

NONCOMMUTATIVE ASPECTS OF OPEN/CLOSED STRINGS VIA FOLIATIONS

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(Received September 11, 2007 — Revised February 18, 2008)

We give a brief summary of algebraic aspects of string theory arising in the noncommutative geometry setting of foliations called string diagrammatics which we introduced jointly with Bob Penner. We furthermore discuss how this gives rise to actions on the Hochschild complex of a Frobenius algebra. We then explain how this leads to new quantum chains for loop spaces and a stabilization in the semi-simple case.

Keywords: foliations, noncommutative geometry, Hochschild cochains, moduli space actions, open/closed string theory, loop space, quantum cohomology, quantum deformations, string topology.

1. Introduction

Over the years there have been various mathematical incarnations of the operations of string theory. These come in many different flavors as we recall in Section 2. Here we will view moving strings as naturally defining surfaces with partially measured foliations. On this geometric/topological level the string interactions are described by operadic structures. These are the mathematical structures that describe gluing operations where both the surfaces and the foliations are glued. The result which is joint with Bob Penner is a combinatorially defined open/closed CFT with partial compactification [1].

This approach is very powerful as it gives the right phenomenology. First, on the degree zero level, that is restricting to connected components, one obtains a new proof of the axioms of open/closed TFT. Second, we obtain the correct BV formalism in the closed sector and interesting new behavior in the open/closed sector. Passing to the homology or operator level, we were able to give an action on the Hochschild co-chains of a Frobenius algebra which had been expected from string topology, D -branes and purely on mathematical grounds [2]. Finally, we give applications in the semi-simple case by defining a quantum loop space and considering stabilizations of the moduli spaces.

2. Moving strings as a noncommutative geometry of measured foliations

The basic picture we have in mind is that strings are either parameterized circles or intervals. As such they have a measure sitting on them and special points, namely 0 in the closed case or the endpoints of the interval in the open case. Now as the strings move, split and recombine they sweep out a surface. Depending on our point of view, we could stop here and we would be considering TFT. We could also include more data as discussed below. The second fundamental aspect of this *a posteriori* actually not-so-naïve picture is that we can let the string system run for a bit, then stop and then let it run again. The final result will be a surface which has been glued from the surface swept out during the first time period and the surface swept out during the second. This is what gives rise to the operadic structure.

In order to be more precise, we need to fix the geometry we are talking about. In the table below, we fix the data we are gluing—which will always be surfaces together with extra data—and specify the theory we are encoding mathematically. In order to simplify things we will for the moment restrict to the closed sector. This means there will be only one distinguished point per boundary component.

| Geometry | Data (roughly) | Theory |
|--|--|------------------------|
| Topological surfaces Σ w/ boundary $\partial\Sigma$ / Cobordisms thereof | $(\Sigma, \partial\Sigma)$ | TFT |
| Surface Σ w/ boundary $\partial\Sigma$ and conformal structure $[g]$ / “Segal operad/category”/ Open moduli space $M_{g,n}$ operad | $(\Sigma, \partial\Sigma, [g])$ | CFT |
| Complex curve C w/ marked points p_i nodes possible $C \in \bar{M}_{g,n}$ | (C, p_1, \dots, p_n) | CohFT GW invariants |
| Surface Σ w/ boundary $\partial\Sigma$ marked points $p_i \in \partial_i\Sigma$ and class of Foliations $[\alpha]$ | $(\Sigma, \partial\Sigma, p_i \in \partial_i\Sigma, [\alpha])$ | Combinatorial CFT |

We will be concerned with the last entry.

In order to do the open/closed version we will have to add more points on the boundary. These points correspond to the ends of open strings and are hence labelled by a set of *D*-brane labels chosen from a set of elementary labels \mathcal{B} . To give the gluing structure we will pass to the power set $\mathcal{P}(\mathcal{B})$ as the labelling set for the points. The label \emptyset will mean “closed string”. If a point has a collection

of labels, we think of this as the intersections of relevant branes. It is allowed that these can all be empty in a realization.

3. String diagrammatics on the space, chain and operator level

Adopting the point of view that the strings have measures, we see that the surface they sweep out actually has a foliation, which has a transversal measure, see Fig. 1. We can think of each point of the string as giving a leaf of this foliation. The strings themselves will also give a foliation transversal to the previous one. Each string has a measure and hence the first foliation has a transversal measure. Going on, we can “squeeze” the leaves together like a curtain to fit into bands on the surface. These bands will now end on an interval between two marked points called windows. As the bands also come with a transversal measure, they have a width. Thus the whole data can be thought of as combinatorially given by a graph on the surface—one edge for each band—whose edges are labelled by positive real numbers that indicate the width. We will enlarge the picture by allowing the graphs to degenerate. This means that the underlying ribbon graph can have a different topological type from the surface it lies on, see [1, 3] for details and examples. We call a foliation admissible if its graph as a graph on the surface has no parallel (homotopic) edges and none of the edges are parallel to a window, where in the homotopy the endpoints are not allowed to pass each other or the marked points. We also exclude the cases in which an edge is parallel to a part of the boundary including a point marked with \emptyset . Finally, we require that there is at least one edge.

3.1. Basic descriptions of the interactions

Simply looking at the string interaction, we stay on the level spaces of foliations of surfaces. Here the string interaction is captured by gluing the surfaces along the interacting strings/boundary components as depicted in Fig. 1. The point of view of [1] is that we only glue if the widths are the same. This corresponds to the dynamic picture mentioned above. The important thing to notice is that in the open gluing, the marked points become punctures, while in the closed gluing they are erased after the gluing. It is this fundamental difference which leads to the correct algebra of interactions.

We can go one step further and consider parameterized families of these operations. What this basically means is that we consider varying weights on the bands. Here the topology of the space is given by varying the weights and erasing a band as its width tends to zero. Just as the gluing space level operations above there are family gluing operations as well.

With a lot of work, [1] we could show that there are operations induced on the homology level as well. What this means is that we obtain vector spaces of operators. These are graded by their degree which is the number of parameters. These operators then define a version of open/closed CFT, see below. The lowest degree 0 corresponds to the connected components and hence to TFT. Using this description gives a new proof for minimal axioms of open/closed TFT [4].

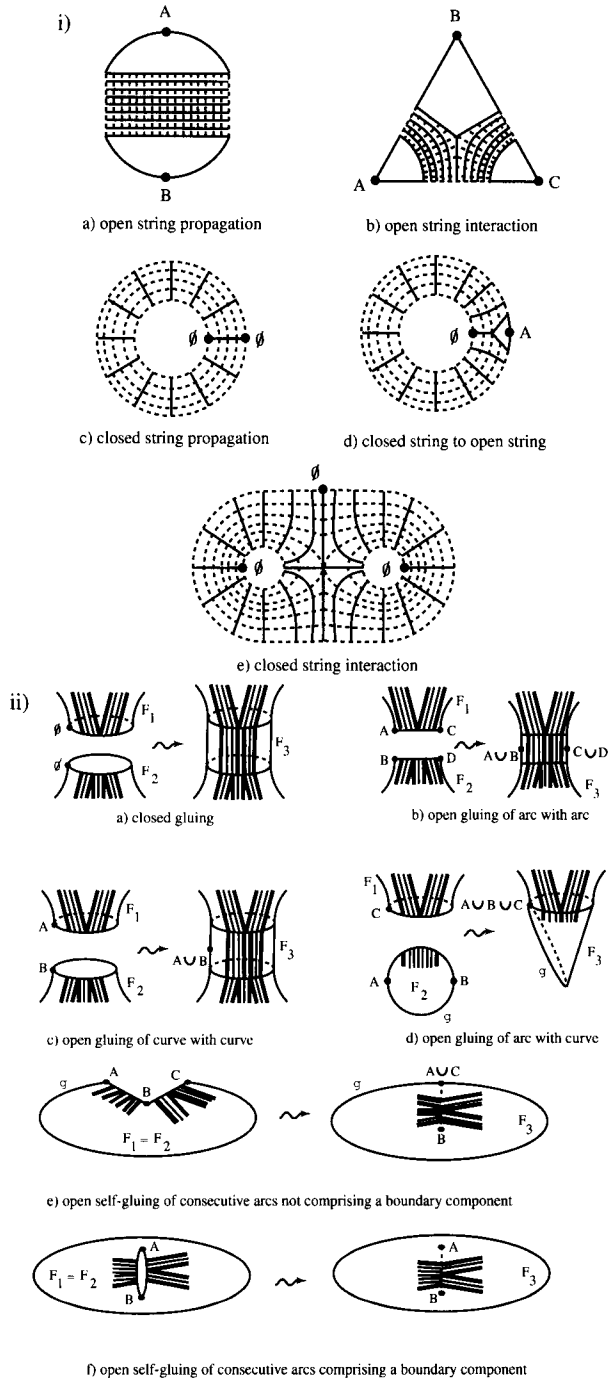


Fig. 1. Moving strings as foliations and glueings

3.2. A mathematical description of the structure

We will call $\widetilde{\text{Arc}}(n, m)$ the space of classes of admissible foliations under orientation-preserving homeomorphisms of brane labelled surfaces with n active closed window and m active open windows. Here we call a window “open” if both endpoints (which may coincide) are labelled by a nonempty D -brane label and a window “closed” if it is on a boundary component with one marked point that is marked by \emptyset . Being active means that the window is hit by leaves of the foliation.

THEOREM 1. *The spaces $\widetilde{\text{Arc}}(n, m)$ of surfaces with partially measured foliations form an \mathbb{R} graded C/O-structure.*

This is a new type of operad-like structure defined in [1]. Being a C/O structure roughly means that we can associatively glue a closed window to a closed one or an open window to an open one under the condition that the weights agree. Here self-gluing of surfaces are allowed, that is the two windows that are glued can either lie on two different surfaces or on the same surface. One of the really surprising results which requires hard work is:

THEOREM 2. *The operations on the space level induce the structure of a bi-modular bi-operad on the homology level.*

This is the gold standard for mathematical descriptions of CFT and we were not able to show this using the previous approach of [3] where we allowed more gluings on the space level using an overall projective scaling of the weights. This in turn however prevented us from defining self-gluing. Of course on the homology level the operations of [3] and [1] agree. Another important result is that we can even lift the operations above to the chain level, which is important for applications to string topology. This fact is actually needed in the proof of the Theorem above which consists of a careful analysis of flows of foliations on the respective surfaces. It should be remarked that on the homology level there is no condition on weights anymore.

3.3. Operators and relations

The homology is linear and hence we can look for (operadic) algebra representations of it. I.e. each element of homology defines an operator in such a way that the relations are preserved under gluing. This means roughly that gluing two families representing homology classes corresponds to concatenating the relevant operators. Notice that homology is graded and these degrees should be preserved. A degree k class can be represented by a k -dimensional family and this should correspond to an operator of degree k . The degree 0 classes correspond to points up to homotopy and are therefore represented by the connected components.

Disregarding the data of the parameterized families of graphs, each surface with marked points on the boundary and a foliation can be decomposed into bi-gons, triangles, annuli with one point on each of the boundaries, once punctured annuli of the same type, discs with two punctures and one marked point on the boundary

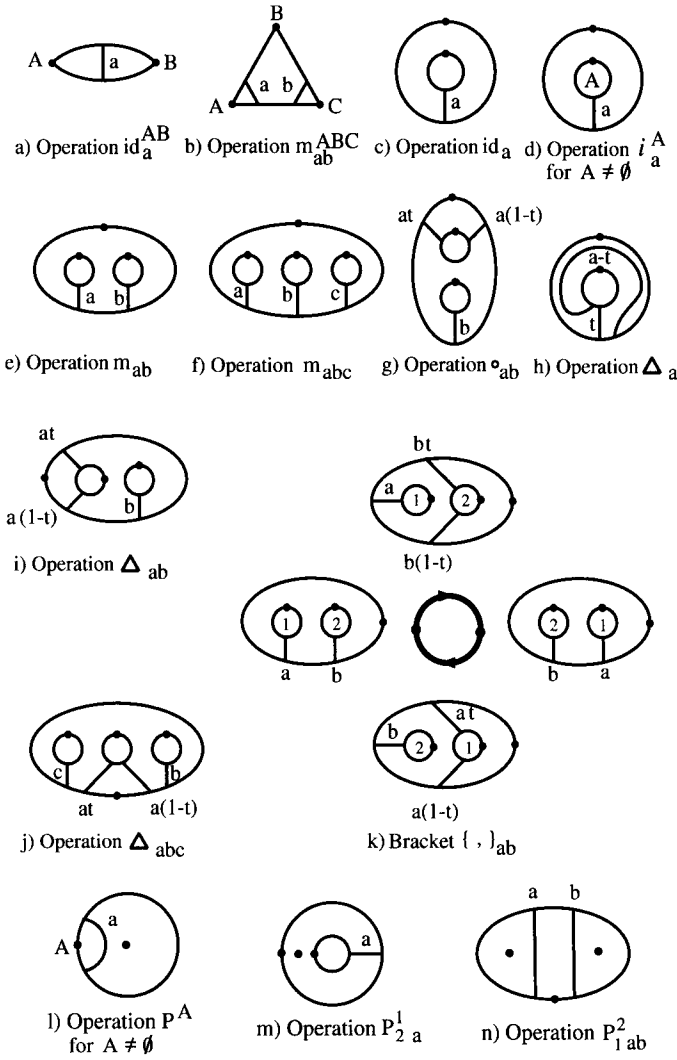


Fig. 2. Operations

and pairs of pants with one marked point on each boundary. The surfaces without punctures are exactly the surfaces of Fig. 2. Moreover the indecomposable families of degree 0 and 1 without punctures are exactly the ones depicted in Fig. 2.

Different decompositions of surfaces with families of foliations give rise to relations. Different pieces can glue together to form the same surface and therefore the composition of the respective operators should agree. In this way one obtains all the expected relations in degree 0. The relations of degree zero involving the open sector and the open/closed interaction are for example depicted in Fig. 3. For the

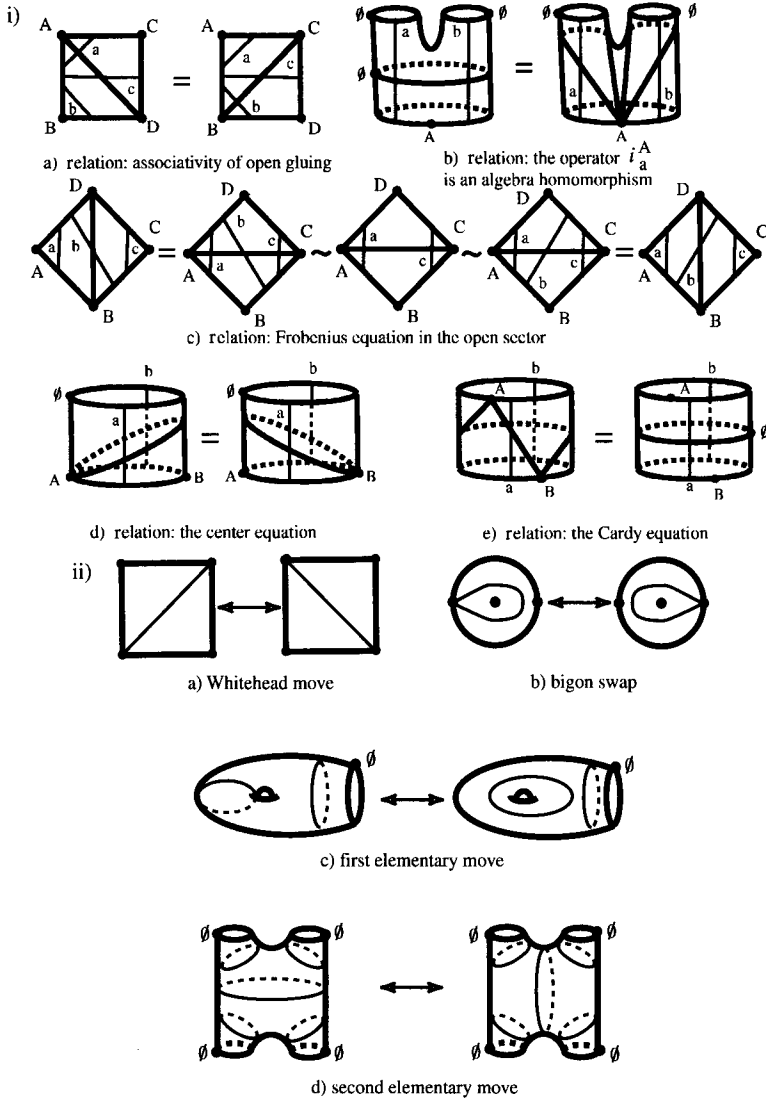


Fig. 3. Moves and relations

standard Frobenius relations on the closed sector and the relations of degree 1 which establish that Δ is a BV operator that is compatible with a natural Gerstenhaber bracket we refer the reader to [1].

THEOREM 3. *An algebra over the modular bi-operad $H_*(\coprod_{n,m} \widetilde{\text{Arc}}(n,m))$ is a pair of vector spaces (C, \mathcal{A}) with the following properties: C is a commutative Frobenius BV algebra (C, m, m^*, Δ) , and $\mathcal{A} = \bigoplus_{(A,B) \in \mathcal{P}(\mathcal{B}) \times \mathcal{P}(\mathcal{B})} \mathcal{A}_{AB}$ is a $\mathcal{P}(\mathcal{B})$ -*

colored Frobenius algebra (see e.g., [5] for the full list of axioms of such a structure). In particular, there are multiplications $m^{ABC} : \mathcal{A}_{AB} \otimes \mathcal{A}_{BC} \rightarrow \mathcal{A}_{AC}$ and a non-degenerate metric on \mathcal{A} which makes each \mathcal{A}_{AA} into a Frobenius algebra.

Furthermore, there are morphisms $i^A : C \rightarrow \mathcal{A}_{AA}$ which satisfy the following equations: letting i^* denote the dual of i , τ_{12} the morphism permuting two tensor factors, and letting A, B be arbitrary non-empty brane-labels, we have

$$i^B \circ i^{A*} = m_B \circ \tau_{12} \circ m_A^* \text{ (Cardy)}, \tag{1}$$

$$i^A(C) \text{ is central in } \mathcal{A}_A \text{ (center)}, \tag{2}$$

$$i^A \circ \Delta \circ i^{B*} = 0 \text{ (BV vanishing)}. \tag{3}$$

These constitute a spanning set of operators and a complete set of independent relations in degree zero. All operations of all degrees supported on indecomposable surfaces are generated by the degree zero operators and Δ .

Here the intriguing equation is the BV vanishing. This is a new feature, which has to do with the partial compactification.

Using the results of transitivity of the four moves on decompositions of constant foliations, see Fig. 3, and the theorem above we also obtain the restriction to an open/closed TFT:

THEOREM 4. *An algebra over the degree 0 part of the operad, that is an open/closed TFT is precisely given by the data (C, \mathcal{A}) which have the following properties: C is a commutative Frobenius algebra, and $\mathcal{A} = \bigoplus_{(A,B) \in \mathcal{P}(\mathcal{B}) \times \mathcal{P}(\mathcal{B})} \mathcal{A}_{AB}$ is a $\mathcal{P}(\mathcal{B})$ -colored Frobenius algebra which satisfies the Cardy and Center equation.*

Hence, as the degree 0 part, we recover the Cardy/Lewellen axiomatic picture ([4], see also [5]) from the point of view of strings yielding foliations. This includes the non-commutative Frobenius algebras of the open string sector and the easy description of the Cardy equation of Fig. 2.

3.4. Moduli space/CFT

In this section, we will restrict ourselves to the closed string sector. As we stated above, we augmented the admissible foliations, by allowing the graphs to be degenerate—that is of a “smaller” topological type than the surface. To be precise a graph of a foliation is called quasi-filling if the complementary regions are either polygons or once punctured polygons. Let $ARC_{\#}(g, n, s)$ denote the space of quasi-filling graphs of closed sector surfaces of genus g with s punctures and n marked points, that is surfaces whose boundaries each only have one marked point labelled by \emptyset .

THEOREM 5 ([2, 6]). *$ARC_{\#}(g, n, s)$ is homotopy equivalent to the decorated moduli space of surfaces of genus g with s punctures and n marked points. In particular, if there are no punctures $ARC_{\#}(g, n, 0) \cong M_{g,n}^{1^n}$ that is it is isomorphic to the moduli space of genus g curves with n punctures and one tangent vector at each puncture.*

In other words, we recover moduli space and hence restricting the operad structure to this subspace we obtain a combinatorial version of CFT. This is however a bit subtle, since on the topological level the statement is true only *cum grano salis*. The mathematically correct statement is as follows.

THEOREM 6. *Using the gluing of [3] the subspaces $ARC_{\#}(g, n, 0)$ form a rational (i.e. densely defined) cyclic operad which induces a cyclic operad structure on the relative chain complex of open cells indexed by marked ribbon graphs (see [2] for full details).*

4. Actions on Hochschild

4.1. Motivation

There are three sources of motivation to look for actions of a chain model of moduli space. Perhaps the most intriguing come from using the logic of Kontsevich–Kapustin–Rozansky [7], which we can rephrase as follows. If the closed string states are thought of as deformations of the open string states and the open string states are represented by a category of D -branes, then the closed strings should be elements of the Hochschild co-chains of the endomorphism algebra of this category. Now thinking on the worldsheet, we can insert closed string states. That is, for a world sheet, we should get a correlator by inserting, say n closed string states. This is what we will have done, if one simplifies to a space filling D -brane and twists to a TCFT.

The second motivation is from string topology [8], where surfaces should act on the homology of the loop space of a manifold. Now it is well established that if the manifold is simply connected, then the homology of the loop space is calculated by the Hochschild complex of co-chains of the manifold. Going one step in the spectral sequence and supposing that the manifold is compact and hence has Poincaré duality, we again expect a cell level action of moduli space.

Lastly just looking at Deligne’s conjecture and its generalizations (see [2] for a full list of references and proofs thereof) we are motivated to include all surfaces.

4.2. Results on actions

As expected there is indeed such an action on the Hochschild co-chains, which can be understood as an action of the discretization of the foliations.

THEOREM 7. *There is an action on the Hochschild co-chains of a Frobenius algebra by the relevant chain complex of ribbon graphs which calculates the co-homology of moduli space. Restricting to a particular subpart and partially compactifying, we obtain an action which is the one of string topology of Chas and Sullivan (possibly up to lower order terms) [2].*

4.3. Some details on the action relevant to the further discussion

Fix a commutative unital Frobenius algebra A with multiplication μ and pairing \langle , \rangle . Set $\int a := \langle a, 1 \rangle$ and let e be the Euler element of A that is $e = \mu\Delta(1)$

where Δ is the adjoint of μ . Using dualization, we need to define correlators: $\langle \phi_1, \dots, \phi_n \rangle_{\Sigma_{g,n}, \Gamma}$ for any cell given by a surface $\Sigma_{g,n}$ with boundary and marked points as before and a ribbon graph Γ representing a foliation $[\alpha]$ with varying weights. Here we think of $\phi_i \in TA \simeq C^*(A, A)$ where TA is the tensor algebra. The action is now roughly given as follows (full details are given in [2]): (1) Duplicate edges so that the number of incoming edges at the vertex $i = \text{deg}(\phi_i)$. We sum over all possibilities to do this, if this is not possible then the operation is zero. (2) Assume the ϕ_i are pure tensors. Pull apart the edges and decorate the pieces of the boundary with the elements of ϕ . Cut along all the edges of the graph and call the set of disjoint pieces of surface P . Let $I(p)$ be the index set of the components a_j of the ϕ_i decorating edges belonging to a piece $p \in P$ and let $\chi(p)$ be the Euler characteristic of the surface p . Notice that the pieces p which possibly have non-trivial topology are a subset of the pieces of the surfaces, that are obtained by cutting along the original edges before duplication. (3) Set $\langle \phi_1, \dots, \phi_n \rangle_{\Sigma_{g,n}, \Gamma} := \prod_{p \in P} \langle \phi_1, \dots, \phi_n \rangle_p$ where

$$\langle \phi_1, \dots, \phi_n \rangle_p = \int \prod_{i \in I(p)} a_i e^{-\chi(p)+1} \tag{4}$$

5. Stabilization, the semi-simple case and a quantum loop space

We wish to point out that the topology of the surface pieces p enters only through the factor $e^{-\chi(p)+1}$. This factor is invertible for $\chi(p) \neq 1$ precisely if the algebra A is semi-simple. Moreover if A is semi-imple with idempotents e_i , set $\lambda_i = \int e_i$ then $e = 1$ if all $\lambda_i = 1$. We will call such an algebra a normalized semi-simple Frobenius algebra. All semi-simple Frobenius algebras can be obtained from a normalized one by scaling the metric. Moreover, any semi-simple finite dimensional algebra can be endowed with a metric that makes it into a normalized semi-simple Frobenius algebra.

For such an algebra the action of a cell (Σ, Γ) equals to the action of the surface where the pieces p are replaced by discs. We call this operation stabilization. Here the boundary pieces of p are cut and glued to an S^1 in a fixed fashion, which keeps the relative order of the pieces intact.

THEOREM 8. For a normalized semi-simple Frobenius algebra the action factors through the stabilization. Moreover the stabilized surfaces form an operad and contain an E_∞ suboperad. Hence the Hochschild co-chains of a normalized semi-simple Frobenius algebra have the structure of an E_∞ algebra.

The first part of this is immediate. For the second we construct the stabilization via a colimit whose maps actually add topology to the pieces p and explicitly give the E_∞ operations as the geometrization of the operations of [9]. If A is not semi-simple the action does not pass through the stabilization. We can however flow the metric to a normalized one which has the effect of changing the co-multiplication.

Also, in the semi-simple case the operations are shifted from the stabilized ones by invertible elements.

5.1. Quantum chains on the Loop space and the quantum string bracket

It is well known that there is a cyclic model for the chains of loop space of a simply connected manifold M given by $C^*(S^*(M), S_*(M))$. Calculating the homology we are led to the E_1 term of the spectral sequence which is isomorphic to $C^*(A, A)$, with $A = H^*(M)$. Now say M is a smooth projective variety, we can quantum deform A to $A_q := H_q^*(M)$, where H_q is the quantum cohomology at the point q . This gives us a model for *quantum chains on the loop space*: $C^*(A_q, A_q)$. As such we have the operations above and in particular the string bracket induced by the Chas–Sullivan PROP.

Now if A_q is semi-simple—this is expected generically if M is a Fano variety that has a system of exceptional sheaves of appropriate length, e.g. \mathbb{P}^n —then we can actually use gravitational descendants to flow to a normalized semi-simple Frobenius algebra [10, 11]

THEOREM 9. *Given a semi-simple point, we can flow to a normalized point (essentially by coupling to gravity) at which the quantum chains on the loop space are an E_∞ algebra. In this situation the quantum deformed string topology bracket vanishes.*

5.2. Connections to family field theories and Mumford–Morita–Miller classes

The correlators on $V = TA \simeq C^*(A, A)$ for a Frobenius algebra define chain level family field theories with partition functions $Z_{g,n} \in \text{Hom}(V^{\otimes n}, \text{Hom}(C^*(M_{g,n}^{1^n}), k))$. One result is that when A is semi-simple, the operations on the degenerate higher genus Penner-boundary actually come from lower genus moduli space via an invertible morphism. We expect to show that we can identify our stabilization with that of Tilmann, Segal and Madsen [12] and prove that the partition function is actually fixed by its genus zero contribution. For this we can pull back co-chains along our stabilization maps of the colimit which induces push-forward on the Hom-duals. This means that we can effectively push the operations to infinite genus. If we have an identification with the usual stabilization we can apply the Mumford conjecture proved in [14]. The upshot will be a classification result for these theories in terms of κ classes along the lines of [10, 11].

In order to compare to GW-invariants, we need to “go to the boundary” of moduli space in the Deligne–Mumford sense. One approach is using frames of graphs à la Fulton–MacPherson (see also [13]) another is to cut the surface along a curve that does not intersect any arcs and then contracting these curves to points. This essentially yields Kontsevich’s compactification. One indication that this approach is natural is given by the observations in Appendix B of [1]. The next step is then to lift the $Z_{g,n}$ from the Kontsevich compactification to the DM compactification. Going beyond this we wish to point out that the underlying algebra TA_q is the algebra governing the quantum cohomology of the symmetric products [15]. These

in turn define the genus g GW invariants as discussed in [16]. This would provide another approach to Teleman's classification result [11] and give further insight to Givental's "loop space approach" [17].

6. Conclusion and outlook

We have discussed how to regard moving strings as providing surfaces with foliations. Using this approach, we have recovered the axiomatics of open/closed TFT, given a new combinatorial CFT and a partial compactification of it which relates to/defines string topology operations and lends itself to give actions on the Hochschild complex of a Frobenius algebra. We also obtained a partial list of axioms for the CFT and partially compactified CFT. In the future we hope that we can get a full axiomatic description.

Furthermore, we were able to give a chain level action of the relevant structures generalizing Deligne's conjectures. Along the same lines of reasoning it should be possible to get open/closed operations on the Hochschild complex of a category or a pair of algebras in the space filling D -brane situation, say in a Landau–Ginzburg model.

Interesting future directions include the study of mapping spaces to varieties/orbifolds. An even more ambitious project is to try to connect our constructions to the Chiral deRham complex or other bundle theories.

Acknowledgements

It is a pleasure to thank our collaborator Bob Penner. We also thank the organizers of the ISQS XVI for providing this wonderful forum.

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