Review Sheet for Mid 2
Math 266
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Disclaimer: This sheet is neither claimed to be complete nor indicative and may contain typos.

1. Homogeneous 2nd Order equations

1.1. Theorems.
Consider

$$L[y] := y'' + p(t)y' + q(t)y, \quad L[y] = g(t) \tag{1}$$

Theorem. If p, q, g continuous on an interval I, then

- there is a unique solution on I for any given set of initial conditions

$$y(t_0) = y_0, y'(t_0) = y'_0 \tag{2}$$

- the *general solution* to the homogeneous equation $L[y] = 0$ is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{3}$$

  where y_1, y_2 are a *fundamental set*, i.e. two linearly independent solutions.
- two solutions y_1, y_2 are a fundamental set on I if and only if the Wronskian $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$ and hence for all $t \in I$. Here

$$W(y_1, y_2)(t) = \left| \begin{pmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{pmatrix} \right| \tag{4}$$

- If $Y(t)$ is *a particular* solution of $L[y] = g(t)$ and y_1, y_2 are a fundamental set for $L[y] = 0$ then *all* solutions of $L[y] = g(t)$ are of the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \tag{5}$$

1.2. Initial Conditions.
Problem: Given a fundamental set y_1, y_2 solve for given initial conditions.

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \tag{6}$$

$$c_1 y'_1(t_0) + c_2 y'_2(t_0) = y'_0 \tag{7}$$

Solution

$$c_1 = \begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix} / (W(y_1, y_2)(t_0)), \quad c_2 = \begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix} / (W(y_1, y_2)(t_0)) \tag{8}$$

1.3. Constant coefficients.

$$ay'' + by' + cy = 0 \tag{9}$$

Solution: Guess: e^{rt}, equation for r:

$$ar^2 + br + c = 0 \tag{10}$$

Solution

$$r_{1,2} = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} \tag{11}$$

roots	general solution
$r_1 \neq r_2$ real	$Ae^{r_1 t} + Be^{r_2 t}$
$r_1 = r_2 = r$ real	$Ae^{rt} + Bte^{rt}$
$r_{1,2} = \lambda \pm i\mu$	$Ae^{\lambda t}\cos(\mu t) + Be^{\lambda t}\sin(\mu t)$

1.4. **Reduction of order.** When to use: You have *one* solution y_1 of $L[y] = 0$ and are looking for a second. Get second solution from $y(t) = v(t)y_1(t)$.

Equation to solve. If

$$y'' + p(t)y' + qy = 0 \tag{12}$$

then solve

$$y_1 v'' + (2y_1' + py_1)v' = 0 \tag{13}$$

(1) Set $u = v'$ and solve first order eq. $y_1 u' + (2y_1' + py_1)u = 0$: get $u(t)$.
(2) Solve $v' = u$ and get $v(t)$.
(3) Get $y(t) = v(t)y_1(t)$.

2. Non–homogeneous equations

2.1. **Sums.**

$$L[y] = g_1(t) + g_2(t) \tag{14}$$

(1) Solve $L[y] = g_1(t)$: get Y_1.
(2) Solve $L[y] = g_2(t)$: get Y_2.
(3) Get particular solution $Y(t) = Y_1(t) + Y_2(t)$

2.2. **Method of Undetermined Coefficients.** Find general solution of

$$ay'' + by' + cy = g(t) \tag{15}$$

(1) Solve the homogeneous equation:get $r_{1,2}$ and fundamental set y_1, y_2.
(2) Depending on $g(t)$ and $r_{1,2}$ make the following guess

$g(t)$	$Y(t)$	s
$a_0 t^n + \cdots + a_n$	$t^s(A_0 t^n + \cdots + A_n)$	multiplicity of 0 as a root
$(a_0 t^n + \cdots + a_n)e^{\alpha t}$	$t^s(A_0 t^n + \cdots + A_n)e^{\alpha t}$	multiplicity of α as a root
$(a_0 t^n + \cdots + a_n)e^{\alpha t}\cos(\beta t)$ or $(a_0 t^n + \cdots + a_n)e^{\alpha t}\sin(\beta t)$	$t^s[(A_0 t^n + \cdots + A_n)e^{\alpha t}\cos(\beta t) +$ $(B_0 t^n + \cdots + B_n)e^{\alpha t}\sin(\beta t)]$	multiplicity of $\alpha + i\beta$ as a root

The multiplicity of x as a root is 0 if $x \neq r_{1,2}$, 1 if $x = r_1 \neq r_2$ or $x = r_2 \neq r_1$ and 2 if $x = r_1 = r_2$.

(3) Plug into eq. and solve for the A_i, B_j: get $Y(t)$.
(4) Get the general solution: $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$

2.3. **Variation of Parameters.**

$$L[y] := y'' + p(t)y' + q(t)y = g(t) \tag{16}$$

p, q, g continuous on I.

When to use: Have fundamental set y_1, y_2 need particular solution $Y(t)$

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\mathrm{d}t + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\mathrm{d}t \tag{17}$$

You can set the integration constants to anything you want. If you keep them you get the general solution.

3. Vibrations: Mass on a Spring

In equilibrium: Mass m hanging on spring with spring constant k. Spring is extended to length L.

$$mg = kL \tag{18}$$

Equation of motion: γ damping constant

$$mu''(t) + \gamma u'(t) + ku(t) = g(t) \tag{19}$$

3.1. **Homogeneous/Free.**

Undamped: $\gamma = 0$

$$mu''(t) + ku(t) = 0 \tag{20}$$

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\omega_0 t - \delta) \tag{21}$$

$$\omega_0^2 = \frac{k}{m} \tag{22}$$

$$A = R\cos(\delta) \qquad B = R\sin(\delta) \tag{23}$$

$$R = \sqrt{A^2 + B^2} \qquad \tan(\delta) = B/A \tag{24}$$

With damping: $\gamma \neq 0$

$$mu''(t) + \gamma u'(t) + ku(t) = 0 \tag{25}$$

Roots: $r_{1,2} = \frac{\gamma}{2m}(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}})$

(26)

$$u(t) = \begin{cases} Ae^{r_1 t} + B^{r_2 t} & \gamma^2 > 4km \text{ overdamped} \\ (A + Bt)e^{-\gamma t/2m} & \gamma^2 = 4km \text{ critically damped} \\ e^{-\gamma t/2m}(A\cos(\mu t) + B\sin(\mu t)) = Re^{-\gamma t/2m}\cos(\mu t + \delta) & \gamma^2 < 4km \end{cases}$$

$\mu = \frac{(4km-\gamma^2)^{\frac{1}{2}}}{2m}$ is the *quasi frequency*, $T = \frac{2\pi}{\mu}$ is the *quasi period*, R and δ as in eqs. (23) and (24).

3.2. **Inhomogeneous/Forced.**

$$mu''(t) + \gamma u'(t) + ku(t) = F_0\cos(\omega t) \tag{27}$$

Special solution.

$$U(t) = R\cos(\omega t + \delta) \tag{28}$$

$$R = \frac{F_0}{\Delta}, \quad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \sin\delta = \frac{\gamma\omega}{\Delta}, \quad \Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}, \quad \omega_0^2 = \frac{k}{m}$$

Resonance at $\omega_0 \approx \omega$ maximum amplitude

$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1 - (\gamma^2/4mk)}} \approx \frac{F_0}{\gamma\omega_0}(1 + \frac{\gamma^2}{8mk}) \tag{29}$$

4. Higher order equations

4.1. **Theorems.** Consider

$$L[y] := y^{(n)} + p_1(t)y^{(n-1)} + \dots p_n(t)y, \quad L[y] = g(t) \tag{30}$$

Theorem If $p_1, \dots, p_n, g$ continuous on an interval I, then

- ▷ there is a unique solution on I for any given set of initial conditions

$$y(t_0) = y_0, y'(t_0) = y_0', \dots, y^{(n-1)}(t_0) = y_0^{(n-1)} \tag{31}$$

- ▷ the *general solution* to the homogeneous equation $L[y] = 0$ is given by

$$y(t) = c_1y_1(t) + c_2y_2(t) + \dots + c_ny_n(t) \tag{32}$$

  where $y_1, \dots, y_n$ are a *fundamental set*, i.e. n linearly independant solutions.
- ▷ n solutions $y_1, \dots, y_n$ are a fundamental set on I if and only if the Wronskian $W(y_1, \dots, y_n)(t_0) \neq 0$ for some $t_0 \in I$ and hence for all $t \in I$. Here

$$W(y_1, \dots, y_n)(t) = \left| \begin{pmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \\ y_1'(t) & y_2'(t) & \dots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \dots & y_n^{(n-1)}(t) \end{pmatrix} \right| \tag{33}$$

- ▷ If $Y(t)$ is *a particular* solution of $L[y] = g(t)$ and $y_1, \dots, y_n$ are a fundamental set for $L[y] = 0$ then *all* solutions of $L[y] = g(t)$ are of the form

$$y(t) = c_1y_1(t) + \dots + c_ny_n(t) + Y(t) \tag{34}$$

4.2. **Constant coefficients: homogeneous case.**

$$a_0y^{(n)} + a_1y^{(n-1)} + \dots + a_ny = 0 \tag{35}$$

Solution: Guess: e^{rt}, equation for r:

$$a_0r^n + a_1r^{n-1} + \dots + a_n = 0 \tag{36}$$

The equation has n possibly complex roots counted with multiplicity

$$a_0r^n + a_1r^{n-1} + \dots + a_n = a_0(r - r_1)\dots(r - r_n) \tag{37}$$

Since the coefficients are real, the roots are either real or appear in complex conjugate pairs. The multiplicity of a fixed root α is the number of factors $(r - \alpha)$ in eq. (37).

roots	summands of general solution
α real root multiplicity s	$e^{\alpha t}, te^{\alpha t}, \dots, t^{s-1}e^{\alpha t}$
complex conjugate pair $(\alpha, \bar{\alpha}) = \lambda \pm i\mu$ of multiplicity s	$e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t), te^{\lambda t}\cos(\mu t), te^{\lambda t}\sin(\mu t), \dots,$ $t^{s-1}e^{\lambda t}\cos(\mu t), t^{s-1}e^{\lambda t}\sin(\mu t)$

4.3. **Integer coefficients, rational roots.** When to use: If $a_i \in \mathbb{Z}$, can try to guess rational roots. If p/q is a root, then q divides a_0 and p divides a_n.

Caveat: Notice this may not give all/any of the roots.

4.4. **Constant coefficients: Inhomogenous case – Undetermined coefficients.** Same as in §2.2 only now the multiplicity s of x can be $0, 1, \dots, n$.