Chapter 6

Arc geometry and algebra: foliations, moduli spaces, string topology and field theory

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Introduction

There has long been an intricate relationship between foliations, arc complexes and the geometry of Teichmüller and moduli spaces [49], [12]. The study of string theory as well as that of topological and conformal field theories has added a new aspect to this theory, namely to study these spaces not only individually, but together all at once. The new ingredient is the idea to glue together surfaces with their additional data. Physically, this can for instance be viewed as stopping and starting time for the generation of the world-sheet. Mathematically, the general idea of gluing structures together in various compatible ways is captured by the theory of operads and PROPs [41], [3], [40]. This theory was originally introduced in algebraic topology to study loop spaces, but has had a renaissance in conjunction with the deepening interaction between string theory and mathematics.

The $\mathcal{A}rc$ operad of [31] specifically provides the mathematical tool for this approach using foliations. Combinatorially, the underlying elements are surfaces with boundaries and windows on these boundaries together with projectively weighted arcs running in between the windows. Geometrically these elements are surfaces with partial measured foliations. This geometric interpretation is the basis of the gluing operation. We glue the surfaces along the boundaries, matching the windows, and then glue the weighted arcs by gluing the respective foliations. The physical interpretation is that the mentioned foliations are transversal to the foliation created by the strings. The details of this picture are given in [32].

The gluing operation, which is completely natural from the foliation point of view, yields a surface based geometric model, for a surprising abundance of algebraic and geometric structures germane to loop spaces, string theory, string topology as well as conformal and topological field theories. Surprisingly this also includes higher-dimensional structures such as the little k-cubes, associahedra, cyclohedra and D-branes. This is true to the slogan that one only needs strings. It gives for instance rise to models for the little discs and framed little discs operads, moduli space and the Sullivan-PROP. These models exist on the topological, the chain and the homology levels. On the chain and homology these operads and PROPs correspond to Gerstenhaber, BV algebras, string topology operations and CFT/string field theory operations. One characteristic feature is that they are very small compared to their classical counterparts. Topologically this means that they are of small dimension. On the chain level this means that they are given by a small cellular model.

A classical result is that the little discs operad detects two-fold loop spaces, consequently so does the arc operad. The classical theory about loop spaces goes further to state that k-fold loop spaces are detected by the little k cubes or any E_k operad. This generalized to $k = \infty$. By using a stabilization and a unital fattening of a natural suboperad of $\mathcal{A}rc$ one obtains a surface model for all these operads. A consequence is a new infinite loop space spectrum coming from the stabilized unital fattened $\mathcal{A}rc$ operad.

Another consequence of the foliation description are natural actions of the chains on the Hochschild cohomology of associative or Frobenius algebras, lifting the Gerstenhaber algebra structure on the cohomology. This type of action was conjectured by Deligne and has been a central theme in the last decade. One important application is that this type of action together with the fact that the little discs are formal as an operad implies Kontsevich's deformation quantization.

There is a vast extension of this chain level action, which gives a version of string topology for simply connected spaces. For this the surface boundaries are classified as "in" or "out" boundaries. This is the type of setup algebraically described by a PROP. In particular, as a generalization of the above results there is a PROP, the Sullivan-PROP, which arises naturally in the foliation picture. Again there is a CW model for it and its chains give an action on the Hochschild co-chains of a Frobenius algebra extending the previous action.

This action further generalizes to a model for the moduli space of surfaces with marked points and tangent vectors at these marked points as it is considered in conformal field theory and string field theory. Both actions are given by a discretization of the $\mathcal{A}rc$ operad and their algebra and combinatorics are geometrically explained by foliations with integer weights.

Finally, there is an open/closed version of the whole theory. This generalizes the actions as well. On the topological level one consequence of this setup is a clear geometric proof of the minimality of the Cardy–Lewellen axioms for open/closed topological field theory using Whitehead moves.

Thus the $\mathcal{A}rc$ operad and its variations provide a wonderful, effective, geometric tool to study and understand the origin of these algebraic structures and give new results and insights. The explicit homotopy BV equation provided by the arc operad given in Figure 11 or the geometric representation of the classical $\cup_i$ products in Figure 17 may serve as an illustration.

We expect that this foliation geometry together with the operations of gluing will provide new results in other fields such as cluster algebras, $2+1$-dimensional TFTs and any other theory based on individual moduli spaces.

Scope

The scope of the text is a subset of the results of the papers [31], [22], [26], [33], [32], [28], [27], [25], [29] and [32]. It is the first time that all the various techniques developed in the above references are gathered in one text. We also make some explicit interconnections that were previously only implicit in the total body of results. The main ones being the stabilization of $\mathcal{A}rc$ and the arc spectrum and the analysis of the S^1 equivariant geometry.

For a more self-contained text, we have added an appendix with a glossary containing basic notions of operads/PROPs and their algebras as well as Hochschild cohomology and Frobenius algebras.

Layout of the exposition

In this theory there are usually two aspects. First and foremost there is the basic geometric idea about the structure, and then secondly there is a more technical mathematical construction to make this idea precise. This gives rise to the basic conundrum in presenting the theory. If one first defines the mathematically correct notions, one has to wait for quite a while before hearing the punch line. If one just presents the ideas, one is left with sometimes a formidable task to make mathematics out of the intuitive notions. We shall proceed by first stating the idea and then giving more details about the construction and, if deemed necessary, end with a comment about the finer details with a reference on where to find them.

The text is organized as follows: In the first section, we introduce the spaces of foliations we wish to consider. Here we also give several equivalent interpretations for the elements of these spaces. There is the foliation aspect, a combinatorial graph aspect and a dual ribbon graph version.

The basic gluing operation underlying the whole theory is introduced in Section 2 as are several slight variations needed later. This leads to the $\mathcal{A}rc$ operad, which is a cyclic operad. We also study its discretization and the chain and homology level operads. For the homology level, we also discuss an alternative approach to the gluing which yields a modular operad structure on the homology level. We furthermore elaborate on the natural S^1-actions and the resulting geometry. The chain and homology levels for instance yield new geometric examples for the so-called $\mathfrak{K}$-modular operads.

Section 3 contains the explicit description of the little discs and framed little discs in this framework. Technically there are suboperads of $\mathcal{A}rc$ that are equivalent, that is quasi-isomorphic, to them. This includes an explicit presentation of the Gerstenhaber and BV structures and their lift to the chain level. In this section we give an explicit geometric representation for the bracket and the homotopy BV equation, see Figures 9 and 11.

Section 4 contains the generalization to the Sullivan-(quasi)-PROP which governs string topology and the definition of the rational operad given by the moduli spaces. There are some fine points as the words "rational" and "quasi-" suggest which are fully explained.

Another fine point is that the operads as presented do not have what is sometimes called a unit. This is not to be confused with the operadic unit that they all possess. This is why we call operads with a unit "pointed". For the applications to string topology and Deligne's conjecture the operads need not be pointed. However, for the applications to loop spaces this is essential. The main point of Section 5 is to give the details of how to include a unit and make the operad pointed. This allows us to fatten the $\mathcal{A}rc$ to include a pointed E_2 operad and hence detect double loop spaces. The other basic technique given in this section is that of stabilization. The result of combining both adding a unit and stabilization leads to the E_k operads, explicit geometric representatives for the $\cup_i$ products (see Figure 17) and a new spectrum, the $\mathcal{A}rc$ spectrum.

The various chain level actions are contained in Section 6. The first part is concerned with Deligne's conjecture and its A_∞, the cyclic and cyclic A_∞ generalization. These are given by actions given by a dual tree picture. The section also contains the action of the chain level Sullivan-PROP and that of moduli space on the Hochschild cochains of a Frobenius algebra. For this we introduce correlation functions based on the discretization of the foliations. As a further application we discuss the stabilization and the semi-simple case.

We close the main text in Section 7 with a very brief sketch of the open/closed theory.

Conventions

We fix a field k. For most constructions any characteristic would actually do, but sometimes we use the isomorphism between $\mathbb{S}_n$ invariants and $\mathbb{S}_n$ co-invariants in which case we have to assume that k is of characteristic 0. There is a subtlety about what is meant by Gestenhaber in characteristic 2. We will ignore this and take the algebra over the operads in question as a definition.

When dealing with operads, unless otherwise stated, we always take $H_*(X)$ to mean $H_*(X,k)$, so that we can use the Künneth theorem to obtain an isomorphism $H^*(X \times Y, k) \simeq H_*(X) \otimes_k H_*(Y)$.

1 The spaces

1.1 The basic idea

As any operad the $\mathcal{A}rc$ operad consists of a sequence of spaces with additional data, such as symmetric group actions and gluing maps. There are two ways in which to view the spaces:

Geometric version. The spaces are projectively weighted families of arcs on surfaces with boundary that end in fixed windows at the boundary considered up to the action of the mapping class group.

An alternative equivalent useful characterization is:

Combinatorial version. The spaces are projectively weighted graphs on surfaces with boundaries, where each boundary has a marked point and these points are the vertices of the graph, again considered up to the action of the mapping class group.

The geometric version can be realized by partial measured foliations which make the gluing natural, while the combinatorial version allows one to easily make contact with moduli space and other familiar spaces and operads, such as the little discs, the Sullivan-PROP, etc.

1.2 Windowed surfaces with partial measured foliations

We will now make precise the geometric version following [31].

1.2.1 Data and notation. Let $F = F^s_{g,r}$ be a fixed oriented topological surface of genus $g \geq 0$ with $s \geq 0$ punctures and $r \geq 1$ boundary components, where $6g - 7 + 4r + 2s \geq 0$. Also fix an enumeration $\partial_1, \partial_2, \dots, \partial_r$ of the boundary components of F once and for all.

Furthermore, in each boundary component ∂_i of F, fix a closed arc $W_i \subset \partial_i$, called a *window*.

The *pure mapping class group* $\mathrm{PMC} = \mathrm{PMC}(F)$ is the group of isotopy classes of all orientation-preserving homeomorphisms of F which fix each $\partial_i - W_i$ pointwise (and fix each W_i setwise).

Define an *essential arc* in F to be an embedded path a in F whose endpoints lie in the windows, such that a is not isotopic rel endpoints to a path lying in ∂F. Two arcs are said to be *parallel* if there is an isotopy between them which fixes each $\partial_i - W_i$ pointwise (and fixes each W_i setwise).

An *arc family* in F is the isotopy class of a non-empty unordered collection of disjointly embedded essential arcs in F, no two of which are parallel. Thus, there is a well-defined action of PMC on arc families.

1.2.2 Induced data. Fix F. There is a natural partial order on arc families given by inclusion. Furthermore, there is a natural order on all the arcs in a given arc family as follows. Since the surface is oriented, so are the windows. Furthermore we enumerated the boundary components. This induces an order, by counting the arcs by starting in the first window in the order they hit this window, omitting arcs that have already been enumerated and then continuing in the same manner with the next window.

This procedure also gives an order $<_i$ to all arcs incident to a specific boundary ∂_i.

1.3 The spaces of weighted arcs

We define $K^s_{g,r}$ to be the semi-simplicial realization of the poset of arcs on $F^s_{g,r}$. This is the simplicial complex which has one simplex for each arc family α with the i-th face maps given by omitting the i-th arc. The dimension of such a simplex is the number of arcs $|\alpha|$ minus 1. Hence the vertices of this complex correspond to the arc families consisting of single arcs.

The space $|K^s_{g,r}|$ has a natural continuous action of $\mathrm{PMC}(F^s_{g,r})$. We define $A^s_{g,r} := |K^s_{g,r}|/\mathrm{PMC}(F^s_{g,r})$. This space is not necessarily simplicial any more, but it remains a CW complex whose cells are indexed by PMC orbits of arc families, which we denote by $[\alpha]$. The number of arcs is invariant under the PMC action. With the notation $|[\alpha]| := |\alpha|$ the dimension of the cell indexed by $[\alpha]$ is $|[\alpha]| - 1$.

We also consider the de-projectivized version $|K^s_{g,r}| \times \mathbb{R}_{>0}$ and its PMC quotient $D^s_{g,r} = A^s_{g,r} \times \mathbb{R}_{>0}$.

1.4 Different pictures for arcs

Depending on the circumstances there are different completely equivalent pictures which we can use to be closer to intuition. There are the following choices for the windows.

I Disjointly embedded arcs with endpoints in windows.

II Shrinking the complement of the (open) window to a point. The two endpoints of the window then are identified and give a distinguished point on the boundary. Arcs still do not intersect pairwise and avoid the marked points on the boundary. This version is particularly adapted to understand the S^1 action (see 2.7) and the operads yielding the Gerstenhaber and BV structures.

III Shrinking the window to a point. The arcs may not be disjointly embedded at the endpoints anymore, but they form an embedded graph. This is a version that is very useful in combinatorial descriptions, e.g. a dual graph approach for moduli spaces.

These are depicted in Figure 1.

We also may choose the following different pictures for the arcs as we discuss in 2.1 in greater detail.

A Arcs with weights.

B Bands of leaves with (transversal) width.

C Bands of leaves with width filling the windows.

The arcs with weights are the quickest method to construct the relevant spaces, the bands-of-leaves picture is what makes the operadic gluing natural. It also greatly helps elucidate the S^1 action and the discretization that acts on the Hochschild complexes.

The cases I, II, III A and I B are depicted in Figure 1, the cases I C and II C are in Figure 2. (In the figures u, v, w denote positive real weights.)

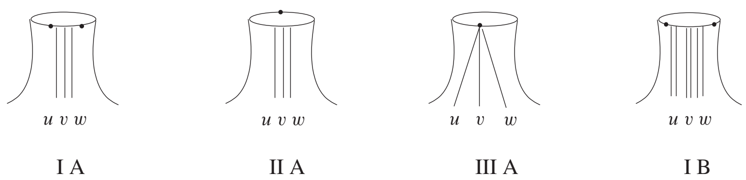


Figure 1. I A. Arcs running to a point on the boundary. II A. Arcs running to a point at infinity. III A. Arcs in a window. I B. Bands in a window.

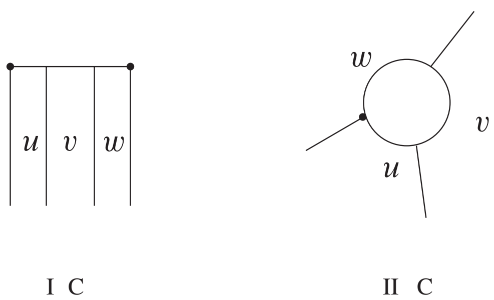


Figure 2. I C. Bands ending on an interval. II C. Bands ending on a circle.

1.4.1 Example. Consider $K^0_{0,2}$, see Figure 3. The 0-simplices are given by (the isotopy class of) a straight arc and all its images under a Dehn twist. Thus the 0 skeleton can be identified with $\mathbb{Z}$. It is possible to embed two arcs which differ by one Dehn twist. Calling these one cells $I_{i\,i+1}$, if the first arc is the i-fold Dehn twist

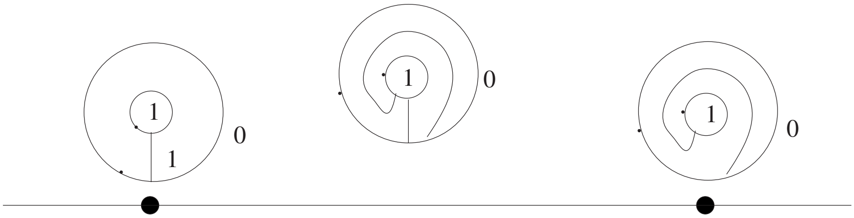


Figure 3. The space $K^0_{0,2} = \mathbb{R}$ as a simplicial space. We indicated two 0-simplices and a connecting 1-simplex.

of 0, we see that $|K^2_{0,2}| = \mathbb{R}$. PMC$(F^0_{0,2})$ is generated by the Dehn twist and hence $A^0_{0,2} = \mathbb{R}/\mathbb{Z} = S^1$.

This S^1 is what underlies the BV geometry, see §3.

1.4.2 Elements as weighted arc families. A weight function wt on an arc family α is a map that associates to each arc of α a positive real number. There is a natural scaling action by $\mathbb{R}_{>0}$ on the set of weight functions on α and we denote by [wt] the equivalence class of a given weight function wt under this action.

An element $a \in |A^s_{g,r}|$ in the realization of $A^s_{g,r}$ lies in a unique open simplex. If α is the arc family indexing this simplex, then using the enumeration of arcs, we can identify the barycentric coordinates with weights on the arcs of α. In this picture, a codimension-one boundary is given by sending one of the weights to zero. We are free to think of the barycentric coordinates as a projective class [wt] of a positive weight function wt on the arcs. In this fashion $a = (\alpha, [\mathrm{wt}])$.

In this picture, the elements of $|K^s_{g,r}| \times \mathbb{R}_{>0}$ are naturally pairs (α, wt) of an arc family together with a weight function.

Taking the quotient by PMC, we get a description of elements of $D^s_{g,r}$ as pairs $([\alpha], \mathrm{wt})$ where $[\alpha]$ denotes the PMC orbit of α. Further taking the quotient with respect to the $\mathbb{R}_{>0}$ action elements of $A^s_{g,r}$ are pairs $([\alpha], [\mathrm{wt}])$.

1.4.3 Weights at the boundary. Taking up the picture above, given $(\alpha, \mathrm{wt}) \in D^s_{g,r}$, we define the weight $\mathrm{wt}(\partial_i)$ at the boundary i of α to be the sum of the weights of the ends of the arcs incident to ∂_i. Notice that in this count, if an arc has both ends on ∂_i its weight counts twice in the sum.

Definition 1.1. A weighted arc family $([\alpha], \mathrm{wt}) \in D^s_{g,r}$ or $([\alpha], [\mathrm{wt}]) \in A^s_{g,r}$ is called exhaustive if $\mathrm{wt}_i(\alpha) \neq 0$ for all i.

We set $\mathcal{A}rc^s_g(r-1) \subset A^s_{g,r}$ and $\mathcal{D}Arc^s_g(r-1) \subset D^s_{g,r}$ to be the subsets of exhaustive elements.

We furthermore set $\mathcal{A}rc(n) = \amalg_{g,s}\mathcal{A}rc_g^s(n)$, $\mathcal{D}Arc(n) = \amalg_{g,s}\mathcal{D}Arc_g^s(n)$ where $\amalg$ is the coproduct given by disjoint union, and finally $\mathcal{A}rc = \amalg_n \mathcal{A}rc(n)$ and $\mathcal{D}Arc(n) = \amalg_n \mathcal{D}Arc(n)$.

The natural $\mathbb{S}_r$ action descends both to $\mathcal{A}rc(r-1)$ and to $\mathcal{D}Arc(r-1)$.

1.5 Quasi-filling families, arc graphs and dual ribbon graphs

We call an arc family *quasi-filling* if the complementary regions are polygons which contain at most one marked point.

1.5.1 Dual (ribbon) graph. Let $\hat{\Gamma}(\alpha)$ be the dual graph in the surface of α. This means there is one vertex for every component of $F \setminus \alpha$ and an edge for each arc of α connecting the two vertices representing the two regions on either side of the arc.

If the graph is quasi-filling, this graph is again an embedded (up to isotopy) naturally ribbon graph. The cyclic order at each vertex is induced by the orientation of the surface. The cycles of the ribbon graph are naturally identified with the boundary components of F. This identification also exists in the non-quasi-filling case. Here the set of oriented edges or flags of $\hat{\Gamma}(\alpha)$ is partitioned into cycles, or, in other words, it is partitioned into a disjoint union of sets each with a cyclic order.

There is an additional structure of a marking where a marking is a fixed vertex for every cycle. This vertex is the vertex corresponding to the region containing the complement of the window.

Combinatorially, the vertices have valence ≥ 2 with only the marked vertices possibly having valence 2.

Given an element $a \in \mathcal{D}Arc$, we also obtain a metric on the dual graph of the underlying arc family, simply by keeping the length of each edge. The geometric realization is obtained by gluing intervals of these given lengths together at the vertices.

1.5.2 Arc graph. It is sometimes convenient to describe the arc families simply as a graph. The basic idea is as follows: given an arc family α on F we define its graph $\Gamma(\alpha)$ to be the graph on F obtained by shrinking each window W_i to a point v_i. The v_i are then the vertices and the arcs of α are the edges. We can think of $\Gamma(\alpha)$ as embedded in F. Again there is some fine print. First the graph only has an embedding up to homotopy. Secondly by changing the window, we changed our initial data, which is fine, but then the arcs are not disjointly embedded anymore. A rigorous geometric interpolation of the two pictures is given in [31].

As an abstract graph, we can also let the vertices be given by the W_i and the edges be given by the set of arcs of α. In §1.6.2, we give a geometric construction of this space.

In the situation $s = 0$ and in the case of a quasi-filling α the data of the marked ribbon graph $\hat{\Gamma}$ is equivalent to α, since one can obtain Γ_α by reversing the dualization.

1.6 Foliation picture

If $([\alpha], \mathrm{wt}) = ([\{a_0, a_1, \ldots, a_k\}], \mathrm{wt}) \in D^s_{g,r}$ is given by weights $(w_0, w_1, \ldots, w_k) \in \mathbb{R}^{k+1}_+$, then we may regard w_i as a transverse measure on a_i, for each $i = 0, 1, \ldots, k$ to determine a "measured train track with stops" and a corresponding "partial measured foliation", as considered in [49].

This works as follows. Fix some complete Riemannian metric ρ of finite area on F, suppose that each a_i is smooth for ρ, and consider for each a_i the "band" B_i in F consisting of all points within ρ-distance w_i of a_i. Since we can scale the metric ρ to $\lambda\rho$, for $\lambda > 1$, we will assume that these bands are pairwise disjointly embedded in F, and have their endpoints lie in the windows. The band B_i about a_i comes equipped with a foliation by the arcs parallel to a_i which are at a fixed ρ-distance to a_i, and this foliation comes equipped with a transverse measure inherited from ρ; thus, each B_i can be regarded as a rectangle of width w_i and some irrelevant length. The foliated and transversely measured bands B_i, for $i = 0, 1, \ldots, k$, combine to give a "partial measured foliation" of F, that is, a foliation of a closed subset of F supporting an invariant transverse measure (cf. [49]). The isotopy class in F rel ∂F of this partial measured foliation is independent of the choice of metric ρ.

1.6.1 Partial parametrization at the boundary. For $i = 1, 2, \ldots, r$, consider $\partial_i \cap \left(\coprod_{j=0}^{k} B_j\right)$, which is empty if α does not meet ∂_i and its intersection with W_i is otherwise a collection of closed intervals in W_i with disjoint interiors. Collapse to a point each component complementary to the interiors of these intervals in W_i to obtain an interval, which we shall denote $\partial_i(\alpha')$. Each such interval $\partial_i(\alpha')$ inherits an absolutely continuous measure μ^i from the transverse measures on the bands. If $\partial_i(\alpha')$ is not empty, scaling the measure to have total weight one, this gives a unique measure preserving map of $c_i^\alpha : \partial_i(\alpha') \to S^1$ where S^1 has the Haar measure.

Further collapsing the complement of the interior of W_i to a point, we get a space $S^1_i(\alpha)$ and c_i^α induces a map from this quotient to S^1 which is a measure preserving homeomorphism that maps the image of the endpoints of W_i to $0 \in S^1 = \mathbb{R}/\mathbb{Z}$. We call this map the parameterized circle at ∂_i.

A pictorial representation can be found in Figure 2.

1.6.2 Loop graph of an arc family: a geometric construction of the dual graph. The loop graph of a weighted arc family $a \in \mathcal{D}Arc$ is the space obtained from $\amalg_i S^1_i / \sim$ where $\sim$ is the equivalence relation which is the transitive closure of the symmetric relation $p \approx q$ if p and q are the endpoints of a leaf. The loop is invariant under the PMC action and we call the resulting space $\mathcal{L}(a)$. There are natural maps $l_i : S^1_i \to \mathcal{L}(a)$, the images are called the i-th circle or lobe. The 0-th circle is also called the outside or output circle, while the circles for $i \neq 0$ are called the input circles.

The loop of the graph is homeomorphic to the geometric realization of the dual graph with its metric. The circles correspond to the cycles and the marked point on each cycle is the image of 0.

2 The gluing and the operad structures

2.0.3 Basic idea. Think of the elements of $A^s_{g,r}$ as partial measured foliations modulo common scaling. The most natural way to do this is in terms of foliations as derived from the theory of "train tracks" (cf. [49])

In this picture, two weighted exhaustive arc families can be naturally composed by fixing one boundary component on each of the surfaces and glue naturally if one glues the underlying surfaces along a pair of fixed boundary components. Concretely on the condition that the weights on the two boundaries agree one produces a weighted arc family on the glued surface from two given arc families by viewing them as foliations. If the families are exhaustive, this can always be achieved by scaling. If one starts with projective weights, one only chooses representatives which satisfy the condition.

This gluing yields the sought after operadic structure.

2.1 Standard gluing for foliations

2.1.1 Basic idea. Given two weighted arc families (α, wt) in $F = F^s_{g,m+1}$ and (β, wt') in $F' = F^t_{h,n+1}$, construct the respective foliation. Now picking one boundary component on each surface, we can glue them and the respective foliations if they have the same weights. This is well defined up to the action of PMC.

More precisely, if the two foliations have the non-zero same weights say $\mathrm{wt}_i(\alpha) = \mathrm{wt}_0(\beta)$ we can glue them to give a foliation on the surface obtained by gluing the boundary i of F to the boundary i of F'. Identifying the glued surface with $F^{s+t}_{g+h,m+n}$ we obtain the weighted family $(\alpha, \mathrm{wt}) \circ'_i (\beta, \mathrm{wt}')$.

If we have two exhaustive weighted families whose weights do not agree, we can use the $\mathbb{R}_{>0}$ action to make them agree and then glue. That is in general: $([\alpha], \mathrm{wt}) \circ_i ([\beta], \mathrm{wt}') := \mathrm{wt}_i(\alpha)([\alpha], \mathrm{wt}) \circ'_i \mathrm{wt}_0([\beta])(\beta, \mathrm{wt}')$ and this descends to $\mathcal{A}rc$.

2.1.2 Gluing weighted arc families. Given two weighted arc families (α, wt) in $F^s_{g,m+1}$ and (β, wt') in $F^t_{h,n+1}$ so that $\mu_i(\partial_i(\alpha')) = \mu_0(\partial_0(\beta'))$, for some $1 \le i \le m$, we will define a weighted arc family $(\gamma, \mathrm{wt}'') := (\alpha, \mathrm{wt}) \circ_i (\beta, \mathrm{wt}')$ for each $1 \le i \le m$ on the surface $F^{s+t}_{g+h,m+n}$ as follows:

First, let's fix some notation: let ∂_i denote the i-th boundary component of $F^s_{g,m+1}$, and let ∂'_0 denote the 0-th boundary component of $F^t_{h,n+1}$. We glue the boundaries together using the maps to S^1 given above. This yields a surface X homeomorphic to $F^{s+t}_{g+h,m+n}$, where the two curves ∂_i and ∂'_0 are thus identified to a single separating curve in X. There is no natural choice of homeomorphism between X and $F^{s+t}_{g+h,m+n}$, but there are canonical inclusions $j : F^s_{g,m+1} \to X$ and $k : F^t_{h,n+1} \to X$.

We enumerate the boundary components of X in the order

$$\partial_0, \partial_1, \dots, \partial_{i-1}, \partial'_1, \partial'_2, \dots \partial'_n, \partial_{i+1}, \partial_{i+2}, \dots \partial_m.$$

The punctures are enumerated simply by enumerating the ones on $\mathcal{F}^s_{g,m+1}$ first.

Choose an orientation-preserving homeomorphism $H\colon X \to F^{s+t}_{g+h,m+n}$ which preserves the labeling of the boundary components as well as those of the punctures, if any.

In order to define the required weighted arc family, consider the partial measured foliations $\mathcal{G}$ in $F^s_{g,m+1}$ and $\mathcal{H}$ in $F^t_{h,n+1}$ corresponding respectively to (α') and (β'). By our assumption that $\mu_i(\partial_i(\alpha')) = \mu_0(\partial_0(\beta'))$, we may produce a corresponding partial measured foliation $\mathcal{F}$ in X by identifying the points $x \in \partial_i(\alpha')$ and $y \in \partial_0(\beta')$ if $c_i^{(\alpha)}(x) = c_0^{(\beta)}(y)$.

The resulting partial measured foliation $\mathcal{F}$ may have *simple closed curve leaves* which we must *simply discard* to produce yet another partial measured foliation $\mathcal{F}'$ in X.

The leaves of $\mathcal{F}'$ thus run between boundary components of X and therefore, as in the previous section, decompose into a collection of bands B_i of some widths w_i, for $i = 1, 2, \dots, I$, for some I. Choose a leaf of $\mathcal{F}'$ in each such band B_i and associate to it the weight w_i given by the width of B_i to determine a weighted arc family (δ') in X which is evidently exhaustive. Let $(\gamma') = H(\delta')$ denote the image in $F^{s+t}_{g+h,m+n}$ under H of this weighted arc family.

Lemma 2.1. *The* $\mathrm{PMC}(F^{s+t}_{g+h,m+n})$*-orbit of* (γ') *is well-defined as* (α') *varies over a* $\mathrm{PMC}(F^s_{g,m+1})$*-orbit of weighted arc families in* $F^s_{g,m+1}$ *and* (β') *varies over a* $\mathrm{PMC}(F^t_{h,n+1})$*-orbit of weighted arc families in* $F^t_{h,n+1}$.

Proof. Suppose we are given weighted arc families

$$(\alpha'_2) = \phi(\alpha'_1) \quad \text{and} \quad (\beta'_2) = \psi(\beta'_1)$$

for $\phi \in \mathrm{PMC}(F^s_{g,m+1})$ and $\psi \in \mathrm{PMC}(F^t_{h,n+1})$, respectively, as well as a pair

$$H_\ell\colon X_\ell \to F^{s+t}_{g+h,m+n}$$

of homeomorphisms as above together with the pairs $j_1, j_2\colon F^s_{g,m+1} \to X_\ell$ and $k_1, k_2\colon F^t_{h,n+1} \to X_\ell$ of induced inclusions, for $\ell = 1, 2$. Let $\mathcal{F}_\ell$, $\mathcal{F}'_\ell$ denote the partial measured foliations and let (δ'_ℓ) and (γ'_ℓ) denote the corresponding weighted arc families in X_ℓ and $F^{s+t}_{g+h,m+n}$, respectively, constructed as above from (α'_ℓ) and (β'_ℓ), for $\ell = 1, 2$.

Let $c_\ell = j_\ell(\partial_0) = k_\ell(\partial'_i) \subseteq X_\ell$, and remove a tubular neighborhood U_ℓ of c_ℓ in X_ℓ to obtain the subsurface $X'_\ell = X_\ell - U_\ell$, for $\ell = 1, 2$. Isotope j_ℓ, k_ℓ off of U_ℓ in the natural way to produce inclusions $j'_\ell\colon F^s_{g,m+1} \to X'_\ell$ and $k'_\ell\colon F^t_{h,n+1} \to X'_\ell$ with disjoint images, for $\ell = 1, 2$.

The mapping class ϕ induces a homeomorphism $\Phi\colon X'_1 \to X'_2$ supported on $j'_1(F^s_{g,m+1})$ so that $j'_2 \circ \phi = \Phi \circ j'_1$, and ψ induces a homeomorphism $\Psi\colon X'_1 \to X'_2$ supported on $k'_1(F^t_{h,n+1})$ so that $k'_2 \circ \psi = \Psi \circ k'_1$. Because they have disjoint supports, Φ and Ψ combine to give a homeomorphism $G'\colon X'_1 \to X'_2$ so that $j'_2 \circ \phi = G' \circ j'_1$ and $k'_2 \circ \psi = G' \circ k'_1$. We may extend G' by any suitable homeomorphism $U_1 \to U_2$ to produce a homeomorphism $G\colon X_1 \to X_2$.

By construction and after a suitable isotopy, G maps $\mathcal{F}_1 \cap X_1'$ to $\mathcal{F}_2 \cap X_2'$, and there is a power τ of a Dehn twist along c_2 supported on the interior of U_2 so that $K = \tau \circ G$ also maps $\mathcal{F}_1 \cap U_1$ to $\mathcal{F}_2 \cap U_2$. K thus maps $\mathcal{F}_1'$ to $\mathcal{F}_2'$ and hence (δ_1') to (δ_2'). It follows that the homeomorphism

$$H_2 \circ K \circ H_1^{-1} \colon F^{s+t}_{g+h,m+n} \to F^{s+t}_{g+h,m+n}$$

maps (γ_1') to (γ_2'), so (γ_1') and (γ_2') are indeed in the same $\mathrm{PMC}(F^{s+t}_{g+h,m+n})$-orbit. □

Notice that although the gluing is local on the boundaries, there is a global effect of gluing the leaves together. For instance there can be bands which have both ends on the same boundary. If these are split, they may recursively cut other bands. An example of such a gluing is given in Figure 4. Alternatively, one can describe the

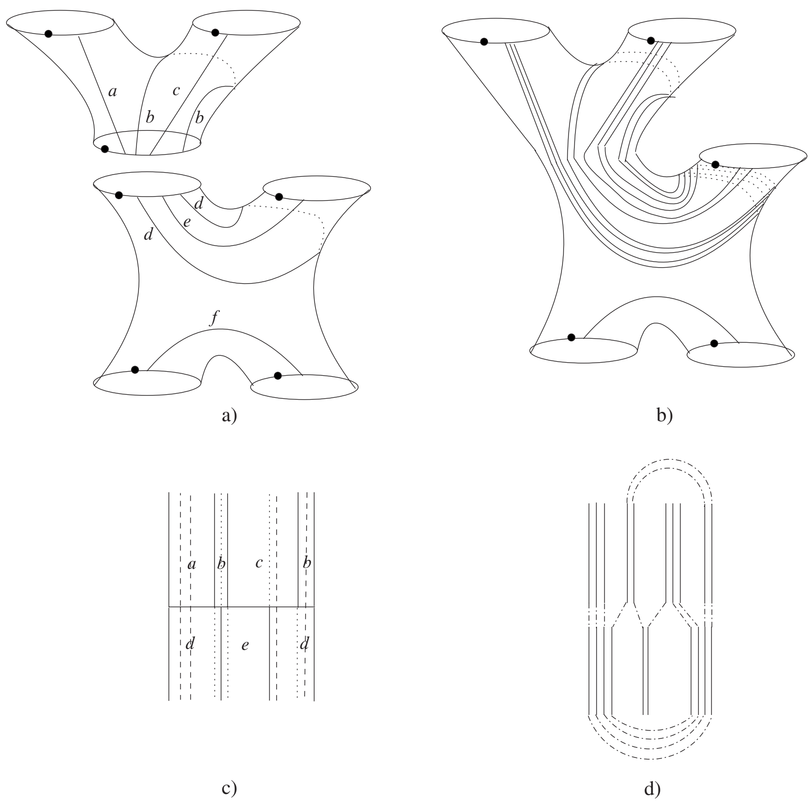


Figure 4. a) The arc graphs which are to be glued assuming the relative weights a, b, c, d and e as indicated by the solid lines in c). b) The result of the gluing (the weights are according to c). c) The combinatorics of cutting the bands. The solid lines are the original boundaries, the dotted lines are the first cuts, and the dashed lines represent the recursive cuts. d) The combinatorics of splitting and joining flags.

gluing procedure purely combinatorially, see [23] for the details. For this one uses a least common partition of the unit interval, duplicates each edge for every cut and then glues the flags or half edges if they are indexed by the same subinterval.

Remark 2.2. An alternative to discarding the simple closed curve leaves is to enlarge the space $A^s_{g,r}$ to include them. This would be in the spirit of V. Jones' planar algebras [18]. We however do not take this route and the applications such as to string topology do not exhibit these type of curves.

2.1.3 Symmetric group actions. On $K^s_{g,r}$ there is a natural action of the symmetric group of r elements $\mathbb{S}_r$ which permutes the labels $0, \dots, r-1$ enumerating the boundary components. It contains a subgroup $\mathbb{S}_{r-1}$ which only permutes the labels $1, \dots, r$ keeping 0 fixed. Like above, after renumbering, we have to choose a homeomorphism to the standard surface. In the figures, this is usually suppressed.

2.1.4 Comments on the details, see [31]. Notice that since we fixed the surfaces $F^s_{g,r}$ the gluing actually depends *a priori* on a choice of homeomorphism of the glued surface. These choices become irrelevant after passing to PMC quotients. Other possibilities are to choose models $F^s_{g,r}$ and compatible morphisms of surfaces glued from these back to the chosen models.

2.1.5 Partial/colored operad structure

Proposition 2.3. *The gluings together with the symmetric group action permuting the labels give a* (*cyclic*) *partial operad structure to the spaces* $D(r-1) := \amalg_{g,s} D^s_{g,r}$. *Moreover this partial operad structure is an* $\mathbb{R}_{\geq 0}$ *colored operad. Here* $\mathbb{R}_{\geq 0}$ *is considered with the discrete topology.*

Proof. Notice that $\mathbb{S}_n$ naturally acts on $D(n)$ via permuting the labels $1, \dots, n$ on the boundaries. Moreover $\mathbb{S}_{n+1}$ acts by permuting the labels on the boundaries $0, \dots, n$. The gluings *if defined* are associative and symmetric group equivariant; for the precise definition of the various actions, see the Appendix. The important point is that the gluing did not depend on the *name* of the boundaries. This is a straightforward check. The additional equation for a cyclic operad is also easy to check. The only obstruction to gluing is that the weights on the two boundaries which are glued are the same. The procedure also works if the two boundaries are both not hit by any arc. Thus assigning the color $\mathrm{wt}(\partial_i) \geq 0$ to each i we obtain an $\mathbb{R}_{\geq 0}$ colored operad. □

2.2 Cyclic operad structure: the scaling approach of [31]

In the gluing operation above, we could compose two weighted arc families if they had the same weight at the designated boundaries. We can get rid of this restriction if we consider exhaustive families by using the $\mathbb{R}_{>0}$ scaling action. Given exhaustive arc families $([\alpha],[\mathrm{wt}]) \in \mathcal{A}rc(n)$ and $([\beta],[\mathrm{wt}']) \in \mathcal{A}rc(m)$

$$([\alpha],\mathrm{wt}) \circ_i ([\beta],\mathrm{wt}') := \mathrm{wt}_i(\alpha)([\alpha],\mathrm{wt}) \circ_i' \mathrm{wt}_0([\beta])(\beta,\mathrm{wt}'). \tag{2.1}$$

Theorem 2.4 ([31]). *The spaces $\mathcal{D}Arc(n)$ form a cyclic operad and this cyclic operad structure descends to $\mathcal{A}rc(n)$, which comprise the $\mathcal{A}rc$ operad.*

Proof. One has to recheck the associativity for this case, but it again works out [31]. □

Remark 2.5. Notice that if we start in $\mathcal{A}rc$ there are unique representatives $([\alpha],\mathrm{wt}) \in D(n)$ and $([\beta],[\mathrm{wt}]) \in D(m)$ such that $\mathrm{wt}(\partial_i) = \mathrm{wt}'(\partial_0) = 1$ and we could have used these to define gluing directly on $\mathcal{A}rc$, but that would have not allowed us to lift the operad structure to $\mathcal{D}Arc$.

2.2.1 Discretization: the suboperad of positive integer weights/multiarcs

Proposition 2.6. *The arc families in $\mathcal{D}Arc$ with positive integer weights form a cyclic suboperad. Using the inclusion $\mathbb{N} \subset \mathbb{R}$, they also form an $\mathbb{N}$ colored cyclic suboperad of the $\mathbb{R}$-colored version of $\mathcal{D}Arc$.*

A useful pictorial realization of an arc family with positive integer weights is to replace an arc of weight k by k parallel arcs.

Alternatively, one adds $k-1$ parallel copies to the arc after say fixing a small rectangular neighborhood of the original arc. We will call these multi-arc families.

These multi-arc families are what is used in the string topology and moduli space actions. There they will appear in their $\mathbb{N}$ colored version.

Another relevant operad structure is the one sums of these elements given as follows: Given an exhaustive arc graph α, with arcs $e_1,\dots e_k$, let $\alpha^{(n_1,\dots,n_k)} \in \mathcal{D}Arc$ for $n_i \in \mathbb{N}$ be defined by $\mathrm{wt}(e_i) = n_i$,

$$\alpha^{\mathbb{N}} = \amalg_{\vec{n}\in\mathbb{N}^k} \alpha^{\vec{n}}. \tag{2.2}$$

Furthermore, for two exhausting arc graphs we set

$$\alpha^{\mathbb{N}} \circ_i \beta^{\mathbb{N}} = \sum_{(\vec{n},\vec{m})}{}^{'} \alpha^{\vec{n}} \circ_i \beta^{\vec{m}}, \tag{2.3}$$

where $\sum'$ runs over the pairs $(\vec{n},\vec{m})$ such that $\mathrm{wt}(\partial_i(\alpha^{\vec{n}})) = \mathrm{wt}(\partial_0(\beta^{\vec{m}}))$, that is, those pairs for which the $\mathbb{N}$-colors match.

Proposition 2.7. *The compositions $\circ_i$ are operadic. Furthermore dropping the superscript $\mathbb{N}$ they give an operad structure to the collection of exhaustive arc graphs α where the operad degree is the number of boundaries of α.* □

2.3 Chains and homology

One basic question is how operads behave with respect to functors of homology and various chain functors. It is the homology level that gives the algebra and the chain level basically the "algebra up to homotopy" level which is relevant for applications from Deligne's conjecture to field theory.

2.3.1 Operads and functors: technical details. The general answer to the question of what kind of functor pushes forward an operad structure is that it should be a weak monoidal one. Let us denote this functor as $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ where both $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories with a product $\otimes$. The condition of being weakly monoidal means among other things (see e.g. [20]) that there are natural morphisms $\mathcal{F}(X,Y): \mathcal{F}(X) \otimes \mathcal{F}(Y) \to \mathcal{F}(X \otimes Y)$. For our operads this means that we get compositions $\circ_i^{\mathcal{F}}$ by using the sequence of maps

$$\circ_i^{\mathcal{F}}: \mathcal{F}(\mathcal{O}(n)) \otimes \mathcal{F}(\mathcal{O}(m)) \xrightarrow{\mathcal{F}(\mathcal{O}(n),\mathcal{O}(m))} \mathcal{F}(\mathcal{O}(n) \otimes \mathcal{O}(m)) \xrightarrow{\mathcal{F}(\circ_i)} \mathcal{F}(\mathcal{O}(m+n-1)).$$

2.4 Singular homology and singular chains

If we take (singular) homology with coefficients in a field k then the Künneth formula guarantees us that the functor $H_*(\ ,k)$ is a strong monoidal functor, that is, there is an isomorphism $H_*(\mathcal{O}(n),k) \otimes H_*(\mathcal{O}(m),k) \simeq H_*(\mathcal{O}(n) \times \mathcal{O}(m),k)$, and this shows that the homology is again a cyclic operad. For singular chains the Eilenberg–Zilber theorem provides us with a weak monoidal functor on the chain level.

Corollary 2.8. *The singular chains and the homologies of $\mathcal{D}\mathcal{A}rc(n)$ and $\mathcal{A}rc(n)$ form cyclic operads.*

2.4.1 Other chains. There may be other chain models besides singular chains we might want to use. For singular chains one has to use the Eilenberg–Zilber theorem. This is for instance easier to track in cubical chains. Mostly we will be interested in either singular or cellular chains. Throughout we will denote singular chains by S_* and cellular chains by CC_*. When dealing with CW complexes one has the extra bonus of proving that the topological compositions indeed give rise to cellular maps.

Additionally in some particular cases one may obtain special chain models that work for a given operad in a special situation, although there is no general *a priori* guarantee that the construction is valid. This then of course can and has to be checked *a posteriori*.

We will just use Chain to denote any operadic chain model. In all the models we consider, one has families representing the chains. We will from now on treat families and leave open the specification of a particular chain model. We will mostly use singular or cellular chains in the following.

2.5 Open/cellular chains

$\mathcal{A}rc_g^s(n)$ (unlike $\mathcal{D}Arc$) is a subspace of the CW complex $A_{g,r}^s$. That complex has cells $C_{[\alpha]}$ indexed by classes of arc graphs α. For $A_{g,r}^s$ and the partial gluing structure, we can use cellular chains. Some of the suboperads/PROPs of $\mathcal{A}rc$ actually have homotopy equivalent CW models whose cellular chains are models for them on the chain level. These will give us the solution to Deligne's conjecture and it generalizations up to and including string topology.

Unfortunately $\mathcal{A}rc$ itself is not outright a CW complex, since the condition of being exhaustive is not necessarily stable under removing arcs. We can however consider the complex $CC_*(A_{g,r}^s, A_{g,r}^s \setminus \mathcal{A}rc_g^s(r-1))$.

Alternatively, $\mathcal{A}rc_g^s(r-1)$ is also the disjoint union of open cells and hence filtered by using the dimension of cells

$$\mathcal{A}rc_g^s(r-1) = \amalg_{[\alpha]:[\alpha] \text{ is exhaustive}} \dot{C}_{[\alpha]} \tag{2.4}$$

where $\dot{C}$ denotes the open cell. And although we cannot use cellular chains, we can work with the free Abelian group generated by the open cells which is denoted by $\mathcal{C}_o^*(\mathcal{A}rc_g^s(r-1))$. Each generator is given by an oriented cell. Such a cell is given in turn by an arc graph. The dimension is the number of arcs minus 1. There is a differential, which deletes arcs, as long as the result is still exhaustive. In order to get an operad structure on $\mathcal{C}_o^*(\mathcal{A}rc)$, we recall the following facts from [23]:

As sets, we have

$$\dot{C}_{[\alpha]} \circ_i \dot{C}_{[\beta]} = \amalg_{\Gamma \in I([\alpha],[\beta])} \dot{C}_{[\gamma]} \tag{2.5}$$

where $I([\alpha],[\beta])$ is a finite index set of arc graphs on the glued surfaces [23] running through all the graphs that appear as the underlying graphs of the composed families.

If α has k arcs and β has l arcs then, if two conditions are met, for any weights, generically the number of arcs in $([\alpha],[\mathrm{wt}]), ([\beta],[\mathrm{wt}'])$ is $k+l-1$. The two conditions are that (1) there are no closed loops and (2) that not both arc families are twisted at the boundary at which they are glued. In these cases, the dimension of the composed cell drops. Overall the composition respects the filtration by dimension. Moreover, the "bad" part, that is the locus, where the families glue together to form families with less than the expected graphs is of codimension at least 1, if the two families are not twisted at the boundary simultaneously. In that case the number of arcs in the composition generically already has one less arc than expected. In the top dimension the composition map is bijective.

The best way to treat the operadic structure is to pass to the associated graded $\mathrm{Gr}\,\mathcal{C}_o^*(\mathcal{A}rc)$ of $\mathcal{C}_o^*(\mathcal{A}rc)$.

In [23] we showed that

Theorem 2.9. *The Abelian groups*

$$\mathcal{C}_o^*(\mathcal{A}rc)(n) = \coprod_{g,s} \mathcal{C}_o^*(Arc_g^s(n))$$

and

$$\mathrm{Gr}\,\mathcal{C}_o^*(\mathcal{A}rc) = \coprod_{g,s} \mathrm{Gr}\mathcal{C}_o^*(Arc_g^s(n)) = \coprod_{g,s} CC_*(A_{g,r}^s, A_{g,r}^s \setminus \mathcal{A}rc_g^s(r-1))$$

are cyclic operads. Here, the compositions are given by

$$\dot{C}(\alpha) \circ_k \dot{C}(\beta) = \sum_{i \in I} \pm \dot{C}(\gamma_i) \tag{2.6}$$

where $\pm$ is the usual sign corresponding to the orientation and I is a subset of $I(\alpha,\beta)$. More precisely: (1) *I is empty if both families are twisted at the glued boundaries,* (2) *I runs over the arc graphs which are not obtained by erasing arcs, and* (3) *in the graded case* Gr*, the arc family also has the expected maximal number of arcs.*

2.5.1 Discretizing the chain level. When dealing with the chain level and the actions, there are two modifications for the discretization (2.3). The first is to add appropriate signs. The signs are given in [24] and are discussed in the appendix. Here we will just use the fact that such appropriate signs exist. The second is to adjust the gluing to fit with that of relative chains, that is to use the modification (1) or both modifications (1) and (2) of Theorem 2.9. On the discrete side, there is again a filtration by the degree of the underlying arc family.

For the action we will use the following.

Theorem 2.10 ([24]). *For α an arc graph with k arcs, let*

$$\mathcal{P}(\alpha) := \sum_{\vec{n} \in \mathbb{N}^k} \pm \alpha^n \tag{2.7}$$

with the appropriate sign. Then for two arc graphs α and β we have: $\mathcal{P}(\alpha \circ_i \beta) = \pm\mathcal{P}(\alpha) \circ_i \mathcal{P}(\beta)$, where $\pm$ denotes again the appropriate sign and the two $\circ_i$ are taken with the same modification. In particular using both modifications for the gluing $\mathcal{P}$ gives an operad morphism from $\mathcal{C}_o^*(\mathcal{A}rc)$ to the operad of $\mathbb{N}$-weighted arc families (modified with appropriate signs, see [24]. *Using the respective associated grading on the discrete $\mathcal{P}$ gives an operad morphism from $CC_*(A_{g,r}^s, A_{g,r}^s \setminus \mathcal{A}rc_g^s(r-1))$ to the associated graded.*

Remark 2.11. Here $\alpha \circ_i \beta$ is the gluing given by gluing the open cells and enumerating the indexing set of the resulting cells. *A priori* this can be any gluing between two boundaries that are hit, but also between two empty boundaries. With the obvious modification this applies as well to gluing an empty boundary to a non-empty one by erasing.

2.6 Modular structure: the approach of [32]

One can ask whether there is a modular operad structure for $\mathcal{A}rc$. The challenge is to add self-gluings. One can readily see that the partial operad structure on the

$D(n)$ is indeed modular. One can glue any two boundary components on connected or disconnected surfaces as foliations in the above manner, *if* the weights on the two boundaries agree. It is not possible in general, however, to scale in order to obtain self-gluings without restrictions, at least at the topological level. The reason for this is that the $\mathbb{R}_{>0}$ action scales the weights on all boundaries simultaneously, so if they do not agree one cannot change them to agree merely by the action.

There is however a flow, which one can use, to make them agree. This is a very intricate procedure which even works in families. It is contained as one result in [32]. We will content ourselves with just stating the main result as it pertains to the discussion here.

Theorem 2.12. *The homologies $H_*(\mathcal{D}Arc(n)) = H_*(\mathcal{A}rc(n))$ form a modular operad using g as the genus grading. Moreover, it is induced from a modular operad structure up to canonical homotopies on the chain level. That structure is obtained from the $\mathbb{R}_{>0}$ colored topological operad structure on $\mathcal{D}Arc(n)$ via flows.*

2.7 S^1 action

We have already seen that $A^0_{0,2} \simeq S^1$. Since all the elements are exhaustive in this case, we obtain that $\mathcal{A}rc^2_0(1) = A^0_{0,2}$. We now show that this is even true as groups, where the composition in $\mathcal{A}rc$ is given by $\circ_1$.

2.7.1 Group structure. Pick two elements Δ_t, Δ_s with $s, t \in [0, 1)$ as depicted in Figure 5. Gluing them together there are two situations, namely (i) $s + t \leq 1$ or (ii) $1 < s + t < 2$. In the first case we immediately see that $\Delta_t \circ_1 \Delta_s = \Delta_{s+t}$; in the second situation we see that after gluing the outer two strands become parallel and indeed $\Delta_t \circ_1 \Delta_s = \Delta_{s+t-1}$.

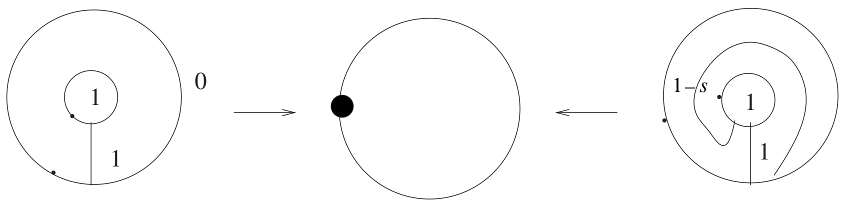
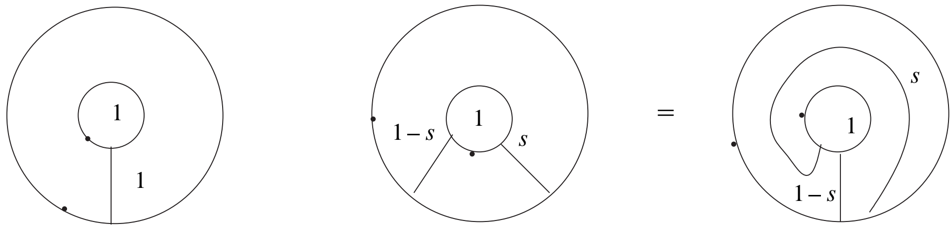


Figure 5. The CW complex $A^0_{0,2} = S^1$. We indicated the base point and a generic element.

Now via gluing there is an $(S^1)^{\times n+1}$-action on each $\mathcal{A}rc^s_g(n)$. Let us enumerate the Cartesian product of $(S^1)^{\times n+1}$ as having factors $0, \dots, n$. Then the factors $i = 1, \dots n$ act via $\rho_i(t)([\alpha], [\mathrm{wt}]) = [\alpha], [\mathrm{wt}]) \circ_i \Delta_t$ and the 0 component acts as $\rho_0(t)([\alpha], [\mathrm{wt}]) = \Delta_t \circ_0 [\alpha], [\mathrm{wt}])$.



I II

Figure 6. I. The identity. II. The arc family δ yielding the BV operator.

2.7.2 S^1 action as twisting and moving the base point. If we look at the arc picture II B or C, in the enumeration in § 1.4, we can nicely describe the geometry of this action. It simply moves the basepoint around the boundary in the direction of its orientation. The distance is given by the transverse measure of the bands as in §2.1. If the basepoint moves into a band, it simply splits it.

Definition 2.13. An arc family is called untwisted at a boundary i if no two arcs are parallel after removing the base point of the boundary I in picture II. Otherwise it is called twisted at i. The elements of $A^s_{g,r}$ and $\mathcal{A}rc^s_g(n)$ are twisted or untwisted if their underlying arc families are. We will also say that an arc family is twisted or untwisted if arcs become parallel or not after removing all basepoints at the boundaries.

In the pictures II B and C we can see the twisting more explicitly. An element is twisted at i if the basepoint of ∂_i is inside a band which is not split at the other end.

An example of a twisted arc family is given by Δ_t. This is even the general case in the following sense.

Lemma 2.14. *Any element of $\mathcal{A}rc^s_g(n)$ lies in the orbit of an untwisted element under the $(S^1)^{n+1}$-action.*

Proof. Just use the action to slide the basepoints at the different boundaries out of any band they might be inside of. In this way we obtain an element which is untwisted at each boundary. Now it can happen that there are arcs which become parallel only after removing both the basepoints of the boundaries they run between. In this case, we can move the points in sync outside of the band. □

Proposition 2.15. *There is an action of $(S^1)^{\times n+1}$ on $\mathcal{A}rc(n)$.*

Proof. The action is given by $(\theta_0,\dots,\theta_n)\alpha = (\cdots((\Delta_{\theta_0}\circ_1\alpha)\circ_1\Delta_{\theta_1})\cdots)\circ_n\Delta_{\theta_n})$. The fact that this is indeed an action follows from the associativity of the operadic compositions. □

We denote the coinvariants by $\mathcal{A}rc(n)_{S^1}$.

2.8 Twist gluing

There is an additional gluing we can perform which yields an odd structure on the chain and the homology level. This is inspired by string field theory and it gives rise to a second type of Gerstenhaber and BV structure on the homology and chain levels. For this we let Δ be the chain given by $I \to \mathcal{A}rc(1)$, $t \mapsto \Delta_t$. Notice that this chain represents a generator of the homology $H^1(\mathcal{A}rc_0^0(1))$.

Definition 2.16. Given two chains $\alpha \in S_*(\mathcal{A}rc(n))$ and $\beta \in S_*(\mathcal{A}rc(m))$ we define the twist gluing $\alpha \bullet_i \beta$ to be the chain obtained from the map $\Delta^n \times I \times \Delta^m \to \mathcal{A}rc(m+n-1)$ given by $\alpha \circ_i \Delta \circ_1 \beta$.

Theorem 2.17. *On the chain and homology level the operations $\circ_i$ induce the structure of an odd cyclic operad. Furthermore, $H_*(\mathcal{A}rc(n))$ is an odd modular operad, also known as a $\mathfrak{K}$-modular operad.*

Likewise the chains on the S^1 co-invariants $\mathcal{A}rc(n)_{(S^1)}$ form an odd cyclic operad and $H_*(\mathcal{A}rc(n)_{S^1})$ is an odd modular operad.

Proof. It is easy to see that on the chain level the twist gluing is of degree 1, which is precisely what we need for the odd versions of the operadic structures. This immediately shows the first two claims. Notice that the S^1 coinvariants do not form an operad by themselves. To define the gluings, we simply choose representatives, glue and take coinvariants again. This is independent of choices, since any two lifts differ by an S^1 action on the boundaries and these get absorbed into the family. □

Corollary 2.18. *The direct sums of cyclic group coinvariants $\bigoplus_n (H_*(\mathcal{A}rc(n)))_{C_{n+1}}$ and $\bigoplus_n \mathrm{Chain}(\mathcal{A}rc(n))_{C_{n+1}}$ carry a Gerstenhaber bracket and that of $\mathbb{S}_{n+1}$ coinvariants $\bigoplus_n H_*(\mathcal{A}rc(n))_{\mathbb{S}_{n+1}}$ carries a BV operator.*

An analogous result holds for the S^1 coinvariants.

2.9 Variations on the gluings

We have already deviated a bit from the original gluing to obtain the modular structure on homology. There are several other variations on the basic gluing, which are necessary and helpful. Usually these do not alter the picture on the level of homology.

2.9.1 Local scaling. In the gluing of $\mathcal{D}Arc$ we scaled both surfaces in order to obtain an associative structure. To make the two weights match, we could also just locally scale the width of *only those bands incident to $\partial_i(F)$ and/or those incident to $\partial_0(F)$*. In fact for gluing disjoint surfaces this is exactly done by the first type of flow in [32]. What happens in this case is that the gluings are not associative any longer, but there is a homotopy between the two different ways to compose. This guarantees a *bona fide* associative structure on homology. It is rather surprising that in several situations, notably that of string topology, there are already strictly associative chain models, – see §3.3 and §4.3.2 below.

2.9.2 Erasing. We have not discussed how to glue a boundary of weight 0 with one of non-zero weight. The natural idea is to simply erase all bands incident to the boundary which is glued. Indeed this is sometimes the answer. There are two *caveats* however. First, this operation is again not associative. Second one has to be careful that iterating such gluings one does not obtain an empty arc family. One such undesirable situation occurs when one tries to use this type of gluing to extend the operad structure to $A^s_{g,r}$. Then one could obtain an empty family, but adding it would entail making the spaces contractible and hence kill all homology information.

However this type of gluing is used in string topology. The trick here is that the empty family does not appear due to the conditions that are placed on the graphs to make them part of the Sullivan-PROP.

2.9.3 Wilting. The last modification is that instead of erasing, one lets the leaves wilt. Technically this can be formalized by adding wilting weights at the boundary. A wilting weight is an assignment of a length in $\mathbb{R}_{\geq 0}$ to each interval of $\partial_i \setminus W_i$. The source of the map c_i is then the full interval with the induced measure. For the source of the map $\mathscr{L}$, we contract the intervals just as before. The effect on the maps l_i is that they are stationary on these intervals.

A foliation interpretation is as follows: on top of the foliation on the surface, we also consider a compatible germ of a trivial transverse measured foliation of a neighborhood of each boundary. Here trivial means that all leaves are homotopic to a meridian of the cylinder. Compatible means that the restriction of the given partial measured foliation of the surface is a sub-foliation. The other leaves are called wilted leaves. Notice that the weight can be zero, which means that the band is empty.

For the gluing of these foliations we use the modified maps c_i and proceed as before. Upon gluing regular leaves to wilted leaves, the leaves wilt and are erased from the surface foliation, but kept for the foliations near the boundaries which are not glued. This gluing is used in section 5.1 to add units.

3 Framed little discs and the Gerstenhaber and BV structures

3.1 Short overview

One extremely important feature of the $\mathcal{A}rc$ operad is that it contains several suboperads that are quasi-isomorphic to classically important operads. The main ones are spineless cacti which are equivalent to the little discs, cacti which are equivalent to the framed little discs and the corrolas which are equivalent to the tight little intervals suboperads.

The first two structures are responsible for Gerstenhaber and BV algebras on the homology level. This is what gives rise to string topology brackets and operators as well as solutions to various forms of Deligne's conjecture. In our approach we get a version of these algebras up to homotopy on the cell level which has all homotopies explicitly given. One nice upshot is the new symmetry of the BV equation which now manifests itself as a completely symmetric 12 term identity which is geometrically nicely described by a Pythagorean like triangle, see Figure 11.

On the topological level these operads give rise to loop space structures. Going a bit further the stabilization of the $\mathcal{A}rc$ operad gives rise to a filtered sequence of so-called E_k, $k \in \mathbb{N} \cup \{\infty\}$, operads. These detect k-fold loop spaces respectively infinite loop spaces, see the Background section 3.1.1 below.

Without the stabilization $\mathcal{A}rc$ contains a E_2 in the form of spineless cacti and an E_1 operad in the form of corollas, both which are defined below.

There is one technical detail, namely, whether or not to include a 0 component in the operad. We will call operads with such a component pointed.[1] For the chain level and the algebraic structures it is enough to have the non-pointed version of E_2. For the topological level and e.g. loop space detection it is necessary to have the pointed versions. For this one has to enlarge the setup by "fattening" the operads. The details are given in §5.1.

Theorem 3.1. *The $\mathcal{A}rc$ operad contains suboperads $\mathcal{C}or$, $\mathcal{C}act$ and $\mathcal{C}acti$. $\mathcal{C}or$ is an E_1 operad that is it is equivalent to the little intervals, $\mathcal{C}act$ is an E_2 operad equivalent to the little discs, $\mathcal{C}acti$ are equivalent to the framed little discs. This is as non-pointed operads.*

Corollary 3.2. (1) *On the topological level: Any algebra over the group completion of ${}^{\mathrm{Fat}}\mathcal{C}or$ has the homotopy type of a loop space. Any algebra over the group completion of ${}^{\mathrm{Fat}}\mathcal{C}act$ has the homotopy type of a double loop space. Any algebra over the group completion of ${}^{\mathrm{Fat}}\mathcal{A}rc$ has the homotopy type of a double loop space.*

(2) *On the homology level: Any algebra over $H_*(\mathcal{C}act)$ is a Gerstenhaber algebra. Any algebra over $H_*(\mathcal{C}acti)$ is a BV algebra. Any algebra over $H_*(\mathcal{A}rc)$ is a BV algebra.*

(3) *On the chain level: Any algebra over* $\mathrm{Chain}(\mathcal{C}or)$ *is a homotopy associative algebra. Any algebra over* $\mathrm{Chain}(\mathcal{C}acti)$ *is a homotopy Gerstenhaber algebra. Any algebra over* $\mathrm{Chain}(\mathcal{C}acti)$ *is a homotopy BV algebra. Any algebra over* $\mathrm{Chain}(\mathcal{A}rc)$ *is a homotopy BV algebra.*

(4) *Cellular chains: For $\mathcal{C}or$, $\mathcal{C}act$ and $\mathcal{C}acti$ there exist CW-models where the up to homotopy structures of the relevant algebras are given by explicit chains.*

[1] Another common name for these are unital operads. This is however confusing, since this could also mean that there is a unit in the 1 component of the operad. This is the case for all the operads we consider.

3.1.1 Background. One role of linear operads, that is those based on (complexes of) vector spaces, is that they can encode certain algebraic structures. Among these are associative, commutative, Lie, but also more complicated algebras like pre-Lie, Gerstenhaber algebras and BV algebras. We will say that an operad represents a type of algebra if the algebras over this operads are precisely of the given type. For a vector space to be an algebra over an operad means that for each element of the operad there is an associated multi-linear operation and these operations are compatible with all the operad structures.

Moreover some of these linear operads are actually the homology of a topological operad. This provides the geometric reason for the appearance of certain types of algebras. If the topological operad acts on a space at the topological level, the homology of this operad acts on the homology of this space. This provides algebraic structures on these homologies. In this type of setup it is clear that one can replace the operad by a different one if they have the same homology operad. The correct notion for this type of equivalence is the one induced by quasi-isomorphism. One of the questions that arises is to what extent linear actions can be lifted to the chain or topological level. On the chain level we are dealing with a dg structure and the algebras are of the type of the algebra over the homology, but only up to homotopy.

The two classical examples we will consider are the little discs and the framed little discs.

Theorem 3.3 ([5], [14]). *An algebra is a Gerstenhaber algebra if and only if it is an algebra over the homology of the little discs operad D_2.*

An algebra is a Gerstenhaber algebra if and only if it is an algebra over the homology of the framed little discs operad fD_2.

On the topological level the relevant theorems are:

Theorem 3.4 ([53], [3], [41]). *A connected space has the homotopy type of a loop space if and only if it is an algebra over the A_∞ operad with base point.*

If a connected space is an algebra over an E_k operad, it has the homotopy type of a k-fold loop space, for $k \in \mathbb{N} \cup \{\infty\}$ with base point.

Here the E_k are the little k-cubes operads and being an E_k operad means that the operad is equivalent to an E_k operad. For instance the little discs are an E_2 operad. There is a subtlety here wether or not to include the base point, see §5.1.

The $\mathcal{A}rc$ operad itself contains E_0, E_1 and E_2 operads as well as the framed versions. To obtain the higher E_k operads one can stabilize as in [27] and §5.1. We will now make the structures present in $\mathcal{A}rc$ explicit. Note that this gives an explicit chain level version of these operads which in turn gives explicit ∞ or better "up to homotopy"-versions of the respective algebras. That is for example an explicit notion of Gerstenhaber algebra up to homotopy or BV algebra up to homotopy. These explicit up to homotopy versions are extremely adapted to describe natural actions. This is one of the "miracles" of the theory: "The geometry of foliations chooses the correct algebraic model".

The Sullivan quasi-PROP is a rigorous incarnations of the idea of Chas–Sullivan on string topology. This is a PROP up to homotopy, which contains homotopy versions of the two suboperads above. It is designed to furnish even more operations on homology of loop spaces. This meshes well with the above results. In particular, we will exhibit such an action by using the Hochschild co-chain approach. The main Theorem being

Theorem 3.5 ([17], [6]). *If M is a simply connected manifold then*

$$H_*(LM) \simeq HH^*(S^*(M), S_*(M)).$$

The fact that the suboperads act are versions of Deligne's Hochschild conjecture. We will give the details below.

3.2 (Framed) little discs and (spineless) cacti

The operad of framed little discs appears as a suboperad as follows.

Definition 3.6. The operad $\mathcal{C}acti$ is the suboperad of $\mathcal{D}Arc$ given by the surfaces with weighted arc families, which satisfy

(1) $g = s = 0$,

(2) there are only arcs which run from ∂_0 to ∂_i, where $i \neq 0$.

The operad $\mathcal{C}act$ called spineless cacti operad is the suboperad of $\mathcal{C}acti$ where additionally

(3) for any two arcs e_1, e_2 incident to ∂_i with $e_2 <_i e_1$ we have $e_1 <_0 e_2$.

The operad $\mathcal{C}or$ is the suboperad of $\mathcal{C}act$ where each element has exactly one arc per boundary $i > 0$ that runs to the boundary 0.

Theorem 3.7 ([21]). *The operad $\mathcal{C}acti$ as well as its image in $\mathcal{A}rc$ are equivalent to the framed little discs operad.*

The operad $\mathcal{C}act$ as well as its image in $\mathcal{A}rc$ are equivalent to the little discs operad.

Remark 3.8. The theorem about $\mathcal{C}acti$ was first stated by Voronov. This and the theorem for $\mathcal{C}act$ were proven in [21]. The method of proof is to show that there is a forgetting map, which essentially fills in the nth boundary component and that this map is a quasi-fibration. The proof itself is quite subtle and lengthy.

Remark 3.9. These are not the original definitions of $\mathcal{C}acti$ and $\mathcal{C}act$. These were given in [63] and [21]. They are however isomorphic operads as shown in [31], [21]. In [31] the images of these operads in $\mathcal{A}rc$ were called $\mathcal{T}ree$ and $\mathcal{LT}ree$.

Immediate consequences are:

Corollary 3.10. (1) *The $\mathcal{A}$rc operad detects loop spaces. That is: a connected space that is an algebra over the $\mathcal{A}$rc operad has the homotopy type of a two-fold loop space.*

(2) *The group completion of the $\mathcal{A}$rc operad has the homotopy type of a double loop space.*

3.2.1 Cactus terminology. One arrives at the usual pictures, resembling succulents, if one considers the images under the loop map $\mathscr{L}$ or the dual graphs. The conditions (1) and (2) translate into the fact that this graph is topologically a planar tree of S^1's – one S^1 for each cycle corresponding to the boundaries $\partial_1, \dots, \partial_n$ – with a marked point on each of them and a global marked point on the "outside circle", which is the cycle corresponding to ∂_0. The cycles are indeed parameterized via the maps l_i or simply by using the weights on the edges as lengths. The cycle ∂_0 traverses each edge exactly once. This is what is combinatorially taken to be the definition of treelike for ribbon graphs.

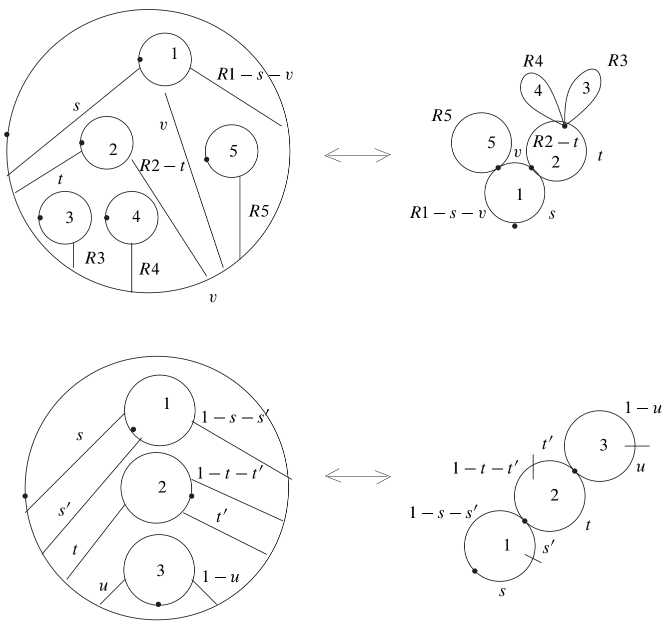


Figure 7. A cactus and a spineless cactus.

The S^1's are usually called lobes and the marked points are indicated by tick marks which are called spines. The marking on the outside circle is usually called the root or global marked point. Spineless cacti have the feature that all the spines are at intersection points and moreover they are at the unique intersection point along the outside circle, where the outside circle first intersects the respective lobe. In the combinatorial version this is just the point before the first arc, which is uniquely determined by the order $<_0$. Hence, the information about the spines is redundant and can be omitted, whence the name spineless.

3.2.2 Bi-crossed structure. The relationship between the little discs and the framed little discs is given by the statement that the framed little discs are a semi-direct product of the little discs operad and the operad built on the circle group S^1 [52].

The corresponding relationship for $\mathcal{C}act$ and $\mathcal{C}acti$ is a bit more complicated. $\mathcal{C}acti$ is a bi-crossed product of $\mathcal{C}act$ and the operad built on the circle group S^1 [21].

This fact was used by Westerland and Salvatore for their further study of actions [66], [51].

The general algebraic structure is lengthy to describe, but its geometrical content becomes clear when one thinks about twists. The condition for $\mathcal{C}act$ means that the elements are untwisted at all boundaries $1, \dots, n$. This together with all the other conditions allows us to unambiguously reconstruct the marked points on the boundary. Now when we glue the twist on the 0-th boundary "propagates" through the surface which is why there is a bi-crossed product.

3.3 Cellular structure

The suboperads $\mathcal{C}act$ and $\mathcal{C}acti$ are not genuine CW complexes per se, but they retract to CW complexes. The CW complexes are given by the condition

$$\mathrm{wt}(\partial_i) = 1 \quad \text{for } i = 1, \dots, n. \tag{*}$$

Given the conditions of Cacti, this automatically makes $\mathrm{wt}(\partial_0) = n$. We denote the respective subspaces of $\mathcal{C}act$ and $\mathcal{C}acti$ by $\mathcal{C}act^1$ and $\mathcal{C}acti^1$. As topological spaces $\mathcal{C}act(n) = \mathcal{C}act^1(n) \times \mathbb{R}^n_{>0}$ and $\mathcal{C}acti(n) = \mathcal{C}acti^1(n) \times \mathbb{R}^n_{>0}$, where the $\mathbb{R}_{>0}$ factors simply keep track of $\mathrm{wt}(\partial_i)$.

Dropping the $\mathbb{R}_{>0}$ factors, the spaces loose their operadic structure, since the condition (*) is not preserved upon gluing. The way out is to use the local scaling version of the gluing.

For two elements $\alpha \in \mathcal{C}acti^1(n)$ and $\beta \in \mathcal{C}acti^1(m)$ we define $\alpha \circ_i^1 \beta$ to be the weighted arc family obtained by scaling all weights of arcs incident to ∂_i of α homogeneously by the factor m.

Proposition 3.11 ([21]). *The operation $\circ_i'$ preserves the condition $(*)$ and it also preserves the conditions of spineless cacti. Therefore, $\circ_i' \colon \mathcal{C}act^1(n) \times \mathcal{C}act^1(m) \to \mathcal{C}act^1(m+n-1)$ and $\circ_i' \colon \mathcal{C}acti^1(n) \times \mathcal{C}acti^1(m) \to \mathcal{C}acti^1(m+n-1)$.*

Theorem 3.12 ([21]). *Both $\mathcal{C}act^1$ and $\mathcal{C}act^i$ are CW complexes.*

The operations $\circ'_j$ are symmetric group invariant, associative up to homotopy and cellular. Moreover, the induced operations $CC_*(\circ'_j)$ induce a bona fide operad structure on the collection of cellular chains.

Finally the induced operad structure on homology agrees with the one induced from $\mathcal{D}Arc$.

3.3.1 Explicit representatives for the bracket and the BV equation. The points in $\mathcal{A}rc_0^0(1) = \mathcal{C}acti(1)$ are parameterized by the circle, which is identified with $[0,1]$, where 0 is identified to 1. As stated above, there is an operation associated to the family δ. For instance, if F_1 is any arc family $F_1 \colon k_1 \to \mathcal{A}rc_0^0$, δF_1 is the family parameterized by $I \times k_1 \to \mathcal{A}rc_0^0$ with the map given by the picture by inserting F_1 into the position 1. By definition,

$$\Delta = -\delta \in C_1(1).$$

In $C_*(2)$ we have the basic families depicted in Figure 8 which in turn yield operations on C_*.

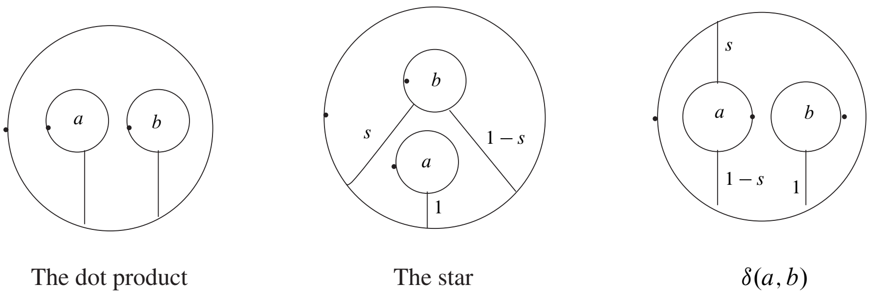


Figure 8. The binary operations.

To fix the signs, we fix the parameterizations we will use for the glued families as follows: we say the families F_1, F_2 are parameterized by $F_1 \colon k_1 \to \mathcal{A}rc_0^0$ and $F_2 \colon k_2 \to \mathcal{A}rc_0^0$ and $I = [0,1]$. Then $F_1 \cdot F_2$ is the family parameterized by $k_1 \times k_2 \to \mathcal{A}rc_0^0$ as defined by Figure 8 (i.e., the arc family F_1 inserted in boundary a and the arc family F_2 inserted in boundary b).

Interchanging labels 1 and 2 and using $*$ as the explicit chain homotopy given in Figure 9 yields the commutativity of $\cdot$ up to chain homotopy:

$$d(F_1 * F_2) = (-1)^{|F_1||F_2|} F_2 \cdot F_1 - F_1 \cdot F_2. \tag{3.1}$$

Notice that the product $\cdot$ is also associative up to chain homotopy.

Likewise $F_1 * F_2$ is defined to be the operation given by the second family of Figure 8 with $s \in I = [0,1]$ parameterized over $k_1 \times I \times k_2 \to \mathcal{A}rc_0^0$.

By interchanging the labels, we can produce a cycle $\{F_1, F_2\}$ as shown in Figure 9 where now the whole family is parameterized by $k_1 \times I \times k_2 \to \mathcal{A}rc_0^0$,

$$\{F_1, F_2\} := F_1 * F_2 - (-1)^{(|F_1|+1)(|F_2|+1)} F_2 * F_1.$$

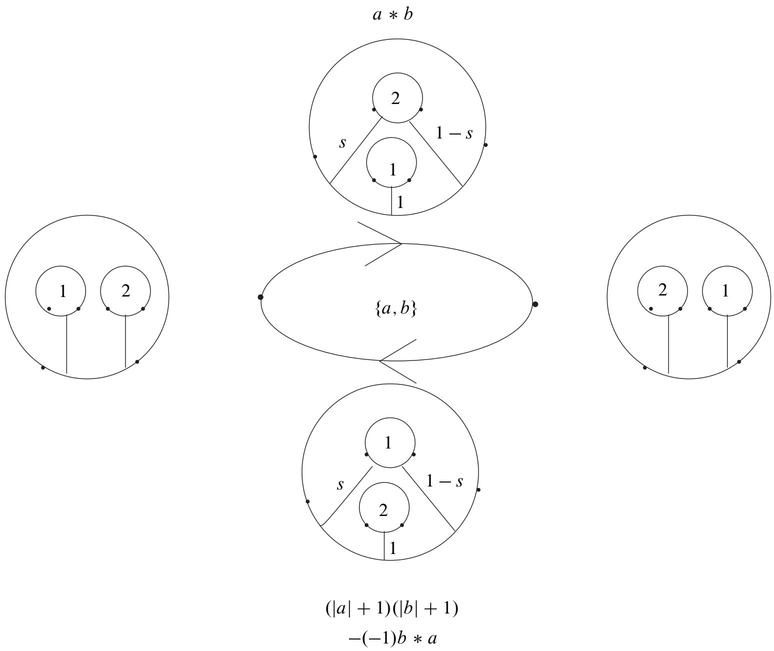


Figure 9. The definition of the Gerstenhaber bracket.

Remark 3.13. We have defined the following elements in C_*:

δ and $\Delta = -\delta$ in $C_1(1)$;

$\cdot$ in $C_0(2)$, which is commutative and associative up to a boundary.

$*$ and $\{-,-\}$ in $C_1(2)$ with $d(*) = \tau \cdot - \cdot$ and $\{-,-\} = * - \tau *$.

Note that $\delta, \cdot$ and $\{-,-\}$ are cycles, whereas $*$ is not.

3.4 The BV operator

The operation corresponding to the arc family δ is easily seen to square to zero in homology. It is therefore a differential and a natural candidate for a derivation or a higher order differential operator. It is easily checked that it is *not a derivation*, but it *is a BV operator*.

Proposition 3.14. *The operator Δ satisfies the relation of a BV operator up to chain homotopy:*

$$\begin{aligned} \Delta^2 &\sim 0, \\ \Delta(abc) &\sim \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{|sa||b|}b\Delta(ac) - \Delta(a)bc \\ &\quad - (-1)^{|a|}a\Delta(b)c - (-1)^{|a|+|b|}ab\Delta(c). \end{aligned} \tag{3.2}$$

Thus, any Arc algebra and any Arc_{cp} algebra is a BV algebra.

Lemma 3.15.

$$\delta(a,b,c) \sim (-1)^{(|a|+1)|b|}b\delta(a,c) + \delta(a,b)c - \delta(a)bc. \tag{3.3}$$

Proof. The proof is contained in Figure 10. Let $a\colon k_a \to \mathcal{A}rc_0^0$, $b\colon k_b \to \mathcal{A}rc_0^0$ and $c\colon k_c \to \mathcal{A}rc_0^0$ be arc families, then the two parameter family filling the square is

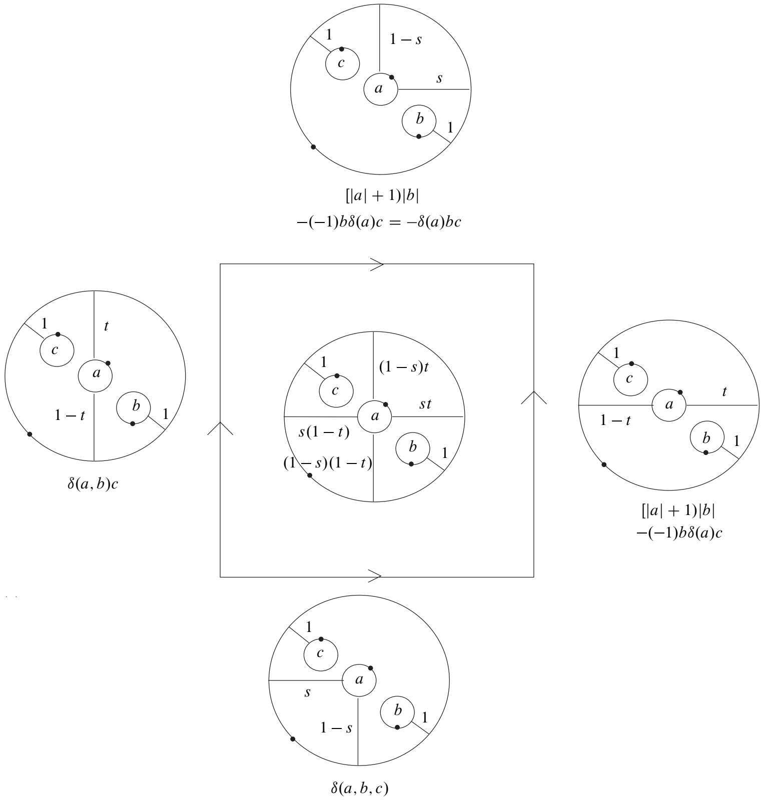


Figure 10. The basic chain homotopy responsible for BV.

parameterized over $I \times I \times k_a \times k_b \times k_c$. This family gives us the desired chain homotopy. □

Given arc families $a: k_a \to \mathcal{A}rc_0^0$, $b: k_b \to \mathcal{A}rc_0^0$ and $c: k_c \to \mathcal{A}rc_0^0$, we consider the two parameter families given in Figure 11, where the families in the rectangles are the depicted two parameter families parameterized over $I \times I \times k_a \times k_b \times k_c$ and the triangle is not filled. Its boundary is the operation $\delta(abc)$.

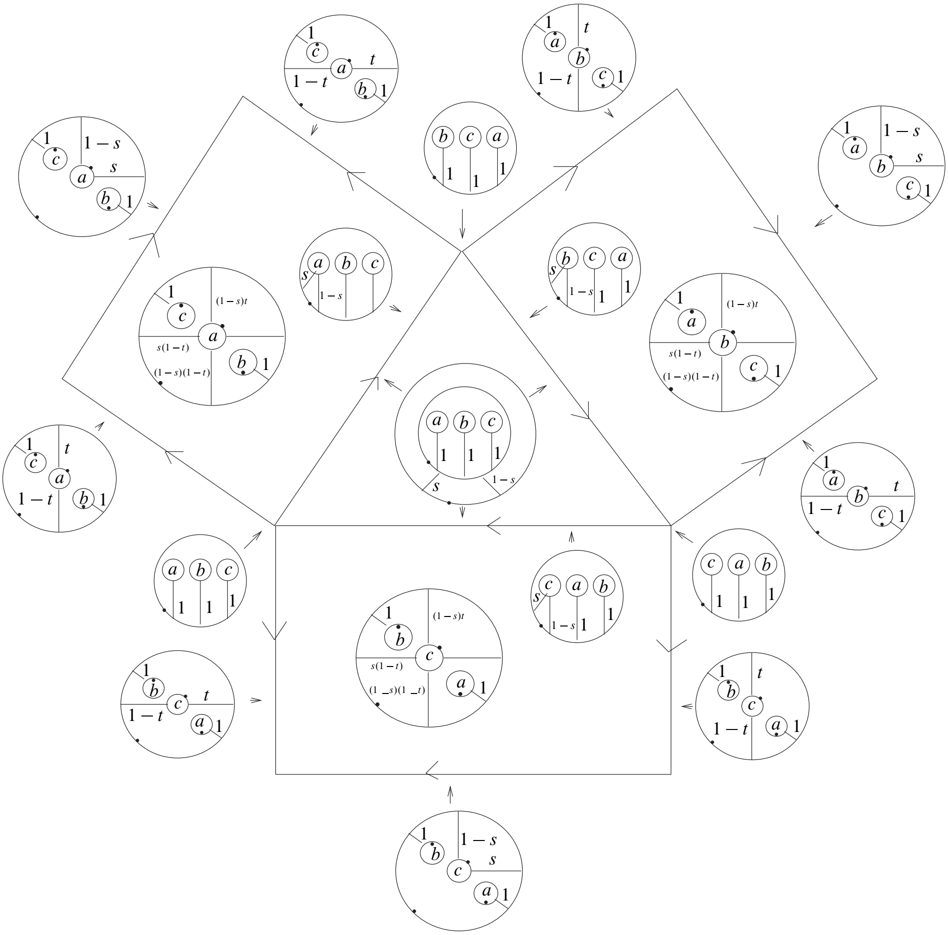


Figure 11. The homotopy BV equation.

From the diagram we get the chain homotopy for BV. The threefold operation consists of three terms from the boundary of the inner triangle, and this is homotopic to nine terms given by the outside sides of the three rectangles. This makes the BV equation a highly symmetric twelve term equation.

Remark 3.16. The fact that the chain operads of $\mathcal{A}rc$ and as we show below $\mathcal{C}act(i)$ or $\mathcal{C}act^1(i)$ all possess the structure of a G(BV) algebra up to homotopy means that for any algebra V over them the algebra as well as Hom_V have the structure of G(BV). If one is in the situation that one can lift the algebra to the chain level, then the G(BV) will exist on the chain level up to homotopy.

Remark 3.17. We would like to point out that the symbol $\bullet$ in the standard super notation of odd Lie-brackets $\{a \bullet b\}$, which is assigned to have an intrinsic degree of 1, corresponds geometrically in our situation to the one-dimensional interval I.

3.5 The associator

It is instructive to do the calculation in the arc family picture with the operadic notation. For the gluing $* \circ_1 *$ we obtain the elements in $C_2(2)$ presented in Figure 12 to which we apply the homotopy of changing the weight on the boundary 3 from 2 to 1 while keeping everything else fixed. We call this normalization.

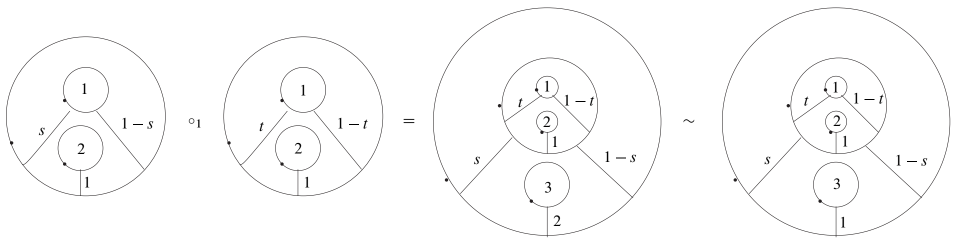


Figure 12. The first iterated gluing of $*$.

Unraveling the definitions for the normalized version yields Figure 13, where in the different cases the gluing of the bands is shown in Figure 14.

The gluing $* \circ_2 *$ in arc families is simpler and yields the gluing depicted in Figure 15 to which we apply a normalizing homotopy – by changing the weights on the bands emanating from boundary 1 from the pair $(2s, 2(1-s))$ to $(s, 1-s)$ using pointwise the homotopy $(\frac{1+t}{2}2s, \frac{1+t}{2}(1-s))$ for $t \in [0,1]$.

Combining Figures 13 and 15 while keeping in mind the parameterizations we can read off the pre-Lie relation:

$$\begin{aligned} &F_1 * (F_2 * F_3) - (F_1 * F_2) * F_3 \\ &\quad \sim (-1)^{(|F_1|+1)(|F_2|+1)}(F_2 * (F_1 * F_3) - (F_2 * F_1) * F_3) \end{aligned} \tag{3.4}$$

which shows that the associator is symmetric in the first two variables and thus following Gerstenhaber [G] we obtain:

Corollary 3.18. *$\{F_1, F_2\}$ satisfies the odd Jacobi identity.*

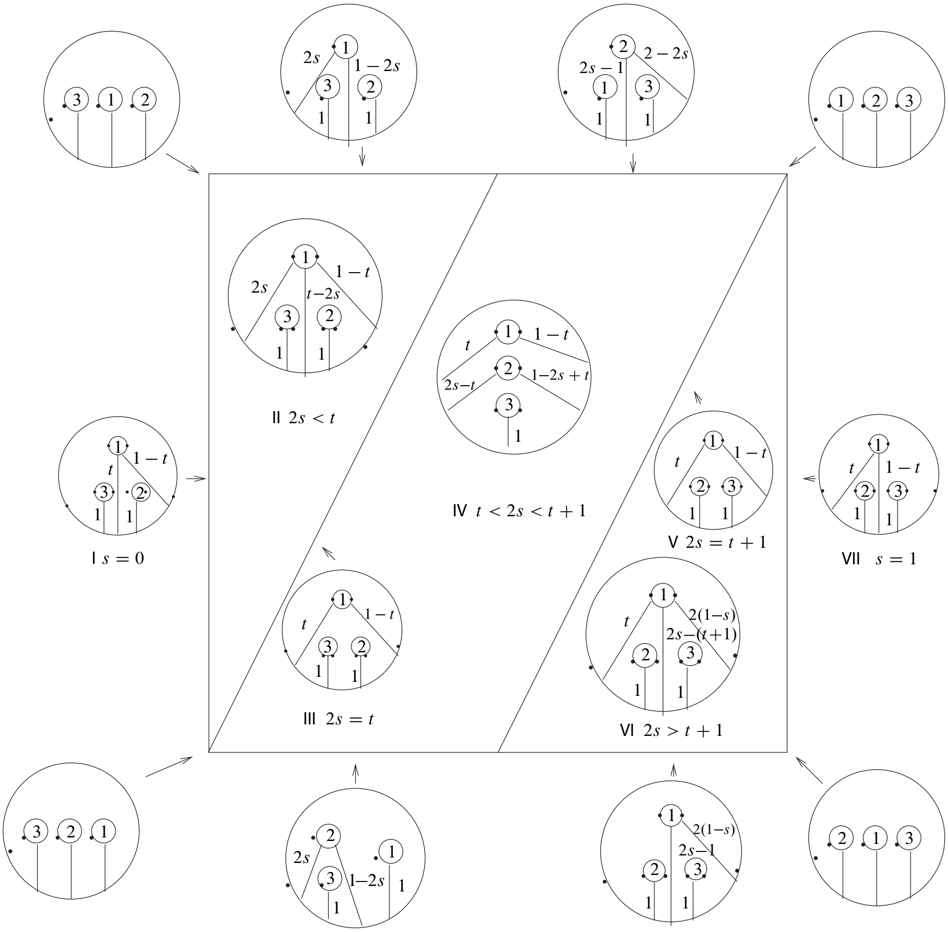


Figure 13. The glued family after normalization.

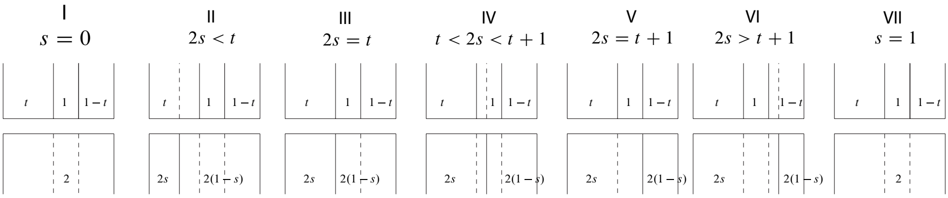


Figure 14. The different cases of gluing the bands.

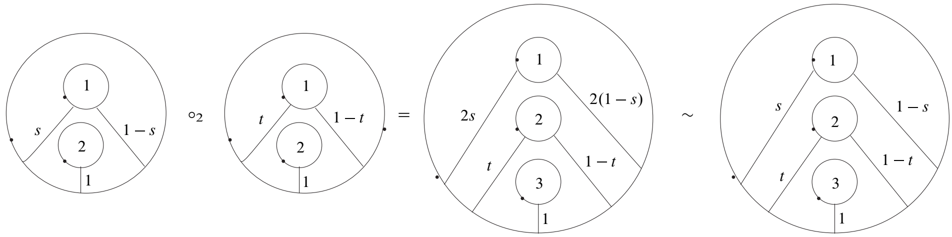


Figure 15. The other iteration of $*$.

4 Moduli space, the Sullivan-PROP and (framed) little discs

One of the applications of the arc operad is to CFT and string topology. In principle, moduli space is a "suboperad" of the arc operad and the Sullivan-PROP is a quasi-PROP generalization that works for a partial compactification of a subset of the arc operad in its ambient spaces $A^s_{g,r}$. This quasi-PROP is also a generalization of two bona fide suboperads of the arc operad which are equivalent to the well known little discs and framed little discs operads. These operads are responsible for the preeminent algebraic structures found in CFT and string topology, the Gerstenhaber bracket and the BV operator.

In order to set up everything completely rigorously for the compositions a little finesse is needed.

4.1 Moduli spaces

Let $M^{1,\dots,1}_{g,r,s}$ be the subset of elements of $A^s_{g,r}$ whose arc families are quasi-fillings. Here the superscript 1 is repeated r times.

If $s = 0$, we simply write $M^{1,\dots,1}_{g,r}$. From the description in terms of the dual ribbon graphs 1.5.1 the following theorem can be obtained using Strebel differentials (see e.g. [23])

Theorem 4.1. *The space $M^{1,\dots,1}_{g,r}$ is proper homotopy equivalent to the moduli space of Riemann surfaces of genus g with n marked points and a tangent vector at each point modulo the free and proper scaling action of $\mathbb{R}_{>0}$ which scales all tangent vectors simultaneously.*

The homotopy equivalence can be lifted to the product with $\mathbb{R}_{>0}$ thus lifting $M^{1,\dots,1}_{g,r}$ to $D^0_{g,r}$ and reversing the quotient by $\mathbb{R}_{>0}$ on the moduli space side.

In fact, using the hyperbolic approach, Penner was able to identify the moduli space corresponding to $M^{1,\dots,1}_{g,r,s}$ for arbitrary s. Define the "moduli space" $M = M(F)$ of the surface F with boundary to be the collection of all complete finite-area metrics of constant Gauss curvature -1 with geodesic boundary, together with a distinguished point p_i in each boundary component, modulo push forward by diffeomorphisms. There is a natural action of $\mathbb{R}_+$ on M by simultaneously scaling each of the hyperbolic lengths of the geodesic boundary components.

Theorem 4.2 ([47]). *The space $M^{1,\dots,1}_{g,r,s}$ is proper homotopy equivalent to the quotient $M/\mathbb{R}_+$.*

Theorem 4.3. *If $3g-2n+3>0$ the $(S^1)^r$ coinvariants of the subspace $M^{1,\dots,1}_{g,r}$ is homeomorphic to the moduli space $M_{g,r}$.*

4.2 Operad structure on moduli spaces

A natural question to ask is whether the operad structure on $\mathcal{A}rc$ can be restricted to the quasi-filling families given by the subspaces $M^{1,\dots,1}_{g,r,s}$. This is not true on the nose. In fact on a codimension-1 set, the gluing of two quasi-filling families might take us to a non-quasi-filling family. Generically this does not happen, though. A careful analysis was given in [23]. The upshot is that if α has k arcs and β has l arcs then generically $\alpha\circ_i\beta$ has $k+l-1$ arcs. And in this case, essentially by an Euler-characteristic argument, the resulting family is again quasi-filling. In order for the number of arcs on $\alpha\circ_i\beta$ to drop we need that two of the points which form the boundary of the bands in the construction §2.1.2 coincide. We introduced new terminology for this type of situation. A rational (cyclic) operad is an operad structure on a dense open subset.

Theorem 4.4. *The collection $M(r-1):=\amalg_g M^{1,\dots,1}_{g,r,s}\subset\mathcal{A}rc(r-1)$ forms a rational cyclic operad.*

Things really work out on the chain level after passing to the associated graded.

4.2.1 Cell level for the moduli spaces. As in the case of the arc operad the moduli space is the disjoint union of open cells $\dot{C}([\Gamma]$ where now there is one cell for any given quasi-filling $[\Gamma]$.

We let $\mathcal{C}^*_o(\mathcal{A}rc^0_\#)$ be the subgroup of $\mathcal{C}^*_o(\mathcal{A}rc)$ generated by the cells corresponding to quasi-filling arc families with no punctures and write $\mathrm{Gr}\,\mathcal{C}^*_o(\mathcal{A}rc^0_\#)$ for the image of this subgroup on $\mathrm{Gr}\,\mathcal{C}^*_o(\mathcal{A}rc)$. Notice that *on this subset* the cells indexed by an *arc graph* Γ can be equivalently thought of as indexed by the *ribbon graph* $\hat{\Gamma}$. The differential that removes arcs in Γ acts on $\hat{\Gamma}$ by contracting the corresponding edge.

Definition 4.5. Following the usual arguments [36], [47], [48], [10], the graph complex of marked ribbon graphs is the Hopf algebra whose primitive elements are connected marked ribbon graphs and whose product is the disjoint union. Its differential is given by the sum of contracting edges $d\Gamma=\sum_{e\in E'(\Gamma)}\pm\Gamma/e$, where $E'(\Gamma)$ is the subset of edges e such that the topological type of Γ coincides with that of Γ/e and the sign is the usual sign.

Theorem 4.6 ([23]). *The complex* $(\mathrm{Gr}\,\mathcal{C}_o^*(\mathcal{A}rc_\#^0)(F_{g,r}^0), d)$ *is isomorphic to the graph complex of marked ribbon graphs. Both can be used to compute* $H^*(\mathrm{PMC}(F_{g,s^0}))$, *the cohomology of the pure mapping class group and the spaces* $H^*(M_{g,r,0}^{1,\dots,1})$.

The induced operad structure on the collection

$$\mathrm{Gr}\,\mathcal{C}_o^*(\mathcal{A}rc_\#^0)(r-1) = \amalg_{g,s}\mathrm{Gr}\,\mathcal{C}_o^*(\mathcal{A}rc_\#^0)(F_{g,r}^0)$$

is a cyclic dg-operad structure which descends as a cyclic operad to $H^*(M_{g,r,0}^{1,\dots,1})$.

These are the cellular operads which give rise to the Hochschild actions for moduli space, see §6.4.

4.3 The Sullivan quasi-PROP

The arc operad or even the spaces $A_{g,r}^s$ are inherently symmetric in all boundaries. This symmetry was a bit broken by designating the boundary 0 as special. The idea is that this is the output boundary, while the other n boundaries are the input boundaries. The full symmetry is restored in the cyclic setting.

Keeping with the in- and output picture, we can add additional information by specifying input and output boundaries on $F_{g,r}^s$. Technically this marks the move from operads to PROPs. In the PROP setting one composes, by gluing all inputs of one element to all outputs of another if their number matches. This setup is used to describe the string topology of Chas and Sullivan [8], [9], [56], [55], [57], [7].

Furthermore in the PROPic setting one usually does not demand that the surfaces are connected. This ensures the existence of so-called horizontal compositions (see §4.3.3 below and the appendix). To spell this out in our situation, using the standard models, we can consider disjoint unions $F_{g_1,r_1}^{s_1} \amalg \cdots \amalg F_{g_k,r_k}^{s_1}$ and consider their PMCs which are the products of the individual PMCs. The number of boundaries r of such a not necessarily connected surface is just the sum of the r_i. An arc family on such a disjoint union is just the disjoint union of arc families on the individual surfaces.

The slightly subtle points are (1) that the disjoint union is not strictly symmetric monoidal and (2) the enumeration of the boundaries. Given a possibly non-connected surface as above with r boundaries, we also fix n, m such that $n + m = r$ and now separately enumerate n of the boundaries from 1 to n calling them input boundaries and enumerate the remaining m boundaries from 1 to m calling them output boundaries.

We let $D(n,m)$ be the set of PMC orbits of weighted arc families on such surfaces. Technically, we again enumerate components by the total genus and the total number of punctures to get a break down of the space into finite-dimensional spaces and then take the colimit. Moreover, the components are indexed by k and further by tuples $(g_1,\dots,g_k)$, $(s_1,\dots,s_k)$, $(r_1,\dots,r_k)$. To get a CW complex we could quotient out by a global scaling action.

The space $D(n,m)$ has an $\mathbb{S}_n \times \mathbb{S}_m$ action which permutes the input and the output boundaries. We let $D^0(n,m)$ be the families on surfaces without punctures.

4.3.1 The spaces

Definition 4.7. The spaces of Sullivan type arc families $\mathcal{S}ull(n,m) \subset D^0(n,m)$ are the subspaces which satisfy that

(1) arcs only run from inputs to the outputs or from the outputs to outputs,

(2) all input windows are hit.

We furthermore define the following subspaces:

(1) the strict Sullivan arc families $\mathcal{S}ull^{\mathrm{st}}(n,m)$ which is the subset of families with arcs only from in to out boundaries.

(2) $\mathcal{S}ull^1(n,m) \subset \mathcal{S}ull^{\mathrm{st}}(n,m)$ where the condition is that the weight of each of the n input boundaries is 1.

These spaces have the following properties:

(1) $\mathcal{S}ull(n,m)$, has an $\mathbb{S}_n \times \mathbb{S}_m$ action which permutes the input and the output boundaries (separately).

(2) $\mathcal{S}ull(n,m)$ retracts onto $\mathcal{S}ull^{\mathrm{st}}(n,m)$ simply by scaling the weights of all the arcs from output to output to zero.

(3) The dual ribbon graphs of the arc graphs of $\mathcal{S}ull(n,m)$ are Sullivan chord diagrams in the sense that the cycles corresponding to the in boundaries can be disjointly embedded up to finitely many points of intersection. These circles are joined by edges corresponding to the arcs going from outputs to outputs. These are not present in $\mathcal{S}ull^{\mathrm{st}}(n,m)$.

Remark 4.8. Notice that as the arc families neither have to be quasi-filling nor exhaustive, these dual graphs do not necessarily determine the topological type of even the number of boundaries of the surface they lie on. One can add this extra information if one chooses to do so. We will continue with the arc graphs since these are unambiguous as the surface they lie on carries this extra information.

Notice that if $\mathcal{D}Arc(n,m)$ again denotes the exhaustive families in $D(n,m)$, then $\mathcal{S}ull(n,m) \not\subseteq \mathcal{D}Arc(n,m)$, and $\mathcal{S}ull(n,m) \not\supseteq \mathcal{D}Arc(n,m)$ but also $\mathcal{S}ull(n,m) \cap \mathcal{D}Arc(n,m) \neq \emptyset$. In the same fashion the quasi-filling families $M(n,m) \subset D(n,m)$ have non-zero intersection but have no containment relation with $\mathcal{D}Arc(n,m)$.

4.3.2 A CW model: $\mathcal{S}ull^1$

Proposition 4.9. *The space $\mathcal{S}ull(n,m)$ deformation retracts to $\mathcal{S}ull^{\mathrm{st}}(n,m)$ and this in turn deformation retracts onto a smaller subspace $\mathcal{S}ull^1(n,m)$ which is a CW complex.*

The cells are indexed by the classes of arc graphs $[\alpha]$.[2] And the dimension of a cell is given by the $|[\alpha]| - n$. To see that these are cells and yield a CW complex, we can proceed as before. Given $[\alpha]$ with n inputs, the arc families with that type and non-zero weights given the restriction are a product of n open simplices. The face maps are given as before by identifying the face with the open simplex corresponding to the family where an arc has been removed. Notice that since the total weight on each input boundary is one, the condition of all input boundaries being hit is stable under taking the boundary of the simplex – some arc always remains. The retraction is simply given as follows. First we can retract by scaling the weights of arcs going form outputs to outputs as before. Since there are no arcs from input to input boundary each remaining arc is incident to a unique boundary component. For each boundary component we now simultaneously scale the weights of all the arcs incident to it to make their sum equal to one. We can do this at each boundary separately or we can do it at all the boundaries at once.

4.3.3 (Quasi)-PROPic gluing. As mentioned above a PROP $\mathcal{P}$ is similar to an operad, but there are two main differences.

The first is that there is a simultaneous gluing of all inputs to all outputs

$$\circ \colon \mathcal{P}(n,m) \otimes \mathcal{P}(m,p) \to \mathcal{P}(n,p)$$

which is associative and equivariant with respect to the various symmetric group actions. The intuitive example is again based on a vector space V over a field k: $\mathrm{Hom}_V(n,m) := \mathrm{Hom}_k(V^{\otimes n}, V^{\otimes m})$ with composition. The composition will be given by a local scaling version of the scaling and its extension by erasing. As we have mentioned before, the local scaling version usually does not produce associative structures and this happens here as well. We do get a structure that is associative up to homotopy, however, which is what we defined to be a topological quasi-PROP.

The second difference is that in the proper definition of a PROP there is also a horizontal composition $\bullet \colon \mathcal{P}(n,m) \otimes \mathcal{P}(k,l) \to \mathcal{P}(n+k, m+l)$ again associative and compatible with $\circ$ and the symmetric group actions. For Hom_V the horizontal composition is given by the tensor product.

In our case of $\mathcal{S}ull$ as well as in most topological examples we add a horizontal composition by taking it to be the disjoint union. For this reason, we enlarged the spaces to include not necessarily connected components.

The composition in $\mathcal{S}ull$ is given as follows. Let $a \in \mathcal{S}ull(n,m)$ and $b \in \mathcal{S}ull(m,p)$ then we first prepare b so that the input boundary i of b has the same weight as the output boundary i of a. We do this by using the local scaling action as before. This action is naturally given by a flow which scales all the weights of the arcs incident to a given input boundary in the same fashion. Again we can prepare all input boundaries simultaneously, since each arc that hits an input boundary hits a unique such boundary. After this preparation step the weights on the boundaries $1, \dots, n$ of

[2] We keep the same notation also in the non-connected case.

a and b that are to be glued coincide and we just glue the surfaces and the foliations at all these boundaries.

This type of gluing is not associative, but the fact that the preparation step is given by a flow ensures that there is a homotopy between two different ways of associating. This is basically done by using the flow in the reverse direction. The condition that there is no arc from outputs to outputs is preserved, since we only glue together arcs that run from inputs to outputs. Also neither the flow nor the gluing changes the total weights on the input boundaries of a so that the spaces $\mathcal{S}ull^1(n,m)$ are stable.

Theorem 4.10 ([23]). *The composition* $\circ\colon \mathcal{S}ull(m,n)\times\mathcal{S}ull(n,p)\to\mathcal{S}ull(m,p)$ *is homotopy associative, symmetric, group invariant and compatible with the horizontal composition* $\amalg\colon \mathcal{S}ull(m,n)\times\mathcal{S}ull(k,p)\to\mathcal{S}ull(m+k,n+p)$*; that is these spaces form a topological quasi-PROP.*

The subspaces $\mathcal{S}ull^1(m,n)$ *form a topological sub-quasi-PROP.*

The cellular chains $CC_*(\mathcal{S}ull^1)$ *form a (strict) PROP.*

The last statement is not straightforward, it relies on an analysis of the gluing maps as in the case of cacti.

Corollary 4.11. *The* $\mathbb{S}_n\times\mathbb{S}_m$ *modules* $H_*(\mathcal{S}ull(n,m),k)$ *form a PROP.*

5 Stops, stabilization and the $\mathcal{A}rc$ spectrum

5.1 Stops: adding a unit

5.1.1 The little discs operad case. When considering the little discs operad $D_2(n)$, one has to be a bit careful whether or not one considers it pointed or not. In practice this means that one either includes $D_2(0)$ in the sequence of the $D_2(n)$ or not. $D_2(0)$ is just the big discs without any little discs inside. Notice that an element in the 0 component of an operad has no inputs. Gluing it into another element decreases the number of inputs by one. In the particular case of D_2, the zero component $D_2(0)$ is just a point and composition with it just erases the little disc it is glued into. This point is taken to be the base point of the operad. We will call D_2 with the 0 component the little discs with base point.

On the homology level, the inclusion of $H_*(D_0,k)=k$ has the effect that the algebras over the little discs with base point are *unital* Gerstenhaber algebras. The unit of the algebra is just the image of $1\in k$, while without the base point the algebras are not required to have a unit.

This is a general phenomenon. Including a contractible 0 component, mostly just a (base) point, to an operad whose algebras are some known type of algebra, restricts the algebras over it to be unital.

On the topological level, especially for detecting loop spaces, the base point is needed as most of the construction works in pointed topological spaces. Indeed in Theorem 3.4 the versions of the E_k operads with base points are needed.

5.2 Adding a unit in the arc formalism

We will show how to add a unit in this sense to the suboperads $\mathcal{G}\mathit{Tree}$ of $\mathcal{A}\mathit{rc}$ as in [27] and also give the generalization, to the PROP $\mathcal{S}\mathit{ull}(n, m)$. The latter generalization has not formally appeared. $\mathcal{G}\mathit{Tree}_g(n) \subset \mathcal{A}\mathit{rc}_g^0(n)$ is defined to be the suboperad which has arcs only running from the input boundaries i to the output boundary 0. This is the condition (2) of Definition 3.6. It was called the operad of Chinese trees in [31], the $\mathcal{G}$ stands for "higher genus". We take $\mathcal{G}\mathit{Tree}(n) = \amalg_g \mathcal{G}\mathit{Tree}(n)$. This is the same space as $\mathcal{S}\mathit{ull}^{\mathrm{st}}(n, 1)/\mathbb{R}_{>0}$.

There is an operadic inclusion of $\mathcal{C}\mathit{acti}$ into $\mathcal{G}\mathit{Tree}$ as the components of genus 0. The straightforward generalization to higher genus of $\mathcal{C}\mathit{act}$ is the suboperad $\mathcal{L}\mathcal{G}\mathcal{T}\mathit{ree}$ which is comprised of the arc families satisfying the condition (3) of Definition 3.6.

5.2.1 Basic idea. As a first approximation to the unit, we could add a 0 component to $\mathcal{C}\mathit{acti}$ by setting $\mathcal{C}\mathit{acti}(0) = \mathrm{pt}$. This point could represents a disc, without an arc family considered as having no inputs, but an output. If we simply erase the foliation upon gluing the result will not be associative. It can be seen that it is homotopy associative, by using a flow argument. That would be enough for the chain and homology level, but to get access to the topological theorems about loop spaces, one needs to have a strict operad structure. This is achieved through the process of wilting. So the unit will be a disc with wilted leaves on the boundary, see §2.9.3. The scaling action of $\mathbb{R}_{>0}$ is retained and thus we can assume the measure or the weight to be 1. When gluing in these discs, the leaves that are wilted are glued in the same fashion as in the standard gluing see §2.9.3.

If we add this point then by gluing, we have to allow replacing any sub-band of leaves of an element of $\mathcal{G}\mathit{Tree}$ by wilted leaves.

In particular, to extend units to all of $\mathcal{G}\mathit{Tree}$, we also have to add one point for each genus $\mathcal{G}\mathit{Tree}_g(0) = \mathrm{pt}$ represented by the surface of genus g with one output boundary and only wilted leaves at that boundary.

5.2.2 Details and results. For any of the spaces considered thus far, we define the fattened version by allowing wilting weights on the outputs as defined in §2.9.3 and denote the result by adding superscript ${}^{\mathrm{Fat}}$ on the left.

Proposition 5.1 ([27]). *The spaces*

$$ {}^{\mathrm{Fat}}\mathcal{G}\mathit{Tree}(n) $$

together with

$$ {}^{\mathrm{Fat}}\mathcal{G}\mathit{Tree}(0) = \amalg_g {}^{\mathrm{Fat}}\mathcal{G}\mathit{Tree}_g(0) $$

where $\mathcal{G}Tree_g(0)$ *consists of a point which represents a surface of genus* g *with one boundary and a wilting weight* (*scaled to* 1) *on that boundary, form an operad under the gluing of* §2.9.3.

The reason why this works is that there are only arcs from the inputs to the output. Upon gluing some of these may wilt, but then the wilted leaves are again only part of the foliation near the output.

In the same manner one can check that the conditions that (1) all the input boundaries are hit and that (2) unless the surface is just a disc, there is at least one arc, are stable under the gluing.

Remark 5.2. We could also lift $\mathcal{G}Tree$ to $\mathcal{D}Arc$, then the 0 component of genus g would be $\mathbb{R}_{>0}$, representing the choice of allowing any weight for the wilted foliations. We would simply get a homotopy equivalent operad.

In this spirit, we let ${}^{\mathrm{Fat}}\mathcal{C}act$ and ${}^{\mathrm{Fat}}\mathcal{C}acti$ be the respective operads whose arc families are in $\mathcal{C}act$ respectively $\mathcal{C}acti$ for $n > 0$ and whose 0 component is $\mathbb{R}_{>0}$.

Theorem 5.3. ${}^{\mathrm{Fat}}\mathcal{C}act$ *is equivalent to the pointed version of the little discs.* ${}^{\mathrm{Fat}}\mathcal{C}acti$ *is equivalent to the pointed version of the framed little discs.*

The first part of the theorem is contained in [27] and the second follows similarly.

Corollary 5.4. ${}^{\mathrm{Fat}}\mathcal{C}act$ *as well as* $\mathcal{G}Tree$ *detect double loop spaces.*

5.2.3 Cacti with stops. If we regard the map loop $\mathcal{L}$ as defined in 2.9.3 the subspaces ${}^{\mathrm{Fat}}\mathcal{C}act$ and ${}^{\mathrm{Fat}}\mathcal{C}acti$ get a very nice geometrical interpretation as cacti with stops. This just means that the parameterization of the outside circle may be constant for certain intervals.

This point of view was explained in [30] and used by Salvatore [51] to provide the details of the announcement made by McClure and Smith [44]. This is the topological version the cyclic Deligne conjecture.

5.2.4 PROP version. The same type of analysis leads to the PROP version of the above proposition.

Proposition 5.5. *The spaces* ${}^{\mathrm{Fat}}\mathcal{S}ull^{\mathrm{st}}(m,n)$ *as well as* $\mathcal{S}ull^{\mathrm{Fat}}(m,n)$ *form a quasi-PROP.*

These statements have not appeared so far, but they follows in the same manner as the operadic counterparts.

5.2.5 The E_1 case. We have treated the full E_2 case. For reference we also give the E_1 restriction. The restriction is given by ${}^{\mathrm{Fat}}\mathcal{C}or$. It is easily seen that $\mathcal{C}or$ are isomorphic to the tight little intervals, that is, partitions of the unit interval, while ${}^{\mathrm{Fat}}\mathcal{C}or$ are isomorphic to the little intervals operad, which is the same as the little 1-cubes.

5.2.6 Further generalizations. One can also fatten the $\mathcal{A}rc$ as a cyclic operad to obtain the unital cyclic operad ${}^{\mathrm{Fat}}\mathcal{A}rc$.

When adding the necessary families obtained by gluing, one quickly realizes by gluing two cylinders that one can produce a cylinder whose boundaries both are only hit by wilted leaves. This then allows one given any arc family to let all its leaves wilt resulting in a surface with only wilted leaves. So in addition to the wilting weights the condition of $\mathcal{A}rc$ that every boundary is hit by an arc is changed to

every boundary is hit by an arc or at least has a non-vanishing wilting weight.

Proposition 5.6. *The spaces $\mathcal{S}t^{\mathrm{Fat}}$ obtained by adding wilting weights and using the above condition form a cyclic operad ${}^{\mathrm{Fat}}\mathcal{A}rc$ called the unital arc operad.*

The proof is straightforward.

5.3 Stabilization

Stabilization is a process in which the $\mathcal{A}rc$ operad or its suboperads are glued together along the non-quasi filling families. We will call such families *unstable*.

5.3.1 Basic idea. The basic idea is that given an unstable family, we just delete any topology from the complementary region. This is easy to grasp if we only have one boundary component in a complementary region. In that case, which we call a genus defect, we wish to just replace the complementary region with a disc. The other case that can appear is that there are more boundary components, we call this a boundary defect. Again, we wish to just forget about such defects. This idea can be made rigorous by taking a colimit over a system of maps, which introduce boundary and genus defects.

The result is then that the operad structure descends and we obtain the stabilized operad. If we restrict to the tree-like setting and add an identity, the resulting operad contains an E_∞ sub-operad. This fact leads to loop space detection and the $\mathcal{A}rc$ spectrum.

5.3.2 Technical details. As shown in [27], for $\mathcal{G}Tree$ all the unstable elements can be obtained by gluing an unstable element from $\mathcal{G}Tree(1)$ to a quasi-filling one. That is, every $a \in \mathcal{G}Tree$ can be decomposed as $a_1 \circ_1 a'$ with $a_1 \in \mathcal{G}Tree(1)$ and a' quasi-filling, and we can furthermore decompose a_1 into a sequence of standard generators and a quasi-filling element. The standard generators are $T_{-a} \circ_1 H_b \circ_1 T_a$ and $T_{-a} \circ_1 G \circ_1 T_a$, where T_a is an element in $\mathcal{G}Tree_0(1)$ and G, H_a are given in Figure 16. Gluing on these generators to the boundary 0 gives maps $st^g_H(a,b)\colon \mathcal{G}Tree_g(n) \to \mathcal{G}Tree_{g+1}(n)$ and $st^g_G(a)\colon \mathcal{G}Tree_g(n) \to \mathcal{G}Tree_{g+1}(n)$.

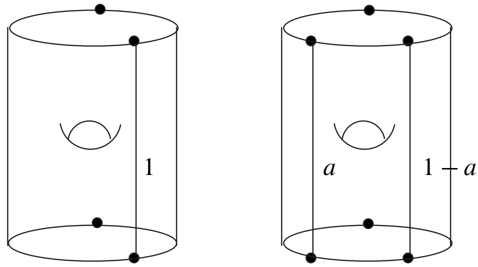


Figure 16. The two basic unstable arc graphs G and H_a.

Definition 5.7. We define $\mathcal{S}t\mathcal{G}Tree(n) := \operatorname{colim}_{\mathcal{S}} \mathcal{G}Tree(n)$ where the colimit is taken over the system of maps $\mathcal{S}$ generated by $st^g_G(a)$ and $st^g_H(b,c)$ with $a, b \in [0,1)$ and $c \in (0,1)$. We will denote the image of a subspace by the prefix $\mathcal{S}t$, e.g. $\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree$.

Theorem 5.8 ([27]). *The operad structure of $\mathcal{G}Tree$ descends to $\mathcal{S}t\mathcal{G}Tree$. Moreover $\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree$ is a suboperad. Furthermore, the elements in $\mathcal{S}t\mathcal{G}Tree$ have a unique quasi-filling representative.*

The proof goes through a standard form argument using the decomposition mentioned above.

5.4 Generalization to all of $\mathcal{A}rc$

The arguments of [27] generalize to the full arc operad by using the colimit over gluing at all boundaries. So for $i = 1, \dots, n$ let $st^g_G(i;a)$ and $st^g_H(i;b,c)$ be the maps of $\mathcal{A}rc_g(n) \to \mathcal{A}rc_{g+1}(n)$ given by $\alpha \mapsto \alpha \circ_i (T_{-a} \circ_1 G \circ_1 T_a)$ and $\alpha \mapsto \alpha \circ_i (T_{-b} \circ_1 H_c \circ_1 T_b)$. For $st^g_G(0;a)$ we use the same definition as for $st^g_G(a)$, now extended to all of $\mathcal{A}rc$ and likewise for $st^g_H(0;a,b)$.

Theorem 5.9. *The spaces $\mathcal{S}t\mathcal{A}rc(n) := \operatorname{colim}_{\mathcal{S}} \mathcal{A}rc(n)$ where the colimit is taken over the system of maps $\mathcal{S}$ generated by $st^g_G(i;a)$ and $st^g_H(i;b,c)$ with $a, b \in [0,1)$ and $c \in (0,1)$ form an operad called the stabilized arc operad.*

5.5 Stabilization and moduli space

The situation about representatives is more complicated, the stabilization with respect to the two types of elements above still allows for non-quasi filling representatives.

The new type of degeneracy comes from being able to find closed curves that do not intersect any of the arcs. We call a maximal choice of a system of such curves which are mutually non-intersecting a curve degeneracy system.

If we cut along the curve degeneracy system, we get elements in the spaces $A^s_{g,r}$, since there will be some boundaries which are not hit. This is a partial operadic decomposition if we allow to glue along empty boundaries. It also becomes operadic, if we allow boundaries with only wilting leaves. By shrinking the curves in the degeneracy system they become double points and the representatives live in a Deligne–Mumford type setup of the space $M^{1,\dots,1}_{g,r,s}$. The precise details will be given elsewhere, and they should be compared to [39].

5.6 Stabilization and adding a unit. The E_∞ and E_k structures

Combining the two procedures, stabilization and adding a unit, we end up with the operad versions ${}^{\mathrm{Fat}}\mathcal{S}t$. Here it is inessential in which order we do the two procedures. If we fatten first, then we wish to point out that in the stabilization step all elements in operad degree 0 become identified to the disc with wilted leaves on the boundary. That is, ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{G}\mathcal{T}ree(0) = \mathrm{pt}$ is a point and so is ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{A}rc(0)$. For the latter operad, the resulting fattened spaces ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{A}rc^s(n)$ are contractible to the representative given by $F^s_{0,n+1}$ with no arcs, but some constant wilting weight on the boundary. Notice that these surfaces are fixed points under the $\mathbb{S}_{n+1}$ action. If we restrict to $s = 0$, the spaces ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{A}rc^0(n)$ are contractible, but as the $\mathbb{S}_n$ action is not free we do not get E_∞ operads. Staying within ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{G}\mathcal{T}ree$ however we obtain an E_∞ operad and along with it a filtration by E_k operads.

We define ${}^{\mathrm{Fat}}\mathcal{L}\mathcal{G}\mathcal{T}ree$ analogously to its non-thickened counterpart.

Theorem 5.10 ([27]). *The operad structure of ${}^{\mathrm{Fat}}\mathcal{G}\mathcal{T}ree$ descends to ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{G}\mathcal{T}ree$ and ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree$ is a suboperad.*

Using the same arguments as in *loc. cit.* one obtains:

Theorem 5.11. *The operad structure of ${}^{\mathrm{Fat}}\mathcal{A}rc$ descends to ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{A}rc$ and ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{G}\mathcal{T}ree$ is a suboperad.* □

The most important structure theorem is then the following.

Theorem 5.12 ([27]). *There exists a filtration of ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree$ by suboperads ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree_k$. The operads ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree_k$ are E_k operads and the operad ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree$ is an E_∞ operad.*

The proof uses Fiedorowicz's theorem, see Berger, Theorem 1.16. For this one constructs so-called cellular E_k operads $k \in \mathbb{N} \cup \{\infty\}$ and according to [1] these are E_k operads.

Corollary 5.13. *The sub-operads $\{\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree_k(n), n > 0\}$ with zero wilted weights are equivalent to $\{C_k(n), n > 0\}$ where C_k are the non-pointed k cubes and the operad $\{\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree, n > 0\}$ is an E_∞ operad without a 0-term.*

Moreover ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree$ acts on ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{A}rc$ and thus:

Theorem 5.14. *The group completion of ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{A}rc$ is homotopy equivalent to an infinite loop space and hence gives rise to an infinite loop space spectrum, the $\mathcal{A}rc$ spectrum.*

An analogous statement holds for ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{G}\mathcal{T}ree$ as stated in [27].

5.6.1 The E_k-operads in detail: the hemispherical decomposition of S^∞. In order to identify the E_k suboperads, one uses so-called degeneracy maps. These are simply given by gluing in the element $*$ of $\mathcal{S}t\mathcal{G}\mathcal{T}ree(0)$ into the i-th position. In particular, if one glues $*$ into *all but* the ith and jth position, one obtains a map $\phi_{ij} : \mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree(n) \to \mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree(2)$. This space retracts to S^∞. We let $E_k^\pm$ be the upper and lower hemisphere of $S^k \subset S^\infty$. As is well known these are the cells of a CW decomposition of S^∞.

Proposition 5.15 ([27]). *$\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree(2)$ retracts to $S^\infty \times \mathbb{R}^2_{>0}$. The preimage of $E_k^\pm \times \mathbb{R}^2_{>0}$ under the retraction are arc families whose underlying graphs for $E_k^+ \times \mathbb{R}^2_{>0}$ are those given in Figure 17, while those for $E_k^- \times \mathbb{R}^2_{>0}$ are simply the image under interchanging the labels 1 and 2. The factor of $\mathbb{R}^2_{>0}$ is simply the weight $\mathrm{wt}(\partial_i)$ on the boundaries 1 and 2.*

This implies that ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree(2)$ retracts onto S^∞ by scaling the wilting weights to zero.

Definition 5.16. $\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree_k(n) = \bigcap_{i,j\in\{1,\dots,n\}} \phi_{ij}^{-1}(S^k \times \mathbb{R}^2_{>0})$. This means these are those stabilized arc families which land in $S^k \times \mathbb{R}^2_{>0} \subset S^\infty \times \mathbb{R}^2_{>0}$ under all the maps $\phi_{i,j}$.

We set ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree_k(n)$ to be those arc families which lie in $\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree_k(n)$ after forgetting the wilting weights, or, equivalently, those families which under the above retraction land in S^k.

The above theorem also asserts that these are indeed suboperads.

5.7 CW decomposition and $\cup_i$ products

For $\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree$, there is a CW model, given simply again by normalizing the weight on the input boundaries to be 1. Since ${}^{\mathrm{Fat}}\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree(n)$ retracts onto its suboperad $\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree(n)$ for $n > 0$, we also get a chain model for the fattened version. If we want to include a 0 component on the chain level, we simply take it to be k, the ground field, represented by the disc with the empty graph and the action given by erasing arcs.

This CW model for $\mathcal{S}t\mathcal{LGT}ree(2)$ is exactly S^∞ in its hemispherical decomposition.

The cells of the upper hemispheres give the chain representatives for the $\cup_i$ products. These are given in Figure 17. Recall that $\cup_{i+1}$ is a homotopy between $\cup_i$ and $\tau_{12}(\cup_i)$. This is again an incarnation of S^∞ with S^0 being the two orders of the multiplication μ. This is made explicit in Figures 18 before the stabilization and 19 after applying stabilization. This may also serve as a good general example of how the stabilization works.

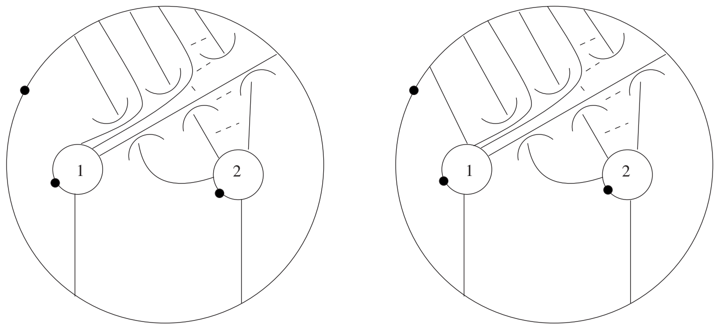


Figure 17. The $\cup_i$ operations for i even and i odd. These are also the cells for upper hemispheres of S^∞ in its hemispherical decomposition.

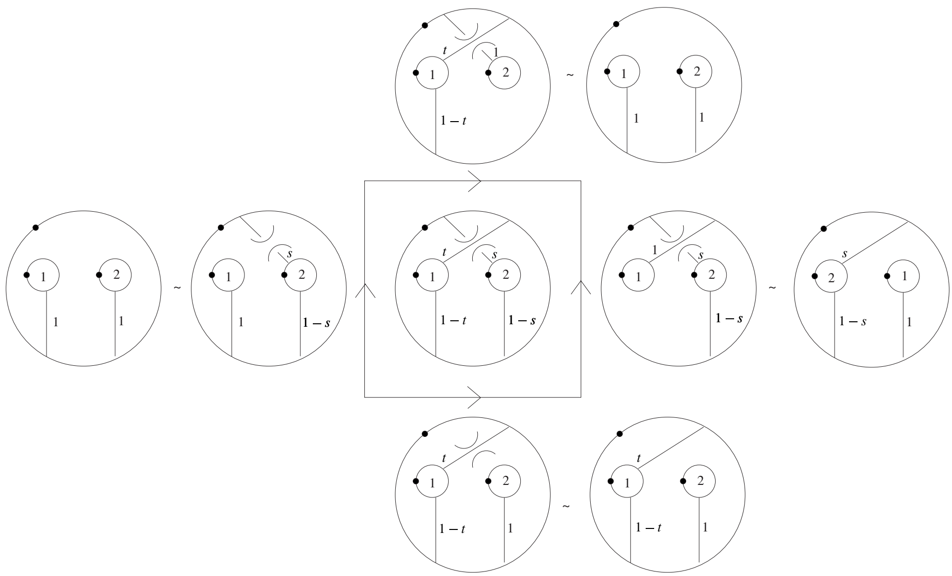


Figure 18. The $\cup_2$ operation and its boundary before stabilization and stable representatives of the boundary components.

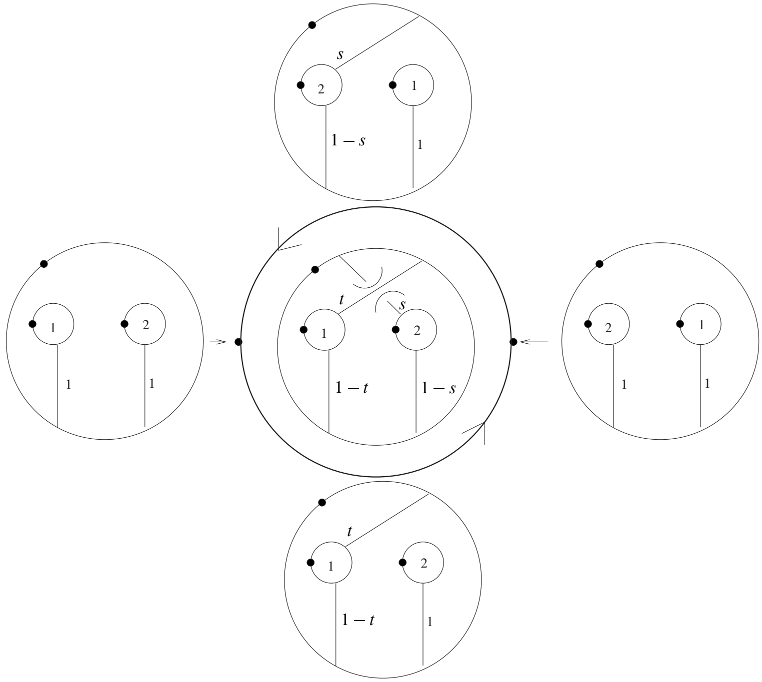


Figure 19. The $\cup_2$ operation after stabilization.

6 Actions

6.1 Algebras

One of the main applications of the $\mathcal{A}rc$ operad and its derivates are chain and homology level actions on the Hochschild complexes of an algebra A. There are several types of algebras one considers. The algebras are either taken to be strictly associative or A_∞. The latter is an algebra with a multiplication $\mu_2\colon A \otimes A \to A$, a differential d and it is associative up to homotopy with all higher homotopies explicitly given by higher multiplications $\mu_n\colon A^{\otimes n} \to A$. We also take the algebras to be unital for simplicity.

The next choice is if these algebras have a suitable duality. In the associative case this means that the algebras are Frobenius algebras, that is they have a non-degenerate symmetric (even) bilinear form $\langle\ ,\ \rangle$ which is invariant: $\langle a, bc\rangle = \langle a, bc\rangle$. In the A_∞ case one postulates the symmetry of all the μ_n: $\langle a_0, \mu_n(a_1, \ldots, a_n)\rangle = \langle a_n, \mu_n(a_0, \ldots a_{n-1})\rangle$. We remark that we can also use the category of graded algebras and their graded duals where we can get a duality if the graded pieces are finite-dimensional.

Since the algebras are unital, we will use the following notation:

$$\int a := \langle a, 1 \rangle \tag{6.1}$$

where 1 is the unit. In this notation, the symmetry and invariance become cyclicity

$$\int a_0 a_1 \dots a_n = \int a_1 \dots a_n a_0. \tag{6.2}$$

The techniques for cases with or without a duality are a bit different. Without a duality, we have an asymmetry between inputs and outputs. To a given action or better an algebra V over an operad or a PROP, we need a map in $\mathrm{Hom}(V^{\otimes n}, V)$ or $\mathrm{Hom}(V^{\otimes n}, V^{\otimes} m)$ for each element of the operad or PROP. This asymmetry is reflected in the geometry by restricting the arc families we can use to give cellular actions. The actions in this case are given by flow charts.

If there is a duality, we can also ask that V has such a duality which is compatible. This means that we can construct maps in $\mathrm{Hom}(V^{\otimes n+1}, k)$ or $\mathrm{Hom}(V^{\otimes n+m}, k)$ which we call *correlation functions*. These are defined with the help of the co-unit $\int$.

We can actually define these correlation functions even in the absence of a duality, but without it there is no way to compose unless we choose extra data, such as special elements or propagators in physics parlance in $V \otimes V$ which are otherwise provided by the Casimir element of the non-degenerate bilinear form, see Appendix A.1.9 for the formulas.

6.2 Deligne's conjecture

Deligne's conjecture is the following statement.

Theorem 6.1. *There is an operadic cell model of the little discs operad that acts on the Hochschild cochains $CH^*(A, A)$ of an associative algebra in such a way that its induced action of the homology of the little discs operad gives rise the known Gerstenhaber algebra structure on $HH^*(A, A)$.*

The theorem has been proved in many variants [35], [58], [44], [62], [37] [43], [2]. Notice that is has two main statements.

(1) There is a chain level action which induces the Gerstenhaber structure on (co)homology, and

(2) the chain level operad is an operadic chain model for the little discs.

The second statement means that (a) the chains compute the right homology and (b) that the induced structure on homology coincides with the one from the little discs. In some of the variants the second statement is not proven, we will call such a solution to the problem a weak solution.

Theorem 6.2. *The operad of spineless cacti are equivalent to the little discs and its chain model $CC_*(\mathcal{C}act^1)$ solves Deligne's conjecture.*

We give an outline of the proof. First, notice that indeed $CC_*(\mathcal{C}act^1)$ is an operadic chain model for spineless cacti; for details see [21]. The action will be given by a flow chart.

6.2.1 Intersection graph trees. Given a cell indexed by an arc family $[\alpha]$ its flow chart $\tau([\alpha])$ is given by the intersection graph of the dual graph. The intersection graph is a bipartite rooted planar tree. Here "rooted planar tree" which is sometimes also called planted planar tree means that there is a marked vertex called root and at each vertex there is a *cyclic* order of the adjacent edges and a *linear* order at the root.[3] A dual graph of an element of $\mathcal{C}act$ is a ribbon graph which has one cycle that contains all edges in exactly one orientation; call this cycle the outside cycle or loop – the one corresponding to the boundary 0. The intersection graph for such a graph has black vertices corresponding to the vertices of the graph. It has white vertices corresponding to the cycles of the graph except for the outside cycle. There is an edge connecting a white and a black vertex if the respective vertex lies on the respective cycle. For an example, see Figure 20.

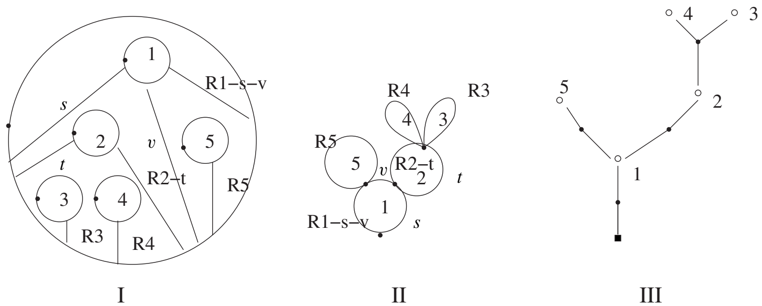


Figure 20. I. An element of $\mathcal{C}act$. II. Its dual graph. III. Its intersection graph.

This graph can be shown to be a tree. The cyclic order at each vertex is induced by the ribbon graph structure. For the black vertices this is the identical order and for the white vertices this is the order induced on the directed edges of a cycle. The root is taken to be the vertex which we called the global zero. That is the one corresponding to the marked point on the boundary 0. Finally, the linear order at this root vertex is given by saying that its first edge is the one corresponding to the first arc on the boundary 0. Since each other vertex has a unique edge pointing towards the root, declaring this to be the last edge gives a linear order of the edges at each vertex.

[3] It is enough to give a linear order on all vertices. At any non-root vertex there is a unique edge going towards the root. This edge is set to be the last one.

Since this data only depends on the incidence conditions of the arcs on the boundary, it is clear that this tree only depends on the class of α. The white vertices are also numbered $1, \dots, n$ corresponding to the boundary components they represent.

6.2.2 Flow chart of the tree. The action is simply given by decorating $\tau([\alpha])$ with elements $f_i \in CH^*(A, A)$ and performing brace operations at each white vertex and multiplication at each black vertex. More precisely, the action is defined for homogeneous elements and then extended by linearity. For the basis element of $CC^*(\mathcal{C}act(m))$ given by a cell indexed by $[\alpha]$ the action on $(f_1, \dots, f_n)$ with $f_i \in CH^{n_i}(A, A)$, $i = 1, \dots, n$ is zero unless $m = n$. In the case where these numbers match: Decorate the white vertex i with f_i. Now $\tau([\alpha])$ is planar and has a flow toward the root. The outermost vertices or leaves are white. We start with these functions. If the flow hits a black vertex, we multiply the incoming functions in the linear order given by this vertex. The product is the outgoing function. If the flow hits a white vertex i, we take the brace operation of f_i with the product of all incoming functions (again in the order dictated by the linear order on the vertex) and make this the outgoing function.

In the example given in Figure 20 the operation would be $f_1\{f_2\{f_3 f_4\}, f_5\}$.

6.2.3 The A_∞-version. The A_∞ version of Deligne's conjecture was first proven in [37]. The action was given by means of a homotopy argument and was not explicit. The basic idea is that there is a naturally acting operad called the minimal operad and that this operad is quasi-isomorphic to the little discs. To build the quasi-isomorphism one goes through a very large model and proves existence of the quasi-isomorphisms by homotopy theory without having to construct them.

In [33] the A_∞ version was proven in a minimal constructive way. The method used is to again employ flow charts given by planted, planar, two-colored, stable trees. The colors on the vertices are again black and white and stability means that each black vertex is at least 3-valent. The white vertices are also numbered. These form the minimal operad of [37] and their action is given by using the multiplication μ_n at a black vertex that is $(n-1)$-valent and the brace operations at the white vertices.

Theorem 6.3. (1) *The trees above with fixed numbering from* $1, \dots, n$ *index the cells of a CW complex* $K^1(n)$.

(2) *The collection of CW complexes* $K^1(n)$ *are a cell model for the little discs operad.*

(3) *The cellular chains of* $K^1(n)$ *form an operad isomorphic to the minimal operad of* [37] *and hence give operations on* $CH^*(A, A)$ *for any* A_∞ *algebra* A *in such a way that the induced action is the usual Gerstenhaber structure on* $HH^*(A, A)$.

The proof of the second statement uses spineless cacti as a reference model. The tertium comparationis is third cell model called K^{ht}, where ht stands for "height". K^{ht} is shown to be a subdivision of K^1 and retractable to spineless cacti.

The connection to the arc picture and thus to moduli spaces, foliations and moduli spaces comes from K^{ht}. It is given in the Appendix of [33]. The basic upshot is that the trees are again the intersection graphs of dual graphs of arc families. Again, the genus is zero and arcs are allowed to run from boundaries $i = 1, \dots, n$ to the boundary 0 *or* from 0 to 0. All boundaries i are hit and there is no arc from 0 to 0 that is homotopic to the union of another arc and a boundary.

Tracing through the definitions, one sees that the arcs from 0 to 0 give rise to edges between two black vertices and the last condition ensures that there are no black vertices of valence less that 3.

One upshot of this treatment is an arc indexed subdivision of Stasheff polytopes and a new subdivision of cyclohedra by cells indexed by arc families, which leads to a new explicit blowup procedure starting at a simplex, see [33].

6.3 The cyclic Deligne conjecture

The cyclic Deligne conjecture, first proven in [26] using spineless cacti and their CW model $\mathcal{C}act^1$, states that

Theorem 6.4 ([26]). *There is an operadic cell model of the framed little discs operad that acts on the Hochschild cochains $CH^*(A, A)$ of a Frobenius algebra in such a way that its restriction to the operadic chain model for the sub-operad of the little discs operad gives rise the known Gerstenhaber algebra structure on $HH^*(A, A)$.*

It was actually conjectured in [59]. A consequence of this statement is

Corollary 6.5. *For a Frobenius algebra $HH^*(A, A)$ has the structure of a BV algebra, for which the induced bracket is the Gerstenhaber bracket.*

This statement was first proved by [45] without the chain level version. During the publication process of [26], several other versions of actions yielding a BV structure on homology providing weak solutions in the sense of §6.2 were produced in [61], [38], [11]. In some of these references actions of even bigger operads or PROPs were constructed. A very recent version is given in in [64].

The proof in [26] again uses a tree picture for the actions. We will give these actions in a different but equivalent guise when discussing the Sullivan-PROP in 6.4.4. This makes the fundamental role of foliations in the solution more apparent.

In the Frobenius case we also have an isomorphism of complexes $CH^*(A, A)$ and $CC^*(A)$, the cyclic cohomology chain complex, see e.g. [24].

6.3.1 The cyclic A_∞-version. A solution to the A_∞ generalization of the cyclic conjecture (and not just a weak one) was announced in [28] and just fully proven in [65]. The method of proof is to extend the action of [33] to the case of trees with spines of marks as in [26].

Theorem 6.6 ([65]). *The cyclic A_∞ conjecture holds.*

The weak statement of this can be found in [38].

6.4 Moduli space action and the Sullivan-PROP also known string topology action

6.4.1 Correlation functions. Fix a commutative unital Frobenius algebra A with multiplication μ and pairing $\langle\ ,\ \rangle$. Set $\int a := \langle a, 1\rangle$ and let e be the Euler element of A, that is,

$$e = \mu\Delta(1)$$

where Δ is the adjoint of μ, see the Appendix.

The actions will be given on $CH := CH^*(A, A)$. Now since A is a Frobenius algebra, then $CH^n(A, A) \simeq A^{\otimes n+1}$ so that we may use an isomorphism of the Hochschild cochains with the tensor algebra $CH^*(A, A) \simeq \overline{TA}$, where $\overline{TA} = \bigoplus_{n=1}^{\infty} A^{\otimes n}$ is the reduced tensor algebra. Furthermore in the graded sense $\overline{TA}$ is Frobenius by using the tensor product of the Frobenius algebra structures.

Thus suitably dualizing, we can represent any $\Phi \in \mathrm{Hom}(CH^{\otimes n}, CH^{\otimes m})$ as an element $Y \in \mathrm{Hom}(\overline{TA}^{\otimes n+m}, k)$ and vice versa. The multi-linear maps Y are called correlators and are fixed on homogenous element of $\overline{TA}$.

We will define *basic correlators* Y depending on the cells of $A^s_{g,r}$. For homogenous elements $\phi_i \in \overline{TA}, i = 0, \dots, r$ these will be multilinear maps to k denoted by $\langle\phi_0, \dots, \phi_r\rangle_{[\alpha]}$ for any cell given by a PMC class $[\alpha]$.

6.4.2 Basic idea. The action is roughly given as follows: fix α an arc graph with k arcs on some $F^s_{g,r}$ and fix homogenous $\phi_i \in \overline{TA}, i = 0, \dots, r-1$.

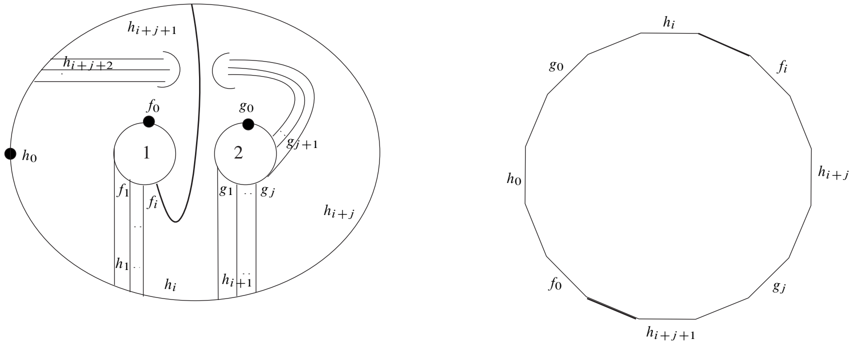
(1) Duplicate edges so that the number of incoming edges at the vertex $i = \deg(\phi_i)$. We sum over all possibilities to do this, if this is not possible then the operation is zero.

(2) Assume the ϕ_i are pure tensors. Pull apart the edges and decorate the pieces of the boundary with the elements of ϕ. Cut along all the edges of the graph and call the set of disjoint pieces of surface P. Let $I(F)$ be the index set of the components a_j of the ϕ_i decorating edges belonging to a piece $S \in P$ and let $\chi(S)$ be the Euler characteristic of the surface S.

(3) For each S integrate over the product over all the boundary decorations and a factor of $e^{-\chi(S)+1}$.

(4) Multiply together all the local contributions of (3).

An example is given in Figure 21.



I II

Figure 21. A partitioned arc graph with decorations by elements of A and one of its decorated polygons. The bold line corresponds to the bold edge.

6.4.3 Technical details. Given any arc graph α on $F = F^0_{g,r}$, let $e_1, \dots, e_k$ be its arcs enumerated in their order. For any tuple of positive numbers $\boldsymbol{n} = (n_1, \dots, n_k)$, $n_i \in \mathbb{N}_{>0}$, let $\alpha^{\boldsymbol{n}}$ be the arc graph obtained from α by replacing each edge e_i by n_i parallel arcs. This can be for instance done in some rectangle with spine e_i.

Consider $\partial F \setminus (\alpha^{\boldsymbol{n}} \cap \partial F) = \amalg_{i=0}^{r-1} \amalg_{j \in J_i} I_{ij}$, which is a disjoint union of intervals or simply the whole boundary components, and J_i indexes the components sitting inside ∂_i. The set J_i has a natural cyclic order, which we upgrade to a linear order by declaring the unique interval containing the marked point of ∂_i to be the first interval.

Likewise, let $F^0_{g,r} \setminus \alpha^{\boldsymbol{n}} = \amalg_{F \in P} F$. Each surface $F \in P$ has a boundary in which each boundary component is a $2k$-gon whose sides alternate between the arcs and pieces of the boundary, or simply the whole boundary, if the boundary has no incident arc.

Consider homogenous elements $\phi_i = \bigotimes_{j \in J_i} \in A^{\otimes J_i}$.

For $S \in P$ let $B(S)$ be the subset of those boundary indices (i, j) such that I_{ij} is a part of the boundary of S,

$$\langle \phi_0, \dots \phi_{r-1} \rangle_S := \int \prod_{(i,j) \in B(S)} a_{ij} e^{-\chi(S)+1}$$

and finally set

$$\langle \phi_0, \dots, \phi_{r-1} \rangle_{F^0_{g,r}, \alpha} := \prod_{S \in P} \langle \phi_1, \dots, \phi_n \rangle_S$$

where

$$\langle \phi_1, \dots, \phi_n \rangle_p = \int \prod_{i \in I(S)} a_i e^{-\chi(S)+1}. \tag{6.3}$$

Combining these correlators with the map $\mathcal{P}$ yields:

Theorem 6.7. *Let A be a Frobenius algebra and let $CH^*(A,A)$ be the Hochschild complex of the Frobenius algebra. Then the cyclic chain operad of the open cells of $\mathcal{A}rc$ acts on $CH^*(A,A)$ via correlation functions.*

In the same fashion all the suboperads, di-operads and sub-PROPs of [23] *act. In particular the total graph complex* $\mathrm{Gr}\,\mathcal{C}^*_o(\mathcal{A}rc^0_\#)$ *of the moduli spaces* $M^{1,\dots,1}_{g,r,s}$ *of pointed curves with fixed tangent vectors at each point acts.*

6.4.4 String topology. For string topology there is a modification of the above basic correlators , which amounts to inserting the unit element in certain places, depending on the choice of in and out boundaries.

Consider $\partial F \setminus (\alpha^{\boldsymbol{n}} \cap \partial F) = \amalg_{i=0}^{r-1} \amalg_{j\in J_i} I_{ij}$ as above. These elements are called angles in [24]. On each boundary there is one angle that contains the marked point. This angle is called the *outer* angle. If I_{ij} has as its endpoints the endpoints of two parallel arcs of $\alpha^{\boldsymbol{n}}$ (and is not outer) it is called a partitioning angle.

All the other angles, that is neither outer and nor partitioning, are called *inner* angles. These are the angles corresponding to the angles of α that are not outer angles.

Let $F^{i/o}$ be a surface together with designated in and out boundaries, as in the Sullivan-PROP situation.

Given $\phi_i \in CH^{n_i}(A,A)$ we let $\tilde{\phi}_i = \phi$ for all in boundaries i, and for an out boundary k, given $\phi_k \in CH^{n_k}(A,A)$, we let $\tilde{\phi}_k \in CH^{|J_k|}(A,A)$ be the element obtained from ϕ_k by inserting the unit 1 in all the positions j for which I_{kj} is an inner angle. This is accomplished by the use of degeneracy maps. Given $\phi_k = \phi_{i_1} \otimes \cdots \otimes \phi_{i_{n_i}}$ set $s_j(\phi_i) = \phi_{i_1} \otimes \cdots \otimes \phi_{i_{j-1}} \otimes 1 \otimes \phi_{i_j} \otimes \cdots \otimes \phi_{i_{n_i}}$.

That is $\tilde{\phi}_k = s_{j_1} \dots s_{j_l} \phi_k$, where $j_1 < \cdots < j_k$ are the positions of the inner angles:

$$\langle \phi_0, \dots, \phi_{r-1} \rangle_{F^{i/o},\alpha} := \langle \tilde{\phi}_0, \dots, \tilde{\phi}_{r-1} \rangle_{F,\alpha}. \tag{6.4}$$

Notice that this expression is 0 unless $n_i = |J_i|$ for all in boundaries and $n_k = |J_k|-$ the number of inner angles.

An example of this type of decoration is given in Figure 22.

Theorem 6.8 ([24]). *The $Y(\alpha)$ defined in equation* (6.4) *give operadic correlation functions for the chain model $CC_*(\mathcal{S}ull^1)$ of the Sullivan-PROP $\mathcal{S}ull$ and induce a dg-action of the dg-PROP $CC_*(\mathcal{S}ull^1)$ on the dg-algebra $\overline{CH}^*(A,A)$ of reduced Hochschild co-chains for a commutative Frobenius algebra A.*

By restricting to genus 0 and $\mathcal{S}ull(n,1)$ one gets an action of $\mathcal{C}acti$; this together with the quasi-isomorphism of $\mathcal{C}acti$ and the framed little discs yields:

Theorem 6.9. *The cyclic Deligne conjecture holds.*

The connection to string topology is as follows. Let M be a simply connected compact manifold M, denote the free loop space by $\mathcal{L}M$ and let $C_*(M)$ and $C^*(M)$ be the singular chains and (co)-chains of M. We know from [18], [6] that $C_*(\mathcal{L}M) =$

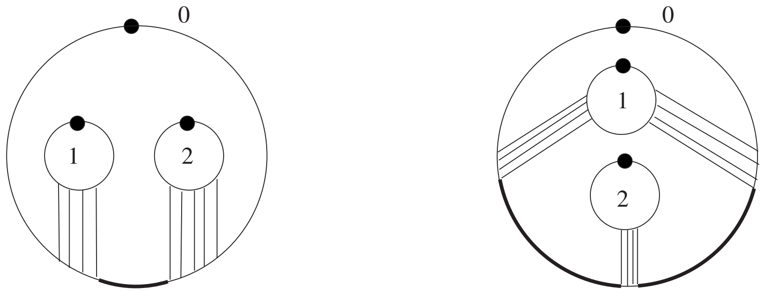


Figure 22. Examples of the partitioned families yielding $\cup$, $\circ_i$ in the string topology/Deligne setting and $\sqcup$ and $\square_i$ in the moduli setting. The outer angles are the ones with the dot, the bold angles are the non-partitioning inner angles are marked in bold.

$CH^*(C^*(M, C_*(M)))$ and $H_*(\mathcal{L}M) \simeq HH^*(C^*(M), C_*(M))$. Moreover $C^*(M)$ is an associative dg algebra with unit, differential d and an integral (M was taken to be a compact manifold) $\int : C^*(M) \to k$ such that $\int d\omega = 0$. By using the spectral sequence and taking field coefficients we obtain operadic correlation functions Y for $\mathcal{C}acti$ on $E^1 = CH^*(H, H)$ which converges to $HH^*(C^*(M))$ and which induces an operadic action on the level of (co)-homology. Except for the last remark, this was established in [26].

Theorem 6.10. *When taking field coefficients, the above action gives a dg action of a dg-PROP of Sullivan chord diagrams $CC_*(\mathcal{S}ull^1)$ on the E^1-term of a spectral sequence converging to $H_*(LM)$, that is, the homology of the free loop space of a simply connected compact manifold and hence induces operations on this loop space.*

Remark 6.11. There is also a lift to co-cycles of a dg-algebra whose cohomology is Frobenius, see [24].

6.4.5 Moduli space and $\mathcal{A}rc$ action. The basic correlators compose completely algebraically, see A.1.9. This will give an *algebraic action*, if one leaves out the modification in the operad composition that the closed leaves are erased as discussed in Remark 2.2. Also in this version, one uses no extra signs in the discretization. Notice that in the string topology gluing, no closed leaves can appear.

For the case of the relative chains of $\mathcal{A}rc$ or in the moduli space case, namely $\mathrm{Gr}\,\mathcal{C}_o^*(\mathcal{A}rc_\#^0)$, we need the signs. In order to get a map of operads, we therefore first have to account for the signs in the operad $\mathrm{Hom}(CH^*)$. This is implicitly done in string topology and the various versions of Deligne's conjecture. Here if one wishes to phrase the fact that the action exists in operadic terms, the target of the morphism of operads is not $\mathrm{Hom}(CH^*)$ but the so-called brace operad $\mathcal{B}race$ which is formed by certain subspaces of the endomorphism operad, but has different sign rules. This is what we called a twisted Hom-operad structure. Moreover in our solution to Deligne's conjecture $\mathcal{B}race$ is in a sense the tautological recipient of the operad map, since it is the isomorphic image of $CC_*(\mathcal{C}act)$. This is however only *a posteriori* and not true in all solutions of Deligne's conjecture – albeit this type of statement is a special feature in all forms of the conjecture coming from chains of the $\mathcal{A}rc$ operad. *A priori* $\mathcal{B}race$ is defined to be spanned by the multiplication and brace operations. The signs then come from a new natural grading. Since any $\mathrm{Hom}(V)$ operad is already graded this is actually a bi-grading. The theorem below gives identification of such a Hom-operad for the moduli space case. That is, there is an isomorphic image of $\mathrm{Gr}\,\mathcal{C}_o^*(\mathcal{A}rc_\#^0)$ of subspaces $\mathcal{MCS}$ in $\mathrm{Hom}(CH^*(A))$ which is naturally (bi)graded, thus providing the signs.

Furthermore, in the case at hand, on the geometric side there is also a modification in the combinatorial gluing coming from the use of cellular chains, namely that if the dimension of the chain is not additive under the gluing, the result is 0. On the geometric side this was handled by passing to the associated graded (see §2.5) and likewise on the algebraic side we analogously pass to the associated graded.

Theorem 6.12. *Let A be a Frobenius algebra and let $CH^* = CH^*(A, A)$ be the Hochschild complex of the Frobenius algebra.*

There are subspaces $\mathcal{MCS} \subset \mathrm{Hom}(CH^*)$, defined in terms of natural operations below, such that for all $\alpha^{\boldsymbol{n}}$ for α a quasi-filling, the correlation functions (6.3) *suitably dualized are elements of $\mathcal{MCS}$.*

Furthermore $\mathcal{MCS}$ has a natural grading, such that the correlation functions $Y(\mathcal{P}(\alpha))$ for $\alpha \in \mathrm{Gr}\,\mathcal{C}_o^*(\mathcal{A}rc_\#^0)$ yield an operad morphism of cyclic operads to $\mathrm{Gr}\mathcal{MCS}$, viz. they are operadic correlation functions with values in $\mathrm{Gr}\mathcal{MCS}$.

Pushing forward the differential, this action becomes dg.

Remark 6.13. Again there is a lift to the chain/cycle level.

Remark 6.14. Unlike in the previous cases, the differential is not the natural one on the Hochschild side. This is actually the case for the arc graphs with arcs running only from input to output. We give the details below.

Although the details have to be spelled out, it is fairly straightforward to obtain:

Claim 6.15. *The above theorem also holds in the modular operad setting.*

And, by dualization, the analogous theorem holds for the PROP setting, that is, for the complex of quasi-filling families with input and output markings.

6.4.6 $\mathcal{MCS}$ and the operations of $\mathcal{C}act$. For completeness we will briefly define $\mathcal{MCS}$. It is given as the subspace generated by three types of operations and certain permutations. These are the maps:

$$\int : TA \to k, \quad \int (a_1 \otimes \cdots \otimes a_n) = \int a_1 \dots a_n,$$
$$\Diamond : TA \to TA \otimes A \otimes TA,$$
$$a_1 \otimes \cdots \otimes a_n \mapsto \sum_i (a_1 \otimes a_{i-1}) \otimes a_i \otimes (a_{i+1} \otimes \cdots \otimes a_n),$$
$$\eta^n : A^{\otimes n} \otimes A^{\otimes n} \to k, \quad \eta(a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_n) = \pm \prod_i \int (a_i b_i),$$

where TA is the tensor algebra and the sign is the usual Koszul sign.

To explain the shuffles, one introduces a new monoidal structure $\boxtimes$ for bimodules [24]. For TA this amounts to the definition $TA \boxtimes TA = TA \otimes A \otimes TA$. This means that $\Diamond$ is a co-product. $\Diamond : TA \to TA \boxtimes TA$. This type of co-product also naturally appeared in [43]. The correlators $Y(\alpha^{\boldsymbol{n}})$ then can be factorized as

$$Y(\alpha^{\boldsymbol{n}}) : TA^{\otimes n} \to \bigotimes_{i=0}^{r} A^{\boxtimes m_i+1} \to k \tag{6.5}$$

where m_i is the number of inner angles at the boundary i. The shuffles are then the shuffles of factors TA and A in the middle part. Note that $\int$ is cyclically invariant, so that only the cyclic order of the tensors matters.

More precisely for any $\alpha^{\boldsymbol{n}}$ with α a quasi-filling, let m_i be the number of inner angles at boundary i, n_i be the entries of $\boldsymbol{n}$; then

$$Y(a^p) = \Big(\bigotimes_{\pi \in \mathrm{Comp}(\alpha)} \int_\pi \otimes \bigotimes_{i=0}^{r} \eta^{n_i-1} \Big) \circ \sigma \circ \bigotimes_{i=0}^{n} \Diamond^{m_i} \tag{6.6}$$

where we identify the complementary regions $\mathrm{Comp}(\alpha)$ of α with a subset of those of $\alpha^{\boldsymbol{n}}$, $\int$ is applied to the cyclically ordered decorations of the fixed polygonal region π as in 6.3, the iterated coproduct defined by $\Diamond^l : TA \to TA^{\boxtimes l+1}$ is the iteration of $\Diamond$ given by $(\Diamond \otimes (\mathrm{id}_A \otimes \mathrm{id}_{TA})^{\otimes l}) \circ \Diamond \otimes (\mathrm{id}_A \otimes \mathrm{id}_{TA})^{\otimes l-1}) \circ \cdots \circ (\Diamond \otimes \mathrm{id}_A \otimes \mathrm{id}_{TA}) \circ \Diamond$, σ is a shuffle of the factors A and TA in the image of the $\Diamond$ operations, that is, $\bigotimes_{i=0}^{r} A^{\boxtimes m_i+1}$.

$\mathcal{MCS}$ is then the space dual to the one generated by operations of the type (6.6). The degree of such an operation is $l = \frac{1}{2}\sum(m_i + 1) - 1$. If the operation comes from some α then this is exactly the dimension of the cell given by α.

For the basic cells from $\mathcal{C}act$ given by the diagrams in Figure 22 we obtain the operations

$$\sqcup : (A \otimes \check{A}^{\otimes n}) \boxtimes (A \otimes \check{A}^{\otimes m}) \xrightarrow{\boxtimes \circ \sigma} A \otimes A \otimes (\check{A}^{\otimes n} \boxtimes \check{A}^{\otimes m}) \xrightarrow{\mu} A \otimes \check{A}^{\otimes n+1+m} \tag{6.7}$$

and

$$\square_i(f,g)(a_1,\dots,a_{n+m+2}) \\ = f(a_1,\dots,a_{i-1},a_i g(a_{i+1},\dots a_{i_m})a_{i+m+1},a_{i+m+2},\dots,a_{n+m+2}). \tag{6.8}$$

These type of operations have also appeared in other work on the Deligne conjecture.

Proposition 6.16. *The operations of the suboperad $\mathcal{C}act$ correspond to the operations $\sqcup$ and $\square_i$ induced by Ξ_2 as defined in* [43].

6.4.7 Stabilization and the action. Given the results of the previous paragraph, we can ask when does the action pass to the stabilization. The answer [25] is that this is the case if and only if A is a normalized semi-simple Frobenius algebra. Semi-simple means that there are generators e_i with the multiplication $e_i e_j = \delta_{ij} e_i$. It follows that $1 = \sum_i e_i$ and if $\lambda_i = \int e_i$ then $e = \sum_i \lambda_i e_i$. The algebra is normalized if in addition $e = 1 \in A$ which is equivalent to all $\int \lambda_i = 1$. Any semi-simple Frobenius algebra can be normalized by rescaling the metric.

Moreover, if the algebra is just semi-simple then all unstable correlators are completely determined by the stable ones.

In both cases the action of the Sullivan-PROP on the homology level is of course trivial, since the Hochschild cohomology of a semi-simple algebra is trivial. The chain level gives such a preferred trivialization. But there is an interesting action of the moduli spaces. This could be related to a similar story of stabilization discussed by Teleman [60]. The connection is given by Gromov–Witten invariants with semi-simple quantum cohomology such as that of projective spaces and conjecturally a class of Fano varieties.

7 Open/closed version

There is an open/closed version of the whole story. This is given in details in [32] and [29]. For this one introduces marked points on the boundaries and brane labels for these as well as marked points in the interior. Here the brane label can be Ø to indicate the closed sector or some element of an indexing set to indicate an open brane label. One major difference is that in that setting, basically due to the Cardy equation, one cannot restrict to the case of no marked point in the interior.

The theory again is completely natural from the foliation aspect. Geometrically there is one simple rule in the background. Points with the closed label are considered as marked points in the surface, but points with an open label are considered as deleted from the surface.

One upshot is a short topological proof of the minimality of the Cardy/Levellen axioms.

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A Glossary

A.1 Operads and PROPs

An operad basically formalizes combinations of flow diagrams. The individual pieces have n inputs and 1 output.

Let $\mathcal{C}$ be one of the following: (chain complexes of dg) vector spaces with tensor product, topological spaces with Cartesian product or chain complexes of Abelian groups with tensor product over $\mathbb{Z}$. In general $\mathcal{C}$ will be a symmetric monoidal category.

Definition A.1 (Short definition). An operad in $\mathcal{C}$ is a sequence of objects $\mathcal{O}(n) \in \mathcal{C}$ together with an $\mathbb{S}_n$-action on $\mathcal{O}(n)$ and morphisms

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(m+n-1), \quad \text{for } i = 1, \dots n,$$

which are equivariant for the symmetric group actions and associative.

Remark A.2. In the full definition, $\mathcal{C}$ should be a symmetric monoidal category. The categories above are such categories.

A.1.1 Functors and operads

Remark A.3. It is clear that (weak) symmetric monoidal functors transform operads to operads. The ones that we care about are H_* and S_* as well as CC_* (cellular chains for CW complexes with cellular maps).

A.1.2 Standard example. The standard example is the one of multivariable functions. From this example one can also make precise how the associativity and symmetric group equivariance can be written in formulas. We will make this concrete in two examples. Let X be a locally compact Hausdorff space. We let $\mathrm{Hom}(X)(n) := C(X^{\times n}, X)$; an element is just a continuous function f of n variables. $f \circ_i g$ just substitues g in the i-th variable of f. If f has n variables and g has m variables then $f \circ_i g$ has $n+m-1$ variables. The $\mathbb{S}_n$-action is given by permuting the variables of the function. Given two permutations σ_n and σ_m there is a unique permutation $\sigma_m \circ_i \sigma_n$ of the new $n+m-1$ variables such that $\sigma_m(f) \circ_{\sigma_m(i)} \sigma_n(g) = \sigma_m \circ_i \sigma_n(f \circ_i g)$. The interested reader is referred to e.g. [40] or encouraged to work out the combinatorial formula.

The associativity states that if we have three functions f, g, h then it does not matter in which way we make the substitutions. Writing down the explicit formula is again a bit subtle, since the indexing of the variables changes. The reader is encouraged to work out the combinatorial formula which is in general given by

A.1.3 Associativity. For three elements $\mathrm{op}_k \in \mathcal{O}(k)$, $\mathrm{op}'_l \in \mathcal{O}(l)$ and $\mathrm{op}''_m \in \mathcal{O}(m)$,

$$(\mathrm{op}_k \circ_i \mathrm{op}'_l) \circ_j \mathrm{op}''_m = \begin{cases} (\mathrm{op}_k \circ_j \mathrm{op}''_m) \circ_{i+m-1} \mathrm{op}'_l & \text{if } 1 \le j < i, \\ \mathrm{op}_k \circ_i (\mathrm{op}'_l \circ_{j-i+1} \mathrm{op}''_m) & \text{if } i \le j < i+l, \\ (\mathrm{op}_k \circ_{i-l+1} \mathrm{op}'_l) \circ_j \mathrm{op}''_m & \text{if } i+l \le j. \end{cases}$$

A.1.4 Remarks and fine print. In general, Example A.1.2 extends in any closed monoidal category $\mathcal{C}$. This means that the sets $\mathrm{Hom}(X,Y)$ can also be regarded as objects in $\mathcal{C}$. If viewed in this way they are called inner homs and denoted $\underline{\mathrm{Hom}}(X,Y)$. To be closed then means that $\mathrm{Hom}(X \otimes Y, Z) \simeq \mathrm{Hom}(X, \underline{\mathrm{Hom}}(Y,Z))$. In this case, for any object X: $\mathrm{Hom}(X)(n) := \mathrm{Hom}(X^{\otimes n}, X)$ forms an operads just as above.

This is the reason we restricted the category of topological spaces to the full subcategory of locally compact Hausdorff spaces. Namely, if we restrict to these spaces the category is closed. This means that $C(X^{\otimes n}, X)$ is again in the category and we have the duality $C(X \times Y, Z) = C(X, C(Y,Z))$. One can make more elaborate constructions to extend this property to a larger set of spaces. For this one has to alter the product structure slightly, see e.g. [42].

A.1.5 Cyclic operads. For a cyclic operad $\mathcal{O}(n)$ has an $\mathbb{S}_{n+1}$-action extending the $\mathbb{S}_n$-action. This action has to satisfy one compatibility equation. For this let * denote the action by the long cycle $(1 \cdots n+1)$; then if $b \in \mathcal{O}(k)$

$$(a \circ_1 b)^* = b^* \circ_k a^* \tag{A.1}$$

The standard example is $\mathrm{Hom}(V)$ where V has a non-degenerate bi-linear form. Then

$$\mathrm{Hom}(V)(n) = V \otimes \check{V}^{\otimes n} \simeq \check{V}^{\otimes n+1} \simeq \mathrm{Hom}(V^{\otimes n+1}, k)$$

has a natural $\mathbb{S}_{n+1}$-action.

In terms of flow diagrams, this means that we can dualize inputs and outputs. We need the particular dualization to have $n+1$ inputs and no output.

A.1.6 Modular operads. The technical definition for modular operads is too long to reproduce here. One can define them by making the following statements precise. They have an underlying cyclic operad. They also have a genus grading, that is they are a collection $\mathcal{O}(n,g)$ each having an $\mathbb{S}_{n+1}$-action. Usually one imposes stability, $3g-3+n>0$.

One should think of elements of $\mathcal{O}(n,g)$ as having $n+1$ inputs. There are two types of compositions:

- Non-self gluings. Just like in cyclic operads glue two inputs on two separate elements together, by dualizing one of them to an output. These are subadditive in n and additive in g.
- Self-gluings. Glue together two inputs of one element. These decrease n by 2 and increases g by one.

These should be compatible, associative and symmetric group-equivariant. A good example to keep in mind is to glue topological surfaces of genus g together at the boundaries.

A.1.7 PROPs. Just like operads PROPs are a collection of symmetric group modules. More precisely, again fix a symmetric monoidal category $\mathcal{C}$.

Definition A.4 (short). A PROP is a collection $\mathcal{O}(m,n)$ of $\mathbb{S}_n\times\mathbb{S}_m$-modules together with

(1) vertical compositions

$$\mathcal{O}(m,n)\otimes\mathcal{O}(n,k)\to\mathcal{O}(m,k),$$

(2) horizontal compositions

$$\mathcal{O}(m,n)\otimes\mathcal{O}(k,l)\to\mathcal{O}(m+k,n+l),$$

which are associative, symmetric group equivariant and compatible.

A paradigmatic example is $\mathcal{O}(m,n)=\mathrm{Hom}(V^{\otimes n},V^{\otimes m})$. The vertical composition is just composition and the horizontal one is given by tensoring the maps.

A.1.8 Algebra over an operad. Let $\mathcal{C}$ be a closed monoidal category.

Definition A.5. An algebra over an operad $\mathcal{O}$ is an object X in the same category and a morphism ρ of operads $\rho\colon\mathcal{O}\to\mathrm{Hom}(X)$.

For *a cyclic operad*, one needs that X has a duality. We will only use algebras over linear cyclic operads.

Definition A.6. An algebra over a linear cyclic operad is a morphism of cyclic operads to $\mathrm{Hom}(V)$, where V is a vector space with non-degenerate bi-linear form.

A.1.9 Operadic correlation functions. Let $\mathcal{C}$ be a linear category. By this we mean that the objects of the category have the property that they are finite-dimensional vector spaces or graded vector spaces which are finite-dimensional in each degree over a field k, for instance finite-dimensional algebras.

Let V be an object of $\mathcal{C}$ and let Δ_i be a basis for V. Fix a non-degenerate bilinear paring $\langle\ ,\ \rangle$, set $g_{ij} = \langle \Delta_i, \Delta_j\rangle$ and g^{ij} the inverse matrix. The Casimir element is $C = \sum_{ij} \Delta_i g^{ij} \Delta_j$. This is independent of the choice of basis. It is the dual of $\langle\ ,\ \rangle$ considered as an element of $\check{V} \otimes \check{V}$.

With the help of this element we can dualize any $\phi \in \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$ via

$$Y(\phi)(v_0 \otimes v_1 \otimes \cdots \otimes v_n\rangle := \langle v_0, \phi(v_1 \otimes \cdots \otimes v_n). \tag{A.2}$$

The composition $\circ_i$ then satisfies

$$\begin{aligned} &Y(\phi \circ_i \psi)(v_0 \otimes \cdots \otimes v_{m+n-1}) \\ &\qquad = \sum_{kl} Y(\phi)(v_0 \otimes \cdots \otimes v_{i-1}\Delta_k \otimes v_{i+1} \otimes \cdots \otimes v_n) \\ &\qquad\qquad g^{kl\otimes} Y(\psi)(\Delta_k \otimes v_{n+1} \otimes \cdots \otimes v_{m+n-1}) \\ &\qquad =: Y(\phi) \circ_i Y(\psi)(v_0 \otimes \cdots \otimes v_{m+n-1}), \end{aligned} \tag{A.3}$$

and this defines the composition on correlators.

Remark A.7. There are invariant ways to write this using tensor products indexed by sets. We used the Casimir to compose. An additional structure is given by replacing the Casimir by some other element, a propagator. This is done in [24] to lift from cohomology to the chain level.

A.2 Standard operads and their algebras

A.2.1 Standard algebras. For the readers' convenience, we list the definition of the algebras we talk about. Let A be a graded vector space over k and let $|a|$ be the degree of an element a. Let's fix char $k = 0$ or at least $\neq 2$.

(1) Pre-Lie algebra. $(A, \circ\colon A \times A \to A)$ such that

$$a \circ (b \circ c) - (a \circ b) \circ c = (-1)^{|a||b|}[a \circ (c \circ b) - (a \circ c) \circ b].$$

(2) Odd Lie. $(A, \{\bullet\}\colon A \otimes A \to A)$.

$$\{a \bullet b\} = (-1)^{|a|-1)(|b|-1)}\{b \bullet a\} \text{ and Jacobi with appropriate signs.}$$

(3) Odd Poisson or Gerstenhaber. $(A, \{\bullet\}, \cdot)$ Odd Lie plus another associative multiplication for which the bracket is a derivation with the appropriate signs.

(4) (dg)BV. $(A, \cdot, \Delta)$. $(A, \cdot)$ associative (differential graded), Δ a differential of degree 1: $\Delta^2 = 0$ and

$$\{a \bullet b\} := (-1)^{|a|}\Delta(ab) - a\Delta(b) - (-1)^{|a|}\Delta(a)b$$

is a Gerstenhaber bracket.

(5) (dg)GBV. This name is used if *a priori* there is a BV operator and a given Gerstenhaber bracket and *a posteriori* the given Gerstenhaber bracket coincides with the one induced by the BV operator.

A.2.2 Typical operads. A list of standard types of operads and their typical examples are given in Table 1.

Table 1. Reference table for types of operads.

Operad	$\mathrm{Hom}(V)$
Odd operad	Naïve suspension of a graded linear operad.
Cyclic operad	$\mathrm{Hom}(V)$, V with symmetric non-degenerate bi-linear form
Anti-cyclic operad	$\mathrm{Hom}(V)$, V with anti-symmetric non-degenerate bi-linear form
Modular operad	$H_*(\bar{M}_{gn})$
$\mathfrak{K}$ modular	Feynman transform of modular operad and $H_*(\bar{M}^{KSV}_{g,n})$[4]

A.3 Hochschild cohomology

Let A be an associative algebra over a field k. We define

$$CH^*(A, A) := \bigoplus_{q \geq 0} CH^q(A, A)$$

with $CH^q(A, A) = \mathrm{Hom}(A^{\otimes q}, A)$.

There are two natural operations

$$\circ_i : CH^p(A, A) \otimes CH^q(A, A) \to CH^{p+q-1}(A, A),$$
$$\cup : CH^n(A, A) \otimes CH^m(A, A) \to CH^{m+n}(A, A),$$

where the first morphism is for $f \in CH^p(A, A)$ and $g \in CH^q(A, A)$:

$$f \circ_i g(x_1, \ldots, x_{p+q-1}) = f(x_1, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+q-1}), x_{i+q}, \ldots, x_{p+q-1}) \tag{A.4}$$

and the second is given by the multiplication

$$f(a_1 \ldots, a_m) \cup g(b_1, \ldots, b_n) = f(a_1 \ldots, a_m) g(b_1, \ldots, b_n) \tag{A.5}$$

[4] $\bar{M}^{KSV}_{g,n}$ is the real blow up of the DM compactification defined in [34].

A.3.1 The differential on CH^*. The Hochschild complex has a differential which is derived from the algebra structure. Given $f \in CH^n(A, A)$ then

$$\begin{aligned}\partial(f)(a_1, \dots, a_{n+1}) \\ := a_1 f(a_2, \dots, a_{n+1}) - f(a_1 a_2, \dots, a_{n+1}) + \cdots \\ \cdots + (-1)^{n+1} f(a_1, \dots, a_n a_{n+1}) + (-1)^{n+2} f(a_1, \dots, a_n) a_{n+1}.\end{aligned} \tag{A.6}$$

Definition A.8. The Hochschild complex is the complex (CH^*, ∂), its cohomology is called the Hochschild cohomology and denoted by $HH^*(A, A)$.

A.3.2 The Gerstenhaber structure. Gerstenhaber [13] introduced the $\circ$ operations: for $f \in CH^p(A, A)$ and $g \in CH^q(A, A)$

$$f \circ g := \sum_{i=1}^{p} (-1)^{(i-1)(q+1)} f \circ_i g. \tag{A.7}$$

He defined the bracket

$$\{f, g\} := f \circ g - (-1)^{(p-1)(q-1)} g \circ f \tag{A.8}$$

and showed that this indeed induces what is now called a Gerstenhaber bracket, i.e. an odd Poisson bracket for $\cup$, on $HH^*(A, A)$. Here odd Poisson bracket means odd Lie bracket and the derivation property of the bracket with shifted (odd) signs.

A.3.3 Reduced Hochschild cohomology. The subcomplex of reduced chains

$$\overline{CH}^*(A, A) \subset CH^*(A, A)$$

is the subcomplex generated by functions f which vanish if one of the variables is 1. That is $f(a_1 \otimes \cdots \otimes a_i \otimes 1 a_{i+1} \otimes \cdots \otimes a_n) = 0$.

This sub-complex is quasi-isomorphic to the full complex, that is its cohomology is $HH^*(A, A)$.

A.4 Frobenius algebras

A Frobenius algebra A is a unital associative algebra with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ which satisfies $\langle ab, c\rangle = \langle a, bc\rangle$.

A.4.1 Comultiplication, Casimirs and Euler elements in Frobenius algebras. For explicit formulas, fix a basis Δ_i of A, set $g_{ij} := \langle \Delta_i, \Delta_j\rangle$ and let g^{ij} be the inverse matrix.

A.4.2 Tensor powers. Let A be a Frobenius algebra, then $A \otimes A$ is also a Frobenius algebra for the multiplication $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bc$ where $|\cdot|$ is the degree in the graded case. The bilinear form is defined by $\langle a \otimes b, c \otimes d \rangle = (-1)^{|b||c|}\langle a, c\rangle\langle b, d\rangle$. Notice this makes the tensor algebra $TA = \bigoplus_{n \geq 0} A^{\otimes n}$ into a *graded* Frobenius algebra. Here we are using the graded dual and the internal degree, and not the total degree for the signs.

A.4.3 Comultiplication. The comultiplication Δ is the adjoint to the multiplication. It is defined by[5]

$$\langle \Delta(a), b \otimes c \rangle = \langle a, bc \rangle. \tag{A.9}$$

The comultiplication satisfies the Frobenius condition

$$(\mu \otimes \mathrm{id})(\mathrm{id} \otimes \Delta) = \Delta\mu = (\mathrm{id} \otimes \mu)(\Delta \otimes \mathrm{id}). \tag{A.10}$$

A.4.4 Casimir. The Casimir element $C \in A \otimes A$ is the dual to the element $\langle\ ,\ \rangle \in \check{A} \otimes \check{A}$. It is also given by $C = \Delta(1) = \sum_{ij} g^{ij}\Delta_i \otimes \Delta_j$ where 1 is the unit.

A.4.5 Euler element. The Euler element of a Frobenius algebra is defined to be $e = \mu\Delta(1) = \sum_{ij} g^{ij}\Delta_i\Delta_j = \sum_{ijk} g^{ij}c^k_{ij}\Delta_k$.

Notice that if $A = H^*(V)$ where V is a compact oriented manifold or variety then $\int_V e = \chi(V)$ and e is the Euler class. In general this type of equation is true if A is Gorenstein.

There is another important equation. Applying equation (A.10):

$$\mu\Delta(a) = \mu\Delta(a1) = \mu\Delta\mu(a \otimes 1)\mu(\mu \otimes \mathrm{id})(a \otimes \Delta(1)) = ae. \tag{A.11}$$

A.5 Reference for symbols

Spaces

$A^s_{g,r}$	space of projectively weighted arc families
$D^s_{g,r}$	space of weighted arc families

Operads/PROPs

$\mathcal{A}rc$	operad of exhaustive (all boundaries are hit) projectively weighted arc families
$\mathcal{D}Arc$	operad of exhaustive weighted arc families
$M^{1,\dots,1}_{g,r,s}$	subspace of quasi-filling arc families, these form a rational operad

[5] In the graded situation, one should of course insert the appropriate signs obtained from switching the middle two factors of A where the above formula is thought of as giving a morphisms from $A^{\otimes 4} \to k$.

$\mathcal{C}acti$	suboperad of $\mathcal{D}Arc$ equivalent to framed little discs, conditions: $g = s = 0$, all arcs from 0 to i
$\mathcal{C}act$	suboperad of $\mathcal{C}acti$ equivalent to little discs, extra condition: linear orders compatible
$\mathcal{C}or$	suboperad of $\mathcal{C}act$ equivalent to little intervals, extra condition: exactly one arc each from boundary i to boundary 0
$\mathcal{G}Tree$	suboperad of $\mathcal{D}Arc$, condition: all arcs from 0 to i
$\mathcal{L}\mathcal{G}\mathcal{T}ree$	suboperad of $\mathcal{G}Tree$, extra condition: linear orders compatible
$\mathcal{S}t\mathcal{O}$	stabilized version of $\mathcal{O}$
${}^{\mathrm{Fat}}\mathcal{O}$	fattened/pointed/unital version of $\mathcal{O}$
$\mathcal{S}ull$	Sullivan quasi-PROP, surfaces with in/out markings, arcs from in to out or out to out

Chain models

$\mathcal{C}act(i)^1$	CW models for $\mathcal{C}act(i)$, condition: all boundaries $i \neq 0$ have weight 1
$\mathcal{S}ull^1$	CW model for $\mathcal{S}ull$: only arcs from "in" to "out" and all in boundaries have weight 1
$\mathcal{C}_o^*(\mathcal{A}rc)$	complex of open cells of $\mathcal{A}rc$
$\mathrm{Gr}\,\mathcal{C}_o^*(\mathcal{A}rc)$	associated graded of $\mathcal{C}_o^*(\mathcal{A}rc)$, the same as $CC_*(A^s_{g,r}, A^s_{g,r} \setminus \mathcal{A}rc^s_g(r-1))$
$\mathrm{Gr}\,\mathcal{C}_o^*(\mathcal{A}rc^0_{\#})$	image of the cells of $M^{1,\dots,1}_{g,r,s}$ in $\mathrm{Gr}\,\mathcal{C}_o^*(\mathcal{A}rc)$

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