Review Sheet for Mid 1
Math 266
Ralph Kaufmann

Disclaimer: This sheet is neither claimed to be complete nor indicative and may contain typos.

1. Types of equations and techniques

1.1. First order Linear Equations.

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = g(t) \tag{1}$$

Solution:

$$\mu(t) := \exp(\int p(t)\mathrm{d}t) \tag{2}$$

$$y(t) = \frac{1}{\mu(t)}(\int \mu(t)g(t)\mathrm{d}t + c) \tag{3}$$

The initial condition determines c. E.g. $y(0) = y_0$.

1.2. Separable Equations.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)/h(y) \tag{4}$$

Solution:

$$h(y)\mathrm{d}y = g(x)\mathrm{d}x \tag{5}$$

$$\int h(y)dy = \int g(x)\mathrm{d}x + c \tag{6}$$

Get implicit solution: $\int h(y)dy - \int g(x)dx = c$.

1.3. Exact Equations.

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0 \tag{7}$$

Criterion:

$$M_y = N_x \tag{8}$$

Conditions: M, N, M_y, N_x continuous on simply connected region (e.g. rectangle).
Solution:

$$Q(x,y) = \int_{x_0}^{x} M(s,y)\mathrm{d}s, \quad \text{defines } Q \text{ and solves } Q_x = M \tag{9}$$

$$\psi(x,y) = Q(x,y) + h(y), \quad \text{defines } \psi \text{ in terms of } Q \text{ and } h \tag{10}$$

$$\psi_y = Q_y + h'(y) = N, \quad \text{gives differential eq. for } h \tag{11}$$

$$h(y) = \int Q_y - N\mathrm{d}y, \quad \text{defines } h \tag{12}$$

$$\psi = c, \quad \text{gives the answer} \tag{13}$$

Get implicit solution.

1.4. **Integrating factors.** Multiply equation (7) by $\mu(x,y)$

$$\mu(x,y)M(x,y)\mathrm{d}x + \mu(x,y)N(x,y)\mathrm{d}y = 0 \tag{14}$$

CRITERION:

$$(\mu M)_y = (\mu N)_x \tag{15}$$

SPECIAL CASE

$\mu(x,y) = \mu(x)$ depends only on x. Test: $(M_y - N_x)/N$ only depends on x.

SOLUTION:

(1)

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{M_y - N_x}{N}\mu \tag{16}$$

gives ODE for μ. Solve!

(2) With $\tilde{M}(x,y) := \mu(x)M(x,y)$, $\tilde{N}(x,y) = \mu(x)N(x,y)$, get new *exact* equation:

$$\tilde{M}(x,y)\mathrm{d}x + \tilde{N}(x,y)\mathrm{d}y = 0 \tag{17}$$

Use algorithm above for exact equations.

2. Qualitative behavior of autonomous equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y) \tag{18}$$

2.1. Phase line.

1. Find constant/equilibrium solutions: $y : f(y) = 0$.
2. Draw vertical line and mark off constant solutions.
3. In each interval between constant solutions determine the sign of $f(y)$ and draw arrow (up if $f(y) > 0$, down if $f(y) < 0$). This means that the graph of y is going up or down for values of y in the respective interval.
4. Constant solutions are:

stable	if both arrows point towards
unstable	if both arrows point away
semi-stable	if one arrow points toward and one away

   from the particular point of constant solution

2.2. **Second order behavior of solutions.** Need to consider $f(y)$ and $f'(y)$ for behavior of graph of $y(x)$

$f(y)f'(y) > 0$	convex
$f(y)f'(y) < 0$	concave
$f(y)f'(y) = 0$	may be an inflection point.

Notice $f(y) = 0$ means constant (equilibrium) solution so $f'(y) = 0$ are the interesting points of inflection.

3. Setting up equations

3.1. Growth. Determine population P as a function of time given r rate of reproduction and k "kill" rate.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP - k \tag{19}$$

Special case: Rate r given indirectly: e.g. Population increases by factor s after each time interval if system is unperturbed.

Solution: unperturbed equation $\frac{\mathrm{d}P}{\mathrm{d}t} = rP$ has solution $P(t) = P_0 e^{rt}$ so $s = P(t+1)/P(t) = e^r$ and hence $r = \ln(s)$.

3.2. Flow. Amount of substance Q in a liquid system with *liquid* inflow rate r_{in} and outflow rate r_{out}. Let $V(t)$ be the volume of system and C the inflow concentration.

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \text{substance in rate} - \text{substance out rate} \tag{20}$$

$$= r_{in}C - \frac{r_{out}}{V(t)}Q \tag{21}$$

Special cases:

1. $V(t) = V_0$ fixed $r_{in} = r_{out}$
2. V_0 initial volume: $V(t) = V_0 + (r_{in} - r_{out})t$

4. Theorems on existence and continuity

4.1. Linear first order equations.

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = g(t) \tag{22}$$

If $p(t)$ and $q(t)$ are continuous on I then solutions exist, are unique and continuous on I.

Discontinuous or singular points can only appear at discontinuities or singular points of $p(t)$ and $g(t)$. You do not have to solve the equation to determine the maximum intervals of definition and continuity of the solution given an initial time t_0.

4.2. Non-linear first order.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) \tag{23}$$

If f and $\partial f/\partial y$ are continuous on rectangle R then solutions with initial conditions (t_0, y_0) exist and are unique in an interval J, s.t. $t_0 \in J$ and all points $(t, y_0) \in R$ for $t \in J$.

Notice J may not be maximal, i.e. it does not have to include all points t with $(t, y_0) \in R$. You have to solve the equation to know the maximal domains of continuity and definition of the solution given an initial point (t_0, y_0) often given as $y(t_0) = y_0$. The important data is t_0.

At points of discontinuity of $\partial f/\partial y$ with f continuous, existence is guaranteed, but more that one solution can exist.