A combinatorial E_∞-algebra structure on cubical cochains and the Cartan–Serre map

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Résumé. Les cochaînes cubiques sont munies d'un produit associatif, dual à la diagonale de Serre, relevant la structure commutative graduée en cohomologie. Dans ce travail, nous introduisons par des méthodes combinatoires explicites une extension de ce produit à une structure E_∞. Comme application, nous prouvons que l'application de Cartan–Serre, qui relie les cochaînes singulières cubiques et simpliciales d'espaces, est un quasi-isomorphisme de E_∞-algèbres.

Abstract. Cubical cochains are equipped with an associative product, dual to the Serre diagonal, lifting the graded commutative structure in cohomology. In this work we introduce through explicit combinatorial methods an extension of this product to a full E_∞-structure. As an application we prove that the Cartan–Serre map, which relates the cubical and simplicial singular cochains of spaces, is a quasi-isomorphism of E_∞-algebras.

Keywords. Cubical sets, cochain complex, cup product, Cartan–Serre map, E_∞-algebras, operads.

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1. Introduction

Instead of simplices, in his groundbreaking work on fibered spaces Serre considered cubes as the basic shapes used to define cohomology, stating that:

> Il est en effet evident que ces derniers se pretent mieux que les simplexes a l'etude des produits directs, et, a fortiori, des espaces fibres qui en sont la generalisation. [Ser51, p.431]

Cubical sets, a model for the homotopy category, were considered by Kan [Kan55; Kan56] before introducing simplicial sets, they are central to non-abelian algebraic topology [BHS11], and have become important in Voevodsky's program for univalent foundations and homotopy type theory [KV20; Coh+17]. Other areas that highlight the relevance of cubical methods are applied topology, where cubical complexes are ubiquitous in the study of images [KMM04], condensed matter physics, where models on cubical lattices are central [Bax85], and geometric group theory [Gro87], where fundamental results have been obtained considering actions on certain cube complexes characterized combinatorially [Ago13].

Cubical cochains are equipped with the *Serre algebra structure*, a lift to the cochain level of the graded ring structure in cohomology. Using an acyclic carrier argument it can be shown that this product is commutative up to coherent homotopies in a non-canonical way. The study of such objects, referred to as E_∞-algebras, has a long history, where (co)homology operations [SE62; May70], the recognition of infinite loop spaces [BV73; May72] and complete algebraic models of the p-adic homotopy category [Man01] are key milestones. The goal of this work is to introduce a description of an explicit E_∞-algebra structure naturally extending the Serre algebra structure, and relate it to one on simplicial cochains extending the Alexander–Whitney algebra structure.

We use the combinatorial model of the E_∞-operad $\mathrm{U}(\mathcal{M})$ obtained from the finitely presented prop $\mathcal{M}$ introduced in [Med20a]. The resulting $\mathrm{U}(\mathcal{M})$-algebra structure on cubical cochains is induced from a natural $\mathcal{M}$-bialgebra structure on the chains of representable cubical sets, which is determined by only three linear maps. To our knowledge, this is the first effective construction of an E_∞-algebra structure on cubical cochains. Non-constructively, this result could be obtained using a lifting argument based on the cofibrancy of

the reduced version of the operad $U(\mathcal{M})$ in the model category of operads [Hin97; BM03], but this existence statement is not very useful in concrete situations. To illustrate the advantages of an effective construction let us consider a prime p. The mod p cohomology of spaces is equipped with natural stable endomorphisms, known as Steenrod operations [SE62]. Following an operadic viewpoint developed by May [May70], in [KM21] we exhibited integral elements in $U(\mathcal{M})$ representing Steenrod operations on the mod p homology of $U(\mathcal{M})$-algebras. Since, as proven in this article, the cochains of a cubical set are equipped with a $U(\mathcal{M})$-algebra structure, we obtain natural cochain level multioperations for cubical sets representing Steenrod operation at every p. This cubical cup-(p, i) products are explicit enough to have been implemented in the open source computer algebra system `ComCH` [Med21a].

We now turn to the comparison between cubical and simplicial cochains. In [Ser51, p. 442], Serre described for any topological space $\mathfrak{Z}$ a natural quasi-isomorphism

$$S_\square^\bullet(\mathfrak{Z}) \to S_\triangle^\bullet(\mathfrak{Z}) \tag{1}$$

between its cubical and simplicial singular cochains, stating this to be a quasi-isomorphism of algebras with respect to the usual structures. We will consider a well known Quillen equivalence

$$\mathsf{sSet} \underset{\mathcal{U}}{\overset{\mathcal{T}}{\perp}} \mathsf{cSet}$$

between simplicial and cubical sets, and construct a natural chain map

$$N_\square^\bullet(\mathcal{U}\,Y) \to N_\triangle^\bullet(Y) \tag{2}$$

for every simplicial set Y. In [Med20a], a natural $U(\mathcal{M})$-algebra structure extending the Alexander–Whitney coalgebra structure was constructed on simplicial sets. With respect to it and the one defined here for cubical sets we have the following results after passing to a sub-E_∞-operad of $U(\mathcal{M})$.

Theorem. *The map presented in Equation (2) is a quasi-isomorphism of E_∞-algebras.*

From this result, stated as Theorem 15, we deduce the following two. The first one concerns the triangulation functor $\mathcal{T}$ and it is stated more precisely as Corollary 16.

Corollary. *There is a natural zig-zag of E_∞-algebra quasi-isomorphisms between the cochains of a cubical set and those of its triangulation.*

The next one concerns the map presented in Equation (1), relating the cubical and simplicial singular cochains of a space, and it is stated more precisely as Corollary 17.

Corollary. *The Cartan–Serre map is a quasi-isomorphism of E_∞-algebras.*

Remark. In this introduction we have used the setting defined by cochains and products since it is more familiar, whereas in the rest of the text we use the more fundamental one defined by chains and coproducts.

Outline

We recall the required notions from homological algebra and category theory in Section 2. The necessary concepts from the theory of operads and props is reviewed in Section 3, including the definition of the prop $\mathcal{M}$. Section 4 contains our main contribution; an explicit natural $\mathcal{M}$-bialgebra structure on the chains of representable cubical sets and, from it, a natural E_∞-coalgebra structure on the chains of cubical sets. The comparison between simplicial and cubical chains is presented in Section 5, where we show that the Cartan–Serre map is a quasi-isomorphism respecting E_∞-structures. We close presenting some future work in Section 6.

Acknowledgment

2. Conventions and preliminaries

2.1 Chain complexes

Throughout this article $\Bbbk$ denotes a commutative and unital ring and we work over its associated closed symmetric monoidal category of differential (homologically) graded $\Bbbk$-modules $(\mathsf{Ch}, \otimes, \Bbbk)$. We refer to the objects and morphisms of this category as *chain complexes* and *chain maps* respectively. We denote by $\mathrm{Hom}(C, C')$ the chain complex of $\Bbbk$-linear maps between chain complexes C and C', and refer to the functor $\mathrm{Hom}(-, \Bbbk)$ as *linear duality*.

2.2 Presheaves

Recall that a category is said to be *small* if its objects and morphisms form sets. We denote the category of small categories by Cat. Given categories B and C with B small we denote their associated *functor category* by $\mathsf{Fun}(\mathsf{B}, \mathsf{C})$. A category is said to be *cocomplete* if any functor to it from a small category has a colimit. If A is small and C cocomplete, then the (*left*) *Kan extension of g along f* exists for any pair of functors f and g in the diagram below, and it is the initial object in $\mathsf{Fun}(\mathsf{B}, \mathsf{C})$ making

$$\begin{array}{ccc} \mathsf{A} & \xrightarrow{\;g\;} & \mathsf{C} \\ {\scriptstyle f}\downarrow & \nearrow \\ \mathsf{B} & \end{array}$$

commute. A Kan extension along the *Yoneda embedding*, i.e., the functor

$$\mathcal{Y}\colon \mathsf{A} \to \mathsf{Fun}(\mathsf{A}^{\mathrm{op}}, \mathsf{Set})$$

induced by the assignment

$$a \mapsto \big(a' \mapsto \mathsf{A}(a', a)\big),$$

is referred to as a *Yoneda extension*. Abusively we use the same notation for a functor and for its Yoneda extension. We refer to objects of $\mathsf{Fun}(\mathsf{A}^{\mathrm{op}}, \mathsf{Set})$ in the image of the Yoneda embedding as *representable*.

3. Operads, props and E_∞-structures

We now review the definition of the finitely presented prop $\mathcal{M}$ introduced in [Med20a] and whose associated operad is a model of the E_∞-operad. Given its small number of generators and relations, is well suited to explicitly define E_∞-structures. We start by recalling some basic material from the theory of operads and props.

3.1 Symmetric (bi)modules

Let $\mathbb{S}$ be the category whose objects are the non-negative integers $\mathbb{N}$ and whose set of morphisms between n and n' is empty if $n \neq n'$ and is otherwise the symmetric group $\mathbb{S}_n$. A *left $\mathbb{S}$-module* (resp. *right $\mathbb{S}$-module* or $\mathbb{S}$-*bimodule*) is a functor from $\mathbb{S}$ (resp. $\mathbb{S}^{\mathrm{op}}$ or $\mathbb{S}^{\mathrm{op}} \times \mathbb{S}$) to Ch. In this paper we prioritize left module structures over their right counterparts. As usual, taking inverses makes both perspectives equivalent. We respectively denote by $\mathsf{Mod}_\mathbb{S}$ and $\mathsf{biMod}_\mathbb{S}$ the categories of left $\mathbb{S}$-modules and of $\mathbb{S}$-bimodules with morphisms given by natural transformations.

Given a chain complex C, we have the following key examples of a left and a right $\mathbb{S}$-module

$$\mathrm{End}^C(n) = \mathrm{Hom}(C, C^{\otimes n}), \qquad \mathrm{End}_C(m) = \mathrm{Hom}(C^{\otimes m}, C),$$

and of an $\mathbb{S}$-bimodule

$$\mathrm{End}_C^C(m, n) = \mathrm{Hom}(C^{\otimes m}, C^{\otimes n}),$$

where the symmetric actions are given by permutation of tensor factors.

The group homomorphisms $\mathbb{S}_n \to \mathbb{S}_1^{\mathrm{op}} \times \mathbb{S}_n$ induce a forgetful functor

$$\mathrm{U} \colon \mathsf{biMod}_\mathbb{S} \to \mathsf{Mod}_\mathbb{S} \tag{3}$$

defined explicitly on an object $\mathcal{P}$ by $\mathrm{U}(\mathcal{P})(n) = \mathcal{P}(1, n)$ for $n \in \mathbb{N}$. The similarly defined forgetful functor to right $\mathbb{S}$-modules will not be considered.

3.2 Composition structures

Operads and *props* are obtained by enriching $\mathbb{S}$-modules and $\mathbb{S}$-bimodules with certain composition structures. Intuitively, these are obtained by abstracting the composition structure naturally present in the left $\mathbb{S}$-module End^C

(or right $\mathbb{S}$-module End_C), naturally an operad, and the $\mathbb{S}$-bimodule End_C^C, naturally a prop. More explicitly, an operad $\mathcal{O}$ is a left $\mathbb{S}$-module with chain maps

$$\mathbb{k} \to \mathcal{O}(1),$$
$$\mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_r) \otimes \mathcal{O}(r) \to \mathcal{O}(n_1 + \cdots + n_r),$$

satisfying relations of associativity, equivariance and unitality. Similarly, a prop $\mathcal{P}$ is an $\mathbb{S}$-bimodule together with chain maps

$$\mathbb{k} \to \mathcal{P}(n, n),$$
$$\mathcal{P}(m, k) \otimes \mathcal{P}(k, n) \to \mathcal{P}(m, n),$$
$$\mathcal{P}(m, n) \otimes \mathcal{P}(m', n') \to \mathcal{P}(m + m', n + n'),$$

satisfying certain natural relations. For a complete presentation of these concepts we refer to Definition 11 and 54 of [Mar08]. We respectively denote the category of operads and props with structure preserving morphisms by Oper and Prop.

Let C be a chain complex, $\mathcal{O}$ an operad, and $\mathcal{P}$ a prop. An $\mathcal{O}$-*coalgebra* (resp. $\mathcal{O}$-*algebra* or $\mathcal{P}$-*bialgebra*) structure on C is a structure preserving morphism $\mathcal{O} \to \mathrm{End}^C$ (resp. $\mathcal{O} \to \mathrm{End}_C$ or $\mathcal{P} \to \mathrm{End}_C^C$). We mention that the linear dual of an $\mathcal{O}$-coalgebra is an $\mathcal{O}$-algebra.

Since the forgetful functor presented in Equation (3) induces a functor

$$\mathrm{U} \colon \mathsf{Prop} \to \mathsf{Oper},$$

any $\mathcal{P}$-bialgebra structure on C

$$\mathcal{P} \to \mathrm{biEnd}_C^C$$

induces a $\mathrm{U}(\mathcal{P})$-coalgebra structure on it

$$\mathrm{U}(\mathcal{P}) \to \mathrm{U}(\mathrm{biEnd}_C^C) \cong \mathrm{coEnd}^C.$$

3.3 E_∞-operads

Recall that a *projective $\mathbb{S}_n$-resolution* of a chain complex C is a quasi-isomorphism $R \to C$ from a chain complex R of projective $\mathbb{k}[\mathbb{S}_n]$-modules. An $\mathbb{S}$-module M is said to be E_∞ if there exists a morphism of $\mathbb{S}$-modules $M \to \underline{\mathbb{k}}$ inducing for each $n \in \mathbb{N}$ a free $\mathbb{S}_n$-resolution $M(n) \to \mathbb{k}$. An operad is said to be an E_∞-*operad* if its underlying $\mathbb{S}$-module is E_∞. A prop $\mathcal{P}$ is said to be an E_∞-*prop* if $\mathrm{U}(\mathcal{P})$ is an E_∞-operad.

3.4 Presentations

The *free prop* construction is the left adjoint to the forgetful functor from props to $\mathbb{S}$-bimodules. Explicitly, the free prop $\mathrm{F}(M)$ generated by an $\mathbb{S}$-bimodule M is constructed using isomorphism classes of directed graphs with no directed loops that are enriched with the following labeling structure. We think of each directed edge as built from two compatibly directed half-edges. For each vertex v of a directed graph Γ, we have the sets $in(v)$ and $out(v)$ of half-edges that are respectively incoming to and outgoing from v. Half-edges that do not belong to $in(v)$ or $out(v)$ for any v are divided into the disjoint sets $in(\Gamma)$ and $out(\Gamma)$ of incoming and outgoing external half-edges. For any positive integer n let $\overline{n} = \{1, \ldots, n\}$ and set $\overline{0} = \emptyset$. For any finite set S, denote the cardinality of S by $|S|$. The labeling is given by bijections

$$\overline{|in(\Gamma)|} \to in(\Gamma), \qquad \overline{|out(\Gamma)|} \to out(\Gamma),$$

and

$$\overline{|in(v)|} \to in(v), \qquad \overline{|out(v)|} \to out(v),$$

for every vertex v. We refer to the isomorphism classes of such labeled directed graphs with no directed loops and m incoming and n outgoing half-edges as (m, n)-*graphs*. We denote the set these form by $\mathfrak{G}(m, n)$. We use graphs immersed in the plane to represent elements in $\mathfrak{G}(m, n)$, with the direction implicitly given from top to bottom and the labeling from left to right. Please consult Figure 1 for an example.
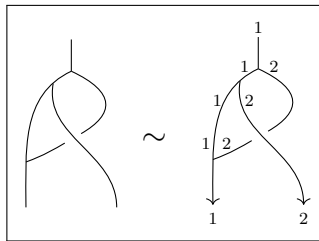

Figure 1: Immersed graph representing a $(1, 2)$-graph.

We consider the right action of $\mathbb{S}_m$ and the left action of $\mathbb{S}_n$ on a (m, n)-graph given respectively by permuting the labels of $in(\Gamma)$ and $out(\Gamma)$. This

action defines the $\mathbb{S}$-bimodule structure on the free prop

$$\mathrm{F}(M)(m,n) \ = \ \bigoplus_{\substack{\Gamma \text{ in} \\ \mathfrak{G}(m,n)}} \ \bigotimes_{\substack{v \text{ in} \\ Vert(\Gamma)}} \ out(v) \otimes_{\mathbb{S}_p} M(p,q) \otimes_{\mathbb{S}_p} in(v), \qquad (4)$$

where we have simplified the notation writing p and q for $\overline{|in(v)|}$ and $\overline{|out(v)|}$ respectively. The differential $\partial_{\mathrm{F}(M)}$ is the extension of that of M to the tensor product (4), and the prop structure is induced by the "identity graphs"

$$||\ldots|$$

together with (relabeled) grafting and disjoint union.

Let G be an assignment of a set $G(m,n)_d$ to each $m,n \in \mathbb{N}$ and $d \in \mathbb{Z}$. Denote by $\mathbb{k}[\mathbb{S}^{\mathrm{op}} \times \mathbb{S}]\{G\}$ the $\mathbb{S}$-bimodule mapping (m,n) to the chain complex with trivial differential and degree d part equal to

$$\mathbb{k}[\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n]\{G(m,n)_d\}.$$

We will denote by $\mathrm{F}(G)$ the free prop generated by this $\mathbb{S}$-bimodule. Let $\partial \colon \mathbb{k}[\mathbb{S}^{\mathrm{op}} \times \mathbb{S}]\{G\} \to \mathrm{F}(G)$ be a morphism of $\mathbb{S}$-bimodules whose canonical extension $\partial \colon \mathrm{F}(G) \to \mathrm{F}(G)$ defines a differential. We denote by $\mathrm{F}_\partial(G)$ the prop obtained by endowing $\mathrm{F}(G)$ with this differential. Let R be a collection of elements in $\mathrm{F}(G)$ and denote by $\langle R \rangle$ the smallest ideal containing R. The prop *generated by G modulo R with boundary ∂* is defined to be $\mathrm{F}_\partial(G)/\langle R \rangle$.

3.5 The prop $\mathcal{M}$

We now recall the E_∞-prop that is central to our constructions.

Definition 1. Let $\mathcal{M}$ be the prop generated by

$$\downarrow, \qquad \curlywedge, \qquad \curlyvee, \tag{5}$$

in $(1,0)_0$, $(1,2)_0$ and $(2,1)_1$ respectively, modulo the relations

$$\curlywedge\text{-}\downarrow, \qquad \downarrow\text{-}\curlywedge, \qquad \curlyvee, \tag{6}$$

with boundary defined by

$$\partial \downarrow = 0, \qquad \partial \curlywedge = 0, \qquad \partial \curlyvee = \downarrow\downarrow\text{-}\downarrow\downarrow. \tag{7}$$

Explicitly, any element in $\mathcal{M}(m, n)$ can be written as a linear combination of the (m, n)-graphs generated by those in (5) via grafting, disjoint union and relabeling, modulo the ideal generated by the relations in (6). Its boundary is determined by (7) using (4).

Proposition 2 ([Med20a, Theorem 3.3]). $\mathcal{M}$ *is an E_∞-prop.*

Remark. The prop $\mathcal{M}$ is obtained from applying the functor of chains to a prop over the category of cellular spaces [Med21b], a quotient of which is isomorphic to the E_∞-operad of stable arc surfaces [Kau09].

4. An E_∞-structure on cubical chains

In this section we construct a natural $\mathcal{M}$-bialgebra structure on the chains of representable cubical sets. These are determined by three natural linear maps satisfying the relations defining $\mathcal{M}$. A Yoneda extension then provides the chains of any cubical set with a natural $U(\mathcal{M})$-coalgebra structure. We begin by recalling the basics of cubical topology.

4.1 Cubical sets

The objects of the *cube category* $\square$ are the sets $2^n = \{0, 1\}^n$ with $2^0 = \{0\}$ for $n \in \mathbb{N}$, and its morphisms are generated by the *coface* and *codegeneracy* maps

$$\delta_i^\varepsilon = \mathrm{id}_{2^{i-1}} \times \delta^\varepsilon \times \mathrm{id}_{2^{n-1-i}} : 2^{n-1} \to 2^n,$$
$$\sigma_i = \mathrm{id}_{2^{i-1}} \times \sigma \times \mathrm{id}_{2^{n-i}} \quad : 2^n \to 2^{n-1},$$

where $\varepsilon \in \{0, 1\}$ and the functors

$$2^0 \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} 2^1 \xrightarrow{\sigma} 2^0$$

are defined by

$$\delta^0(0) = 0, \qquad \delta^1(0) = 1, \qquad \sigma(0) = \sigma(1) = 0.$$

More globally, the category $\square$ is the free strict monoidal category with an assigned internal bipointed object. We refer to [GM03] for a more leisurely exposition and variants of this definition.

We denote by $\mathrm{Dgn}(2^m, 2^n)$ the subset of morphism in $\square(2^m, 2^n)$ of the form $\sigma_i \circ \tau$ with $\tau \in \square(2^m, 2^{n+1})$.

The category of *cubical sets* $\mathsf{Fun}(\square^{\mathrm{op}}, \mathsf{Set})$ is denoted by cSet and the representable cubical set $\mathcal{Y}(2^n)$ by $\square^n$. For any cubical set X we write, as usual, X_n instead of $X(2^n)$.

4.2 Cubical topology

Consider the topological n-cube

$$\mathbb{I}^n = \big\{ (x_1, \dots, x_n) \mid x_i \in [0, 1] \big\}.$$

The assignment $2^n \to \mathbb{I}^n$ defines a functor $\square \to \mathsf{Top}$ with

$$\delta_i^\varepsilon(x_1, \dots, x_n) = (x_1, \dots, x_i, \varepsilon, x_{i+1}, \dots x_n),$$
$$\sigma_i(x_1, \dots, x_n) = (x_1, \dots, \widehat{x_i}, \dots, x_n).$$

Its Yoneda extension is known as *geometric realization*. It has a right adjoint $\mathrm{Sing}^\square \colon \mathsf{Top} \to \mathsf{cSet}$ referred to as the *cubical singular complex* satisfying

$$\mathrm{Sing}^\square(\mathfrak{Z})_n = \mathsf{Top}(\mathbb{I}^n, \mathfrak{Z})$$

for any topological space $\mathfrak{Z}$.

4.3 Cubical chains

The functor of (*normalized*) *chains* $\mathrm{N} \colon \mathsf{cSet} \to \mathsf{Ch}$ is the Yoneda extension of the functor $\square \to \mathsf{Ch}$ defined next. It assigns to an object 2^n the chain complex having in degree m the module

$$\frac{\Bbbk\{\square(2^m, 2^n)\}}{\Bbbk\{\mathrm{Dgn}(2^m, 2^n)\}}$$

and differential induced by

$$\partial(\mathsf{id}_{2^n}) = \sum_{i=1}^n (-1)^i \left(\delta_i^1 - \delta_i^0 \right).$$

To a morphism $\tau\colon 2^n \to 2^{n'}$ it assigns the chain map

$$
\begin{aligned}
\mathrm{N}(\square^n)_m &\longrightarrow \mathrm{N}(\square^{n'})_m \\
\left(2^m \to 2^n\right) &\longmapsto \left(2^m \to 2^n \xrightarrow{\tau} 2^{n'}\right).
\end{aligned}
$$

The chain complex $\mathrm{N}(\square^n)$ is isomorphic to both: $\mathrm{N}(\square^1)^{\otimes n}$ and the cellular chains on the topological n-cube with its standard CW structure $\mathrm{C}(\mathbb{I}^n)$. We use the isomorphism $\mathrm{N}(\square^n) \cong \mathrm{C}(\mathbb{I}^1)^{\otimes n}$ when denoting the elements in the basis of $\mathrm{N}(\square^n)$ by $x_1 \otimes \cdots \otimes x_n$ with $x_i \in \{[0], [0,1], [1]\}$.

For a topological space $\mathfrak{Z}$, the chain complex $\mathrm{N}(\mathrm{Sing}^\square \mathfrak{Z})$ is referred to as the *cubical singular chains* of $\mathfrak{Z}$.

4.4 Serre coalgebra

We now recall the *Serre coalgebra structure*, a natural (counital and coassociative) coalgebra structure on cubical chains.

By a Yoneda extension, to define this structure it suffices to describe it on the chains of representable cubical sets $\mathrm{N}(\square^n)$. For $\mathrm{N}(\square^1)$ we have

$$
\begin{aligned}
\epsilon\big([0]\big) &= 1, & \Delta\big([0]\big) &= [0] \otimes [0], \\
\epsilon\big([1]\big) &= 1, & \Delta\big([1]\big) &= [1] \otimes [1], \\
\epsilon\big([0,1]\big) &= 0, & \Delta\big([0,1]\big) &= [0] \otimes [0,1] + [0,1] \otimes [1].
\end{aligned}
$$

The Serre coalgebra structure on a general $\mathrm{N}(\square^n)$ is define using the isomorphism $\mathrm{N}(\square^n) \cong \mathrm{N}(\square^1)^{\otimes n}$ and the monoidal structure on the category of coalgebras. Explicitly, the structure maps are given by the compositions

$$
\epsilon\colon \mathrm{N}(\square^1)^{\otimes n} \xrightarrow{\epsilon^{\otimes n}} \Bbbk^{\otimes n} \to \Bbbk
$$

and

$$
\Delta\colon \mathrm{N}(\square^1)^{\otimes n} \xrightarrow{\Delta^{\otimes n}} \left(\mathrm{N}(\square^1)^{\otimes 2}\right)^{\otimes n} \xrightarrow{\sigma_{2n}^{-1}} \left(\mathrm{N}(\square^1)^{\otimes n}\right)^{\otimes 2},
$$

where σ_{2n} in $\mathbb{S}_{2n}$ is the (n,n)-shuffle mapping the first and second "decks" to odd and even values respectively. An explicit description of σ_{2n} is presented in Equation (13).

Remark. Similarly to how the Alexander–Whitney coalgebra can be interpreted geometrically as the sum of all complementary pairs of front and back faces of a simplex, this coproduct is, up to signs, also given by the sum of complementary pairs of front and back faces of a cube.

For later reference we record a useful description of the value of Δ on the top dimensional basis element of $N(\square^n)$.

Lemma 3. *For any $n \in \mathbb{N}$,*

$$\Delta\big([0,1]^{\otimes n}\big) = \sum_{\lambda \in \Lambda} (-1)^{\mathrm{ind}\,\lambda} \left(x_1^{(\lambda)} \otimes \cdots \otimes x_n^{(\lambda)} \right) \otimes \left(y_1^{(\lambda)} \otimes \cdots \otimes y_n^{(\lambda)} \right),$$

where each λ in Λ is a map $\lambda \colon \{1, \ldots, n\} \to \{0, 1\}$ with $\lambda(i)$ interpreted as

$$0: \; x_i^{(\lambda)} = [0,1], \qquad\qquad 1: \; x_i^{(\lambda)} = [0],$$
$$y_i^{(\lambda)} = [1], \qquad\qquad\qquad y_i^{(\lambda)} = [0,1],$$

and $\mathrm{ind}\,\lambda$ is the cardinality of $\{i < j \mid \lambda(i) > \lambda(j)\}$.

4.5 Degree 1 product

Let $n \in \mathbb{N}$. For $x = x_1 \otimes \cdots \otimes x_n$ a basis element of $N(\square^n)$ and $\ell \in \{1, \ldots, n\}$ we write

$$x_{<\ell} = x_1 \otimes \cdots \otimes x_{\ell-1},$$
$$x_{>\ell} = x_{\ell+1} \otimes \cdots \otimes x_n,$$

with the convention
$$x_{<1} = x_{>n} = 1 \in \mathbb{Z}.$$

We define the *product* $* \colon N(\square^n)^{\otimes 2} \to N(\square^n)$ by

$$(x_1 \otimes \cdots \otimes x_n) * (y_1 \otimes \cdots \otimes y_n) = (-1)^{|x|} \sum_{i=1}^{n} x_{<i}\,\epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i})\,y_{>i},$$

where the only non-zero values of $x_i * y_i$ are

$$[0] * [1] = [0,1], \qquad [1] * [0] = -[0,1].$$

Example. Since in $N(\square^3)$ we have that

$$\partial \big([0] \otimes [0] \otimes [0]\big) = \partial \big([1] \otimes [1] \otimes [1]\big) = 0$$

and

$$\begin{aligned}
\partial &\big([0] \otimes [0] \otimes [0] * [1] \otimes [1] \otimes [1]\big) \\
&= \partial \big([0,1] \otimes [1] \otimes [1] + [0] \otimes [0,1] \otimes [1] + [0] \otimes [0] \otimes [0,1]\big) \\
&= [1] \otimes [1] \otimes [1] - [0] \otimes [0] \otimes [0],
\end{aligned}$$

we conclude that in general $*$ is not a cycle in the appropriate Hom complex, so it does not descend to homology. This product should be understood as an algebraic version of a consistent choice of path between points in a cube. In our case, as illustrated in Figure 2, the chosen path is given by the union of segments parallel to edges of the cube.
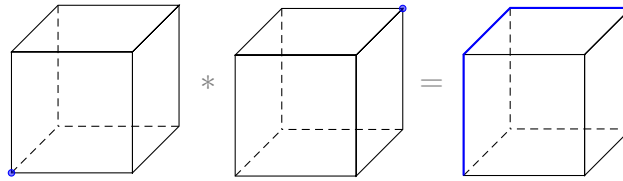

Figure 2: Geometric representation of $\big([0] \otimes [0] \otimes [0] * [1] \otimes [1] \otimes [1]\big)$ where we are using the width-depth-height order.

4.6 $\mathcal{M}$-bialgebra on representable cubical sets

Lemma 4. *The assignment*

$$\Big\downarrow \mapsto \epsilon, \quad \curlywedge \mapsto \Delta, \quad \curlyvee \mapsto *,$$

induces a natural $\mathcal{M}$-bialgebra structure on $N(\square^n)$ for every $n \in \mathbb{N}$.

Proof. We need to show that this assignment is compatible with the relations

$$\curlyveedot = 0, \quad \curlywedge\text{-}\vert = 0, \quad \vert\text{-}\curlywedge = 0,$$

and

$$\partial \,\Big\downarrow = 0, \qquad \partial \curlywedge = 0, \qquad \partial \curlyvee = \,\Big|\Big| \,\text{-}\,\Big|\Big|\,.$$

For the rest of this proof let us consider two basis elements of $N(\square^n)$

$$x = x_1 \otimes \cdots \otimes x_n \qquad \text{and} \qquad y = y_1 \otimes \cdots \otimes y_n.$$

Since the degree of $*$ is 1 and $\epsilon([0,1]) = 0$, we can verify the first relation easily:

$$\varepsilon(x * y) = \sum (-1)^{|x|} \epsilon(y_{<i}) \epsilon(x_{<i}) \otimes \epsilon(x_i * y_i) \otimes \epsilon(x_{>i}) \epsilon(y_{>i}) = 0.$$

For the second relation we want to show that $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$. Since

$$(\epsilon \otimes \mathrm{id}) \circ \Delta([0]) = \epsilon([0]) \otimes [0] = [0],$$
$$(\epsilon \otimes \mathrm{id}) \circ \Delta([1]) = \epsilon([1]) \otimes [1] = [1],$$
$$(\epsilon \otimes \mathrm{id}) \circ \Delta([0,1]) = \epsilon([0]) \otimes [0,1] + \epsilon([0,1]) \otimes [1] = [0,1],$$

we have

$$(\epsilon \otimes \mathrm{id}) \circ \Delta(x_1 \otimes \cdots \otimes x_n) =$$
$$\sum \pm \left(\epsilon\big(x_1^{(1)}\big) \otimes \cdots \otimes \epsilon\big(x_n^{(1)}\big) \right) \otimes \left(x_1^{(2)} \otimes \cdots \otimes x_n^{(2)} \right)$$
$$= x_1 \otimes \cdots \otimes x_n,$$

where the sign is obtained by noticing that the only non-zero term occurs when each factor $x_i^{(0)}$ is of degree 0. The third relation is verified analogously. The fourth and fifth are precisely the well known facts that ϵ and Δ are chain maps. To verify the sixth and final relation we need to show that

$$\partial(x * y) \,+\, \partial x * y \,+\, (-1)^{|x|} x * \partial y \,=\, \epsilon(x) y \,-\, \epsilon(y) x.$$

We have

$$x * y = \sum (-1)^{|x|} x_{<i} \,\epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i}) \, y_{>i}$$

and

$$\partial(x * y) = \sum (-1)^{|x|} \,\partial x_{<i} \,\epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i}) \, y_{>i}$$
$$+ \sum (-1)^{|x|+|x_{<i}|} \, x_{<i} \,\epsilon(y_{<i}) \otimes \partial(x_i * y_i) \otimes \epsilon(x_{>i}) \, y_{>i}$$
$$- \sum (-1)^{|x|+|x_{<i}|} \, x_{<i} \,\epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i}) \,\partial y_{>i}.$$

Since $|x| = |x_{<i}| + |x_i| + |x_{>i}|$ and $\epsilon(x_{>i}) \neq 0 \Leftrightarrow |x_{>i}| = 0$ as well as $\partial(x_i * y_i) \neq 0 \Rightarrow |x_i| = 0$ we have

$$\partial(x * y) = \sum (-1)^{|x|} \, \partial x_{<i} \, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i}) \, y_{>i}$$
$$+ \sum x_{<i} \, \epsilon(y_{<i}) \otimes \partial(x_i * y_i) \otimes \epsilon(x_{>i}) \, y_{>i} \qquad (8)$$
$$- \sum x_{<i} \, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i}) \, \partial y_{>i}.$$

We also have

$$\partial x * y = \sum (-1)^{|x|-1} \, \partial x_{<i} \, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i}) \, y_{>i}$$
$$+ \sum (-1)^{|x|-1+|x_{<i}|} x_{<i} \, \epsilon(y_{<i}) \otimes \partial x_i * y_i \otimes \epsilon(x_{>i}) \, y_{>i}$$
$$+ \sum (-1)^{|x|-1+|x_{<i}|} x_{<i} \, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(\partial x_{>i}) \, y_{>i}.$$

Since

$$\epsilon(\partial x_{>i}) = 0, \quad \partial x_i \neq 0 \Leftrightarrow |x_i| = 1,$$

we have

$$\partial x * y = \sum (-1)^{|x|-1} \, \partial x_{<i} \, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i}) \, y_{>i}$$
$$+ \sum x_{<i} \, \epsilon(y_{<i}) \otimes \partial x_i * y_i \otimes \epsilon(x_{>i}) \, y_{>i}. \qquad (9)$$

We also have

$$(-1)^{|x|} x * \partial y = \sum x_{<i} \, \epsilon(\partial y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i}) \, y_{>i}$$
$$+ \sum (-1)^{|y_{<i}|} x_{<i} \, \epsilon(y_{<i}) \otimes x_i * \partial y_i \otimes \epsilon(x_{>i}) \, y_{>i}$$
$$+ \sum (-1)^{|y_{<i}|+|y_i|} x_{<i} \, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i}) \, \partial y_{>i},$$

which is equivalent to

$$(-1)^{|x|} x * \partial y = \sum x_{<i} \, \epsilon(y_{<i}) \otimes x_i * \partial y_i \otimes \epsilon(x_{>i}) \, y_{>i}$$
$$+ \sum x_{<i} \, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i}) \, \partial y_{>i}. \qquad (10)$$

Putting identities (8), (9) and (10) together, we get

$$\partial(x \otimes y) \, + \, \partial x * y \, + \, (-1)^{|x|} x * \partial y$$
$$= \sum \epsilon(y_{<i}) \, x_{<i} \otimes \big(\partial(x_i * y_i) + \partial x_i * y_i + x_i * \partial y_i \big) \otimes \epsilon(x_{>i}) \, y_{>i}.$$

Since

$$\partial(x_i * y_i) \; + \; \partial\, x_i * y_i \; + \; x_i * \partial\, y_i = \epsilon(x_i)y_i \; - \; \epsilon(y_i)x_i,$$

we have

$$\partial(x * y) \; + \; \partial\, x * y \; + \; (-1)^{|x|} x * \partial\, y =$$
$$\sum \epsilon(y_{<i})\, x_{<i} \otimes \epsilon(x_{\geq i})y_{\geq i} \; - \; \epsilon(y_{\leq i})\, x_{\leq i} \otimes \epsilon(x_{>i})y_{>i}$$
$$= \epsilon(x)y - \epsilon(y)x,$$

as desired, where the last equality follows from a telescopic sum argument. $\square$

4.7 E_∞-coalgebra on cubical chains

Lemma 4 defines a functor from the cube category to that of $\mathcal{M}$-bialgebras. This category is not cocomplete so we do not expect to have an $\mathcal{M}$-bialgebra structure on arbitrary cubical sets. For example, consider the chains on the cubical set X whose only non-degenerate simplices are $v, w \in X_0$. By degree reasons $v * w = 0$ for any degree 1 product $*$ in $\mathrm{N}(X)$. The third relation in $\mathcal{M}$ would then imply the contradiction $0 = w - v$. Since categories of coalgebras over operads are cocomplete we have the following.

Theorem 5. *The Yoneda extension of the composition of the functor* $\square \to$ $\mathsf{biAlg}_{\mathcal{M}}$ *defined in Lemma 4 with the forgetful functor* $\mathsf{biAlg}_{\mathcal{M}} \to \mathsf{coAlg}_{\mathrm{U}(\mathcal{M})}$ *endows the chains of a cubical set with a natural E_∞-coalgebra extension of the Serre coalgebra structure.*

4.8 Cohomology operations

In [Ste47], Steenrod introduced natural operations on the mod 2 cohomology of spaces, the celebrated *Steenrod squares*

$$\mathrm{Sq}^k \colon \; \mathrm{H}^{-n} \xrightarrow{\hspace{2cm}} \mathrm{H}^{-n-k}$$
$$[\alpha] \longmapsto \big[(\alpha \otimes \alpha)\Delta_{n-k}\big],$$

via an explicit construction of natural linear maps $\Delta_i \colon \mathrm{N}(X) \to \mathrm{N}(X) \otimes \mathrm{N}(X)$ for any simplicial set X, satisfying up to signs the following homological relations

$$\partial \circ \Delta_i + \Delta_i \circ \partial = (1 + T)\Delta_{i-1}, \tag{11}$$

with the convention $\Delta_{-1} = 0$. These so-called *cup-i coproducts* appear to be fundamental. We mention two results supporting this claim. In higher category theory they define the nerve of n-categories [Med20b] as introduced by Street [Str87]; and, in connection with K- and L-theory, the Ranicki–Weiss assembly [RW90] can be used to show that chain complex valued presheaves over a simplicial complex X can be fully faithfully modeled by comodules over the symmetric coalgebra structure they define on $\mathrm{N}(X)$ [Med22b].

In the cubical case, cup-i coproducts were defined in [Kad99] and [KP16]. The formulas used by these authors are similar to those introduced in [Med23] for the simplicial case, a dual yet equivalent version of Steenrod's original. A new description of cubical cup-i coproducts can be deduced from our E_∞-structure. We first present it in a recursive form

$$\begin{aligned}
\Delta_0 &= \Delta, \\
\Delta_i &= (* \otimes \mathsf{id}) \circ (23)(\Delta_{i-1} \otimes \mathsf{id}) \circ \Delta.
\end{aligned} \tag{12}$$

A closed form formula for Δ_i uses the $\left(\left\lceil \frac{i+2}{2} \right\rceil, \left\lfloor \frac{i+2}{2} \right\rfloor\right)$-shuffle permutation $\sigma_{i+2} \in \mathbb{S}_{i+2}$ mapping the first and second "decks" to odd and even integers respectively. Explicitly, this shuffle permutation is defined by

$$\sigma_{i+2}(\ell) = \begin{cases} 2\ell - 1 & \ell \leq \left\lceil \frac{i+2}{2} \right\rceil, \\ 2(\ell - \left\lceil \frac{i+2}{2} \right\rceil) & \ell > \left\lceil \frac{i+2}{2} \right\rceil. \end{cases} \tag{13}$$

Let $\Delta^0 = *^0 = \mathsf{id}$ and define for any $k \in \mathbb{N}$

$$\begin{aligned}
*^{k+1} &= * \circ (*^k \otimes \mathsf{id}), \\
\Delta^{k+1} &= (\Delta^k \otimes \mathsf{id}) \circ \Delta.
\end{aligned} \tag{14}$$

With this notation it can be checked that Equation (12) is equivalent to

$$\Delta_i = \left(*^{\left\lceil \frac{i+2}{2} \right\rceil} \otimes *^{\left\lfloor \frac{i+2}{2} \right\rfloor}\right) \circ \sigma_{i+2}^{-1} \Delta^{i+1}. \tag{15}$$
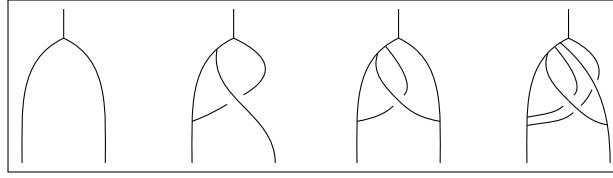
Figure 3: Graphs representing cup-i coproducts.

The first four cup-i coproducts are the images in the endomorphism operad of cubical (and simplicial) chains of the elements $U(\mathcal{M})$ represented by the graphs in Figure 3.

It is not known if the cup-i coproducts defined in Equation (15) agree with those previously constructed, for which a comparison is also missing. This highlights the value of a potential axiomatic characterization of cubical cup-i coproducts as it exists in the simplicial case [Med22a].

As already mentioned, cup-i coproducts represent the Steenrod squares at the chain level, which are primary operations in mod 2 cohomology. To obtain secondary cohomology operations one studies the cohomological relations these operations satisfy, for example the Cartan and Adem relations [SE62]. To do this at the cubical cochain level, as it was done in [Med20c; BMM21] for the simplicial case, the operadic viewpoint is important, so our E_∞-structure on cubical cochains invites the construction of cochain representatives for secondary operations in the cubical case.

For p an odd prime, Steenrod also introduced operations on the mod p cohomology of spaces using the homology of symmetric groups [Ste52; Ste53]. Using the operadic framework of May [May70], we described in [KM21] elements in $U(\mathcal{M})$ representing multicooperations defining Steenrod operations at any prime. In particular, as proven in this work, these so-called *cup-(p, i) coproducts* are defined on cubical chains and are expressible, similarly to Equation (15), in terms of Δ, the permutations of factors, and $*$. The aforementioned construction of cubical cup-(p, i) coproducts has been implemented in the open source computer algebra system `ComCH` [Med21a].

5. The Cartan–Serre map

Let us consider, with their usual CW structures, the topological simplex $\mathbb{A}^n$ and the topological cube $\mathbb{I}^n$. In [Ser51, p. 442], Serre described a quasi-isomorphism of coalgebras between the simplicial and cubical singular chains of a topological space. It is given by precomposing with a canonical cellular map $\mathfrak{cs}\colon \mathbb{I}^n \to \mathbb{A}^n$ also considered in [EM53, p.199] where it is attributed to Cartan.

The goal of this section is to deduce from a more general categorical statement that this comparison map between singular chains of a space is a quasi-isomorphism of E_∞-coalgebras.

5.1 Simplicial sets

We denote the *simplex category* by $\triangle$, the category $\mathsf{Fun}(\triangle^{\mathrm{op}}, \mathsf{Set})$ of *simplicial sets* by sSet, and the representable simplicial set $\mathcal{Y}\big([n]\big)$ by $\triangle^n$. As usual, we denote an element in $\triangle^n_m$ by a non-decreasing tuple $[v_0, \ldots, v_m]$ with $v_i \in \{0, \ldots, n\}$. The *Cartesian product* of simplicial sets is defined by the product of functors. The *simplicial n-cube* $(\triangle^1)^{\times n}$ is the nth-fold Cartesian product of $\triangle^1$ with itself.

We will use the following model of the topological n-simplex:

$$\mathbb{A}^n = \big\{(y_1, \ldots, y_n) \in \mathbb{I}^n \mid i \leq j \Rightarrow y_i \geq y_j\big\},$$

whose cell structure associates $[v_0, \ldots, v_m]$ with the subset

$$\Big\{\big(\underbrace{1, \ldots, 1}_{v_0}, \underbrace{y_1, \ldots y_1}_{v_1 - v_0}, \ldots, \underbrace{y_m, \ldots y_m}_{v_m - v_{m-1}}, \underbrace{0, \ldots, 0}_{n - v_m}\big) \mid y_1 \geq \cdots \geq y_m\Big\}. \quad (16)$$

The spaces $\mathbb{A}^n$ define a functor $\triangle \to \mathsf{CW}$ with

$$\sigma_i(x_1, \ldots, x_n) = (x_1, \ldots, \widehat{x}_i, \ldots, x_n)$$
$$\delta_0(x_1, \ldots, x_n) = (1, x_1, \ldots, x_n),$$
$$\delta_i(x_1, \ldots, x_n) = (x_1, \ldots, x_i, x_i, \ldots, x_n),$$
$$\delta_n(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0).$$

Its Yoneda extension is the *geometric realization* functor. It has a right adjoint $\mathrm{Sing}^\triangle\colon \mathsf{Top} \to \mathsf{sSet}$ referred to as the *simplicial singular complex* satisfying

$$\mathrm{Sing}^\triangle(\mathfrak{Z})_n = \mathsf{Top}(\mathbb{A}^n, \mathfrak{Z})$$

for any topological space $\mathfrak{Z}$.

The functor of (*normalized*) *chains* $\mathrm{N}^\triangle \colon \mathsf{sSet} \to \mathsf{Ch}$ is the composition of the geometric realization functor and that of cellular chains. We denote the composition $\mathrm{N}^\triangle \circ \mathrm{Sing}^\triangle$ by $\mathrm{S}^\triangle$ and omit the superscript $\triangle$ if no confusion may result from doing so. For any $n \in \mathbb{N}$, the *Alexander–Whitney coalgebra structure* on $\mathrm{N}(\triangle^n)$ is given by

$$\Delta\big([v_0, \ldots, v_m]\big) = \sum_{i=0}^{m} [v_0, \ldots, v_i] \otimes [v_i, \ldots, v_m],$$

and

$$\epsilon\big([v_0, \ldots, v_m]\big) = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

The degree 1 product $* \colon \mathrm{N}(\triangle^n)^{\otimes 2} \to \mathrm{N}(\triangle^n)$ is defined by

$$[v_0, \ldots, v_p] * [v_{p+1}, \ldots, v_m] =$$
$$\begin{cases} (-1)^{p+|\sigma|} \big[v_{\sigma(0)}, \ldots, v_{\sigma(m)}\big] & \text{if } v_i \neq v_j \text{ for } i \neq j, \\ 0 & \text{if not,} \end{cases}$$

where σ is the permutation that orders the totally ordered set of vertices and $(-1)^{|\sigma|}$ is its sign. As shown in [Med20a, Theorem 4.2] the assignment

$$\big\downarrow \mapsto \epsilon, \quad \curlywedge \mapsto \Delta, \quad \curlyvee \mapsto *,$$

defines a natural $\mathcal{M}$-bialgebra on the chains of representable simplicial sets, and, by forgetting structure, also a natural $\mathrm{U}(\mathcal{M})$-coalgebra. For any simplicial set, a natural $\mathrm{U}(\mathcal{M})$-coalgebra structure on its chains is defined by a Yoneda extension.

5.2 The Eilenberg–Zilber maps

For any permutation $\sigma \in \mathbb{S}_n$ let

$$\mathfrak{i}_\sigma \colon \mathbb{\triangle}^n \to \mathbb{I}^n$$

be the inclusion defined by $(x_1, \ldots, x_n) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. If e is the identity permutation, we denote $\mathfrak{i}_e$ simply as $\mathfrak{i}$. The maps $\{\mathfrak{i}_\sigma\}_{\sigma \in \mathbb{S}_n}$ define

a subdivision of $\mathbb{I}^n$ making it isomorphic to $\left|(\triangle^1)^{\times n}\right|$ in CW. Using this identification, the identity map induces a cellular map

$$\mathfrak{e}\mathfrak{z}\colon \mathbb{I}^n \to \left|(\triangle^1)^{\times n}\right|.$$

We denote the induced chain map by

$$\mathrm{EZ}\colon \mathrm{N}(\square^n) \to \mathrm{N}\left((\triangle^1)^{\times n}\right).$$

For any topological space $\mathfrak{Z}$, the cubical map

$$\mathcal{U}\,\mathrm{Sing}^\triangle(\mathfrak{Z}) \to \mathrm{Sing}^\square(\mathfrak{Z})$$

is defined, using the adjunction isomorphism

$$\mathsf{sSet}\left((\triangle^1)^{\times n}, \mathrm{Sing}^\triangle(\mathfrak{Z})\right) \cong \mathsf{Top}\left(|(\triangle^1)^{\times n}|, \mathfrak{Z}\right),$$

by the assignment

$$\left(|(\triangle^1)^{\times n}| \xrightarrow{f} \mathfrak{Z}\right) \mapsto \left(\mathbb{I}^n \xrightarrow{\mathfrak{e}\mathfrak{z}} |(\triangle^1)^{\times n}| \xrightarrow{f} \mathfrak{Z}\right).$$

We denote the induced chain map by

$$\mathrm{EZ}_{\mathrm{S}(\mathfrak{Z})}\colon \mathrm{N}^\square\left(\mathcal{U}\,\mathrm{Sing}^\triangle(\mathfrak{Z})\right) \to \mathrm{S}^\square(\mathfrak{Z}).$$

5.3 The Cartan–Serre maps

The cellular map

$$\mathfrak{c}\mathfrak{s}\colon \mathbb{I}^n \to \mathbb{A}^n$$

is defined by

$$\mathfrak{c}\mathfrak{s}(x_1, \ldots, x_n) = (x_1,\ x_1 x_2,\ \ldots,\ x_1 x_2 \cdots x_n).$$

We denote its induced chain map by

$$\mathrm{CS}\colon \mathrm{N}(\square^n) \to \mathrm{N}(\triangle^n).$$

The chain map

$$\mathrm{CS}_{\mathrm{S}(\mathfrak{Z})}\colon \mathrm{S}^\triangle(\mathfrak{Z}) \to \mathrm{S}^\square(\mathfrak{Z})$$

between the singular chain complexes of a topological space $\mathfrak{Z}$ is defined by

$$\mathrm{CS}_{\mathrm{S}(\mathfrak{Z})}(\mathbb{A}^n \to \mathfrak{Z}) = (\mathbb{I}^n \xrightarrow{\mathfrak{c}\mathfrak{s}} \mathbb{A}^n \to \mathfrak{Z}).$$

These maps were considered in [Ser51, p. 442] where it was stated that $\mathrm{CS}_{\mathrm{S}(\mathfrak{Z})}$ is a natural quasi-isomorphisms of coalgebras. We will prove this in §5.6 showing in fact that it is a quasi-isomorphism of E_∞-coalgebras.

5.4 No-go results

Since CS is shown to be a coalgebra map in Lemma 8 and EZ is well known to be one, one may hope for higher structures to be preserved by these maps. We now provide some examples constraining the scope of these expectations.

Example. We will show that EZ does not preserve $U(\mathcal{M})$-structures. More specifically, that in general

$$EZ^{\otimes 2} \circ \Delta_1 \neq \Delta_1 \circ EZ$$

where

$$\Delta_1 = (\ast \otimes \mathsf{id}) \circ (\mathsf{id} \otimes (12)\Delta) \circ \Delta$$

is the cup-1 coproduct presented in Equation (15). Up to signs, on one hand we have

$$\Delta_1\big([01][01]\big) = [01][01] \otimes [1][01] \; + \; [01][1] \otimes [01][01]$$
$$+ \; [0][01] \otimes [01][01] \; + \; [01][01] \otimes [01][0].$$

Therefore,

$$EZ^{\otimes 2} \circ \Delta_1\big([01][01]\big) = \big(011 \times 001 + 001 \times 011\big) \otimes 11 \times 01$$
$$+ \; 01 \times 11 \otimes \big(011 \times 001 + 001 \times 011\big)$$
$$+ \; 00 \times 01 \otimes \big(011 \times 001 + 001 \times 011\big)$$
$$+ \; \big(011 \times 001 + 001 \times 011\big) \otimes 01 \times 00.$$

On the other hand, we have

$$\Delta_1[0,1,2] = [0,1,2] \otimes [0,1] + [0,2] \otimes [0,1,2] + [0,1,2] \otimes [1,2].$$

Therefore,

$$\Delta_1 \circ EZ\big([01][01]\big) = \Delta_1\big(011 \times 001 + 001 \times 011\big)$$
$$= 011 \times 001 \otimes 01 \times 00 \; + \; 01 \times 01 \otimes 011 \times 001$$
$$+ \; 011 \times 001 \otimes 11 \times 01 \; + \; 001 \times 011 \otimes 00 \times 01$$
$$+ \; 01 \times 01 \otimes 001 \times 011 \; + \; 001 \times 011 \otimes 01 \times 11.$$

We conclude that

$$\mathrm{EZ}^{\otimes 2} \circ \Delta_1\big([01][01]\big) \neq \Delta_1 \circ \mathrm{EZ}\big([01][01]\big)$$

since, for example, the basis element $01 \times 11 \otimes 011 \times 001$ appears in the left sum but not in the right one.

Example. We will show that the Cartan–Serre map does not preserve $\mathcal{M}$-structures. More specifically, that in general

$$\mathrm{CS}(x * y) \neq \mathrm{CS}(x) * \mathrm{CS}(y).$$

Consider $x = [1][1]$ and $y = [0][01]$. On one hand we have that

$$\mathrm{CS}\big([1][1]\big) * \mathrm{CS}\big([0][01]\big) = 0$$

since $\mathrm{CS}\big([0][01]\big) = 0$. On the other hand we have, up to a signs, that

$$\mathrm{CS}\Big(([1][1]) * ([0][01])\Big) = \mathrm{CS}\Big([01][01]\Big) = [012],$$

which establishes the claim.

The reason for this incompatibility is that $*$ in the simplicial context is commutative, which is not the case in the cubical one.

Example. We will show that the Cartan–Serre map does not preserve $\mathrm{U}(\mathcal{M})$-structures. More specifically, that in general

$$\mathrm{CS} \circ \widetilde{\Delta}_1 \neq \widetilde{\Delta} \circ \mathrm{CS}$$

where

$$\widetilde{\Delta}_1 = (* \otimes \mathsf{id}) \circ (12)(\mathsf{id} \otimes (12)\Delta) \circ \Delta.$$

On one hand we have that

$$\mathrm{CS}\Big(\widetilde{\Delta}_1\big([01][01]\big)\Big) = T\Delta_1\big([012]\big),$$

and on the other that

$$\widetilde{\Delta}_1 \circ \mathrm{CS}\big([01][01]\big) = \Delta_1\big([012]\big),$$

which establishes the claim.

In §5.6 we will show that CS is a morphism of E_∞-coalgebras. To do so we now introduce an E_∞-suboperad of $\mathrm{U}(\mathcal{M})$ where the incompatibility resulting from the lack of commutativity of $*$ in the cubical context is dealt with.

5.5 Shuffle graphs

Consider $k = k_1 + \cdots + k_r$. A $(k_1, \ldots, k_r)$-shuffle σ is a permutation in $\mathbb{S}_k$ satisfying

$$\sigma(1) < \cdots < \sigma(k_1),$$
$$\sigma(k_1 + 1) < \cdots < \sigma(k_1 + k_2),$$
$$\vdots$$
$$\sigma(k - k_r + 1) < \cdots < \sigma(k).$$

The (*left comb*) *shuffle graph* associated to such σ is the $(1, k)$-graph
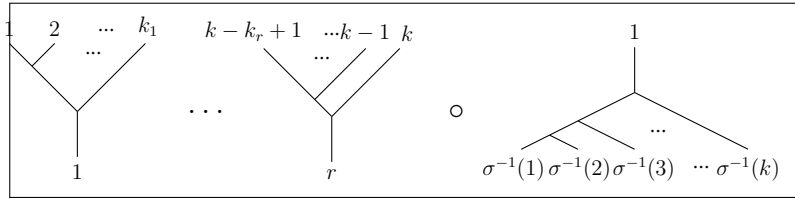
presented as a composition of (left comb) self-graftings of the generators $\curlyvee$ and $\curlywedge$. With the notation introduced in Equation (14), the $\mathrm{U}(\mathcal{M})$-coalgebra sends the shuffle graph associated to σ to

$$\left(*^{k_1} \otimes \cdots \otimes *^{k_r} \right) \circ \sigma^{-1} \Delta^{k-1}.$$

Example. All the graphs in Figure 3 are shuffle graphs. In fact, all the cup-i coproducts presented in Equation (15) are induced from shuffle graphs, whereas

$$\widetilde{\Delta}_1 = (* \otimes \mathsf{id}) \circ (12)(\mathsf{id} \otimes (12)\Delta) \circ \Delta$$
$$= (* \otimes \mathsf{id}) \circ (123)\Delta^2,$$

used in the previous section to probe the limits of the structure preserving properties of CS, is not.

The operad $\mathrm{U}(\mathcal{M}_{sh})$ is defined as the suboperad of $\mathrm{U}(\mathcal{M})$ (freely) generated by shuffle graphs. Explicitly, any element in $\mathrm{U}(\mathcal{M}_{sh})(r)$ is represented by a linear combination of $(1, r)$-graphs obtained by grafting these. The same proof used in [Med20a, p.5] to show that $\mathrm{U}(\mathcal{M})$ is an E_∞-operad can be used to prove the same for $\mathrm{U}(\mathcal{M}_{sh})$.

5.6 E_∞-coalgebra preservation

We devote this subsection to the proof of the following key result.

Theorem 6. *The chain map* $\mathrm{CS}\colon \mathrm{N}(\square^n) \to \mathrm{N}(\triangle^n)$ *is a quasi-isomorphism of* $\mathrm{U}(\mathcal{M}_{sh})$*-coalgebras.*

We start by stating an alternative description of the CS map.

Lemma 7. *Let* $x = x_1 \otimes \cdots \otimes x_n \in \mathrm{N}(\square^n)_m$ *be a basis element with* $x_{q_i} = [0,1]$ *for all* $\{q_1 < \cdots < q_m\}$*. If there is* $x_\ell = [0]$ *with* $\ell < q_m$ *then* $\mathrm{CS}(x) = 0$*, otherwise*

$$\mathrm{CS}(x) = \big[q_1 - 1,\ \ldots\ ,\ q_m - 1,\ p(x) - 1\big]$$

where $p(x) = \min\{\ell \mid x_\ell = [0]\}$ *or* $p(x) = n + 1$ *if this set is empty.*

Proof. This can be directly verified using the cell structure of $\triangle^n$ described in Equation (16). $\qquad\square$

Lemma 8. *The chain map* $\mathrm{CS}\colon \mathrm{N}(\square^n) \to \mathrm{N}(\triangle^n)$ *is a quasi-isomorphism of coalgebras.*

Proof. The chain map CS is a quasi-isomorphism compatible with the counit since it is induced from a cellular map between contractible spaces. We need to show it preserves coproducts. By naturality it suffices to verify this on $[0,1]^{\otimes n}$. Recall from Lemma 3 that

$$\Delta\big([0,1]^{\otimes n}\big) = \sum_{\lambda \in \Lambda} (-1)^{\mathrm{ind}\,\lambda} \left(x_1^{(\lambda)} \otimes \cdots \otimes x_n^{(\lambda)}\right) \otimes \left(y_1^{(\lambda)} \otimes \cdots \otimes y_n^{(\lambda)}\right),$$

where the sum is over all choices for each $i \in \{1, \ldots, n\}$ of

$$\begin{aligned}
x_i^{(\lambda)} &= [0,1], & \text{or} & & x_i^{(\lambda)} &= [0], \\
y_i^{(\lambda)} &= [1], & & & y_i^{(\lambda)} &= [0,1].
\end{aligned}$$

By Lemma 7, the summands above not sent to 0 by $\mathrm{CS} \otimes \mathrm{CS}$ are those basis elements for which $x_i^{(\lambda)} = [0]$ implies $x_j^{(\lambda)} = [0]$ for all $i < j$. For any one such summand, its sign is positive and its image by $\mathrm{CS} \otimes \mathrm{CS}$ is $[0, \ldots, k] \otimes [k, \ldots, n]$ where $k + 1 = \min\{i \mid x_i^{(\lambda)} = [0]\}$ or $k = n$ if this set is empty. The summands $[0, \ldots, k] \otimes [k, \ldots, n]$ are precisely those appearing when applying the Alexander–Whitney coproduct to $\mathrm{CS}\big([0,1]^{\otimes n}\big) = [0, \ldots, n]$. This concludes the proof. $\qquad\square$

We will consider the basis of $\mathrm{N}(\square^n)$ as a poset with

$$(x_1 \otimes \cdots \otimes x_n) \leq (y_1 \otimes \cdots \otimes y_n)$$

if and only if $x_\ell \leq y_\ell$ for each $\ell \in \{1, \ldots, n\}$ with respect to

$$[0] < [0, 1] < [1].$$

As we prove next, an example of ordered elements are the tensor factors of each summand in the iterated Serre diagonal.

Lemma 9. *Writing*

$$\Delta^{k-1}\big([0,1]^{\otimes n}\big) = \sum \pm\, x^{(1)} \otimes \cdots \otimes x^{(k)}$$

with each $x^{(\ell)}$ a basis element of $\mathrm{N}(\square^n)$, we have

$$x^{(1)} \leq \cdots \leq x^{(k)}$$

for every summand.

Proof. This can be proven using a straightforward induction argument whose base case follows from inspecting Lemma 3. $\square$

Lemma 10. *Let x, y and z be basis elements of $\mathrm{N}(\square^n)$. If both $x \leq z$ and $y \leq z$ then either $(x * y) = 0$ or every summand in $(x * y)$ is $\leq z$.*

Proof. Recall that

$$(x_1 \otimes \cdots \otimes x_n) * (y_1 \otimes \cdots \otimes y_n) = (-1)^{|x|} \sum_{\ell=1}^{n} x_{<\ell}\, \epsilon(y_{<\ell}) \otimes x_\ell * y_\ell \otimes \epsilon(x_{>\ell})\, y_{>\ell}.$$

By assumption $x_{<\ell} \leq z_{<\ell}$ and $y_{>\ell} \leq z_{>\ell}$ for every $\ell \in \{1, \ldots, n\}$. If $x_\ell * y_\ell \neq 0$ then $x_\ell * y_\ell = [0, 1]$ and either $x_\ell = [1]$ or $y_\ell = [1]$ which implies $z_\ell = [1]$ as well, so $x_\ell * y_\ell \leq z_\ell$. $\square$

Lemma 11. *If x and y are basis elements of $\mathrm{N}(\square^n)$ satisfying $x \leq y$ then*

$$\mathrm{CS}(x * y) = \mathrm{CS}(x) * \mathrm{CS}(y). \tag{17}$$

Proof. We present this proof in the form of three claims. We use Lemma 7, the assumption $x \leq y$, and the fact that the join of basis elements in $\mathrm{N}(\triangle^n)$ sharing a vertex is 0 without explicit mention.

Claim 1. If $\mathrm{CS}(x) = 0$ or $\mathrm{CS}(y) = 0$ then for every $i \in \{1, \ldots, n\}$

$$\mathrm{CS}\left(x_{<i}\,\epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i})\,y_{>i}\right) = 0. \tag{18}$$

Assume $\mathrm{CS}(x) = 0$, that is, there exists a pair $p < q$ such that $x_p = [0]$ and $x_q = [0, 1]$, then (18) holds since:

1. If $i > q$, then x_p and x_q are part of $x_{<i}$.

2. If $i = q$, then $x_q * y_q = 0$ for any y_q.

3. If $i < q$, then $\epsilon(x_{>i}) = 0$.

Similarly, if there is a pair $p < q$ such that $y_p = [0]$ and $y_q = [0, 1]$, then (18) holds since:

1. If $i < p$, then y_p and y_q are part of $y_{>i}$.

2. If $i = p$, then $x_i = [0]$ and $x_i * y_i = 0$.

3. If $i > p$, then either $x_i * y_i = 0$ or $x_i * y_i = [0, 1]$ and $x_p = [0]$.

This proves the first claim and identity (17) under its hypothesis.

Claim 2. If $\mathrm{CS}(x) \neq 0$ and $\mathrm{CS}(y) \neq 0$ then

$$\mathrm{CS}(x * y) = \mathrm{CS}\left(x_{<p_x}\epsilon(y_{<p_x}) \otimes\; x_{p_x} * y_{p_x} \otimes \epsilon(x_{>p_x})\,y_{>p_x}\right)$$

if $p_x = \min\left\{i \mid x_i = [0]\right\}$ is well-defined and $x * y = 0$ if not.

Assume p_x is not well-defined, i.e., $x_i \neq [0]$ for all $i \in \{1, \ldots, n\}$. Given that $x \leq y$ we have that $[0] < x_i$ implies $x_i * y_i = 0$, and the claim follows in this case.

Assume p_x is well-defined. We will show that for all $i \in \{1, \ldots, n\}$ with the possible exception of $i = p_x$ we have

$$\mathrm{CS}\left(x_{<i}\,\epsilon(y_{<i}) \otimes\; x_i * y_i \otimes \epsilon(x_{>i})\,y_{>i}\right) = 0 \tag{19}$$

This follows from:

1. If $i < p_x$ and $x_i = [1]$ then $y_i = [1]$ and $x_i * y_i = 0$.

2. If $i < p_x$ and $x_i = [0, 1]$ then $x_i * y_i = 0$ for any y_i.

3. If $i > p_x$ then Lemma 7 implies the claim since $x_{p_x} = [0]$ and $x_i * y_i \neq 0$ iff $x_i * y_i = [0, 1]$.

Claim 3. If $CS(x) \neq 0$ and $CS(y) \neq 0$ then (17) holds.

Let us assume that $\{i \mid x_i = [0]\}$ is empty, which implies the analogous statement for y since $x \leq y$. Since neither of x nor y have a factor $[0]$ in them, Lemma 7 implies that the vertex $[n]$ is in both $CS(x)$ and $CS(y)$, which implies $CS(x) * CS(y) = 0$ as claimed.

Assume now that $p_x = \{i \mid x_i = [0]\}$ is well defined, and let $\{q_1 < \cdots < q_m\}$ with $x_{q_i} = [0, 1]$ for $i \in \{1, \ldots, m\}$. Since $CS(x) \neq 0$ Lemma 7 implies that $p_x > q_m$, so $\epsilon(x_{>p_x}) = 1$ and Claim 2 implies

$$CS(x * y) = CS\left(x_{<p_x} \epsilon(y_{<p_x}) \otimes x_{p_x} * y_{p_x} \otimes y_{>p_x}\right).$$

We have the following cases:

1. If $\epsilon(y_{<p_x}) = 0$ then there is q_i such that $y_{q_i} = [0, 1]$ so $[q_i - 1]$ is in both $CS(x)$ and $CS(y)$.

2. If $\epsilon(y_{p_x}) \neq 0$ and $y_{p_x} \in \{[0], [0, 1]\}$ then $x_{p_x} * y_{p_x} = 0$ and $[p_x - 1]$ is in both $CS(x)$ and $CS(y)$.

3. If $\epsilon(y_{p_x}) \neq 0$ and $y_{p_x} = [1]$ let $\{\ell_1 < \cdots < \ell_k\}$ be such that $y_{\ell_j} = [0, 1]$ and let $p_y > \ell_k$ be either $n + 1$ or $\min\{j \mid y_j = \{0\}\}$ then

$$\begin{aligned}
CS(x * y) &= CS\left(x_{<p_x} \otimes x_{p_x} * y_{p_x} \otimes y_{>p_y}\right) \\
&= [q_1 - 1, \ldots, q_m - 1, p_x - 1, \ell_1 - 1, \ldots, \ell_k - 1, p_y - 1] \\
&= CS(x) * CS(y).
\end{aligned}$$

This concludes the proof. $\qquad\square$

Combining the previous two lemmas we obtain the following.

Lemma 12. *Let $x^{(1)} \leq \cdots \leq x^{(k)}$ be basis elements of $N(\square^n)$. Then,*

$$CS \circ *^{k-1}\left(x^{(1)} \otimes \cdots \otimes x^{(k)}\right) = *^{k-1} \circ CS^{\otimes k}\left(x^{(1)} \otimes \cdots \otimes x^{(k)}\right).$$

We are now ready to present the argument establishing that CS is an E_∞-coalgebra map.

Proof of Theorem 6. Since $\mathrm{U}(\mathcal{M}_{sh})$ is generated by elements represented by shuffle graphs, we only need to show that for any $(k_1, \ldots, k_r)$-shuffle σ with $k = k_1 + \cdots + k_r$ the following holds

$$\mathrm{CS}^{\otimes r}(*^{k_1} \otimes \cdots \otimes *^{k_r}) \circ \sigma^{-1}\Delta^{k-1} = (*^{k_1} \otimes \cdots \otimes *^{k_r}) \circ \sigma^{-1}\Delta^{k-1} \circ \mathrm{CS}.$$

By naturality, it suffices to prove this identity for $[0,1]^{\otimes n}$. According to Lemma 9

$$x^{(1)} \leq \cdots \leq x^{(k)}$$

for every summand in

$$\Delta^{k-1}\big([0,1]^{\otimes n}\big) = \sum \pm\, x^{(1)} \otimes \cdots \otimes x^{(k)}.$$

Since σ is a shuffle permutation, Lemma 12 implies that

$$\mathrm{CS}^{\otimes r}(*^{k_1} \otimes \cdots \otimes *^{k_r}) \circ \sigma^{-1}\Delta^{k-1}\big([0,1]^{\otimes n}\big)$$
$$= (*^{k_1} \otimes \cdots \otimes *^{k_r}) \circ \sigma^{-1}\, \mathrm{CS}^{\otimes k} \circ \Delta^{k-1}\big([0,1]^{\otimes n}\big).$$

As proven in Lemma 8, CS is a coalgebra map, which concludes the proof. $\square$

5.7 Categorical reformulation

The assignment $2^n \mapsto (\triangle^1)^{\times n}$ defines a functor $\square \to \mathsf{sSet}$ with

$$\delta_i^\varepsilon \colon (\triangle^1)^{\times n} \to (\triangle^1)^{\times (n+1)}$$
$$\sigma_i \colon (\triangle^1)^{\times (n+1)} \to (\triangle^1)^{\times n}$$

given by inserting $[\varepsilon, \ldots, \varepsilon]$ as the ith factor and removing the ith factor respectively. Its Yoneda extension, referred to as triangulation functor, is denoted by

$$\mathcal{T} \colon \mathsf{cSet} \to \mathsf{sSet}.$$

This functor admits a right adjoint

$$\mathcal{U} \colon \mathsf{sSet} \to \mathsf{cSet}$$

defined, as usual, by the expression

$$\mathcal{U}(Y)(2^n) = \mathsf{sSet}\big((\triangle^1)^{\times n}, Y\big).$$

We mention that, as proven in [Cis06, § 8.4.30], the pair $(\mathcal{T}, \mathcal{U})$ defines a Quillen equivalence when sSet and cSet are considered as model categories.

Definition 13. The simplicial map $\mathsf{cs} \colon (\triangle^1)^{\times n} \to \triangle^n$ is defined by

$$[\varepsilon_0^1, \ldots, \varepsilon_m^1] \times \cdots \times [\varepsilon_0^n, \ldots, \varepsilon_m^n] \mapsto [v_0, \ldots, v_m]$$

where $v_i = \varepsilon_i^1 + \varepsilon_i^1 \varepsilon_i^2 + \cdots + \varepsilon_i^1 \cdots \varepsilon_i^n$.

Please observe that the maps $\mathfrak{cs}$ and $|\mathsf{cs}| \circ \mathfrak{e}_3$ agree.

Definition 14. Let Y be a simplicial set. The map

$$\mathrm{CS}_Y \colon \mathrm{N}^\triangle(Y) \to \mathrm{N}^\square(\mathcal{U}\, Y)$$

is the linear map induced by sending a simplex $y \in Y_n$ to the composition

$$(\triangle^1)^{\times n} \xrightarrow{\mathsf{cs}} \triangle^n \xrightarrow{\xi_y} Y$$

where $\xi_y \colon \triangle^n \to Y$ is the simplicial map determined by $\xi_y\big([n]\big) = y$.

Theorem 15. *For any simplicial set Y the map $\mathrm{CS}_Y \colon \mathrm{N}^\triangle(Y) \to \mathrm{N}^\square(\mathcal{U}\, Y)$ is a quasi-isomorphism of $\mathrm{U}(\mathcal{M}_{sh})$-coalgebras which extend respectively the Alexander–Whitney and Serre coalgebra structures.*

Proof. This is a direct consequence of Theorem 6 following from a standard category theory argument, which we now present. Consider the isomorphism

$$\mathrm{N}(\mathcal{U}\, Y) \cong \bigoplus_{n \in \mathbb{N}} \mathrm{N}(\square^n) \otimes \Bbbk\Big\{\mathsf{sSet}\big((\triangle^1)^{\times n}, \triangle^n\big)\Big\}\Big/ \sim$$

and the canonical linear inclusions:

$$\mathrm{N}(\square^n) \longrightarrow \bigoplus_{m \in \mathbb{N}} \mathrm{Hom}\big(\mathrm{N}(\square^m), \mathrm{N}(\square^n)\big)$$

$$(2^m \xrightarrow{\delta} 2^n) \longmapsto \big(\mathrm{N}(\square^m) \xrightarrow{\mathrm{N}(\delta)} \mathrm{N}(\square^n)\big)$$

and

$$\bigoplus_{n\in\mathbb{N}}\Bbbk\Big\{\mathsf{sSet}\big((\triangle^1)^{\times n},\triangle^n\big)\Big\} \longrightarrow \bigoplus_{n\in\mathbb{N}}\mathrm{Hom}\Big(\mathrm{N}\big((\triangle^1)^{\times n}\big),\,\mathrm{N}(\triangle^n)\Big)$$

$$\big((\triangle^1)^{\times n}\xrightarrow{f}\triangle^n\big) \longmapsto \Big(\mathrm{N}\big((\triangle^1)^{\times n}\big)\xrightarrow{\mathrm{N}(f)}\mathrm{N}(\triangle^n)\Big).$$

We can use these and the naturality of EZ to construct the following chain map which is an isomorphism onto its image.

$$\mathrm{N}(\mathcal{U}\,Y) \longrightarrow \bigoplus_{n\in\mathbb{N}}\mathrm{Hom}\big(\mathrm{N}(\square^n),\,\mathrm{N}(Y)\big)$$

$$(\delta\otimes f) \longmapsto \big(\mathrm{N}(f)\circ\mathrm{EZ}\circ\mathrm{N}(\delta)\big).$$

Let Γ be an element in $\mathrm{U}(\mathcal{M}_{sh})(r)$ and denote by $\Gamma^\square\colon \mathrm{N}(\mathcal{U}\,Y)\to\mathrm{N}(\mathcal{U}\,Y)^{\otimes r}$ and $\Gamma^\triangle\colon \mathrm{N}(Y)\to\mathrm{N}(Y)^{\otimes r}$ its image in the respective endomorphism operads. Using the naturality of $\Gamma^\square$, we have that $\Gamma^\square(\delta\otimes f)$ corresponds to $(\mathrm{N}(f)\circ\mathrm{EZ})^{\otimes r}\circ\Gamma^\square\circ\mathrm{N}(\delta)$. On the other hand, the map CS_Y corresponds to

$$\mathrm{N}(Y)_n \longrightarrow \mathrm{N}(\mathcal{U}\,Y)_n$$

$$y \longmapsto \big(\mathrm{N}(\xi_y)\circ\mathrm{CS}\big)$$

where $\xi_y\colon \triangle^n\to Y$ is determined by $\xi_y\big([n]\big)=y$, and we used that $\mathrm{CS}=\mathrm{N}(\mathsf{cs})\circ\mathrm{EZ}$ to ensure the above assignment is well defined. The image of $\Gamma^\triangle(y)$ corresponds to $\mathrm{N}(\xi_y)^{\otimes r}\circ\Gamma^\triangle\circ\mathrm{CS}$. So the claim follows from the identity

$$\Gamma^\square\big(2^n\otimes(\xi_y\circ\mathsf{cs})\big) = (\mathrm{N}(\xi_y\circ\mathsf{cs})\circ\mathrm{EZ})^{\otimes r}\circ\Gamma^\square$$

$$= \mathrm{N}(\xi_y)^{\otimes r}\circ\mathrm{CS}^{\otimes r}\circ\Gamma^\square$$

$$= \mathrm{N}(\xi_y)^{\otimes r}\circ\Gamma^\triangle\circ\mathrm{CS}$$

where we used that $\mathrm{CS}^{\otimes r}\circ\Gamma^\square=\Gamma^\triangle\circ\mathrm{CS}$ as proven in Theorem 6. $\qquad\square$

Corollary 16. *For any cubical set X*

$$\mathrm{N}^\square(X)\xrightarrow{\mathrm{N}^\square(\xi_X)}\mathrm{N}^\square(\mathcal{U}\,\mathcal{T}\,X)\xleftarrow{\mathrm{CS}_{\mathcal{T}X}}\mathrm{N}^\triangle(\mathcal{T}\,X),$$

where ξ is the unit of adjunction, is a natural zig-zag of quasi-isomorphisms of $\mathrm{U}(\mathcal{M}_{sh})$-coalgebras which extend respectively the Serre and Alexander–Whitney coalgebra structures.

Proof. The map $\mathrm{CS}_{\mathcal{T}X}$ is a quasi-isomorphism of $\mathrm{U}(\mathcal{M}_{sh})$-coalgebras by Theorem 15, whereas $\mathrm{N}^\square(\xi_X)$ is also one since it is induced from a cubical map that is a weak-equivalence. $\qquad\square$

Corollary 17. *The singular simplicial and cubical chains of a topological space $\mathfrak{Z}$ are quasi-isomorphic as $\mathrm{U}(\mathcal{M}_{sh})$-coalgebras which extend respectively the Alexander–Whitney and Serre coalgebra structures. More specifically, the map*

$$\mathrm{CS}_{\mathrm{S}(\mathfrak{Z})}\colon \mathrm{S}^\triangle(\mathfrak{Z}) \to \mathrm{S}^\square(\mathfrak{Z})$$

is a quasi-isomorphism of $\mathrm{U}(\mathcal{M}_{sh})$-coalgebras.

Proof. It can be verified using that $\mathfrak{cs} = |\mathsf{cs}| \circ \mathfrak{e}_{\mathfrak{Z}}$ that this map factors as

$$\mathrm{CS}_{\mathrm{S}(\mathfrak{Z})}\colon \mathrm{S}^\triangle(\mathfrak{Z}) \xrightarrow{\mathrm{CS}_{\mathrm{Sing}^\triangle(\mathfrak{Z})}} \mathrm{N}^\square\big(\mathcal{U}\,\mathrm{Sing}^\triangle(\mathfrak{Z})\big) \xrightarrow{\mathrm{EZ}_{\mathrm{S}(\mathfrak{Z})}} \mathrm{S}^\square(\mathfrak{Z})$$

where the first map was proven in Theorem 15 to be a quasi-isomorphism of $\mathrm{U}(\mathcal{M}_{sh})$-coalgebras, and the second, introduced in § 5.2, is also one since it is induced from a cubical map whose geometric realization is a homeomorphism. $\qquad\square$

6. Future work

In the fifties, Adams introduced in [Ada56] a comparison map

$$\Omega\,\mathrm{S}^\triangle(\mathfrak{Z}, z) \to \mathrm{S}^\square(\Omega_z\mathfrak{Z})$$

from his cobar construction on the simplicial singular chains of a pointed space $(\mathfrak{Z}, z)$ to the cubical singular chains on its based loop space $\Omega_z\mathfrak{Z}$. This comparison map is a quasi-isomorphism of algebras, which was shown by Baues [Bau98] to be one of bialgebras by considering Serre's cubical coproduct. In [MR21] the E_∞-coalgebra structure defined here is used to generalize Baues' result, by showing that Adams' comparison map is a quasi-isomorphism of E_∞-bialgebras or, more precisely, of monoids in the category of $\mathrm{U}(\mathcal{M})$-coalgebras.

For a closed smooth manifold M, in [FMS21] a canonical vector field was used to compare multiplicatively two models of ordinary cohomology. On one hand, a cochain complex generated by manifolds with corners over M, with partially defined intersection; on the other, the cubical cochains of a cubulation of M with the Serre product. With the explicit description introduced here of an E_∞-structure on cubical cochains, we expect to build on this multiplicative comparison and enhance geometric cochains [FMS22] with compatible representations of further derived structure.

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