MA 571 Fall 2014. Problems for the first midterm.

The midterm will consist of three to five problems mostly chosen from the following three lists.

- Definitions and examples
- Assigned exercises and theorems from the book
- Qualifying exam problems

On the exam, it's not enough just to have the right idea or to say something that's approximately correct—you must give a solution which is mathematically correct and is explained in a clear and logical way to get full points.

Unless otherwise stated, you may use anything in Munkres' book in your solution—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.

Definitions and examples

1. Give the definition of the subspace topology.
2. Give the definition of the product topology.
3. How does the product topology behave w.r.t. the subspace topology and the order topology?
4. Give the definition of a quotient map. Give a quotient map that is open, but not closed, and one that is neither open nor closed.
5. Give the definition of a limit point and the definition of a sequence converging to a point.
6. Give the definition of a continuous function.
7. Give the definition of a metric, the metric topology and a metrizable space.

Theorems and exercises

1. Prove that a function to a product space is continuous if and only if its components are.
2. Prove that a subspace is closed if and only if it contains all its limit points.
3. Prove that the projection maps for a product are open maps.
4. Prove that $Bd(A) = \emptyset$ if and only if A is open and closed.
5. Prove that a metric space satisfies the first countability axiom.
6. Prove that $\mathbb{R}^\omega$ is not metrizable in the box topology.
7. Show that the diagonal map is not continuous in the box topology, but it is in the product topology.
8. Prove the sequence lemma.
9. Give an example of a surjective map of spaces that is not a quotient map.
10. Prove that if f_n is a sequence of functions $X \to \mathbb{R}$ considered as elements of $X^{\mathbb{R}}$ with the product topology, then $f_n \to f$ if and only if for each $x \in X$ the sequence $f_n(x)$ converges to the point $f(x)$.
11. Prove that if f_n is a sequence of functions $X \to \mathbb{R}$ considered as elements of $X^{\mathbb{R}}$ with the topology induced by the uniform metric $\bar{\rho}$, then $f_n \to f$ if and only if for the sequence of functions f_n uniformly converges to the point f. (Recall that $f_n : X \to Y$, with Y a metric space, uniformly converges to f if for any $\epsilon > 0$ there exists an integer N such that for all $n > N$ and $x \in X$: $d_y(f_n(x), f(x)) < \epsilon$)
12. Give an example of a quotient space of a Hausdorff space that is not Hausdorff.
13. Formulate and prove the universal property for quotient maps.
14. Formulate and prove the universal property for the product topology.

Qualifying exam problems

1. Let X_α be a family of topological spaces.

   For each α, let A_α be a subset of X_α.

   Prove that
$$\overline{\prod_\alpha A_\alpha} = \prod_\alpha \overline{A_\alpha}.$$

2. Let X_α be a family of topological spaces.

   For each α, suppose that we are given a point b_α of X_α.

   Let $Y = \prod_\alpha X_\alpha$, with the product topology. Let $\pi_\alpha : Y \to X_\alpha$ be the projection.

   Prove that the set

   $$S = \{y \in Y \mid \pi_\alpha y = b_\alpha \text{ except for finitely many } \alpha\}$$

   is dense in Y (that is, its closure is Y).

3. Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X; thus an element of Y is a sequence of 0's and 1's.

   For each $n \geq 1$, let $y_n \in Y$ be the element $(1, 1, \ldots, 1, 0, 0, \ldots)$, with n 1's at the beginning and all other entries 0. Let $y \in Y$ be the element with all 1's. **Prove** that the set $\{y_n\}_{n\geq 1} \cup \{y\}$ is closed. Give a clear explanation. Do not use a metric.

4. **Prove** that the countable product

   $$\prod_{n=1}^{\infty} \mathbb{R}$$

   (with the product topology) has the following property: there is a countable family $\mathcal{F}$ of neighborhoods of the point

   $$\mathbf{0} = (0, 0, \ldots)$$

   such that for every neighborhood V of $\mathbf{0}$ there is a $U \in \mathcal{F}$ with $U \subset V$.

   Note: the book proves that $\prod_{n=1}^{\infty} \mathbb{R}$ is a metric space, but you may not use this in your proof. Use the definition of the product topology.

5. Let X and Y be topological spaces and let $f : X \to Y$ be a function with the property that

   $$f(\overline{A}) \subset \overline{f(A)}$$

   for all subsets A of X.

   Prove that f is continuous.

6. Let X be a topological space and A a subset of X. Suppose that

   $$A \subset \overline{X - \overline{A}}.$$

   Prove that $\overline{A}$ does not contain any nonempty open set

7. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. **Prove** that

   $$f(\overline{A}) \subset \overline{f(A)}$$

   for all subsets A of X.

8. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Let G_f (called the *graph* of f) be the subspace $\{ (x, f(x)) \,|\, x \in X \}$ of $X \times Y$. **Prove** that if Y is Hausdorff then G_f is closed.

9. Let S^1 denote the circle

$$\{x^2 + y^2 = 1\}$$

in $\mathbb{R}^2$. Define an equivalence relation on S^1 by

$$(x, y) \sim (x', y') \Leftrightarrow (x, y) = (x', y') \text{ or } (x, y) = (-x', -y')$$

(you do *not* have to prove that this is an equivalence relation). **Prove** that the quotient space $S^1/\sim$ is homeomorphic to S^1.

One way to do this is by using complex numbers.

10. Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ defined by $(x, y) \sim (x', y')$ if and only if there is a nonzero t with $(x, y) = (tx', ty')$. **Prove** that the quotient space $\mathbb{R}^2/\sim$ not Hausdorff.

11. A subset $A \subset X$ is called dense if its closure $\bar{A} = X$. If f and g are continuous functions on a topological space X with values in a Hausdorff space Y and f and g agree on a dense subset of X, then $f = g$.

12. Formulate and prove the ϵ–δ criterion for a map between two metrizable spaces $f : X \to Y$.