet space.

Now, let $F \in \mathcal{C}_r$ satisfy (1.9). Pointwise multiplication by the Q_i and by $e^{\beta_i t}$ define continuous mappings of $\mathcal{C}_r$ into itself. Hence, from (1.9), $F' \in \mathcal{C}_r$. It then follows by differentiation of (1.9) and induction that $F^{(n)} \in \mathcal{C}_r$ for all n. Hence, $F \in \tilde{\mathcal{V}}_r$.

For $F \in \tilde{\mathcal{V}}_r$, let $M(F)$ be the mapping of $(-\infty, 0]$ into $\tilde{\mathcal{V}}_r$ defined by

$$M(F)(t) : s \to F(t+s) \tag{1.21}$$

for $t \in (-\infty, 0]$. It is easily seen that in fact $M(F) \in \mathcal{C}_r(\tilde{\mathcal{V}})$. Furthermore, if F satisfies (1.9), then

$$M(F)'(t) = Q_0 M(F)(t) + \sum_{i=1}^{d} e^{\beta_i t} \tilde{Q}_i M(F)(t)$$

where

$$\tilde{Q}_i = e^{\beta_i s} Q_i.$$

It follows from Theorem (1.12) that $M(F)$ has an asymptotic expansion as a $\tilde{\mathcal{V}}$ valued map. It is easily seen that if F's asymptotic expansion is as in (1.2), then

$$M(F)(t) \sim \sum_{\alpha \in I} e^{\alpha t} e^{\alpha s} M(F_\alpha)(t).$$

Since $\dfrac{d}{ds}$ is continuous on $\tilde{\mathcal{V}}$, it follows that

$$M(F)^{(n)}(t) \sim \sum_{\alpha \in I} e^{\alpha t} \frac{d^n}{d\,s^n} \left(e^{\alpha s} M(F_\alpha)\right)(t).$$

Our result follows by letting $t = 0$ in the above formula. $\square$

It follows from Proposition (1.20) and Lemma (1.5), that we may formally substitute F's asymptotic expansion (1.2) into (1.9) and equate coefficients of $e^{\alpha t}$ for $\alpha \in I$. We find that for $\alpha \in I$,

$$F_\alpha'(t) + \alpha F_\alpha(t) = Q_0 F_\alpha(t) + \sum_{i=1}^{m} \sum_{\beta \in I, \beta + \beta_i = \alpha} Q_i F_\beta(t). \tag{1.22}$$

We put a partial ordering on I by saying that $\gamma \succeq \alpha$ if $\gamma - \alpha \in I_o$.

Definition 1.23: *Let $F \sim \check{F}$ as in (1.2). We say that $F_\alpha(t)$ is a leading term and α a leading exponent if α is minimal in I under $\succeq$ with respect to the property that $F_\alpha(t) \neq 0$.*

From the definition of I, for all $\alpha \in I$, there is an i such that $\alpha \succeq \alpha_i$. Since the set of α_i is finite, it follows that each α dominates a leading exponent.

Let α be a leading exponent. Then (1.22) implies that

$$F'_\alpha(t) + \alpha F_\alpha(t) = Q_0 F_\alpha(t). \tag{1.24}$$

Since Q_0 is finitely triangularizable, the solution to this differential equation is

$$F_\alpha(t) = e^{(Q_0 - \alpha I)t} F_\alpha(0).$$

Hence, $F_\alpha(0)$ uniquely determines $F_\alpha(t)$. Since $F_\alpha(t)$ is a polynomial, there is an N such that

$$0 = F_\alpha^{(N)}(0) = (Q_0 - \alpha I)^N F_\alpha(0).$$

Hence, $\alpha = \alpha_i$ for some i and $F_\alpha(0) \in \mathcal{V}^i$. Thus *all of the leading exponents come from the α_i*. It also follows that if Q_0 is diagonalizable, then the $F_\alpha(t)$ are constant for all leading exponents α. In fact we have the following:

Proposition 1.25. *The asymptotic expansion of F is uniquely determined by the elements $F_{\alpha_i}(0)$.*

Proof According to the above discussion, the given data is sufficient to determine the leading terms. If there is an α such that $F_\alpha(t)$ is not determined, then there is a minimal such α. But then (1.22) shows that $F_\alpha(t)$ satisfies a differential equation of the form

$$(\frac{d}{dt} + (Q_0 - \alpha I))F_\alpha(t) = G(t)$$

where G is known. Since α is not one of the α_i, the differential operator on the left side of this equality has no kernel in the space of $\mathcal{V}$ valued polynomials, showing that F_α is uniquely determined. □

Definition 1.26: *Let F satisfy (1.9). Then the set of terms in the asymptotic expansion of the form $F_{\alpha_i}(0)$ is referred to as the set of boundary values for F and is denoted $BV(F)$.*

It should be noted that if α_i is a leading exponent, then $F_{\alpha_i}(0)$ is a non-zero boundary value but not conversely–not all non-zero boundary values $F_{\alpha_i}(0)$ need be leading terms. They will be leading terms if either (a) α_i is minimal with respect to the partial ordering on I or (b) $\alpha_i \succ \alpha_j$ implies $F_{\alpha_j}(t) = 0$.

In the next section we will need to consider asymptotic expansions in several variables. Let

$$\mathcal{V}(d) = C^\infty((-\infty,0]^d,\mathcal{V})$$

with the topology of uniform convergence of functions and their derivatives on compact subsets of $(-\infty,0]^d$. For $F \in \mathcal{V}(d)$, we define $\tilde{F} \in C^\infty((-\infty,0],\mathcal{V}(d-1))$ by

$$\tilde{F}(t_1)(t_2,\ldots,t_d) = F(t_1,t_2,\ldots,t_d). \tag{1.27}$$

We define $\mathcal{C}_r(d) \subset \mathcal{V}(d)$ inductively by

$$\mathcal{C}_r(d) = \mathcal{C}_r((-\infty,0],\mathcal{C}_r(d-1)).$$

We define multiple asymptotic expansions inductively as follows:

Definition 1.28: *Let $F \in \mathcal{C}_r(d)$. We say that F has a d-variable asymptotic expansion if*

(a) *$\tilde{F}$ has a $\mathcal{C}_r(d-1)$ valued asymptotic expansion*

$$\tilde{F}(t_1) \sim \sum_{\alpha_1\in I_1}\sum_{0}^{n_{\alpha_1}} t_1^n e^{\alpha_1 t_1} G_{\alpha_1,n}$$

where $I_1 \subset \mathbb{C}$.

(b) *Each $G_{\alpha_1,n}$ has a $d-1$-variable, $\mathcal{V}$ valued asymptotic expansion*

$$G_{\alpha_1,n}(t) \sim \sum_{\alpha\in I(\alpha_1)}\sum_{|N|\le n(\alpha)} t^N e^{\alpha\cdot t} F_\alpha$$

where $t \in (-\infty,0]^{d-1}$ and, for each $\alpha_1 \in I_1$, $I(\alpha_1) \subset \mathbb{C}^{n-1}$.

In this case we write

$$\begin{aligned} F(t) &\sim \sum_{\alpha\in I}\sum_{|N|\le m(\alpha)} t^n e^{\alpha\cdot t} F_{\alpha,n} \\ &= \sum_{\alpha\in I} e^{\alpha\cdot t} F_\alpha(t) \end{aligned} \tag{1.29}$$

where

$$\begin{aligned} I &= \{(\alpha_1,\ldots,\alpha_d) \in \mathbb{C}^d \mid (\alpha_2,\ldots,\alpha_d) \in I(\alpha_1)\}, \\ m(\alpha) &= \max\{n_{\alpha_1}, n(\alpha_2,\ldots,\alpha_n)\}. \end{aligned}$$

Let $\alpha,\beta \in I$. We say that $\alpha \in I$ is minimal if $\operatorname{re}\alpha < \operatorname{re}\beta$ in the lexicographic ordering, for all $\beta \in I$, $\beta \ne \alpha$. If I is the index set for an asymptotic expansion and $I \in \mathbb{R}^d$ then I always has a minimal element, although I might not have a minimal element in general. The following proposition follows from induction on Proposition (1.8).

Proposition 1.30. *Let F have an asymptotic expansion as in (1.29) and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a minimal element of I. Suppose also that F_α is independent of t. Then*

$$\lim_{t_d \to -\infty} \lim_{t_{d-1} \to -\infty} \ldots \lim_{t_1 \to -\infty} e^{-\alpha \cdot t} F(t) = F_\alpha$$

where the limit converges in $\mathcal{V}$.

We also note the following, which follows by induction from Lemma (1.6).

Lemma 1.31. *Let $r \in \mathbb{R}$. The set $I(r)$ of $\alpha \in I$ with re $\alpha_i < r$, $1 \leq i \leq d$, is finite.*

Section 2: Homogeneous Domains

In this section, we discuss those structural features of Siegel domains that we use. These results are, for the most part, well known. Our basic references are [GPV] and [Vin], although we will at times refer the reader to some of our papers where the results are presented in similar notation to our current needs. In particular, the summary given pn p. 86-91 and p. 94-97 of [DHP2] covers many of the essentials. The reader should not interpret such references as a claim of originality on our behalf.

Any bounded, homogeneous domain in $\mathbb{C}^n$ (and hence, every Hermitian symmetric space of non-compac type) may be realized as a Siegel domain of either type I or II. Explicitly, let $\mathcal{M}$ be a finite dimensional real vector space with dimension $n_{\mathcal{M}}$ and let $\Omega \subset \mathcal{M}$ be an open, convex cone that does not contain straight lines. The subgroup of $\mathrm{Gl}(\mathcal{M})$ that leave Ω invariant is denoted G_Ω. We say that Ω is homogeneous if G_Ω acts transitively on Ω via the usual representation of $\mathrm{Gl}(\mathcal{M})$ on $\mathcal{M}$. (We denote this representation by ρ.) In this case, Vindberg showed that there is a a triangular subgroup S of G_Ω that acts simply transitively on Ω. This subgroup may be assumed to contain the dilation maps

$$\delta(t) : v \to tv \tag{2.1}$$

for all $t > 0$.

Suppose further that we are given a complex vector space $\mathcal{Z}$ and a Hermitian symmetric, bi-linear mapping $B_\Omega : \mathcal{Z} \times \mathcal{Z} \to \mathcal{M}_c$. We shall assume that

$$(a)\, B_\Omega(z, z) \in \overline{\Omega} \text{ for all } z \in \mathcal{Z},$$
$$(b)\, B_\Omega(z, z) = 0 \text{ implies } z = 0.$$

The Siegel domain $\mathcal{D}$ associated with this data is defined as

$$\mathcal{D} = \{(z_1, z_2) \in \mathcal{Z} \times \mathcal{M}_c : \text{ im } z_2 - B_\Omega(z_1, z_1) \in \Omega\}. \tag{2.2}$$

The domain is said to be type I or II, depending upon whether or not $\mathcal{Z}$ is trivial. The terms "tube type" and "type I" are synonyms.

The Bergman-Shilov boundary $\mathcal{B}$ of $\mathcal{D}$ is defined as

$$\mathcal{B} = \{(z_1, z_2) \in \mathcal{Z} \times \mathcal{M}_c \mid \text{ im } z_2 = B_\Omega(z_1, z_1)\}.$$

This is the principal open subset of the Shilov boundary referred to in the introduction.

Suppose further that we are given a complex linear algebraic representation σ of S in $\mathcal{Z}$ such that

$$B_\Omega(\sigma(s)z, \sigma(s)w) = \rho(s)B_\Omega(z, w) \text{ for all } z, w \in \mathcal{Z}. \tag{2.3}$$

The group S then acts on $\mathcal{D}$ by

$$s(z, w) = (\sigma(s)z, \rho(s)w). \tag{2.4}$$

We let $\mathcal{M}$ act on $\mathcal{D}$ by translation:

$$x(z, w) = (z, w + x), \ x \in \mathcal{M}. \tag{2.5}$$

Finally, we let $\mathcal{Z}$ act by

$$z_0(z, w) = (z + z_0, w + 2iB_\Omega(z, z_0) + iB_\Omega(z_0, z_0)). \tag{2.6}$$

These actions generate a completely solvable group L which acts simply transitively on $\mathcal{D}$. Specifically, the group N_b generated by the actions (2.5) and (2.6) is isomorphic with $\mathcal{Z} \times \mathcal{M}$ with the product

$$(z_1, m_1)(z_0, m_0) = (z_1 + z_0, m_1 + m_0 + 2 \text{ im } B_\Omega(z_1, z_0)). \tag{2.7}$$

Then L is the semi-direct product $N_b \times_s S$ where the S action on N_b is defined by formula (2.4).

The above product is the Campbell-Hausdorff product on N_b defined by the Lie bracket

$$[(z_1, m_1), (z_0, m_0)] = (0, 4 \text{ im } B_\Omega(z_1, z_0)). \tag{2.8}$$

A Siegel domain that has the structures defined above is referred to as *homogeneous.* It is a fundamental result that every bounded homogeneous domain in $\mathbb{C}^n$ is biholomorphic to a homogeneous Siegel domain. ([GPV]) It is important to note that $\mathcal{D}$ contains a type I domain $\mathcal{D}_o$ as a closed submanifold which is defined by $z_1 = 0$. The subgroup

$$T = \mathcal{M}S \tag{2.9}$$

acts simply transitively on $\mathcal{D}_o$.

We will also use a slight variant on the above construction. Suppose that in addition to the above data we are given a real vector space $\mathcal{X}$ and an $\mathcal{M}$ valued symmetric real-bi-linear form R_Ω satisfying conditions (a) and (b) below condition (2.1). Let $\mathcal{D} \subset \mathcal{X}_c \times \mathcal{Z} \times \mathcal{M}_c$ be the set of points $(x+iy, z, w)$ such that

$$\operatorname{im} w - R_\Omega(x,x) - B_\Omega(z,z) \in \Omega. \tag{2.10}$$

Such domains are bi-holomorphic with Siegel II domains. To see this, extend R_Ω to an $\mathcal{M}_c$-valued, Hermitian-linear, mapping R^c_Ω on $\mathcal{Z}' = \mathcal{X}_c$. Let ϕ be the bi-holomorphism of $\mathcal{Z}' \times \mathcal{Z} \times \mathcal{M}_c$ into itself defined by

$$\phi(z', z, w) = (z', z, 2w - iR^c_\Omega(z', \overline{z}')).$$

Then, as the reader can check, ϕ transforms $\mathcal{D}$ onto the Siegel II domain defined by Ω, $\mathcal{Z}' \times \mathcal{Z}$, and $R^c_\Omega + B_\Omega$.

Let $c_o \in \Omega$ be a fixed base point. We use $b_o = (0, ic_o) \in \mathcal{D}$ as the base point for $\mathcal{D}$. The map $g \to g \cdot b_o$ identifies L and $\mathcal{D}$. We also identify $\mathcal{L}$ with the real tangent space of $\mathcal{L}$ at b_o.

Let $\mathcal{P}$ be the complex subalgebra of $\mathcal{L}_c$ corresponding to T^{01} and let $J : \mathcal{L} \to \mathcal{L}$ be the complex structure so that $\mathcal{P}$ is the $-i$ eigenspace of J. Then J satisfies the "J-algebra" identity:

$$J([X,Y] - [JX, JY]) = [JX, Y] + [X, JY]. \tag{2.11}$$

Also

$$\begin{aligned} J &: \mathcal{Z} \to \mathcal{Z}, \\ J &: \mathcal{S} \to \mathcal{M}, \\ J &: \mathcal{M} \to \mathcal{S}. \end{aligned}$$

It follows that $\mathcal{S}$ and $\mathcal{M}$ are isomorphic as linear spaces. In fact, from the comments following Lemma (2.1) of [DHP2],

$$\begin{aligned} JX &= -d\rho(X)c_o \qquad X \in \mathcal{S}, \\ m &= d\rho(Jm)c_o \qquad m \in \mathcal{M}, \\ JX &= iX \qquad X \in \mathcal{Z} \end{aligned} \tag{2.12}$$

where 'i' is the complex multiplication of $\mathcal{Z}$, '$d\rho$' is the representation of $\mathcal{S}$ obtained by differentiating ρ and c_o is the base point in Ω.

We shall require a description of an L-invariant Riemannian structure on the domain. Koszul ([Kl], Formula 4.5) showed that the Bergman structure is defined by a scalar product of the form

$$g(X,Y) = \mu([JX, Y]) \tag{2.13}$$

where μ is an explicitly described element of $\mathcal{M}^* \subset \mathcal{L}^*$. *We assume only that $\mu \in \mathcal{M}^*$ is such that (2.13) defines an L-invariant Kähler structure on $\mathcal{D}$.*

Since g is J-invariant,

$$\mu([JX, JY]) = -\mu([J^2X, Y]) = \mu([X,Y])$$

The sclar product g is the real part of the Hermitian scalar product on $\mathcal{L}_c$ defined by

$$g_{\mathrm{Her}}(X,Y) = g(X,Y) + ig(X,JY).$$

We will also make use of the Hermitian scalar product g_c on $\mathcal{L}_c$ defined by

$$g_c(Z,W) = \frac{1}{2} g(Z,\overline{W}) \tag{2.14}$$

where g is extended to $\mathcal{L}_c$ by complex bi-linearity.

In [DHP2], we describe a particular decomposition

$$\mathcal{S} = \mathcal{A} + \mathcal{N}_S$$

where $\mathcal{A}$ is a maximal, $\mathbb{R}$-split torus in $\mathcal{S}$ and $\mathcal{N}_S$ is the unipotent radical of $\mathcal{S}$. The *rank* d of $\mathcal{D}$ is, by definition, the dimension of $\mathcal{A}$. This splitting has the property that for all $A \in \mathcal{A}$, the operators ad A are symmetric with respect to g on $\mathcal{L}$. In particular, we may decompose $\mathcal{L}$ into a direct sum of joint eigenspaces for the adjoint action of $\mathcal{A}$.

An element $\lambda \in \mathcal{A}^*$ is said to be a *root* of $\mathcal{A}$ if there is a non-zero element $X \in \mathcal{L}$ such that

$$[A, X] = \lambda(A)X$$

for all $A \in \mathcal{A}$. For $\lambda \in \mathcal{A}^*$, the set of X that satisfy the above equation is denoted $\mathcal{L}_\lambda$ and is referred to as the *root space* for λ. Then

$$[\mathcal{L}_\lambda, \mathcal{L}_\beta] \subset \mathcal{L}_{\lambda+\beta}. \tag{2.15}$$

There is an ordered basis $\lambda_1, \lambda_2, \ldots, \lambda_d$ for $\mathcal{A}^*$ consisting of roots for which the root space of λ_i is a one dimensional subspace $\mathcal{M}_{ii}$ of $\mathcal{M}$. All of the other roots are one of the following types

(a) $\beta_{ij} = (\lambda_i - \lambda_j)/2$ where $i < j$,
(b) $\tilde{\beta}_{ij} = (\lambda_i + \lambda_j)/2$,
(c) $\lambda_i/2$.

We let $\Delta_{\mathcal{S}}$ be the set of roots of type (a), $\Delta_{\mathcal{M}}$ be the set of roots of type (b) and $\Delta_{\mathcal{Z}}$ be the set of roots of type (c).

The root spaces for roots of types (a), (b), and (c) are belong, respectively, to $\mathcal{S}$, $\mathcal{M}$ and $\mathcal{Z}$ and are denoted, respectively, by $\mathcal{S}_{ij}$, $\mathcal{M}_{ij}$ and $\mathcal{Z}_i$, which is a complex subspace of $\mathcal{Z}$. We let $d_{ij} = d_{ji}$ denote the dimension of $\mathcal{M}_{ij}$, which for $i < j$, is also the dimension of $\mathcal{S}_{ij}$. We let f_i be the dimension (over $\mathbb{C}$) of $\mathcal{Z}_i$. In the irreducible symmetric case, the d_{ij} are constant as are the f_i, although these dimensions are not constant in general. In particular, some may be 0.

We define

$$\mathcal{N}_S = \sum_{1 \le i < j \le d} \mathcal{S}_{ij}.$$

The operator J maps each $\mathcal{S}_{ij}$ onto $\mathcal{M}_{ij}$. We note for future reference that from (2.15)

$$[\mathcal{Z}_i, \mathcal{Z}_j] \subset \mathcal{M}_{ij}. \tag{2.16}$$

The ordered basis of $\mathcal{A}$ that is dual to the basis formed by $\{\lambda_i\}$ is denoted $\{A_i\}$ and the span of A_i is denoted $\mathcal{S}_{ii}$. For each i we let $E_i = -JA_i \in \mathcal{M}_{ii}$. Then

$$[A_i, E_i] = E_i. \tag{2.17}$$

For each $1 \leq i \leq d$, we set

$$\mu_i =< E_i, \mu >= g(A_i, A_i) = g(E_i, E_i). \tag{2.18}$$

The element

$$E = \sum_1^r E_i$$

plays a special role:

$$JE = \sum_1^r A_i.$$

It follows that

$$\begin{aligned} \text{ad } JE\big|_{\mathcal{M}} &= I, \\ \text{ad } JE\big|_{\mathcal{Z}} &= I/2. \end{aligned} \tag{2.19}$$

The first equality tells us that JE is the infinitesimal generator of the one parameter subgroup $t \to \delta(t)$. Since

$$\delta(t)c_o = tc_o$$

we see that $d\rho(JE)c_o = c_o$. Hence

$$E = -J(JE) = d\rho(JE)c_o = c_o.$$

Thus, E is the base point of Ω. In particular, $E \in \Omega$.

It follows from formulas (2.11) and (2.19) that for $m \in \mathcal{M}$ and $X \in \mathcal{S}$,

$$\begin{aligned} m &= [Jm, E], \\ X &= J[X, E]. \end{aligned} \tag{2.20}$$

We say that a permutation σ of the indecies $\{1, 2, \ldots, d\}$ is *compatible* if

$$\Delta_{\mathcal{S}} = \{(\lambda_{\sigma(i)} - \lambda_{\sigma(j)})/2 \mid |1 \leq i < j \leq d\}$$

This is equivalent with saying that for $i < j$, $(\lambda_{\sigma(j)} - \lambda_{\sigma(i)})/2$ is not a root. If σ is compatible, then we may replace the sequence λ_i with $\lambda_{\sigma(i)}$ in the preceding discussion. This has the effect of replacing $\mathcal{M}_{ij}$ and $\mathcal{S}_{ij}$ with $\mathcal{M}_{\sigma(i)\sigma(j)}$ and $\mathcal{S}_{ij}$ with $\mathcal{S}_{\sigma(i)\sigma(j)}$ respectively.

Definition 2.21: *We say that λ_i is singular if $(\lambda_i - \lambda_j)/2$ is not a root for all $j > i$. We say that the root sequence is terminated if there is an index d_τ such that the set of singular roots is just $\{\lambda_i \mid d_\tau \leq i \leq d\}$. We refer to d_τ as the point of termination. We say that $\mathcal{D}$ is non-tube like if $d_\tau = d$ and $\lambda_i/2$ is a root for all $1 \leq i \leq d$.*

Lemma 2.22. *There is a compatible permutation σ such that $\{\lambda_{\sigma(i)}\}$ is terminated.*

Proof Our lemma follows from the simple observation that if λ_i is singular where $i < d$, then the permutation that interchanges i and $i+1$ is compatible.

From now on, we assume that the λ_i are terminated. This has the consequence that $\mathcal{S}_{ij} = 0$ if $d_\tau \leq i < j \leq d$.

We define,

$$\begin{aligned} \mathcal{S}_{1*} &= \sum_{1 \leq m} \mathcal{S}_{1m}, \\ \mathcal{N}_{1*} &= \sum_{1<m} \mathcal{S}_{1m}, \\ \mathcal{M}_{1*} &= \sum_{1<m} \mathcal{M}_{1m}, \\ \mathcal{S}_{>1} &= \sum \mathcal{S}_{ij} \qquad (1 < i \leq j \leq r), \\ \mathcal{M}_{>1} &= \sum \mathcal{M}_{ij} \qquad (1 < i \leq j \leq r), \\ \mathcal{Z}_{>1} &= \sum_{2 \leq i \leq f} \mathcal{Z}_i. \end{aligned} \tag{2.23}$$

Then $\mathcal{S}_{1*}$ is a Lie ideal in $\mathcal{S}$ and $\mathcal{S}_{>1}$ is a complimentary Lie subalgebra. Also, $\mathcal{M}_{1*}$ is ad $(\mathcal{S})$ invariant. We identify $\mathcal{M}_{>1}$ with the quotient $\mathcal{M}/(\mathbb{R}E_1 + \mathcal{M}_{1*})$. The image $\Omega_{>1}$ in $\mathcal{M}_{>1}$ of the cone Ω is a cone which is homogeneous under $S/S_{1*} = S_{>1}$. In fact, Ω is the orbit of $c_{>1}$ in $\mathcal{M}_{>1}$ under $\mathcal{S}_{>1}$ where

$$c_{>1} = \sum_2^d E_i.$$

The data $B_\Omega\big|(\mathcal{Z}_{>1} \times \mathcal{Z}_{>1})$, $\mathcal{M}_{>1}$ and $\Omega_{>1}$ defines a Siegel domain on which

$$L_{>1} = (\mathcal{Z}_{>1} \times \mathcal{M}_{>1}) \times_s S_{>1} \subset L$$

acts simply transitively.

The group

$$L_{1*} = (\mathcal{Z}_1 \times \mathcal{M}_{1*}) \times_s S_{1*}$$

also acts simply transitively on a Siegel domain. Explicitly, for $X, Y \in \mathcal{S}_{1*}$, there is a scalar $R(X,Y)$ such that

$$[X, [Y, E_1]] = R(X,Y)E_1.$$

Similarly, for $z, w \in \mathcal{Z}_1$,

$$B_\Omega(z,w) = B_\Omega^o(z,w)E_1$$

where B^o_Ω is a $\mathbb{C}$-valued Hermitian form on $\mathcal{Z}_1$. Then L_{1*} acts simply transitively on the Siegel II domain $\mathcal{D}_{1*} \subset ((\mathcal{S}_{1*})_c \times \mathcal{Z}_1 \times \mathbb{C})$ defined below formula (2.10) by these forms. This domain is in fact equivalent with the unit ball in $\mathbb{C}^{d_1+f_1+1}$.

We note the following (well known) description of the open S-orbits on $\mathcal{M}$. Lacking a good reference, we include the proof. Note that it follows that $E = E_\Omega$, yielding yet another notation for the base point $c_o \in \Omega$.

Proposition 2.24. *Each open ρ-orbit $\mathcal{O}$ in $\mathcal{M}$ contains a unique point of the form*

$$E_{\mathcal{O}} = \sum_1^d \epsilon_i E_i \tag{2.25}$$

where $\epsilon_i = \pm 1$.

Proof We reason by induction on the dimension d of $\mathcal{A}$. If $d = 1$, then $\mathcal{M} = \mathbb{R}$ and $S = \mathbb{R}^+$, so the result is clear.

Now suppose that the theorem is true for all ranks less than d.

Now, let $\mathcal{O} \subset \mathcal{M}$ be an open S-orbit and let $M \in \mathcal{O}$. We claim first that there is a unique $n \in N_{1*}$ such that

$$\rho(n)M = aE_1 + M_o$$

where $M_o \in \mathcal{M}_{>1}$ and $a \in \mathbb{R}$. To see this, write

$$M = aE_1 + W + M_o \tag{2.26}$$

where $a \in \mathbb{R}$, $W \in \mathcal{M}_{1*}$ and $M_o \in \mathcal{M}_{>1}$.

Let $N \in \mathcal{N}_{1*}$. Then, ad (N) maps $\mathcal{M}_{>1}$ into $\mathcal{M}_{1*}$ and $\mathcal{M}_{1*}$ into $\mathcal{M}_{11}$. Thus,

$$\begin{aligned} \rho(\exp N)M =& aE_1 + \text{ ad } (N)W + \frac{\text{ad } (N)^2}{2} M_o \\ &+ [W + \text{ ad } (N)M_o] + M_o \end{aligned} \tag{2.27}$$

where the term in brackets is the $\mathcal{M}_{1*}$ component of $\rho(\exp N)M$. We need to show that there is a unique $N \in \mathcal{N}_1$ that makes this term zero. This will be true if ad $(M_o)|\mathcal{N}_{1*}$ has rank k where $k = \dim \mathcal{M}_{1*} = \dim \mathcal{N}_{1*}$.

To show this is, note that from the following identity, the set $\mathcal{X}$ of all $X \in \mathcal{M}_{>1}$ such that rank (ad $(X)|\mathcal{N}_{1*}) = k$, is $S_{>1}$-invariant and is non-empty since it contains E_1.

$$\text{ad } (\rho(s)X) = \rho(s) \text{ ad } (X)\rho(s^{-1}).$$

Hence, $\mathcal{X}$ is a Zarski-dense, open subset of $\mathcal{M}_{>1}$ which must, therfore, intersect the image of $\mathcal{O}$ in $\mathcal{M}_{>1}$, which is just the $S_{>1}$ orbit of M_o. Our claim follows.

Thus, we may assume that W in formula (2.26) is zero. From the inductive hypothesis, there is a unique $s_1 \in S_{>1}$ such that

$$\rho(s_1)M_o = \sum_2^d \epsilon_i E_i$$

where $\epsilon_i = \pm 1$. Thus, we may assume that M_o has this form.

Finally, we note that in (2.26), $a \neq 0$ since otherwise, $[A_1, M_o] = 0$, which implies that the dimension of the S-orbit of M is less than that of $\mathcal{M}$. This allows us to transform M_o into a point of the form stipulated in the proposition using a unique element of the one-parameter subgroup generated by A_1. Our proposition follows. □

Lemma 2.28. *Let $\mathcal{O}$ be an open ρ orbit in $\mathcal{M}$ and let $E_{\mathcal{O}} \in \mathcal{O}$ be as in Proposition (2.24). Let dm denote Lebesgue measure on $\mathcal{M}$ and let ds be a fixed Haar measure on S. Then there is a constant $C_{\mathcal{O}}$ such that*

$$\int_{\mathcal{O}} f(m)\, dm = C_{\mathcal{O}} \int_S \chi_\rho(s) f(\rho(s) E_{\mathcal{O}})\, ds$$

for all integrable functions f on $\mathcal{O}$.

Proof Let $\Lambda(f)$ be the value of the quantity on the left of the above equality. Then, for all $s_o \in S$,

$$\Lambda(f \circ \rho(s_o)) = \chi_\rho(s_o^{-1})\Lambda(f).$$

The quantity on the right side of the above equality satisfies the same invariance property. It follows from the uniqueness of Haar measure that the left and right sides are equal up to a multiplicative constant that depends only on the orbit in question. We normalize ds so that this constant is 1 for Ω. □

Remark: It can be shown that $C_{\mathcal{O}}$ is independent of $\mathcal{O}$. We will not, however, need this fact.

Our main application of the above proposition will be to orbits of ρ's contragredient representation, ρ^* in $\mathcal{M}^*$. The root functionals of $\mathcal{A}$ on $\mathcal{M}^*$ are the negatives of those on $\mathcal{A}$. Hence the corresponding ordered basis for $\mathcal{A}^*$ is $-\lambda_d, -\lambda_{d-1}, \ldots, -\lambda_1$ and the corresponding ordered basis for $\mathcal{A}$ is $-A_d, -A_{d-1}, \cdots - A_1$.

We define elements $E_j^* \in \mathcal{M}^*$ by

$$< E_i, E_j^* >= \delta_{ij}\mu_i.$$

We use the element

$$E^* = \sum_j E_j^*$$

as the base point for Ω^*. (It is known that this element belongs to Ω^*.) Given an open ρ^* orbit $\mathcal{O}$, the element corresponding to $E_{\mathcal{O}}$ in Proposition (2.24) will be denoted $E_{\mathcal{O}}^*$.

If $\mathcal{L}_o$ is any vector subspace of L, we set

$$\mathcal{P}_{\mathcal{L}_o} = \text{span}_{\mathbb{C}}\{X + iJX \mid X \in \mathcal{L}_o\}$$

Then $\mathcal{P}$ splits as

$$\mathcal{P} = \mathcal{P}_{\mathcal{T}} \oplus \mathcal{P}_{\mathcal{Z}}.$$

Our first use of these constructs will be to prove the following:

Proposition 2.29. *The submanifold $\mathcal{D}_o$ is totally geodesic in $\mathcal{D}$.*

Proof Let X and Y be vector fields on $\mathcal{D}$ that are tangent to $\mathcal{D}_o$ on $\mathcal{D}_o$. To show that $\mathcal{D}_o$ is totally geodesic, it suffices to show that $\nabla_X Y$ is also tangent to $\mathcal{D}_o$. By homogeneity, it suffices to prove this at the base point b_o for left-invariant vector fields on L.

Let

$$Z = (X - iJX)/2 \text{ and } W = (Y - iJY)/2.$$

Then Z and W belong to $\mathcal{Q}$ where

$$\mathcal{Q} = \overline{\mathcal{P}}$$

Then

$$\begin{aligned} \nabla_X Y &= \nabla_{Z+\overline{Z}}(W + \overline{W}) \\ &= \nabla_Z W + \nabla_{\overline{Z}}\overline{W} + \nabla_Z \overline{W} + \nabla_{\overline{Z}} W. \end{aligned} \tag{2.30}$$

It suffices to show that each of these terms is in $\mathcal{T}_c$.

In [DHP2], we computed a formula for the connection on left-invariant vector fields on $\mathcal{D}$. To state this formula, let $\mathcal{Q}_{\mathcal{T}}$ and $\mathcal{Q}_{\mathcal{Z}}$ to be, respectively, the conjugates of $\mathcal{P}_{\mathcal{T}}$ and $\mathcal{P}_{\mathcal{Z}}$. Let π_Q be the projection to $\mathcal{Q}$ along $\mathcal{P}$. For each $Z \in \mathcal{Q}$, we define an operator $M(\overline{Z}) : \mathcal{Q} \to \mathcal{Q}$ by

$$M(\overline{Z})(W) = \pi_Q([\overline{Z}, W]).$$

We also define $M^*(Z) : \mathcal{Q} \to \mathcal{Q}$ by

$$g_c(M^*(Z)W_1, W_2) = g_c(W_1, M(\overline{Z})W_2).$$

where W_1 and W_2 range over $\mathcal{Q}$. These operators extend uniquely to operators (still denoted M and M^*) which map $\mathcal{L}_c$ into itself and satisfy

$$\begin{aligned} \overline{M(Z)W} &= M(\overline{Z})\overline{W}, \\ \overline{M^*(Z)W} &= M^*(\overline{Z})\overline{W}. \end{aligned}$$

The significance of M and M^* is that they describe the connection. Specifically, on p. 85, *loc. cit.*, we showed that for Z and W in $\mathcal{Q}$,

$$\begin{aligned} \nabla_{\overline{Z}} W &= M(\overline{Z})W, \\ \nabla_{\overline{Z}}(\overline{W}) &= -M^*(Z)\overline{W}. \end{aligned}$$

From formula (2.30), and the observation that the connection is real, the statement that $\mathcal{D}_o$ is totally geodesic will follow if we can show that for $Z \in \mathcal{Q}_{\mathcal{T}}$, $M(\overline{Z})$ and $M^*(Z)$ both map $\mathcal{Q}_{\mathcal{T}}$ into $\mathcal{Q}_{\mathcal{T}}$. The first statement follows from the fact that $\mathcal{T}_c$ is a subalgebra and the second follows from the following easily verified observations, where the orthogonal compliment is with respect to g_c in $\mathcal{Q}$.

$$\mathcal{Q}_{\mathcal{T}}^{\perp} = \mathcal{Q}_{\mathcal{Z}},$$
$$[\mathcal{Q}_{\mathcal{T}}, \mathcal{Q}_{\mathcal{Z}}] \subset \mathcal{Z}.$$

□

Next we compute the Laplace-Beltrami operator $\Delta_{\mathcal{D}}$ for $\mathcal{D}$. We choose a g-orthonormal basis X_{ij}^{α} for each $\mathcal{M}_{ij}$ and let $Y_{ij}^{\alpha} = JX_{ij}^{\alpha}$ be the corresponding orthogonal basis for $\mathcal{S}_{ij}$, where $1 \leq \alpha \leq d_{ij} = \dim(\mathcal{M}_{ij})$. We assume that this basis is chosen so that $X_{ii}^{\alpha} = \mu_i^{-1/2} E_i$. Hence $Y_{ii}^{\alpha} = \mu_i^{-1/2} A_i$.

Similarly, we choose a $\mathbb{C}$-basis X_j^{α} for $\mathcal{Z}$ where $1 \leq \alpha \leq f_j = \dim_{\mathbb{C}}(\mathcal{Z}_j)$ that is orthonormal with respect to g_{Her} and let $Y_j^{\alpha} = JX_j^{\alpha}$ so that the X_j^{α}, together with the Y_j^{α} form a real orthonormal basis for $\mathcal{Z}$.

From [O], p. 86, $\Delta_{\mathcal{D}} F$ is the contraction of $\nabla^2 F$. Hence

$$\begin{aligned}
\Delta_{\mathcal{D}} f = &- \sum_{\alpha, i \leq j} \nabla^2 f(X_{ij}^{\alpha}, X_{ij}^{\alpha}) + \nabla^2 f(Y_{ij}^{\alpha}, Y_{ij}^{\alpha}) \\
&- \sum_{\alpha, i} \nabla^2 f(X_i^{\alpha}, X_i^{\alpha}) + \nabla^2 f(Y_i^{\alpha}, Y_i^{\alpha}) \\
= &[A_o - \sum_{\alpha, i \leq j} (X_{ij}^{\alpha})^2 + (Y_{ij}^{\alpha})^2 - \sum_{\alpha, i} (X_i^{\alpha})^2 + (Y_i^{\alpha})^2] f
\end{aligned} \tag{2.31}$$

where

$$\begin{aligned}
A_o = &\sum_{\alpha, i \leq j} \nabla_{X_{ij}^{\alpha}} X_{ij}^{\alpha} + \nabla_{Y_{ij}^{\alpha}} Y_{ij}^{\alpha} \\
&+ \sum_{\alpha, i} \nabla_{X_i^{\alpha}} X_i^{\alpha} + \nabla_{Y_i^{\alpha}} Y_i^{\alpha}.
\end{aligned}$$

Lemma 2.32. *The component of $\Delta_{\mathcal{D}}$ which is tangent to A is*

$$D = \sum_i \mu_i^{-1} (A_i^2 - (1 + d_i + f_i) A_i) \tag{2.33}$$

where $d_i = \sum_{j > i} d_{ij}$

Proof It is clear from (2.31) that the second order term of Δ is as stated. To compute the first order term, we note that since Δ is formally self adjoint with respect to the Riemannian volume form, the operator in formula (2.31) must be formally self adjoint with respect to left invariant Haar measure on L. Let χ_L be the modular function for L. Then the formal adjoint of a left-invariant vector field X is

$$X^* = -X - d\chi_L(X).$$

It follows from formula (2.31) that

$$\begin{aligned}\Delta_{\mathcal{D}} &= \Delta_{\mathcal{D}}^* \\ &= \Delta_{\mathcal{D}} - 2A_o \\ &- 2\sum_{\alpha,i\le j} d\chi_L(X_{ij}^\alpha)X_{ij}^\alpha + d\chi_L(Y_{ij}^\alpha)Y_{ij}^\alpha - 2\sum_{\alpha,i} d\chi_L(X_i^\alpha)X_i^\alpha + d\chi_L(Y_i^\alpha)Y_i^\alpha.\end{aligned}$$

Note that there is no constant term since $\Delta_{\mathcal{D}}$ annihilates constants. Thus, since $d\chi_L$ is trivial on the nilradical and $Y_{ii} = \mu_i^{-1/2}A_i$, the above equality simplifies to

$$\Delta_{\mathcal{D}} = \Delta_{\mathcal{D}} - 2A_o - 2\sum_i \mu_i^{-1} d\chi_L(A_i)A_i.$$

Our lemma follows since

$$\begin{aligned}-d\chi_L(A_i) &= \text{ Tr } \text{ ad } A_i \\ &= \sum_{j<k} d_{jk}\,\frac{\lambda_j-\lambda_k}{2}(A_i) + \sum_{j\le k} d_{jk}\,\frac{\lambda_j+\lambda_k}{2}(A_i) + \sum_j 2f_j\,\frac{\lambda_j}{2}(A_i) \\ &= 1 + \sum_{j<k} d_{jk}\lambda_j(A_i) + f_i = 1 + d_i + f_i.\end{aligned}$$

□

Lemma 2.34. *Let $E_{\mathcal{P}} = JE - iE \in \mathcal{P}$. Then*

$$M(E_{\mathcal{P}})Z = \begin{matrix} Z & (Z \in \mathcal{Q}_{\mathcal{T}}) \\ \frac{Z}{2} & (Z \in \mathcal{Q}_{\mathcal{Z}}). \end{matrix}$$

Proof Let $Z \in \mathcal{Q}_{\mathcal{T}}$. Then $Z = X - iJX$ where $X \in \mathcal{S}$. Hence

$$\begin{aligned}[E_{\mathcal{P}}, Z] &= [JE - iE, X - iJX] \\ &= [JE - iE, X + iJX] - 2i[JE - iE, JX] \\ &= -2i[JE, JX] \quad \mod \mathcal{P} \\ &= -2iJX \quad \mod \mathcal{P} \\ &= (X - iJX) - (X + iJX) \quad \mod \mathcal{P} \\ &= X - iJX \quad \mod \mathcal{P}.\end{aligned}$$

Thus, $M(E_{\mathcal{P}})$ is the identity on $\mathcal{Q}_{\mathcal{T}}$.

Since $\mathcal{M}$ centralizes $\mathcal{Z}$, for $Z \in \mathcal{Q}_{\mathcal{Z}}$,

$$M(E_{\mathcal{P}})Z = [\overline{E}_{\mathcal{P}}, Z] = [JE, Z].$$

Our lemma follows from formula (2.19). □

Corollary 2.35.

$$R(E_{\mathcal{P}}, \overline{E}_{\mathcal{P}})Z = \begin{matrix} -2Z & (Z \in \mathcal{Q}_{\mathcal{T}}) \\ -Z & (Z \in \mathcal{Q}_{H}) \end{matrix}.$$

Proof This follows immediately from the following formula which is a special case of Theorem (1.9), p. 86 of [DHP2]. (Note that from the previous lemma, $M^*(\overline{E}_{\mathcal{P}}) = M(E_{\mathcal{P}})$.)

$$R(\overline{E}_{\mathcal{P}}, E_{\mathcal{P}}) = -M^*(\overline{E}_{\mathcal{P}})M(E_{\mathcal{P}}) + M(E_{\mathcal{P}})M^*(\overline{E}_{\mathcal{P}}) - M^*(M(E_{\mathcal{P}})\overline{E}_{\mathcal{P}}) - M(M(\overline{E}_{\mathcal{P}})E_{\mathcal{P}}).$$

The following result is the main step in the characterization of $\mathcal{H}^2_{\mathrm{HJK}}$.

Theorem 2.36. *The Laplace-Beltrami operator for $\mathcal{D}_o$ is a linear combination of Hua operators on $\mathcal{D}$.*

Proof Let Δ_o be the differential operator on L defined by

$$\Delta_o f = -g_c(\mathrm{HJK}\ (f)E_{\mathcal{P}}, E_{\mathcal{P}}))$$

where $E_{\mathcal{P}}$ is as above. The identity

$$g_c(R(Z, \overline{W})X, Y) = g_c(R(X, \overline{Y})Z, W)$$

shows that

$$\Delta_o f = -\sum C_{ij} \nabla^2 f(Z_i, \overline{Z}_j)$$

where

$$C_{ij} = g_c(R(E_{\mathcal{P}}, \overline{E}_{\mathcal{P}})\overline{Z}_i, \overline{Z}_j)$$

and where Z_i is an g_c-orthonormal basis of $\mathcal{P}$.

If we choose this basis so that $\{Z_1, \ldots, Z_n\} \subset \mathcal{Q}_{\mathcal{T}}$ and $\{Z_{n+1} \ldots Z_d\} \subset \mathcal{Q}_{\mathcal{Z}}$, we see that

$$\begin{aligned} \Delta_o f &= -\sum_{1}^{n} 2\nabla^2 f(Z_i, \overline{Z}_i) - \sum_{n+1}^{d} \nabla^2 f(Z_i, \overline{Z}_i) \\ &= \Delta_{\mathcal{D}_o} f + \Delta_{\mathcal{D}} f. \end{aligned}$$

(Note that from Proposition (2.29) the $\mathcal{D}_o$ connection is obtained by restriction from the $\mathcal{D}$ connection.) Hence

$$\Delta_{\mathcal{D}_o} = \Delta_o - \Delta_{\mathcal{D}}.$$

This proves the lemma since, from Proposition (1.4) of [DHP2], $\Delta_{\mathcal{D}}$ is a Hua operator, while Δ_o is, by definition, a Hua-operator. □

For later purposes, we will require an explicit description of $\Delta_{\mathcal{D}_o} - \Delta_{\mathcal{D}}$. From formulas (2.31) and (2.33) and the analogous formulas for $\Delta_{\mathcal{D}_o}$, we see that

$$\Delta_{\mathcal{D}_o} - \Delta_{\mathcal{D}} = \Delta_H - A'_o \tag{2.37}$$

where

$$\Delta_H = \sum_{\alpha,i} (X_i^\alpha)^2 + (Y_i^\alpha)^2 \tag{2.38}$$

and

$$A'_o = \sum_i \frac{f_i}{\mu_i} A_i. \tag{2.39}$$

Section 3: Hua Boundary Values

We will apply the results from Section 1 to the eigenvalue problem for the "strongly diagonal Hua operators" as defined in [DHP2] (Theorems (2.18) and (3.6).) It follows from (2.10) and (2.16) of [DHP2] that X_{ii} and Y_{ii} in [DHP2] equal what we have called E_i and A_i respectively, while $c_i = (A_i, A_i) = \mu_i$. Then X_{ij}^α and Y_{ij}^α in [DHP2] equal our $\mu_i^{1/2} X_{ij}^\alpha$ and $\mu_i^{1/2} Y_{ij}^\alpha$ respectively. The X_j^α and Y_j^α from [DHP2] correspond to our elements of the same name.

Thus, in our current notation, in the tube case the strongly diagonal Hua operators are

$$HJK_k^T = \mu_k^{-1}(\Delta_k - \frac{d_k + 2}{\mu_k} A_k - \sum_{i<k} \frac{d_{ik}}{\mu_i} A_i)$$

where $d_k = \sum_{k<j} d_{kj}$ and

$$\begin{aligned}\Delta_k = {} & 2\mu_k^{-1}(A_k^2 + E_k^2) + \\ & \sum_{i<k,\alpha} (Y_{ik}^\alpha)^2 + (X_{ik}^\alpha)^2 + \sum_{k<j,\alpha} (Y_{kj}^\alpha)^2 + (X_{kj}^\alpha)^2.\end{aligned} \tag{3.1}$$

In the general Siegel II case, the diagonal Hua operator are defined by

$$HJK_k = HJK_k^T - \frac{f_k}{\mu_k^2} A_k + \mu_k^{-1}(\sum_\alpha (X_k^\alpha)^2 + (Y_k^\alpha)^2) \tag{3.2}$$

where HJK^T_m is as in (3.1). We consider the above equalities as defining elements of $\mathfrak{A}(\mathcal{L})$ which then act as left invariant differential operators on $C^\infty(L)$.

Actually, we will need to consider these operators acting on more general spaces which are most easily described in terms of (right) induced representations. Specifically, suppose that G is a Lie group and G_o a closed subgroup. Let π_o be a differentiable representation of G_o in a Fréchet space $\mathcal{V}$. Let $C^\infty(G,\pi_o)$ be the subspace of $C^\infty(G,\mathcal{V})$ consisting of those functions F such that

$$F(g_o g) = \pi_o(g_o)F(g)$$

for all $g \in G$ and $g_o \in G_o$. We give $C^\infty(G,\mathcal{V})$ the topology of uniform convergence of functions and their derivatives on compact subsets of G and give $C^\infty(G,\pi_o)$ the subspace topology.

We define the C^∞, right-induced, representation $\pi_G^\infty = \mathrm{ind}^\infty \pi_o = \mathrm{ind}^\infty(G_o, G, \pi_o)$ of G acting on $C^\infty(G,\pi_o)$ by

$$\pi_G(g_1)F(g) = F(gg_1).$$

We make use of several simple observations which are well known and easily checked. First, suppose that G_o is normal in G and G_1 is a closed subgroup such that $G_oG_1 = G$. Then restriction defines a Fréchet space isomorphisim

$$C^\infty(G,\pi_o) \to C^\infty(G_1, \pi_o\big|G_o \cap G_1) \tag{3.3}$$

which intertwines the G_1 actions. Furthermore

$$\pi_G(g_2)F(g) = \pi_o(gg_2g^{-1})F(g)$$

for all $g_2 \in G_2$ and $g \in G$. If $X \in \mathcal{G}_2$, then

$$\pi_G(X)F(g) = \pi_o(\text{ Ad }(g)X)F(g). \tag{3.4}$$

(We typically use the same symbole to represent the representation of the Lie algebra obtained by differentiating a representation of the corresponding Lie group.)

Now, suppose that π_o is a differentiable representation of N_L on $\mathcal{V}$. We identify A with $\mathbb{R}^d$ via the mapping $t \to a(t)$ where fot $t = (t_1, \dots, t_d)$,

$$a(t) = \exp(\sum_i t_iA_i).$$

The isomorphism (3.3) then identifies $C^\infty(L,\pi_o)$ with $C^\infty(\mathbb{R}^d,\mathcal{V})$. We say that $F \in C^\infty(L,\pi_o) = C^\infty(\mathbb{R}^d,\mathcal{V})$ is diagonally Hua-harmonic if F is annihilated by the image of the strongly-diagonal Hua system under π_L.

Cases of particular interest are:

(a) π_o is the right regular representation of N_L in $\mathcal{V} = C^\infty(N_L)$. Then π_L is the right regular representation of L in $C^\infty(L)$.
(b) π_o is the right regular representation of N_L in the space of distributions $\mathcal{V} = \mathcal{D}(N_L)$ on N_L.
(c) π_o is the right regular representation of N_L in the space of Schwartz distributions $\mathcal{V} = \mathcal{S}'(N_L)$ on N_L.

The spaces $\mathcal{V}$ in (b) and (c) are particularly important. Specifically, for $F \in C^\infty(L)$, let $\tilde{F} : L \to \mathcal{D}(N_L)$ be defined by

$$< \phi, \tilde{F}(g) >= \int_{N_L} \phi(n)F(ng)\, dg$$

where $\phi \in C_c^\infty(N_L)$. Then $\tilde{F} \in C^\infty(L, \pi_o)$ where π_o is the right regular representation of N_L in $\mathcal{D}(N_L)$. Furthermore, F is diagonally Hua-harmonic if and only if $\tilde{F}$ is. From the example on .p 282 of [War], there are positive costants C and r' such that

$$e^{\tau(x)} \leq C\| \text{ Ad } (x)\|^{r'}$$

where $\|\cdot\|$ denotes the operator norm with respect to any conveniently chosen norm on $\mathcal{L}$. It follows that if F satisfies (0.1), then

$$\tilde{F}\big|A \in \mathcal{C}_r(d)(\mathcal{S}'(N_L))$$

where $\mathcal{C}_r(d)$ is as defined below formula (1.27). Similarly, if F satisfies (0.2), then

$$\tilde{F}\big|A \in \mathcal{C}_r(d)(\mathcal{D}(N_L))$$

Let

$$H_i = \frac{\mu_i^2}{2}\pi_L(HJK_i).$$

Then, according to (3.4), as an operator on $C^\infty(\mathbb{R}^d, \mathcal{V})$

$$\begin{aligned} H_i &= D_i + e^{2t_i}\pi_o(E_i^2) + e^{t_i}\pi_o(\mathcal{Z}_i) \\ &+ \sum_{j>i} e^{t_i - t_j}\pi_o(\mathcal{Y}_{ij}) + e^{t_i+t_j}\pi_o(\mathcal{X}_{ij}) \\ &+ \sum_{1\leq j<i} \frac{\mu_i}{\mu_j} e^{t_j-t_i}\pi_o(\mathcal{Y}_{ji}) + e^{t_j+t_i}\pi_o(\mathcal{X}_{ji}) \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} D_i &= \frac{\partial^2}{\partial t_i^2} - \gamma_i \frac{\partial}{\partial t_i} - \sum_{1\leq j<i} \frac{d_{ji}\mu_i}{2\mu_j}\frac{\partial}{\partial t_j} \\ \mathcal{Y}_{ij} &= \frac{\mu_i}{2}\sum_\gamma (Y_{ij}^\gamma)^2 \\ \mathcal{X}_{ij} &= \frac{\mu_i}{2}\sum_\gamma (X_{ij}^\gamma)^2 \\ \mathcal{Z}_i &= \frac{\mu_i}{2}\sum_\gamma (X_i^\gamma)^2 + (Y_i^\gamma)^2 \end{aligned}$$

and

$$\gamma_i = \frac{d_i + f_i + 2}{2}. \tag{3.6}$$

(We define $\mathcal{X}_{ij} = \mathcal{Y}_{ij} = 0$ if $(\lambda_i - \lambda_j)/2 \notin \Delta_{\mathcal{S}}$. Similarly, we set $\mathcal{Z}_i = 0$ if the space $\lambda_i/2 \notin \Delta_{\mathcal{Z}}$.)

For $i = 1, \ldots, d$ let $\rho_i \geq 0$ and $G^i \in \mathcal{C}_r(d)$ be given. We are interested in studying the system

$$H_i F = \rho_i F + G^i \qquad i = 1, 2, \ldots, d \tag{3.7}$$

for $F \in \mathcal{C}_r(d)$.

Let notation be as in (2.23). From the comments following (2.23), $L_{>1}$ may be identified with a Siegel domain. Let HJK $_{>1}$ be the corresponding Hua system for $L_{>1}$ and

$$H_i^o = \frac{\mu_i^2}{2} \pi_G(\text{HJK }_{>1})_{i-1}$$

where $i \geq 2$ and we embed $\mathfrak{A}(\mathcal{L}_{>1})$ into $\mathfrak{A}(\mathcal{L})$ in the obvious manner. Formulas (3.1) and (3.2) imply

$$H_i = H_i^o - \delta_i \frac{\partial}{\partial t_1} + e^{t_1 - t_i} \pi_o(\mathcal{Y}_{1i}) + e^{t_1 + t_i} \pi_o(\mathcal{X}_{1i}). \tag{3.8}$$

Our main result is:

Theorem 3.9. *Let $F \in \mathcal{C}_r(d)$ satisfy (3.7) where the G^i have a $\mathcal{V}$ valued asymptotic expansion over $(-\infty, 0]^d$. Then F has an asymptotic expansion over $(-\infty, 0]^d$.*

Proof Let $A_0 \in A$ be the subgroup defined by $t_1 = 0$ and let A_1 be defined by $t_i = 0$ for all $i > 1$. Let $L_1 = A_0 N_L$ and define

$$\pi_1 = \text{ind}^\infty(N_L, L_1, \pi_o)$$

realized in $\mathcal{W} = C^\infty(\mathbb{R}^{d-1}, \mathcal{V})$. Then

$$\pi_L = \text{ind}^\infty(L_1, L, \pi_1) \tag{3.10}$$

which we realize in $C^\infty(A_1, \mathcal{W}) = C^\infty(\mathbb{R}, \mathcal{W})$ using the correspondence (3.3). Thus, F and G^1 correspond to the elements $\tilde{F}$ and $\tilde{G}^1$ in $C^\infty(\mathbb{R}, \mathcal{W})$ defined as in formula (1.27). Actually, $\tilde{F}$ and $\tilde{G}$ are valued in $\mathcal{C}_s(d-1)$ for some s. Let

$$\mathcal{C}_\infty(d-1) = \cup_{k=0}^\infty \mathcal{C}_{-k}(d-1)$$

given the direct limit topology. It is clear from formula (3.4) that for all $X \in \mathcal{L}_1$, $\pi_1(X)$ acts continuously on $\mathcal{C}_\infty(d-1)$. From Definition (1.28), $\tilde{G}^1$ has an asymptotic expansion as an $\mathcal{C}_\infty(d-1)$ valued map.

Equation (3.7), with $i = 1$, is equivalent with the $\mathcal{C}_\infty(d-1)$ valued ordinary differential equation $D\tilde{F} = \tilde{G}^1$ where

$$D = \frac{d^2}{dt_1^2} - \gamma_1 \frac{d}{dt_1} + e^{t_1} P_1 + e^{2t_1} P_2 - \rho_1, \tag{3.11}$$

and

$$P_1 = \pi_1(\mathcal{Z}_1 + \sum_{1<j\le d_1} \mathcal{Y}_{1j} + \mathcal{X}_{1j}),$$
$$P_2 = \pi_1(E_1^2).$$

Lemma 3.12. *$\tilde{F}' \in \mathcal{C}_s((-\infty, 0], \mathcal{C}_\infty(d-1))$ for some s.*

Proof Let

$$H(t) = e^{-\gamma_1 t}\tilde{F}'(t).$$

Then

$$\begin{aligned} H'(t) &= e^{-\gamma_1 t}(\tilde{F}''(t) - \gamma_1 \tilde{F}'(t)) \\ &= e^{-\gamma_1 t}\tilde{G}^1(t) - e^{-\gamma_1 t}\left(e^t P_1 + e^{2t}P_2 - \rho_1\right)\tilde{F}. \end{aligned}$$

Hence

$$\begin{aligned} H(t) = H(0) &- \int_0^t e^{-\gamma_1 s}\tilde{G}^1(s)\,ds \\ &- \int_0^t \left(e^{(1-\gamma_1)s}P_1 + e^{(2-\gamma_1)s}P_2 - \rho_1 e^{-\gamma_1 s}\right)\tilde{F}(s)\,ds. \end{aligned}$$

Let ρ be any continuous semi-norm on $\mathcal{C}_\infty(d-1)$. Applying the triangle inequality for ρ to the preceding inequality, and using the continuity of the P_i on $\mathcal{C}_\infty(d-1)$ together with $\tilde{F} \in \mathcal{C}_r((-\infty, 0], \mathcal{C}_\infty(d-1))$, we see that $H \in \mathcal{C}_s((-\infty, 0], \mathcal{C}_\infty(d-1))$ for some s. □

The equation $D\tilde{F} = \tilde{G}^1$ is equivalent with the $\mathcal{C}_\infty(d-1)\times\mathcal{C}_\infty(d-1)$ valued first order system

$$\frac{dY}{dt_1} = M_0 Y + e^{t_1}M_1 Y + e^{2t_1}M_2 Y + Z \tag{3.13}$$

where

$$Y = \begin{bmatrix} \tilde{F} \\ \tilde{F}' \end{bmatrix},$$
$$M_0 = \begin{bmatrix} 0 & 1 \\ \rho_1 & \gamma_1 \end{bmatrix},$$
$$M_1 = \begin{bmatrix} 0 & 0 \\ -P_1 & 0 \end{bmatrix},$$
$$M_2 = \begin{bmatrix} 0 & 0 \\ -P_2 & 0 \end{bmatrix},$$
$$Z = \begin{bmatrix} 0 \\ \tilde{G}^1 \end{bmatrix}.$$

Also Z has an expansion since $\tilde{G}^1$ does.

Theorem (1.12), along with Lemma (3.12), implies that Y has an asymptotic expansion. Projection onto the first component shows that $\tilde{F}$ has an asymptotic expansion. Let

$$\begin{aligned} \tilde{F}(t_1) &\sim \sum_{\alpha\in I_1} e^{\alpha t_1}\tilde{F}_\alpha(t_1),\\ \tilde{G}_1(t_1) &\sim \sum_{\alpha\in I_1} e^{\alpha t_1}\tilde{G}^1_\alpha(t_1). \end{aligned} \tag{3.14}$$

For $i>1$

$$H_i = -\delta_i\frac{\partial}{\partial t_1} + e^{t_1}Q_i + H_i^o \tag{3.15}$$

where $\delta_i = \mu_i d_{1i}/(2\mu_1)$ and

$$Q_i = \pi_1(\mathcal{Y}_{1i}) + \pi_1(\mathcal{X}_{1i}).$$

Applying H_i term-by-term to (3.14) shows that for each $\alpha\in I_1$,

$$(-\delta_i\frac{d}{dt_1} - \delta_i\alpha + H_i^o - \rho_i)\tilde{F}_\alpha = -Q_i\tilde{F}_{\alpha-1} + \tilde{G}^1_\alpha. \tag{3.16}$$

Write

$$\begin{aligned} \tilde{F}_\alpha(t_1) &= \sum_0^{n_\alpha}\tilde{F}_{\alpha,n}t_1^n,\\ \tilde{G}^1_\alpha(t_1) &= \sum_0^{n_\alpha}\tilde{G}^1_{\alpha,n}t_1^n. \end{aligned}$$

Then

$$(H_i^o - \delta_i\alpha - \rho_i)\tilde{F}_{\alpha,n} = n\delta_i\tilde{F}_{\alpha,n+1} - Q_i\tilde{F}_{\alpha-1,n} + \tilde{G}^1_{\alpha,n}. \tag{3.17}$$

In particular,

$$(H_i^o - \delta_i\alpha - \rho_i)\tilde{F}_{\alpha,0} = -Q_i\tilde{F}_{\alpha-1,0} + \tilde{G}^1_{\alpha,0}. \tag{3.18}$$

We will show that each of the $\tilde{F}_{\alpha,k}$ has an asymptotic expansion. If α is any exponent, then there is an $n\in\mathbb{N}_o$ such that $\alpha_o=\alpha-n$ is an exponent, but α_o-k is not for any $k\in\mathbb{N}_o$. In particular, $\tilde{F}_{\alpha_o-1,0}=0$. Hence, from (3.18), $\tilde{F}_{\alpha_o,0}$ satisfies the Hua system on $L_{>1}$ relative to the eigenvalues $\delta_i\alpha_o+\rho_i$. Since $\tilde{F}_{\alpha_o,0}\in\mathcal{C}_\infty(d-1)$, it belongs to $\mathcal{C}_s(d-1)$ for some s. Hence we may assume by induction that $\tilde{F}_{\alpha_o,0}$ has an asymptotic expansion over $(-\infty,0]^{d-1}$ with exponents from some set $I(\alpha_o)\subset\mathbb{C}^{d-1}$. If $\delta_i\neq 0$ for some i, we may solve formula (3.17) for $\tilde{F}_{\alpha_o,n+1}$, concluding, by induction, that $\tilde{F}_{\alpha_o,k}$ has an asymptotic expansion. If all of the $\delta_i=0$, then the existence of an asymptotic expansion for $\tilde{F}_{\alpha_o,k}$ follows as in the $k=0$ case. Hence, $\tilde{F}_{\alpha_o}$ also has such an expansion.

It now follows from formula (3.18) and induction on k, that for all $k\in\mathbb{N}_o$, $\tilde{F}_{\alpha_o+k}$ has an asymptotic expansion, proving our theorem. □

Our next goal is to define the boundary values of a solution. *For the remainder of this section we assume that F satisfies the hypotheses of Theorem (3.9) where all of the $G^i = 0$.*

Let $\mathcal{E} \subset \mathcal{A}^*$ be the set of exponents for F so that

$$F(t) \approx \sum F_\alpha(t) e^{<t,\alpha>} \qquad \alpha \in \mathcal{E} \tag{3.19}$$

where the F_α are non-zero, $\mathcal{V}$ valued polynomial functions on $A = \mathbb{R}^d$.

Given a constant coefficient differential operator D on $C^\infty(A)$, we define a polynomial (the *characteristic polynomial*) on $\mathcal{A}^*$ by

$$D(e^{<t,\alpha>}) = p_D(\alpha) e^{<t,\alpha>}.$$

Let $p_i = p_{D_i}$. Then for

$$\alpha = \sum \alpha_i \lambda_i,$$

$$p_i(\alpha) = \alpha_i^2 - \gamma_i \alpha_i - \rho_i - \sum_{1 \le j < i} \frac{d_{ji}\mu_i}{2\mu_j} \alpha_j. \tag{3.20}$$

Let

$$\mathcal{E}_0 = \{\alpha \mid p_i(\alpha) = 0, i = 1, \ldots, d\}.$$

Notice that p_i depends only on α_j, $j \le i$. It follows that we may compute the elements of $\mathcal{E}_0$ inductively. Specifically, we compute the α_{i+1} by solving the equation

$$p_{i+1}(\alpha_{i+1}\lambda_{i+1} + \sum_1^i \alpha_i \lambda_i) = 0$$

where the terms in the summation range over the (known) roots of $p_1, \ldots, p_i$. In particular, $\mathcal{E}_o$ has at most 2^d elements.

Let $\mathcal{P}(\mathbb{R}^d, \mathcal{V})$ be the space of $\mathcal{V}$ valued polynomials on $\mathbb{R}^d$.

Definition 3.21: *The boundary value map for F is the function $BV : \mathcal{E}_o \to \mathcal{P}(\mathbb{R}^d, \mathcal{V})$ defined by $BV(F)(\alpha) = F_\alpha$.*

Remark: The above definition is not entirely consistent with Definition (1.26) where the boundary map is valued in $\mathcal{V}$ rather than $\mathcal{P}(\mathbb{R}^d, \mathcal{V})$. Note, however, that when we convert an n^{th} order equation to a first order system, our boundary map will in fact be valued in $\mathcal{V}^n$. Specifically, if F solves an n^{th} order equation, then its α^{th} boundary value is the element of $\mathcal{V}^n$ whose k^{th} component is $\frac{d^k}{dt^k}(e^{\alpha t} F_\alpha)(0)$. Thus, the real difference between (1.26) and (3.21) the number of terms of $F_\alpha(t)$ utilized. Of course, if $F_\alpha(t)$ has degree 0, which is the generic case, there is essentially no difference.

Our goal is to prove that F is uniquely determined by $BV(F)$. We first note the following lemma.

Lemma 3.22. *Suppose that D is a constant coefficient differential operator on $C^\infty(\mathbb{R}^d)$ which does not annihilate constants. Then D is injective on the space of polynomial functions on $\mathbb{R}^d$.*

Proof This is a simple consequence of the observation that for any homogeneous polynomial P of degree d

$$D(P) = D(1)P + \text{terms of lower degree.}$$

We leave the details to the reader. □

Let

$$\Delta = \text{span}_{\,2\mathbb{Z}}(\Delta_{\mathcal{S}} \cup \Delta_{\mathcal{M}} \cup \Delta_{\mathcal{Z}}),$$
$$\Delta^+ = \text{span}_{\,2\mathbb{N}_o}(\Delta_{\mathcal{S}} \cup \Delta_{\mathcal{M}} \cup \Delta_{\mathcal{Z}}).$$

where $\Delta.$ is as described below (2.15).

The following proposition proves that F is uniquely determined by its boundary values.

Proposition 3.23. $\mathcal{E} \subset \mathcal{E}_o + \Delta^+$. *Also* $F = 0$ *if and only if* $BV(F) = 0$.

Proof It follows from Proposition (1.20) and the proof of Theorem (3.9) that (3.19) may be differentiated term-by-term. Applying the Hua system to (3.19) yields the equality

$$\begin{aligned} D_i^\alpha F_\alpha = & -\pi_o(E_i^2)F_{\alpha-2\lambda_i} - \pi_o(\mathcal{Z}_i)F_{\alpha-\lambda_i} \\ & - \sum_{j>i} \pi_o(\mathcal{Y}_{ij})F_{\alpha-(\lambda_i-\lambda_j)} + \pi_o(\mathcal{X}_{ij})F_{\alpha-(\lambda_i+\lambda_j)} \\ & - \sum_{1\le j<i} \frac{\mu_i}{\mu_j}(\pi_o(\mathcal{Y}_{ji})F_{\alpha-(\lambda_j-\lambda_i)} + \pi_o(\mathcal{X}_{ij})F_{\alpha-(\lambda_i+\lambda_j)}) \end{aligned} \tag{3.24}$$

where

$$D_i^\alpha F = e^{-<t,\alpha>} D_i(e^{<t,\alpha>}F).$$

Note that (3.24) expresses $D_i^\alpha F_\alpha$ as a linear combination of terms $F_{\alpha-\beta}$ with $\beta \in \Delta^+$. Lemma (1.31) shows that there is a $\beta \in \Delta^+$ with the property that $\alpha' = \alpha - \beta \in \mathcal{E}$ but $\alpha' - \gamma \notin \mathcal{E}$ for any $\gamma \in \Delta^+$. Hence, from (3.24),

$$D_i^{\alpha'} F_{\alpha'} = 0$$

for all i. It follows from Lemma (3.22) that if $\alpha' \notin \mathcal{E}_o$, $D_i^{\alpha'}$ is injective on the space of polynomials contradicting $\alpha' \in \mathcal{E}$; hence $\alpha' \in \mathcal{E}_o$, proving $\mathcal{E} \subset \mathcal{E}_o + \Delta^+$.

The preceding argument shows that if $\mathcal{E}$ is non-empty, then $\mathcal{E} \cap \mathcal{E}_o$ is also non-empty. Hence, if $F_\alpha = 0$ for all $\alpha \in \mathcal{E}_o$, then $F_\alpha = 0$ for all α. We must show that then $F = 0$.

Rank 1 Case

For $\omega \in \mathcal{V}^*$ and $g \in L$, let

$$F_\omega(g) =< F(g), \omega > . \tag{3.25}$$

Then F_ω is a $\mathbb{C}$-valued Hua-harmonic function. It suffices to show that $F_\omega = 0$ for all $\omega \in \mathcal{V}^*$. Thus it suffices to consider scalar valued solutions.

Let $G : N_L \times \mathbb{R}^+ \to \mathbb{C}$ be defined by

$$G(n,t) = \begin{matrix} F(n \exp((\log t)A_1)) & t > 0 \\ 0 \quad t \leq 0 & \end{matrix} .$$

Then G vanishes to infinite order at 0, showing that G is C^∞ on $N_L \times \mathbb{R}$. We apply Theorem 2 of [BG] with

$$\mathcal{P} = H_1 - \rho_1,$$

$m = k = 2$, $p = 0$. Comparison with equation 1 in [BG] shows that the hypotheses of [BG] are met. It follows, then, that G is zero on a neighborhood of e in $N_L \times \mathbb{R}$. Since $\mathcal{P}$ is analytic-hypoelliptic, it follows that F is zero, proving our result in the rank one case.

Rank d Case

We assume by induction that the result is known for all lower ranks. We repeat the discussion leading up to (3.14). Let α_o be a leading exponent for $\tilde{F}$. Then, as before, $\tilde{F}_{\alpha_o,0}$ satisfies the Hua system on $L_{>1}$ relative to the eigenvalues $\delta_i\alpha_o + \rho_i$. The set of roots of the correspondinng characteristic polynomials are

$$\mathcal{E}_o' = \{(\alpha_2, \dots, \alpha_d) \in \mathbb{C}^{d-1} \mid (\alpha_o, \alpha_2, \dots, \alpha_d) \in \mathcal{E}\}.$$

and the boundary value map is

$$BV'(F_{\alpha_o,0})(\alpha_2, \dots, \alpha_d)(t_2, \dots, t_d) = F_\alpha(0, t_2, \dots, t_d) \tag{3.26}$$

where $\alpha = (\alpha_o, \alpha_2, \dots, \alpha_d)$.

Then, $BV(F) = 0$ implies $BV'(F_{\alpha_o,0}) = 0$; hence, from the inductive hypothesis, $\tilde{F}_{\alpha_o,0} = 0$. If any one of the $\delta_i \neq 0$, we can iterate formula (3.17) to show that $\tilde{F}_{\alpha_o} = 0$. If all of the $\delta_i = 0$, then (3.16) shows that $\tilde{F}_{\alpha_o}(t_1)$ satisfies the Hua system on $L_{>1}$ for all $t_1 \in \mathbb{R}$. Also

$$BV'(F_{\alpha_o}(t_1))(\alpha_2, \dots, \alpha_d)(t_2, \dots, t_d) = e^{\alpha_o t_1} F_\alpha(t_1, t_2, \dots, t_d) \tag{3.27}$$

which implies once again that $\tilde{F}_{\alpha_o} = 0$.

Hence, there are no leading terms in the (one variable) asymptotic expansion of $\tilde{F}$, showing that $\tilde{F}$ is asymptotic to 0. To see that F itself is zero, notice that $H_1 \in \mathfrak{A}(\mathcal{L}_{1*})$. From (3.3) and (3.10)

$$\pi_L \big| L_{1*} = \mathrm{ind}^\infty(N_{1*}, L_{1*}, \pi_1 \big| N_{1*}).$$

Our argument is finished by repeating the $d = 1$ argument using $\mathcal{P} = H_1 - \rho_1$ and $\mathcal{V} = \mathcal{H}(\pi_1)$. $\square$

Remark: The proof of Proposition (3.23) allows us, in principal, to construct a mapping (the Poisson transformation) for which $F = P(BV(F))$. Specifically, we assume that the Poisson transformation is known for all ranks less than d. This allows us to construct $\tilde{F}_{\alpha_o,0}$ using (3.26). If at least one $\delta_i \neq 0$, we then use (3.17) to construct $\tilde{F}_{\alpha_o}$. If all of the $\delta_i = 0$, then we use (3.27) to construct F_{α_o}. Thus, we need only know the Poisson transformation for the single equation

$$(H_1 - \rho_1)F = 0.$$

Notice that $H_1 \in \mathfrak{A}(\mathcal{L}_{1*})$. Reasoning as in the proof of Proposition (3.23), it suffices to consider H_i acting on $C^\infty(L_{1*})$. As noted below formula (2.23), L_{1*} acts simply transitively on the unit ball $\mathcal{B}$ in $\mathbb{C}^{d_1+f_1+1}$. Formula (huaeigen) shows that

$$H_1 = \frac{\mu_1^2}{2} \mathrm{HJK}\ _1$$

where HJK $_1$ is the first diagonal Hua operator *for the unit ball.* In [P1], we defined an explicit integral transformation (the N-transformation) which transforms this operator into the image of the Casmir operator of $\mathrm{Sl}(2,\mathbb{R})$ acting in the representation space of a certain unitary representation of the universal covering group $\tilde{\mathrm{Sl}}(2,\mathbb{R})$. (See formula 24, *loc. cit.*.) We also computed a general formula for the Poisson kernel for this operator. Our formula assumed that one avoids certain "singular" eigenvalues, but these assumptions are unnecessary since the Casmir operator on $\tilde{Sl}(2,\mathbb{R})$ is well understood.

From this point on we make the additional assumption that all of the $\rho_i = 0$.

In this case $0 \in \mathcal{E}_o$. The element F_0 is the boundary value studied in [DHP]. The following theorem generalizes one of the main results of [DHP2] to the case of unbounded solutions.

Theorem 3.28. *For all* $1 \leq i < j \leq d$,

$$\pi_o(\mathcal{Y}_{ij})F_0 = 0.$$

In particular, if $\pi_o(N_S)F_0$ *is a bounded subset of* $\mathcal{V}$ *then* $\pi_o(n)F_0 = F_0$ *for all* $n \in N_S$.

The second statement follows from the first: we note first that by an argument similar to that done in the proof of Proposition (3.23), we may assume that $\mathcal{V} = C^\infty(N_S)$. Then [DH] implies that all bounded solutions to

$$\sum_{i<j} \mathcal{Y}_{ij} F = 0$$

are constant on left cosets of N_S, as desired.

For the proof of the first statement, we will do a detailed analysis of F's asymptotic expansion. We prove somewhat more than required due to the needs of the next section. Let

$$\begin{aligned} \beta_i &= \lambda_i - \lambda_{i+1} \quad i < d, \\ \beta_d &= \lambda_d. \end{aligned} \tag{3.29}$$

Every element of Δ^+ is a linear combination, with positive coefficients, of the basis defined by the β_i. Specifically

$$\begin{aligned} \lambda_i - \lambda_j &= \beta_i + \beta_{i+1} + \cdots + \beta_{j-1}, \\ \lambda_i + \lambda_j &= \beta_i + \beta_{i+1} + \cdots + \beta_{j-1} + 2\beta_j + \cdots + 2\beta_d. \end{aligned} \tag{3.30}$$

Let

$$\Lambda = \mathcal{E}_o \cap \operatorname{span}_{\mathbb{R}}\{\lambda_i \mid d_\tau \leq i \leq d\}$$

where τ is as in Definition (2.21). For $d_\tau \leq i \leq d$, p_i depends only on the i^{th} variable and those with index less than d_τ. Thus, if $\alpha \in \Lambda$,

$$0 = p_i(\alpha) = \alpha_i^2 - \alpha_i \gamma_i = \alpha_i(\alpha_i - \gamma_i).$$

Hence,

$$\Lambda = \{\sum \alpha_i \lambda_i \mid \alpha_i \in \{0, \gamma_i\}, d_\tau \leq i \leq d\}. \tag{3.31}$$

Lemma 3.32. *Let $\beta = \nu_1\beta_1 + \cdots + \nu_d\beta_d$ belong to $\mathcal{E}$ where the $\nu_i \in \mathbb{C}$ are such that $\gamma_i - \nu_i \notin -\mathbb{N}_o$, $1 \leq i < d_\tau$. Then $\beta \in \Lambda + \Delta^+$.*

Proof From Proposition (3.23), $\beta - \gamma \in \mathcal{E}_o$ for some $\gamma \in \Delta^+$. We replace β with $\beta - \gamma$, which still satisifies our hypotheses. It suffices to show that $\nu_i = 0$ for $1 \leq i < d_\tau$. If not, let ν_i be the first non-zero coefficient. Since p_i depends only on the first i variables

$$0 = p_i(\beta) = \nu_i^2 - \nu_i\gamma_i = \nu_i(\nu_i - \gamma_i). \tag{3.33}$$

Hence, $\nu_i = \gamma_i$, which contradicts $\gamma_i - \nu_i \notin -\mathbb{N}_o$, proving our lemma. □

A similar argument proves the following.

Corollary 3.34. *If $F_0 \neq 0$, then 0 is the minimal element of $\mathcal{E}$ in the sense defined above (1.30). Furthermore, F_0 is independent of t.*

Proof Let

$$\beta = \nu_1\beta_1 + \cdots + \nu_d\beta_d$$

belong to $\mathcal{E}$. As in the proof of Lemma (3.32) we may assume that $\beta \in \mathcal{E}_o$. Let k be the first index such that $\nu_k \neq 0$. As in the proof of smaller, $\nu_k = \gamma_k > 0$, proving minimality.

The independence of t follows from induction as in the proof of (3.9) together with the comments immediately preceding Proposition (1.25). □

Theorem (3.28) follows immediately from the following result.

Proposition 3.35. *Let $i < j < l$ and $\alpha = n_l\beta_l + \cdots + n_d\beta_d$ where $n_i \in \mathbb{N}_o$ and $n_j \leq 1$ for $j < d_\tau$. Then*

$$\pi_o(\mathcal{Y}_{ij})F_\alpha = 0 = F_{\lambda_i - \lambda_j + \alpha}.$$

If $\alpha = 0$, the above holds for all $1 \leq i < j \leq d$.

Proof Let

$$\epsilon = \lambda_i - \lambda_j + \alpha.$$

From formula (3.6), $\gamma_i > 1$ for $1 \leq i < d_\tau$. Hence the assumptions of Lemma (3.32) apply to $\epsilon - \gamma$ for any $\gamma \in \Delta^+$.

Case 1: $d_\tau \leq i$

Then $(\lambda_i - \lambda_j)/2$ is not a root. Hence $\mathcal{Y}_{ij} = 0$ and the first equality follows. Since $\epsilon \notin \Delta^+ + \Lambda$, Lemma (3.32) shows that $F_\epsilon = 0$ as well, proving our theorem in this case.

Case 2: $i < d_\tau$**,** $j = i + 1$

Then

$$\epsilon = \beta_i + \alpha$$

and the expansion of ϵ in the basis (3.29) contains no β_{i+1} component. It follows from Lemma (3.32) and (3.30) that for $i < m$, $\epsilon - (\lambda_i \pm \lambda_m) \notin \mathcal{E}$ unless $m = i+1$ and $\pm = -$ while for $m < i$, $\epsilon - (\lambda_m \pm \lambda_i) \notin \mathcal{E}$. It is clear also that $\epsilon - m\lambda_i \notin \mathcal{E}$ for $m > 0$. Hence (3.24), with α replaced by ϵ, reduces to a single term implying

$$D_i^\epsilon F_\epsilon = -\pi_o(\mathcal{Y}_{ij})F_\alpha.$$

Similarly, (3.24) reduces to a single term with i replaced by $j = i + 1$ implying

$$D_j^\epsilon F_\epsilon = -\frac{\mu_j}{\mu_i}\pi_o(\mathcal{Y}_{ij})F_\alpha.$$

Hence

$$D_j^\epsilon F_\epsilon = \frac{\mu_j}{\mu_i} D_i^\epsilon F_\epsilon$$

which is equivalent with

$$(D_j - \frac{\mu_j}{\mu_i} D_i)(e^{<t,\epsilon>} F_\epsilon) = 0. \tag{3.36}$$

From Lemma (3.22), for F_ϵ to be non-zero, ϵ must be a root of the characteristic polynomial. Hence

$$p_j(\epsilon) = \frac{\mu_j}{\mu_i} p_i(\epsilon). \tag{3.37}$$

From formula (3.20) and $j = i + 1$

$$p_i(\lambda_i - \lambda_j + \alpha) = 1 - \gamma_i,$$
$$p_j(\lambda_i - \lambda_j + \alpha) = 1 + \gamma_j - \frac{d_{ij}\mu_j}{2\mu_i}.$$

Substitution into (3.37) shows that if $F_\epsilon \neq 0$ then

$$\mu_i^{-1}(1 - \gamma_i + \frac{d_{ij}}{2}) = \mu_j^{-1}(1 + \gamma_j). \tag{3.38}$$

However, from (3.6) the term on the left is non-positive and that on the right is positive. This proves our proposition in this case.

General Case:

Now suppose by induction that

$$\pi_o(\mathcal{Y}_{lm}) F_\alpha = 0 = F_{\lambda_l - \lambda_m + \alpha}$$

for all l and m such that $0 < m - l < j - i$. Then

$$\epsilon - (\lambda_i - \lambda_k) = \lambda_k - \lambda_j + \alpha$$

which, for $i < k < j$ is not an exponent due to the inductive hypothesis. For $j < k$, this term is not an exponent due to Lemma (3.32). Lemma (3.32) also shows that none of $\epsilon - \lambda_i$, $\epsilon - 2\lambda_i$ and $\epsilon - (\lambda_i + \lambda_j)$ are exponents. Thus, (3.24) implies

$$D_i^\alpha F_\epsilon = -\mathcal{Y}_{ij} F_\alpha. \tag{3.39}$$

Now we apply (3.24) with α replaced by ϵ and i replaced by j. Then for $m < j$

$$\epsilon - (\lambda_m - \lambda_j) = \lambda_i - \lambda_m + \alpha$$

which is not an exponent for $m \neq i$ due to Lemma (3.32) $(m < i)$ and the inductive hypothesis $(i < m)$.

For $j \leq m$

$$\epsilon - (\lambda_j - \lambda_m) = \lambda_i - 2\lambda_j + \lambda_m + \alpha$$

which is not an exponent due to Lemma (3.32). Lemma (3.32) also shows that none of $\epsilon - \lambda_j$, $\epsilon - 2\lambda_j$ and $\epsilon - (\lambda_j + \lambda_m)$ are exponents.

Thus

$$D_j^\epsilon F_\epsilon = -\frac{\mu_j}{\mu_i}\mathcal{Y}_{ij}F_\alpha. \tag{3.40}$$

Our result follows just as in the $j = i + 1$ case. □

We can now recover the Johnson-Korányi result:

Theorem 3.41. *Suppose that $\mathcal{D} = G/K$ is a symmetric, tube domain. Then every Hua-harmonic function F on G/K is the Poisson integral of a hyperfunction over the Shilov boundary.*

Proof Our proof is based on the argument beginning at the top of p. 4 of [BV]. Specifically, we write F as a limit of left K-finite functions F_k on G/K. Since the Hua system is invariant, each of the F_k is Hua-harmonic. The F_k are Poisson integrals of K-finite functions f_k over the Furstenberg boundary where the f_k converge to a hyperfunction f whose Poisson integral is F. Since the f_k are continuous on K, they are bounded. It follows from (3.34) and (1.30) that $f_k = (F_k)_0$. Then Theorem (3.28) shows that $\pi_o(N_S)f_k = f_k$. The same must therefore be true of f, showing that f projects to the Shilov boundary, as desired. □

Remark: The same argument shows that the results of [DHP2] imply the Johnson-Korányi result.

Corollary 3.42. *Let*

$$\beta = \beta_{i_1} + \beta_{i_2} + \cdots + \beta_{i_k} + n_{k+1}\lambda_{i_{k+1}} + \cdots + n_d\lambda_d$$

where $1 \le i_1 < i_2 < \cdots < i_m = d$ *and* $d_\tau \le i_{k+1}$. *Then* $\beta \notin \mathcal{E}$ *unless* $i_j = i_1 + j - 1$ *for all* $1 \le j \le k+1$, *in which case*

$$\beta = \lambda_{i_1} + (n_{k+1} - 1)\lambda_{i_{k+1}} + n_{k+2}\lambda_{i_{k+2}} + \cdots + n_d\lambda_d.$$

Proof Let $j \le k+1$ be maximal with respect to $i_l = i_1 + l - 1$ for all $1 \le l \le j$. If $j \le k$, then

$$\beta = \lambda_{i_1} - \lambda_{i_1+j} + (\beta_{i_{j+1}} + \beta_{i_{j+2}} + \cdots + \beta_{i_k} + n_{k+1}\lambda_{i_{k+1}} + \cdots + n_d\lambda_d)$$

where $i_{j+1} > i_1 + j$. Proposition (3.35), with α equal to the term in parentheses, proves that $F_\beta = 0$. Hence, $j = k+1$, proving our corollary. □

Section 4: The Boundary Representation

In this section we collect a number of representation theoretic facts which we need. Our basic reference is [War].

In Section 3 we discussed right-induced C^∞ representations. In this section we need left-induced unitary representations. Let G be a Lie group, G_o a closed subgroup, and let π be a continuous unitary representation of G_o in a Hilbert space $\mathcal{H}(\pi)$, which we denote simply by $\mathcal{H}$.

We define a character χ on G_o by

$$\chi(h) = (\chi_{G_o}/\chi_G)(h)$$

where χ_G and χ_{G_o} are, respectively, the modular functions for left-invariant Haar measure on G and G_o.

The representation $\operatorname{ind}(\pi)$ of G induced from π acts in a subspace space $\mathcal{H}(\operatorname{ind}(\pi))$ of $\mathcal{H}$-valued functions on G which satisfy

$$f(gh) = \chi^{1/2}(h)\pi(h^{-1})f(g) \tag{4.1}$$

for all $g \in G$ and $h \in G_o$. For such f,

$$\|f(gh)\|_{\mathcal{H}} = \chi^{1/2}(h)\|f(g)\|_{\mathcal{H}}.$$

It is well know that there is a unique G invariant functional I defined on the set of continuous, compactly supported modulo G_o, functions on G satisfying the above covariance condition. Then $\mathcal{H}(\operatorname{ind}(\pi))$ is the completion of set of functions for which $\|f\| = I(\|f\|_{\mathcal{H}}) < \infty$.

The representation acts on such functions according to

$$\operatorname{ind}(\pi)(g_o)f(g) = f(g_o^{-1}g).$$

When we wish to explicitly indicate the dependence on G and G_o we will write $\operatorname{ind}(G_o, G, \pi)$ instead of $\operatorname{ind}(\pi)$.

If there is a closed subgroup G_1 of G which is a complement to G_o then,

$$\|f\|^2 = \int_{G_1} \|f\|_{\mathcal{H}}(t)\,dt$$

where dt is left invariant Haar measure on G_1. Hence $\mathcal{H}(\operatorname{ind}(\pi))$ is just $L^2(G_1, dt, \mathcal{H}(\pi))$.

Recall that if π is a continuous representation of G in a Hilbert space $\mathcal{H}$, then $C^\infty(\pi)$ denotes the set of vectors $\mathcal{H}$ for which $g \to \pi(g)v$ is differentiable as a $\mathcal{H}$ valued map, given the topology of uniform convergence on compact subsets of G of such functions and all of their derivatives. We let $C^{-\infty}(\pi)$ denote the anti-dual space to $C^\infty(\pi)$. (i.e. the space of continuous *conjugate-linear* functionals.) We use the scalar product to embed $\mathcal{H}$ linearly into $C^{-\infty}(\pi)$. The contragrediant representation to $\pi\big|C^\infty(\pi)$ defines a continuous (in fact *differentiable*) extension of

π to $C^{-\infty}(\pi)$ which we continue to denote by π. The representation of the universal enveloping algebra $\mathfrak{A}(\mathcal{G})$ on $C^{-\infty}(\pi)$ obtained by differentiating π is denoted by π as well.

Let $\pi_G = \mathrm{ind}(\{e\}, G, 1)$, the unitary left regular representation of G. It is well known that $C^\infty(\pi_G) \subset C^\infty(G)$. We require the following result which, while probably well known, we have not been ale to find in the literature.

Proposition 4.2. *If G is unimodular, then $C^\infty(\pi_G) \subset L^\infty(G)$.*

Proof Let X_1, X_2, $\ldots$, X_n be a basis for the Lie algebra of G and let

$$D = X_1^2 + X_2^2 + \cdots + X_n^2. \tag{4.3}$$

For each natural number k, let

$$f_k = (I - \pi_G(D))^k f.$$

According to Theorem 3.2 of [NS] there is a function $h_k \in L^1(G)$, independent of f, such that

$$f = \pi_G(h_k) f_k = h_k * f_k$$

Furthermore, Corollary 3.2 of [NS] states that if $k = [n/4]+1$, $h_k \in L^2(G)$. But, on a unimodular group, the convolution of two L^2 functions is an L^∞ function. This proves the proposition. $\square$

Now let

$$\pi_b = \mathrm{ind}(S, L, 1).$$

In this case,

$$\chi(s) = \chi_\rho(s)\chi_\sigma(s)$$

where

$$\chi_\rho(s) = \det \rho(s) \text{ and } \chi_\sigma(s) = \det \sigma(s).$$

Since $L = N_b S$, we will extend χ_ρ and χ_σ to all of L by declaring them to be trivial on N_b.

We may identify $\mathcal{H}(\pi_b)$ with $L^2(N_b)$, in which case

$$\pi_b(sh_o)f(h) = \chi(s)^{-1/2} f(h_o^{-1}h^s) \tag{4.4}$$

where $s \in S$, $h_o \in N_b$, and $h^s = s^{-1}hs$.

We begin by describing the primary decomposition of π_b. For this, for each $\beta \in \mathcal{M}^*$, let

$$\chi^\beta(m) = e^{i<m,\beta>}.$$

Let

$$\pi^\beta = \operatorname{ind}(\mathcal{M}, L, \chi^\beta).$$

In this case, the norm is given by

$$\|f\|_\beta^2 = \int_{\mathcal{Z}\times S} |f(z,0,s)|^2 \, dz\, ds < \infty$$

where dz is Lebesgue measure in $\mathcal{Z}$.

It follows from Proposition (2.24) that there are 2^d open, $\rho^*(S)$ orbits in $\mathcal{M}^*$ where d is the rank of $\mathcal{D}$. Furthermore, since the action is algebraic, the union of these orbits is dense in $\mathcal{M}^*$. For each such open orbit $\mathcal{O}$, let $\beta_{\mathcal{O}} \in \mathcal{O}$ be the explicit representative described in Proposition (2.24).

Proposition 4.5.

$$\pi_b = \oplus \sum_{\mathcal{O}} \pi^{\beta_{\mathcal{O}}}.$$

Proof From the theorem on inducing in stages, both π_b and π^β are induced from the analogous representations on T. The general result will follow from the tube case since inducing preserves direct sums. *Thus, we assume that* $\mathcal{Z} = 0$.

Let $\beta = \beta_{\mathcal{O}}$ for some fixed orbit $\mathcal{O}$. For $f \in \mathcal{H}(\pi_b)$ and $g \in T$, we define

$$f^\beta(g) = C_{\mathcal{O}}^{-1/2} \int_{\mathcal{M}} f(gm) e^{i<\beta,m>}\, dm \tag{4.6}$$

where dm is Lebesgue measure on $\mathcal{M}$ and $C_{\mathcal{O}}$ is as in Proposition (2.28). Then, for all $m \in \mathcal{M}$ and $g \in L$,

$$f^\beta(gm) = \chi^\beta(m^{-1}) f^\beta(g). \tag{4.7}$$

which is (4.1) for π^β.

To prove our proposition, it suffices to show that

$$\|f\|^2 = \sum_{\mathcal{O}} \|f^{\beta_{\mathcal{O}}}\|^2$$

where the norm on the left is the $\mathcal{H}(\pi_b)$ norm and those on the right are the $\mathcal{H}(\pi^{\beta_{\mathcal{O}}})$ norms.

Formula (4.1), together with a change of variables shows that for $s \in S$

$$\begin{aligned} C_{\mathcal{O}}^{1/2} f^\beta(s) &= \chi_\rho(s)^{1/2} \int_{\mathcal{M}} f(sms^{-1}) e^{i<\beta,m>}\, dm \\ &= \chi_\rho(s)^{-1/2} f^\wedge(-\rho^*(s)\beta). \end{aligned} \tag{4.8}$$

From Proposition (2.28) (with ρ^* in place of ρ)

$$\int_S |f^{\beta_{\mathcal{O}}}(s)|^2\,ds = C_{\mathcal{O}}^{-1}\int_S |f^\wedge(-\rho^*(s)\beta_{\mathcal{O}})|^2 \chi_\rho(s)^{-1}\,ds$$
$$= \int_{\mathcal{O}} |f^\wedge(-\beta)|^2\,d\beta.$$

It now follows from Plancherel's theorem on $\mathcal{M}$ that

$$\sum_{\mathcal{O}} \|\mathcal{W}^{\beta_{\mathcal{O}}}(f)\|^2 = \|f\|^2$$

which proves our proposition. □

The following lemma shows that in the tube case, the decomposition from Proposition (4.5) is the irreducible decomposition.

Lemma 4.9. *Suppose that $\beta \in \mathcal{M}^*$ is such that the orbit $\mathcal{O}_\beta = \rho^*(S)\beta$ is open in $\mathcal{M}^*$. Then*

$$\pi_T^\beta = \mathrm{ind}(\mathcal{M}, T, \chi^\beta)$$

is irreducible. Furthermore, if $\gamma \in \mathcal{M}^*$ also generates an open orbit $\mathcal{O}_\gamma$, then π_T^β is equivalent with π_T^γ if and only if $\mathcal{O}_\beta = \mathcal{O}_\gamma$.

Proof This all follows directly from Mackey theory. Since $\mathcal{M}$ is normal in T, π^β will be irreducible if and only if the isotropy subgroup of χ^β is trivial under the conjugation action of T on $\mathcal{M}^\wedge$. This is equivalent with saying that the isotropy subgroup of β is trivial under the co-adjoint action of S on $\mathcal{M}^*$. However, the dimension of $\mathcal{O}_\beta$ is the same as that of S, showing that the isotropy subgroup is discrete. Since S is completely solvable, this subgroup must then be trivial, showing irreducibility. The statement about equivalence follows directly from Mackey theory. □

In the non-tube case, the π^β are reducible. Specifically from the theorem on inducing in stages,

$$\pi^\beta = \mathrm{ind}(N_b, L, \pi_{N_b}^\beta)$$

where

$$\pi_{N_b}^\beta = \mathrm{ind}(\mathcal{M}, N_b, \chi^\beta).$$

Let $\mathcal{K}_\beta \subset \mathcal{M}$ be the kernel of β. Then, $\mathcal{K}_\beta$ is central in N_b and $H_\beta = N_b/\mathcal{K}_\beta$ is a Heisenberg group. The representation $\pi_{N_b}^\beta$ is trivial on $\mathcal{K}_\beta$ and, modulo $\mathcal{K}_\beta$, defines a representation of H_β that is inducible from a character of the center. Such a representation of a Heisenberg is always an infinite multiple of an irreducible representation. Thus, we may write

$$\pi^{\beta}_{N_b} = \infty \cdot \Pi^{\beta}_{N_b}$$

where $\Pi^{\beta}_{N_b} \in N_b{}^{\wedge}$. It follows from an argument very similar to that done in the proof of Lemma (4.9) that

$$\Pi^{\beta} = \operatorname{ind}(N_b, L, \Pi^{\beta}_{N_b})$$

is irreducible and

$$\pi_b = \oplus \sum_{\beta_{\mathcal{O}}} \infty \cdot \Pi^{\beta_{\mathcal{O}}} \tag{4.10}$$

defines the irreducible decomposition of π_b.

Now assume that $\beta = \beta_{\mathcal{O}}$ for some open orbit $\mathcal{O}$. There is a convenient realization of Π^{β} as a subrepresentation of π^{β}. We first extend β to $\mathcal{N}_b$ by declaring it to be zero on $\mathcal{Z}$.

Next, we will describe a positive polarization for β. Let X^{α}_j and Y^{α}_j be the basis of $\mathcal{Z}$ described above formulas (2.31). For $1 \le \alpha \le d_j$, $1 \le j \le r$ we define

$$Z^{\alpha}_{\pm j} = X^{\alpha}_j \mp iY^{\alpha}_j.$$

We define

$$\mathcal{P}_{\beta} = \mathcal{M}_c + \operatorname{span}_{\mathbb{C}}\{Z^{\alpha}_{\epsilon_j j}\} \qquad (1 \le j \le d,\, 1 \le \alpha \le d_j)$$

where

$$\beta = \sum_{1}^{d} \epsilon_j E^*_j.$$

Then $\mathcal{P}_{\beta}$ is a complex subalgebra of $\mathcal{L}_c$.

Lemma 4.11. *The subalgebra $\mathcal{P}_{\beta}$ is a totally complex, positive, polarization for β-i.e.*

(a) $[\mathcal{P}_{\beta}, \mathcal{P}_{\beta}] \subset \ker \beta$,
(b) $\mathcal{P}_{\beta} + \overline{\mathcal{P}_{\beta}} = (\mathcal{Z} \times \mathcal{M})_c$,
(c) $\mathcal{P}_{\beta} \cap \overline{\mathcal{P}_{\beta}} = \mathcal{M}_c$,
(d) For all $Z \in \mathcal{P}_{\beta}$,

$$i\beta([Z, \overline{Z}]) > 0.$$

Proof Properties (b) and (c) are clear. For (a), note that from the containment (2.16)

$$[X_j^\alpha, Y_j^\beta] = c_j(\alpha,\beta)E_j \tag{4.12}$$

for some scalar $c_j(\alpha,\beta)$. Formula (2.13) shows that

$$\begin{aligned} c_j(\alpha,\beta)\mu_j &= -g(X_j^\alpha, X_j^\beta) \\ &= -\delta_{\alpha,\beta}. \end{aligned}$$

Hence

$$c_j(\alpha,\beta) = -\mu_j^{-1}\delta_{\alpha,\beta}.$$

Similarly,

$$\begin{aligned} [X_j^\alpha, X_j^\beta] &= 0, \\ [Y_j^\alpha, Y_j^\beta] &= 0. \end{aligned} \tag{4.13}$$

It follows that $[Z_{\epsilon_j j}^\alpha, Z_{\epsilon_j j}^\beta] = 0$, for all α and β. Part (a) now follows from the containment (2.16) along with the observation that β is trivial on $\mathcal{M}_{ij}$.

For (d), we compute

$$\begin{aligned} [Z_{\epsilon_j j}^\alpha, \overline{Z}_{\epsilon_j j}^\alpha] &= [X_j^\alpha - i\epsilon_j Y_j^\alpha, X_j^\alpha + i\epsilon_j Y_j^\alpha] \\ &= -2i\mu_j^{-1}\epsilon_j E_j. \end{aligned} \tag{4.14}$$

Hence

$$i < [\overline{Z}_{\epsilon_j j}^\alpha, Z_{\epsilon_j j}^\alpha], \beta >= 2\mu_j^{-1}\epsilon_j^2 < E_j, E_j^* >= 2.$$

The required positivity follows. □

It now follows from Theorems 3.1 (p. 167) and 3.7 (p. 174) of [BE] that the subspace $\mathcal{H}_w^\beta$ of functions f in $\mathcal{H}(\pi^\beta)$ that satisfy

$$(r(Z) + i\beta(Z))f = 0 \tag{4.15}$$

for all $Z \in \mathcal{P}_\beta$ is a closed, invariant, irreducible, non-zero, subspace of π^β on which π^β is equivalent to Π^β. *From now on* Π^β *refers to this explicit realization of* Π^β.

We will require an explicit (and well known) description of the elements of $\mathcal{H}_\omega^\beta$. For this, we introduce a function $f_o : N_b \to \mathbb{C}$ defined by

$$f_o(z,m) = e^{-\phi(z,z) - i<m,\beta>} \tag{4.16}$$

where

$$\phi(z,w) =< B_\Omega(z,w), E^* > .$$

The following lemma follows directly from (2.13) and (2.8).

Lemma 4.17. *For* z *and* w *in* $\mathcal{Z}$

$$\phi(z,w) = \frac{1}{4} g_{\mathrm{Her}}((z,0),(w,0)).$$

If $h \in L^2(S)$ and $f \in \mathcal{H}(\pi^{\beta}_{N_b})$, we define

$$h \otimes f(s(z,m)) = h(s) f(z,m) \tag{4.18}$$

which is an element of $\mathcal{H}(\pi^{\beta})$.

Lemma 4.19. *For any function $h \in L^2(S)$ the function $h \otimes f_o$ belongs to $\mathcal{H}^{\beta}_{\omega}$.*

Proof We must show that $g \otimes f_o$ satisfies (4.15). For this, let $w \in \mathcal{Z}_j$. Then, from (4.16) and formula (2.7)

$$\begin{aligned} f_o((z,m)(w,0)) &= f_o(z+w, m + 2 \text{ im } B_{\Omega}(z,w)) \\ &= f_o(z,m) e^{-\phi(w,w) - \tau(z,w)} \end{aligned} \tag{4.20}$$

where

$$\tau(z,w) = 2 \text{ re } < B_{\Omega}(z,w), E^* > + 2i \text{ im } < B_{\Omega}(z,w), \beta > .$$

Note that if $z_k \in \mathcal{Z}_k$, $B_{\Omega}(z_k, w) \in (\mathcal{M}_{jk})_c$. Thus

$$< B_{\Omega}(z_k,w), \beta > = \delta_{jk} \epsilon_j < B_{\Omega}(z_k,w), E^* > .$$

Hence

$$\tau(z,w) = \begin{matrix} 2\phi(z,w) & (\epsilon_j = 1) \\ 2\overline{\phi}(z,w) & (\epsilon_j = -1) \end{matrix}. \tag{4.21}$$

Since ϕ is anti-holomorphic in w our lemma follows. $\square$

Using Lemma (4.19), we can produce a dense set of elements of $\mathcal{H}^{\beta}_{\omega}$. Specifically, for $(z,w) \in (\mathcal{Z} \times \mathcal{M})$, let $z_{\alpha,\,j}(z,w) \in \mathbb{C}$ denote the (α, j) coordinate of z with respect to the basis $\{X^{\alpha}_j\}$. We also set

$$z_{\alpha,\,-j} = \overline{z}_{\alpha,\,j}.$$

For each double sequence of non-negative integers

$$N = \{N(\alpha,j)\}_{1 \le \alpha \le f_j \; 1 \le j \le d}$$

we define

$$z^N = \Pi_{\alpha,\,j} (z_{\alpha,\,\epsilon_j j})^{N(\alpha,j)}.$$

Then we have the following proposition:

Proposition 4.22. *For all $h \in L^2(S)$ and all sequences N as described above, the family of functions below is orthogonal with dense span in $\mathcal{H}^\beta_\omega$.*

$$\{h \otimes \overline{z}^N f_o\}.$$

Using (3.3), we may identify $L^2(S)$ with the representation space of

$$\pi^\beta_T = \mathrm{ind}(\mathcal{M}, T, \chi^\beta).$$

We leave the following lemma, which depends on the centrality of $\mathcal{M}$ in $\mathcal{N}_b$, to the reader.

Lemma 4.23. *For all $t \in T$ and $h \in L^2(S)$*

$$(\pi^\beta_T(t)h) \otimes \overline{z}^n f_o = \Pi^\beta(t)(h \otimes \overline{z}^n f_o).$$

The functions $\overline{z}^N f_o$ play an important role in the function theory of N_b because they describe the eigenspace decomposition of certain differential operators.

Lemma 4.24.

$$\pi^\beta_{N_b}(\left(X^\alpha_j\right)^2 + \left(Y^\alpha_j\right)^2)(\overline{z}^N f_o) = -(2N(\alpha, j) + 1)\overline{z}^N f_o. \tag{4.25}$$

Proof We note that

$$\left(X^\alpha_j\right)^2 + \left(Y^\alpha_j\right)^2 = Z^\alpha_j \overline{Z}^\alpha_j - i[X^\alpha_j, Y^\alpha_j].$$

Thus, from formulas (4.1) and (4.12), the term on the left in (4.25) equals

$$\begin{aligned}&(\pi^\beta_{N_b}(Z^\alpha_j\overline{Z}^\alpha_j) + < [X^\alpha_j, Y^\alpha_j], \beta >)(\overline{z}^N f_o)\\ &= (\pi^\beta_{N_b}(Z^\alpha_j\overline{Z}^\alpha_j) - \epsilon_j)(\overline{z}^N f_o).\end{aligned}$$

In the coordinates defined by the X^α_j basis, modulo $\mathcal{M}$, $\frac{1}{2}\,\pi^\beta_{N_b}(Z^\alpha_j)$ is holomorphic differentiation while $\frac{1}{2}\,\pi^\beta_{N_b}(Z^\alpha_{-j})$ is anti-holomorphic differentiation. Hence

$$\begin{aligned}\pi^\beta_{N_b}(\overline{Z}^\alpha_{\epsilon_j j})\overline{z}^N &= 2N(\alpha, j)\overline{z}^{N-\Lambda(\alpha,j)}\\ \pi^\beta_{N_b}(Z^\alpha_{\epsilon_j j})\overline{z}^N &= 0.\end{aligned} \tag{4.26}$$

where $\Lambda(\alpha, j)$ is the sequence which is zero for all indecies except (α, j) where it is 1.

On the otherhand,

$$f_o((z,m)^{-1}) = \overline{f}_o(z,m).$$

Thus, it follows from formula (4.20) that for $w \in \mathcal{Z}_j$

$$\begin{aligned} f_o((w,0)(z,m)) &= \overline{f}_o((-z,-m)(-w,0)) \\ &= f_o(z,m)e^{-\phi(w,w)-\overline{\tau}(z,w)} \\ &= f_o(z,m)e^{-\phi(w,w)-\tau(w,z)}. \end{aligned} \tag{4.27}$$

Recall that the X_j^α are g_{Her} orthogonal. Hence, from Lemma (4.17) and formula (4.21),

$$\tau(w,z) = \frac{1}{2}\sum w_{\alpha,\,\epsilon_j j}\,\overline{z}_{\alpha,\,\epsilon_j j}.$$

Thus, differentiating formula (4.27) with respect to w at $w = 0$ shows that

$$\begin{aligned} &\pi^\beta_{N_b}(\overline{Z}^\alpha_{\epsilon_j j})f_o = 0, \\ &\pi^\beta_{N_b}(Z^\alpha_{\epsilon_j j})f_o = -\overline{z}_{\alpha,\,\epsilon_j j}f_o. \end{aligned} \tag{4.28}$$

Hence

$$\begin{aligned} \pi^\beta_{N_b}(\overline{Z}^\alpha_{\epsilon_j j})\overline{z}^N f_o &= 2N(\alpha,j)\overline{z}^{N-\Lambda(\alpha,j)}f_o \\ \pi^\beta_{N_b}(Z^\alpha_{\epsilon_j j})\overline{z}^N f_o &= -\overline{z}^{N+\Lambda(\alpha,j)}f_o. \end{aligned}$$

If $\epsilon_j = 1$ then

$$\begin{aligned} \pi^\beta_{N_b}\big(\big(X_j^\alpha\big)^2 + \big(Y_j^\alpha\big)^2\big)(\overline{z}^N f_o) &= \pi^\beta_{N_b}(Z_j^\alpha \overline{Z}_j^\alpha - 1)(\overline{z}^N f_o) \\ &= (-2N(\alpha,j) - 1)(\overline{z}^N f_o) \end{aligned}$$

and the lemma follows.

If $\epsilon_j = -1$ then we use the identity

$$\begin{aligned} \big(X_j^\alpha\big)^2 + \big(Y_j^\alpha\big)^2 &= \overline{Z}_j^\alpha Z_j^\alpha + i[X_j^\alpha, Y_j^\alpha] \\ &= Z^\alpha_{\epsilon_j j}\overline{Z}^\alpha_{\epsilon_j j} + i[X_j^\alpha, Y_j^\alpha] \end{aligned}$$

to prove the lemma as before. □

Section 5: $\mathcal{H}^2_{\mathrm{HJK}}$.

Throughout this section, $\mathcal{D}$ is assumed to be non-tube like, as defined in Definition (2.21) in Section 3. We identify A with $\mathcal{A}$ using the exponential mapping and $\mathcal{A}$ with $\mathbb{R}^d$ using the basis $A_1, A_2, \ldots, A_d$. The general element a of A is denoted

$$a = a(t) = \exp(t_1 A_1 + \ldots t_d A_d).$$

We consider the map $a \to (t_1, \ldots, t_d)$ *as defining coordinates on* A.

As mentioned in the introduction, the Hua system has a Poisson kernel on an open dense subset of the Shilov boundary of $\mathcal{D}$. Specifically, there is a finite, positive measure dp on $L/S = N_b$ such that every bounded Hua-harmonic function F may be expressed in the form

$$F(g) = \int_{L/S} f(gh)\, dp(h) \tag{5.1}$$

where $f \in L^\infty(L/S)$ is uniquely determined by F. We refer to f in (5.1) as the *boundary value function* of F, dp as the Poisson measure and we say that F is the *Poisson integral* of f. In fact, we showed in [DHP2] that L/S is a boundary for the Laplace-Beltrami operator and that we may use the corresponding Poisson measure as dp.

Under the identification $L/S = N_b$, $dp = P\, dh$ where dh is Haar measure on N_b and $P \in L^2(N_b) \cap L^1(N_b)$. Under the identification of N_b and L/S, for $h, h_o \in N_b$ and $s \in S$

$$f(h_o sh) = f(h_o shs^{-1}).$$

We may identify $L^2(N_b)$ with the representation space of π_b. Formula (4.4) shows then that (5.1) is equivalent with

$$F(g) = \chi(g)^{-1/2}(\pi_b(g^{-1})f, P) = \chi(g)^{-1/2}(f, \pi_b(g)P). \tag{5.2}$$

It follows from [Pou] Proposition 1.1, p. 92 that $v \in C^\infty(\pi_b)$ if and only if the matrix elements $g \to < \pi(g)v, w >$ are C^∞ on L for all $w \in \mathcal{H}(\pi_b)$. Hence, from the ellipticity of the Laplace-Beltrami operator, $P \in C^\infty(\pi_b)$.

Let $\delta \in C^{-\infty}(\pi_b)$ be evaluation at e:

$$< f, \delta >= f(e).$$

The following is a representation theoretic formulation of the statement that the Poisson kernel is an approximate identity.

Lemma 5.3. *In the weak topology on* $C^{-\infty}(\pi_b)$

$$\lim_{t_d \to -\infty} \lim_{t_{d-1} \to -\infty} \cdots \lim_{t_1 \to -\infty} \chi(a)^{-1/2} \pi_b(a) P = \delta.$$

Proof Let $f \in C^\infty(\pi_b)$. From (5.2), for $a \in A$

$$\chi(a)^{-1/2}(f, \pi_b(a)P) = \int_{N_b} f(aha^{-1}) P(h)\, dh.$$

Since the eigenvalues of ad A_1 in $\mathcal{Z} + \mathcal{M}$ are all non-negative

$$\lim_{t_1 \to -\infty} \text{Ad}\, (\exp t_1 A_1) h = e_1(h)$$

converges uniformly on compact subsets of N_b. Hence, for all $h \in N_b$,

$$\lim_{t_1 \to -\infty} f(aha^{-1})P(h) = f(\hat{a}e_1(h)\hat{a}^{-1})P(h)$$

where $\hat{a} = a(0, t_2, \ldots, t_d)$.

Since the restriction of π_b to N_b is the regular representation of N_b, it follows from Proposition (4.2) that f is bounded. Hence, the dominated convergence theorem shows that the above limit converges in $L^1(N_b)$. Our Lemma follows by iterating this argument and integrating, noting that

$$\lim_{t_d \to -\infty} \lim_{t_{d-1} \to -\infty} \ldots \lim_{t_1 \to -\infty} \text{Ad }(a)h = e.$$

□

In the Hermitian-symmetric tube case, all Poisson integrals over L/S are Hua-harmonic. This, however, is the only case in which this is true. Let $\mathcal{U}$ be the set of $f \in L^2(N_b)$ for which (5.2) defines a Hua-harmonic function. The ellipticity of the Hua system shows that $\mathcal{U}$ is a closed π_b-invariant subspace of $L^2(N_b)$. We refer to $\mathcal{U}$ as the *space of L^2-boundary values for the Hua system.* We define $\mathcal{H}^2_{\text{HJK}}$ to be the space of all functions F as in (5.2) where $f \in \mathcal{U}$. We remark that

$$\mathcal{H}^2_\omega \subset \mathcal{H}^2_{\text{HJK}}$$

where $\mathcal{H}^2_\omega$ denotes the holomorphic $\mathcal{H}^2$ space for $\mathcal{D}$. In particular, it follows that $\mathcal{U}$ is non-trivial.

The main result of this section is the following theorem, which generalizes the main result of [BBDHPT].

Theorem 5.4. *If $\mathcal{D}$ is non-tube like then*

$$\mathcal{H}^2_{\text{HJK}} = \mathcal{H}^2_\omega + \overline{\mathcal{H}^2_\omega}.$$

For the proof, it follows from formula (4.10) that $\pi_b|\mathcal{U}$ is a direct sum of multiples of the representations $\Pi^{\beta_\mathcal{O}}$ for certain open orbits $\mathcal{O}$. Let $\beta = \beta_\mathcal{O}$ for one such orbit. As in Section 4, we realize Π^β in $\mathcal{H}^\beta_\omega$. For each intertwining operator

$$U : \mathcal{H}^\beta_\omega \to \mathcal{U}$$

let $\delta_U \in C^{-\infty}(\Pi^\beta)$ be defined by

$$< f, \delta_U > = < U(f), \delta > .$$

Then, from formula (4.4), for $s \in S$,

$$\Pi^\beta(s)\delta_U = \chi(s)^{-1/2}\delta_U. \tag{5.5}$$

Note that δ_U determines U since

$$U(f)(g) = < \pi_b(g^{-1})U(f), \delta > = < \Pi^\beta(g^{-1})f, \delta_U > \tag{5.6}$$

Let $\mathcal{D}_\beta$ be the set of all δ_U where U varies over the space continuous intertwining operators from $\mathcal{H}^\beta_\omega$ into $\pi_b\big|\mathcal{U}$.

The following proposition proves that $\pi_b\big|\mathcal{U}$ is the product of exactly two irreducible representations. Theorem (5.4) follows since $\mathcal{H}^2_\omega$ and $\overline{\mathcal{H}^2_\omega}$ are two closed, invariant subspaces of $\pi_b\big|\mathcal{U}$.

Proposition 5.7. *The set $\mathcal{D}_\beta$ is non-zero only if $\beta = \pm E^*$, in which case $\mathcal{D}_\beta$ is one dimensional.*

For the proof, let $P_U = (U)^*(P)$ where $(U)^* : L^2(N_b) \to \mathcal{H}_\omega$ is the adjoint of U. We note that for all $f \in \mathcal{H}^\beta_\omega$,

$$F : g \to (f, \Pi^\beta(g)P_U)\chi(g)^{-1/2} = (U(f), \pi_b(g)P)\chi(g)^{-1/2} \tag{5.8}$$

defines a Hua-harmonic function. Let

$$\mathcal{V} = C^{-\infty}(\Pi^\beta).$$

For $g \in L$, let $\tilde{P}(g) \in \mathcal{V}$ be defined by

$$< f, \tilde{P}(g) > = (f, \Pi^\beta(g)P_U)\chi(g)^{-1/2}.$$

Then for $n \in N_L$ and $g \in G$,

$$\tilde{P}(ng) = \Pi^\beta(n)\tilde{P}(g).$$

Hence, $\tilde{P}$ belongs to the representation space $\pi_L = \mathrm{ind}^\infty(N_L, L, \pi_o)$ where $\pi_o = \Pi^\beta\big|N_L$ acting on $\mathcal{V}$. We realize this representation in $C^\infty(\mathbb{R}^d, \mathcal{V})$ using (3.3).

It is easily seen that $\tilde{P}$ satisfies (3.7) with $\rho_i = 0$ and $G_i = 0$. Furthermore, Lemma (5.3) shows that

$$\lim_{t_d \to -\infty} \lim_{t_{d-1} \to -\infty} \cdots \lim_{t_1 \to -\infty} \tilde{P}(t) = \delta_U \tag{5.9}$$

in the weak topology on $\mathcal{V}$. In particular, $\tilde{P} \in \mathcal{C}_0(d)$ where $\mathcal{C}_r(d)$ is as defined above Definition (1.28).

From Theorem (3.9), we obtain an asymptotic expansion

$$\tilde{P}(t) \sim \sum \tilde{P}_\alpha(t)e^{<a,\alpha>} \qquad \alpha \in \mathcal{E} \tag{5.10}$$

where the $\tilde{P}_\alpha$ are $\mathcal{V}$ valued polynomials on $\mathbb{R}^d$.

The key observation in the proof of Proposition (5.7) is that from (5.9), Proposition (1.30), and Corollary (3.34),

$$\tilde{P}_0 = \delta_U. \tag{5.11}$$

We assume that the notation of (3.7) is still in effect. Formulas (2.38), (3.4), and Theorem (2.36) imply

$$\begin{aligned} \left(-A'_o + 2\sum e^{t_i}\tilde{\mathcal{Z}}_i\right)\tilde{P} = 0 \\ \left(D + \sum_i \mu_i^{-1} e^{t_i}\tilde{E}_i^2 + 2\sum_{i<j}\mu_i^{-1}(e^{t_i-t_j}\tilde{\mathcal{Y}}_{ij} + e^{t_i+t_j}\tilde{\mathcal{X}}_{ij})\right)\tilde{P} = 0 \end{aligned} \tag{5.12}$$

where A'_o is as in formula (2.39), D is as in formula (ano) and we set $\tilde{X} = \Pi^\beta(X)$ for $X \in \mathfrak{A}(\mathcal{N}_L)$.

Proposition 5.13. *For* $1 \leq l \leq d$

$$\begin{aligned} \tilde{P}_{\lambda_l+\lambda_d} &= 4(f_l f_d)^{-1}\tilde{\mathcal{Z}}_l\,\tilde{\mathcal{Z}}_d\tilde{P}_0 \quad (l \neq d) \\ \tilde{P}_{2\lambda_d} &= 2f_d^{-2}\tilde{\mathcal{Z}}_d^2\tilde{P}_0. \end{aligned} \tag{5.14}$$

Proof Note that

$$\Delta_{N_b} = 2\sum \mu_i^{-1}\tilde{\mathcal{Z}}_i.$$

We apply the first equality in (5.12) to the asymptotic expansion (5.10) and equate terms with the same exponent. We find

$$(A'_o + < A'_o, \alpha >)\tilde{P}_\alpha = 2\sum_{1\leq i\leq d}\mu_i^{-1}\tilde{\mathcal{Z}}_i\tilde{P}_{\alpha-\lambda_i}. \tag{5.15}$$

Proposition (3.22) shows that if $0 \neq < A'_o, \alpha >$, then $\tilde{P}_\alpha$ is independent of t if all of the $\tilde{P}_{\alpha-\lambda_i}$ are,

In particular, for $\alpha = \lambda_l$, we find (using Corollary (3.42) and Lemma (3.32)) that

$$f_l\tilde{P}_{\lambda_l} = 2\tilde{\mathcal{Z}}_l\,\tilde{P}_0.$$

Then, using $\alpha = \lambda_l + \lambda_d$ with $l \neq d$:

$$\begin{aligned} (\frac{f_l}{\mu_l} + \frac{f_d}{\mu_d})\tilde{P}_{\lambda_l+\lambda_d} &= 2\mu_l^{-1}\tilde{\mathcal{Z}}_l\tilde{P}_{\lambda_d} + 2\mu_d^{-1}\tilde{\mathcal{Z}}_d\tilde{P}_{\lambda_l} \\ &= 4\left(\frac{\mu_l}{f_l} + \frac{\mu_d}{f_d}\right)(\mu_d\mu_l)^{-1}\tilde{\mathcal{Z}}_l\,\tilde{\mathcal{Z}}_d\tilde{P}_0. \end{aligned}$$

Our lemma follows since

$$\left(\frac{f_l}{\mu_l}+\frac{f_d}{\mu_d}\right)^{-1}\left(\frac{\mu_l}{f_l}+\frac{\mu_d}{f_d}\right)=\frac{\mu_l\,\mu_d}{f_l\,f_d}.$$

Finally, using $\alpha = 2\lambda_l$

$$\begin{aligned}2f_l\tilde{P}_{2\lambda_l} &= 2\tilde{\mathcal{Z}}_l\tilde{P}_{\lambda_l}\\ &= 4f_l^{-1}\tilde{\mathcal{Z}}_l^2\tilde{P}_0.\end{aligned}$$

which proves our lemma. □

Proposition 5.16. *For $1 \le l < d$ there is an element $M_l \in (\mathcal{M}_{ld})^2 \subset \mathfrak{A}(\mathcal{L})$ such that*

$$\begin{aligned}\tilde{P}_{\lambda_l+\lambda_d} &= -(\tilde{E}_l\tilde{E}_d+\tilde{M}_l)\tilde{P}_0,\\ \tilde{P}_{2\lambda_d} &= -\frac{1}{2}(\tilde{E}_l^2)\tilde{P}_0.\end{aligned}$$

Proof We apply the second formula in (5.12) to the asymptotic expansion of $\tilde{P}$ and equate terms with the same exponent finding

$$\begin{aligned}e^{-<a,\alpha>}D(\tilde{P}_\alpha e^{<a,\alpha>}) = &-\sum\mu_i^{-1}\tilde{E}_i^2\tilde{P}_{\alpha-2\lambda_i}-2\sum_{1\le i<k\le d}\mu_i^{-1}\tilde{\mathcal{X}}_{ik}\tilde{P}_{\alpha-(\lambda_i+\lambda_k)}\\ &-2\sum_{1\le i<k\le d}\mu_i^{-1}\tilde{\mathcal{Y}}_{ik}\tilde{P}_{\alpha-(\lambda_i-\lambda_k)}\end{aligned} \tag{5.17}$$

where (from formula (2.31) and Lemma (2.33))

$$D=\sum_i\mu_i^{-1}(A_i^2-(1+d_i)A_i).$$

The characteristic polynomial for D is

$$p(\alpha)=\sum_i\mu_i^{-1}(\alpha_i^2-(1+d_i)\alpha_i).$$

For $\alpha = 2\lambda_d$, and $i \le j$, neither $\alpha-(\lambda_i-\lambda_j)$ nor $\alpha-(\lambda_i+\lambda_j)$ is an exponent unless $i = j = d$, in which case (5.17) reduces to

$$2\mu_d^{-1}\tilde{P}_{2\lambda_d}=-\mu_d^{-1}\tilde{E}_d^2\tilde{P}_0$$

as desired.

Now, let $\alpha = \lambda_l + \lambda_d$ where $l < d$. For $i \leq j$,

$$\alpha - (\lambda_i + \lambda_j) = (\lambda_l - \lambda_i) - (\lambda_j - \lambda_d).$$

Lemma (3.32) shows that for this term to be an exponent we must have $l \leq i$ and $j = d$.

Also,

$$\alpha - (\lambda_i - \lambda_j) = (\beta_l + \cdots + \beta_{d-1} + 2\lambda_d) - (\beta_i + \cdots + \beta_{j-1}).$$

Corollary (3.42), Lemma (3.32), and $d = d_\tau$ show that the above expression is not an exponent unless $i = l$ (so $\alpha = \lambda_j + \lambda_d$). Hence (5.17) reduces to

$$p(\lambda_l + \lambda_d)\tilde{P}_{\lambda_l+\lambda_d} = -2\mu_l^{-1}\tilde{\mathcal{X}}_{ld}\tilde{P}_0 - 2\sum_{l<j}\mu_l^{-1}\tilde{\mathcal{Y}}_{lj}\tilde{P}_{\lambda_j+\lambda_d}.$$

Since $\mathcal{X}_{ld} \in (\mathcal{M}_{ld})^2$, this term may be ignored.

Assume by induction that we have proven the result for $l+1 \leq j \leq d$. It follows from (5.5) and (5.11) that for $l < j$, $\tilde{Y}^\alpha_{lj}\tilde{P}_0 = 0$. Hence, for $l < j < d$

$$\begin{aligned}\tilde{Y}^\alpha_{lj}\tilde{P}_{\lambda_j+\lambda_d} &= -\tilde{Y}^\alpha_{lj}(\tilde{E}_j\tilde{E}_d + \tilde{M}_j)\tilde{P}_0\\ &= -\Pi^\beta\left(\text{ ad } Y^\alpha_{lj}(E_jE_d + M_j)\right)\tilde{P}_0\\ &= -\Pi^\beta\left(X^\alpha_{lj}E_d + \text{ ad } Y^\alpha_{lj}(M_j)\right)\tilde{P}_0.\end{aligned}$$

(Note that from (2.15), $[Y^\alpha_{lj}, E_d] = 0$.) Repeating the same argument using $[Y^\alpha_{lj}, X^\alpha_{lj}] = \mu_l^{-1}E_l$, and summing over α, shows that

$$2\mu_l^{-1}\tilde{\mathcal{Y}}_{lj}\tilde{P}_{\lambda_j+\lambda_d} = -\mu_l^{-1}\Pi^\beta(d_{lj}E_lE_d\tilde{P}_0 + \sum_\alpha \text{ ad } (Y^\alpha_{lj})^2(M_j))\tilde{P}_0.$$

Note that $(\text{ ad } Y^\alpha_{lj})^2$ maps $\mathcal{M}^2_{jd}$ into $\mathcal{M}^2_{ld}$.

A similar argument shows

$$2\mu_l^{-1}\tilde{\mathcal{Y}}_{ld}\tilde{P}_{2\lambda_d} = -\mu_l^{-1}(d_{ld}\tilde{E}_l\tilde{E}_d + \sum_\alpha(\tilde{X}^\alpha_{ld})^2)\tilde{P}_0.$$

Summing the previous two formulas over j and using (5.17), we see that

$$p(\lambda_l + \lambda_d)\tilde{P}_{\lambda_l+\lambda_d} = \mu_l^{-1}d_l(\tilde{E}_l\tilde{E}_d + \tilde{M}_l)\tilde{P}_0$$

where $M_l \in \mathcal{M}^2_{ld}$. Our proposition follows since

$$p(\lambda_l + \lambda_d) = -d_l\mu_l^{-1}.$$

□

Next, we will decompose $\tilde{P}_\alpha$ according to the decomposition from Proposition (4.22). We remind the reader For any functional $\phi \in C^{-\infty}(\Pi^\beta)$ and a multi-index N as in Proposition (4.22), we define a distribution ϕ^N on S by

$$< f, \phi^N > = < f \otimes \overline{z}^N f_o, \phi > .$$

It is easily seen that $\phi = 0$ if and only if $\phi_N = 0$ for all N.

Proposition 5.18. *For all N there is a constant K_N such that $\tilde{P}_0^N = K_N \chi^{-1/2}|S$. In particular $\tilde{P}_0^N$ is a C^∞ function.*

Proof For $\phi \in C_c^\infty(T)$ and $s \in S$, let

$$\tilde{\phi}(s) = \int_{\mathcal{M}} \phi(sm) e^{i<m,\beta>} \, dm. \tag{5.19}$$

Then $\tilde{\phi} \in C_c^\infty(S)$ and

$$\tilde{Q}_o : \phi \to < \tilde{\phi} \otimes \overline{z}^N f_o, \tilde{P}_0 >$$

is a distribution on T. From Lemma (4.23), (5.5), and (5.19)

$$L_T(s)\tilde{Q}_o = \chi(s)^{-1/2}\tilde{Q}_o,$$
$$R_T(m)\tilde{Q}_o = e^{i<m,\beta>}\tilde{Q}_o$$

where $\in S$, $m \in \mathcal{M}$ and L_T and R_T are, respectively, the left and right regular representations of T. It follows from Theorem 5.2.2.1 of [War] that there is a constant K_N such that

$$< \tilde{\phi}, \tilde{P}_0^N > = K_N \int_{S\mathcal{M}} \phi(sm)\chi(s)^{-1/2} e^{-i<m,\beta>}$$
$$= < \tilde{\phi}, \chi^{-1/2}|S >$$

proving our proposition. □

Lemma 5.20. $((\tilde{\mathcal{Z}}_l \tilde{\mathcal{Z}}_d)\tilde{P}_0)^N \in C^\infty(S)$.

Proof

For $X \in \mathcal{Z}$ there are C^∞ functions $\phi_{\alpha,i}$ on S such that for all $s \in S$,

$$\text{Ad }(s^{-1})X = \sum \phi_{\alpha,i}(s) Z_{\alpha,\epsilon_i i} + \overline{\phi}_{\alpha,i}(s)\overline{Z}_{\alpha,\epsilon_i i}$$

where the notation is as stated above (4.11).

Let $h \in C_c^\infty(S)$. From the formulas below (4.28), for each multi-index N, there is a finite sequence of multi-indecies N_i and functions $\psi_i \in C_c^\infty(S)$ such that

$$\begin{aligned}\Pi^\beta(X)(h \otimes \overline{z}^N f_o)(sn) &= h(s)[\pi^\beta_{N_b}(\text{ Ad } (s^{-1})X)(\overline{z}^N f_o)](n) \\ &= \sum_i (h\psi_i \otimes \overline{z}^{N_i} f_o)(sn).\end{aligned}$$

Iterating this formula shows that a similar equality holds with $\mathcal{Z}_l\mathcal{Z}_d$ in place of X. Applying this to $\tilde{P}_0$ shows that $((\tilde{Z}_l\tilde{Z}_d)\tilde{P}_0)^N$ is a sum of terms $(\tilde{P}_0^M\psi_M) \otimes (\overline{z}^N f_o)$ where $\psi_M \in C^\infty(S)$ and the M range over a finite set of multi-indecies, proving the lemma. $\square$

Proposition (5.7) follows immediately from the following lemma, proving Theorem (5.4).

Lemma 5.21. *If $\beta \neq \pm E^*$, then $K_N = 0$ for all N. If $\beta = \pm E^*$, then $K_N \neq 0$ if and only if $N = 0$.*

Proof From Proposition (5.16) and Lemma (4.23),

$$\tilde{P}^N_{\lambda_l+\lambda_d} = -K_N(1 - \frac{1}{2}\delta_{ld})\pi^\beta_T(E_lE_d + M_l)\chi^{-1/2}.$$

Furthermore, for $a \in A$,

$$\text{Ad } a^{-1}(\mathcal{M}^2_{ld}) \subset \mathcal{M}^2_{ld} \subset \ker \beta.$$

Hence, if $a = a(t)$, where $t \in \mathbb{R}^d$,

$$\pi^\beta_T(E_l)\chi^{-1/2}(a) = i < \text{ Ad } (a^{-1})(E_l), \beta > \chi^{-1/2}(a) = i\mu_l\epsilon_l e^{-t_l}\chi^{-1/2}(a).$$

Thus,

$$\tilde{P}^N_{\lambda_l+\lambda_d}(a) = K_N(1 - \frac{1}{2}\delta_{ld})\mu_l\mu_d\epsilon_l\epsilon_d e^{-t_d-t_l}\chi^{-1/2}(a). \tag{5.22}$$

On the otherhand, from Proposition (5.13)

$$\tilde{P}_{\lambda_l+\lambda_d} = (1 - \frac{1}{2}\delta_{ld})4\mu_l\mu_d(f_lf_d)^{-1}e^{-t_d-t_l}\pi^\beta_{N_b}(\mathcal{Z}_l\,\mathcal{Z}_d)\tilde{P}_0.$$

Thus, from Lemma (4.24), and formula (4.4), for all $a \in A$,

$$\begin{aligned}\tilde{P}^N_{\lambda_l+\lambda_d}(a) &= (1 - \frac{1}{2}\delta_{ld})4\mu_l\mu_d(f_lf_d)^{-1}e^{-t_d-t_l}[\pi^\beta_{N_b}(\tilde{\mathcal{Z}}_l\tilde{\mathcal{Z}}_d)\tilde{P}_0]^N(a) \\ &= K_N(1 - \frac{1}{2}\delta_{ld})4\mu_l\mu_d(f_lf_d)^{-1}e^{-t_d-t_l}\sum_{j=1}^{f_l}\sum_{k=1}^{f_d}(2N(l,j)+1)(2N(d,k)+1)\chi^{-1/2}(a).\end{aligned}$$

Equating the above expression with (5.22) we find that if $K_N \neq 0$

$$\epsilon_l \epsilon_d = (N(l)+1)(N(d)+1)$$

where

$$N(k) = f_\alpha^{-1} \sum_{1 \leq j \leq f_\alpha} 2N(k,j).$$

This implies that ϵ_l and ϵ_d have the same sign and $N(l) = 0$ for all l. Hence, $N = 0$ and $\beta = \pm E^*$, as desired.

Conversely, we know that the holomorphic and anti-holomorphic functions are Hua-harmonic. These spaces must correspond to $\beta = \pm E^*$. It follows that $K_0 \neq 0$ in these cases. □

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