3rd Review Sheet
Math 266
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Disclaimer: This sheet is neither claimed to be complete nor indicative.

1. Types of equations and techniques

1.1. **Laplace transform.** We use the notation $\mathcal{L}(f(t)) = F(s)$

$$\mathcal{L}(f^{(n)}(t) = s^n F(S) - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0) \tag{1}$$

if $f, f', \ldots, f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$ and the functions are suitably bounded.

To solve differential equations:

(1) Laplace transform equation to get algebraic equation.
(2) Solve algebraic equation.
(3) Do inverse transform.

Use of step functions.

Example:

$$f(t) = \begin{cases} f_1(t) & t < c \\ f_2(t) & c \le t \end{cases}$$

$$f(t) = f_1(t) + u_c(t)[f_2(t) - f_1(t)] \tag{2}$$

Techniques. Convolution, shifts, δ–functions, and so forth are all in given table.

2. Systems of linear equations

2.1. **Matrices: basic facts.** A is an $n \times n$ matrix. We assume real coefficients.

Eigenvectors $\vec{x}$ is an *Eigenvector* with *Eigenvalue* r if

$$A\vec{x} = r\vec{x} \tag{3}$$

A has a basis of Eigenvectors if there are n linear independent Eigenvectors. $\vec{x}^{(1)} \ldots \vec{x}^{(n)}$. If $\vec{x}^{(i)} = (x_{1i}, \ldots, x_{ni})$ let $T = (x_{ij})$ the matrix whose column vectors are the Eigenvectors. Then

$$T^{-1}AT = D = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix} \tag{4}$$

Characteristic polynomial. $\chi(r) = \det(A - r\mathbf{I})$

Eigenvalues/roots $\chi(A) = (r - r_1)^{s_1} \ldots (r - r_l)^{s_k}$. Note, the r_i may be complex.

If A is real then if r is a complex root, then so is $\bar{r}$. Complex roots occur in pairs $r_{l,l+1} = \lambda \pm i\mu$.

Multiplicities. s_i is the *algebraic multiplicity* of r_i.

The *geometric multiplicity* of r_i is $dim(ker(A - r_i\mathbf{I}))$. This is the maximal number of linearly independent Eigenvectors with Eigenvalue r_i.

2.2. **Constant Coefficients.**

$$\vec{x}' = A\vec{x}$$

Fact: If $\vec{\xi}$ is an Eigenvector of A with Eigenvalue r then

$$x(\vec{t}) = \vec{\xi}e^{rt} \tag{5}$$

is a solution. Linearly independent Eigenvectors give linearly independent solutions.

If r is complex. We get two real solutions. Write $\xi = \vec{a} + i\vec{b}$ with $\vec{a}$ and $\vec{b}$ real.

$$\vec{x}^{(l)}(t) = e^{\lambda t}[\vec{a}\cos(\mu t) - \vec{b}\sin(\mu t)] \tag{6}$$

$$\vec{x}^{(l+1)}(t) = e^{\lambda t}[\vec{b}\cos(\mu t) + \vec{a}\sin(\mu t)] \tag{7}$$

2.3. **The n=2 cases (see Figures).**

2.3.1. *A is diagonalizable.* This means that there are two linearly independent Eigenvectors $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ with Eigenvalues r_1, r_2.

Assume $r_1 \neq r_2$. The *phase plane* will be as follows.

$r_{1,2} < 0$	asymptotically stable node	solutions converge to axis of largest Eigenvalue
$r_{1,2} > 0$	unstable node	solutions diverge. Axis of largest Eigenvalue is an asymptote as $t \to \infty$. (arrow reversal of previous situation)
r_1, r_2 opposite sign	saddle point	solutions diverge. Axis of largest Eigenvalue is asymptote as $t \to \infty$.
$r_{1,2} = \lambda \pm i\mu, \lambda < 0$	asymptotically stable spiral	solutions spiral to origin
$r_{1,2} = \lambda \pm i\mu, \lambda > 0$	unstable spiral	solutions diverge

OTHER BEHAVIORS:

$r_{1,2} = \pm i\mu, \lambda = 0$	closed ellipses	solutions bounded
$r_1 > 0$ degenerate	unstable improper node	
$r_1 < 0$ degenerate	stable improper node	

2.4. **Repeated roots.** If $\vec{\xi}$ is an Eigenvector with geometric multiplicity 1 and algebraic multiplicity 2.

$$A\vec{\xi} = r\vec{\xi} \tag{8}$$

Solve

$$(A - r\mathbf{I})\vec{\eta} = \vec{\xi} \tag{9}$$

Solutions of differential equation

$$\vec{x}^{(1)}(t) = \vec{\xi}e^{rt} \tag{10}$$

$$\vec{x}^{(2)}(t) = \vec{\xi}te^{rt} + \vec{\eta}e^{rt} \tag{11}$$

2.5. **Matrix notations.**

2.5.1. *Fundamental matrix.* Let $x^{(1)}(t), \ldots, x^{(n)}(t)$ be a fundamental set of solutions. Set

$$\mathbf{\Psi} = \left(x^{(1)}(t) \ldots x^{(n)}(t)\right) \tag{12}$$

the matrix whose column vectors are the $x^{(i)}(t)$. This is called *a fundamental matrix.*

Then

$$\mathbf{\Psi}' = A\mathbf{\Psi} \tag{13}$$

2.5.2. *Matrix exponential.* A matrix solution is $\mathbf{\Psi} = exp(At)$.

2.5.3. *Diagonalizble A.* If $A = TDT^{-1}$ where $T = (\xi_0 \dots \xi_n)$ is the matrix of Eigenvectors. Then

$$\mathbf{\Psi} = Texp(Dt)$$

is a solution.

2.5.4. *General A (over $\mathbb{C}$).* Use Jordan normal form $J = T^{-1}JT$ the solution is

$$\mathbf{\Psi} = Texp(Jt)$$

2.6. **Nonhomogeneous equations.**

$$\vec{x}' = A\vec{x} + \vec{g}(t) \tag{14}$$

2.6.1. *Diagonalizable A.* If A is diagonalizable $D = T^{-1}AT$, with T the matrix of Eigenvectors. Set

$$\vec{x} = T\vec{y} \tag{15}$$

Solve

$$\vec{y}' = D\vec{y} + T^{-1}\vec{g}(t) \tag{16}$$

This is a system which is decoupled. Solve the system to get $\vec{y}$. Plug solution $\vec{y}$ into eq. (15) to get $\vec{x}$.

2.6.2. *Undetermined Coefficients.* Just like in the case of one equation, we can guess solutions.

In case there are no multiplicities (multiplicity of root is zero), use the forms from the case of one equation, *but with each constant replaced by a constant vector.* E.g. $\vec{a}e^{5t}$ or $\vec{a}t^2 + \vec{b}t + \vec{c}$.

In case of $e^{\lambda t}$ with λ of multiplicity 1 use

$$\vec{a}te^{\lambda t} + \vec{b}e^{t} \tag{17}$$

2.6.3. *Laplace transform.* The Laplace transform of (14) is

$$s\vec{x}(s) - \vec{x}(0) = A\vec{x}(s) + \vec{g}(s)$$

A solution is

$$(sId - A)\vec{x}(s) = \vec{g}(s) + \vec{x}(0)$$

or

$$\vec{x}(s) = (sId - A)^{-1})(\vec{g}(s) + \vec{x}(0))$$

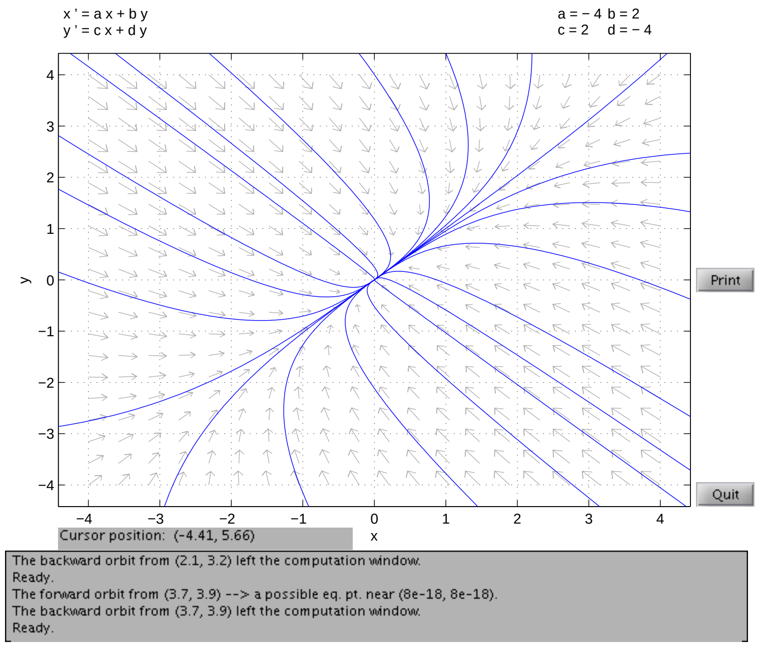

FIGURE 1. Stable node

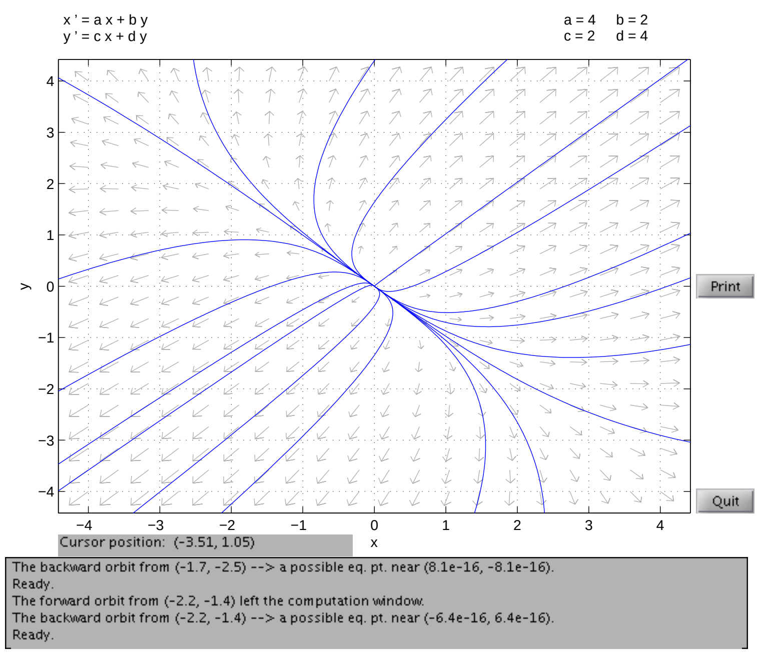

FIGURE 2. Unstable node

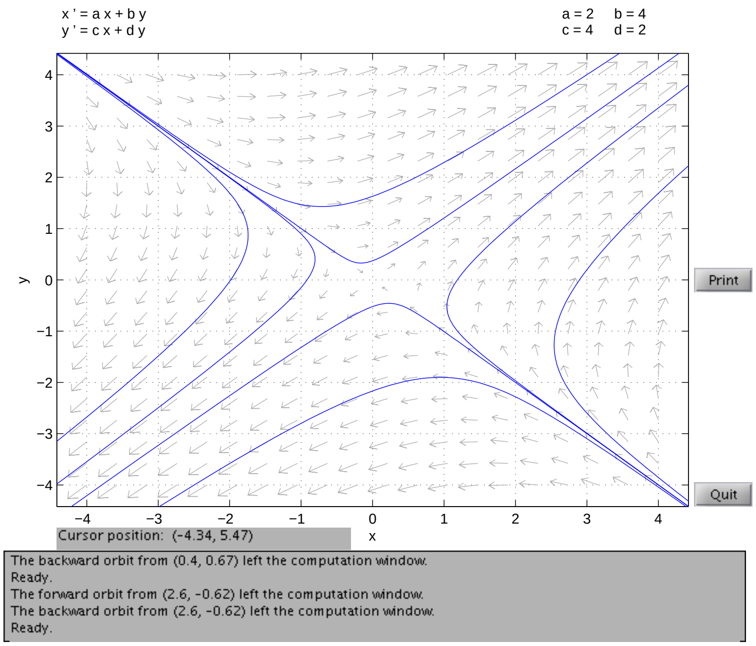

FIGURE 3. saddle

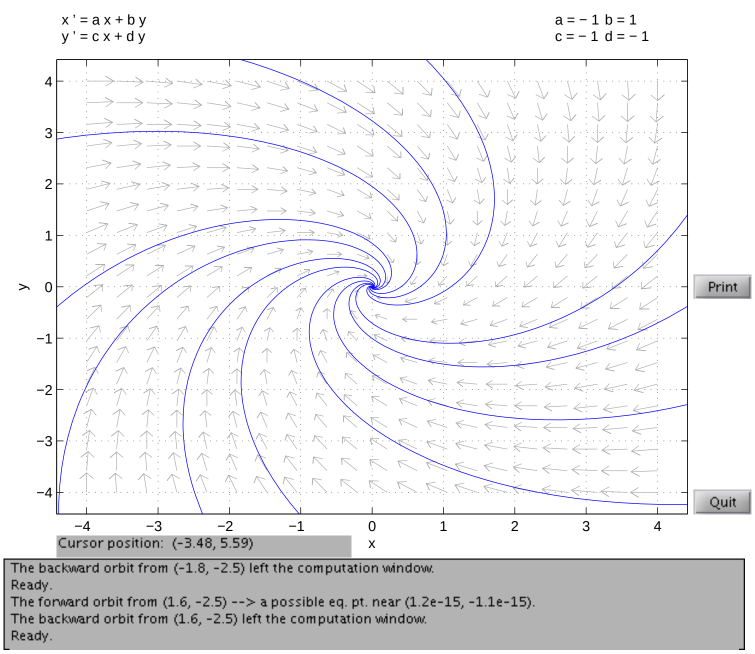

FIGURE 4. Stable spiral

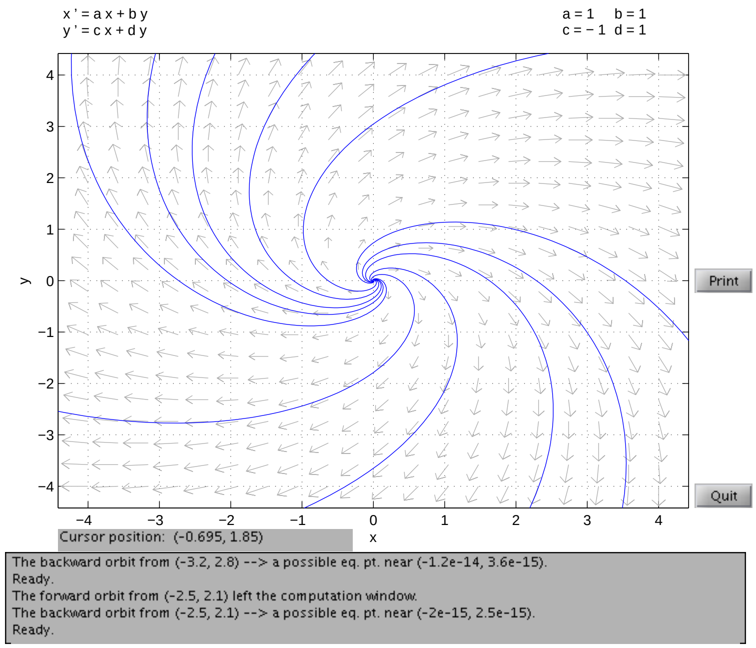

FIGURE 5. Unstable spiral

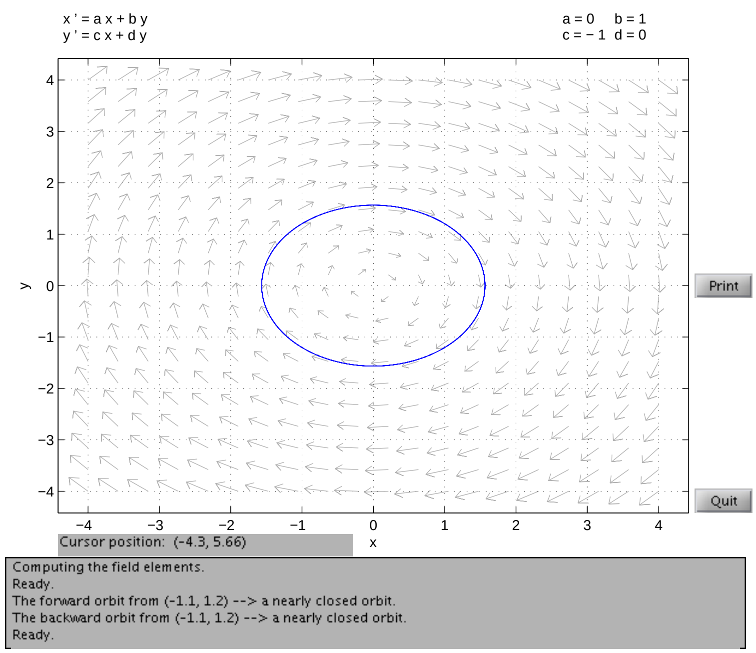

FIGURE 6. Closed trajectories

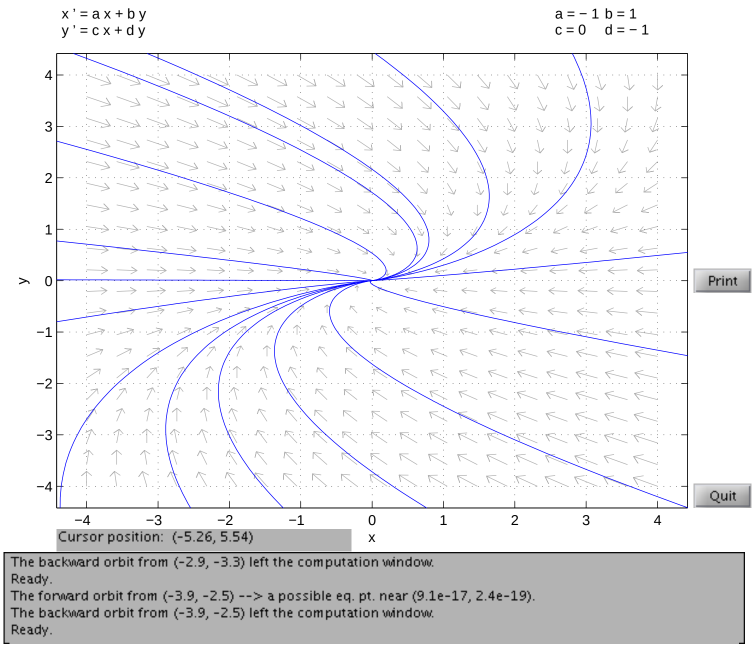

FIGURE 7. Improper stable node

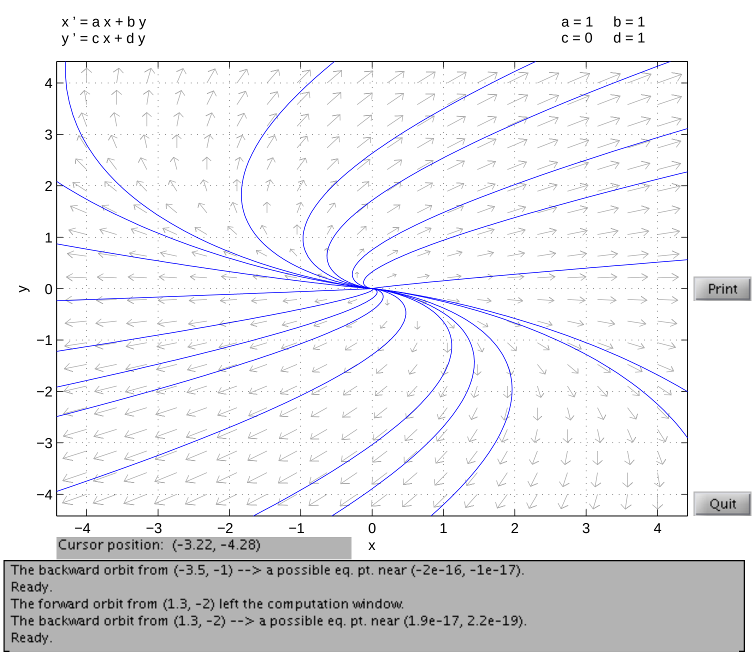

FIGURE 8. Improper unstable node