Projective representations from quantum enhanced graph symmetries

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Abstract. We define re–gaugings and enhanced symmetries for graphs with group labels on their edges. These give rise to interesting projective representations of subgroups of the automorphism groups of the graphs. We furthermore embed this construction into several higher levels of generalization using category theory and show that they are natural in that language. These include projective representations of the re–gauging groupoid and a novel generalization to all symmetries of the graph.

Introduction

In [4], we developed a method of re–gaugings and actions of enhanced graphs symmetries for labelled graphs. The upshot were projective representations that are of interest in condensed matter physics. In those applications there is an underlying geometry at work, but the method itself is more general and can be generalized or reduced to a combinatorial group theoretic framework, which we will present. Presently, we will label the edges of a graph by elements of a group G and we give a presentation of the actions that is precise and concise. The precision is needed, since there are several actions (both left and right) at work which need to be disentangled. We present a new result on the action of general symmetries.

We will also recall the particular projective representations we found in [4] since their occurrence "in nature" as natural symmetries might be of interest to group theorists as well as physicists and give an example of the new result on the action of general symmetries. Lastly, we give a new presentation of our constructions in the language of categories and show that they become very natural.

This paper is at the same time more general and more specific than [4]. In *loc. cit.* the labels were invertible elements of a C^*–algebra, here they live in a general group. The geometry of [4] is then recovered by specializing the group to invertible functions on tori, or more generally invertible elements in a not necessarily commutative C^* algebra. We also give a more technically precise account of the actions. On the other hand, [4] deals with the possibility of an actual groupoid representation, that is invertible morphisms between different vector spaces, while here, we work in the situation where there is only one underlying vector space. The categorical interpretation is entirely new.

1. Combinatorics and graphs

In this section, we give the details of the following construction of representations: Fix a connected graph Γ with k vertices and a labelling lab of its directed edges by group elements of a fixed group G such that edges in opposite orientation are labelled by inverse group elements. We define a group of quantum automorphisms $Aut_q(\Gamma, lab)$ of such a Γ–labelled graph. This is a subgroup of the automorphisms of the graph, which preserves the labels up to re-gauging, defined below. We show that after fixing an ordered rooted spanning tree (ORST) τ of the graph, there is a natural way to attach a $k \times k$ matrix with coefficients in G to each quantum automorphism. In the case that G is Abelian these matrices give a projective representation of the group of quantum automorphisms of (Γ, lab), into the group of $k \times k$ matrix with coefficients in G. The co–cycle is explicitly given and defines a group extension $\widehat{Aut_q}(\Gamma, lab)$. Thus applying any representation to G, we get a projective representation of $Aut_q(\Gamma, lab)$ and a representation of $\widehat{Aut_q}(\Gamma, lab)$. A different choice of ORST gives rise to a projective representation.

1.1. Graphs, paths and spanning trees

In this paragraph, we fix the notations we will be using. This is necessary to be able to be precise later on. A graph is a collection of vertices V, flags or half–edges F, a boundary map $\partial : F \to V$ which attaches to each half–edge its vertex and a fixed point free involution $\imath : F \to F$ on F. An edge is then a pair of half–edges $\{f_1, f_2\}$ constituting an orbit of $\imath$: $f_1 = \imath(f_2)$. We denote the set of edges by E. Then ∂ associates to each edge the set of its endpoints. An orientation of an edge is the choice of ordering of its two flags. Each edge has two possible orientations, which we call opposite. For an oriented edge $\vec{e} = (f_1, f_2)$ we set $s(e) = \partial(f_1)$ and $t(e) = \partial(f_2)$ and call them source and target. Notice that for small loops, whose flags are incident to the same vertex, both orientations have the same sources and targets. This is why we chose to use the more elaborate way to present graphs above. We can also identify an orientation of an edge by the choice of the first half edge. In this way the set F is naturally the set of all oriented edges. The bijection is given by $f \leftrightarrow (f, \imath(f))$. Using this bijection, we define $s(f) = f$ and $t(f) = \imath(f)$ and from now on think of flags as oriented edges. In this notation $\imath$ flips the orientation if $f \leftrightarrow (f, \imath(f)) = \vec{e}$ then $\imath(f) \leftrightarrow (\imath(f), f) =: \imath(\vec{e}) =: \overleftarrow{e}$.

1.1.1. Spanning trees, orders and action of the permutation groups

A spanning tree of a graph is a subgraph that is a tree (i.e. its realization is contractible) whose vertices are all the vertices of the tree. A rooted spanning tree (RST) is such a spanning tree together with the choice of a root. The edges in a rooted tree have a natural orientation by directing the edges *away* from the root; see Figure 1. In any rooted tree τ, any vertex v has a unique shortest path to and from the root vertex v_{rt}, which we will call $\gamma^\tau_{v_{rt}v}$ and $\gamma^\tau_{vv_{rt}} = (\gamma^\tau_{v_{rt}v})^{-1}$.

An order on a graph is a bijection $ord : V \to \{1, \ldots, |V|\}$. A compatible order for a rooted spanning tree has to have $ord(v_{rt}) = 1$. Denote $v = ord^{-1}$ and write $v(i) = v_i$ then $ord(v_i) = i$, and for a rooted spanning tree with a compatible order $v_{rt} = v_1$[1]. An ordered rooted spanning tree (ORST) is a spanning tree together with a compatible order. Notice that an order for a spanning tree gives an order for the graph. The permutation group $\mathbb{S}_k$, $k = |V|$, naturally acts on orders and inverse orders via $\sigma(v) = v \circ \sigma^{-1}$. Setting $v' = \sigma(v)$ this means that $v'_i = v_{\sigma^{-1}(i)}$. Furthermore, $\sigma \in \mathbb{S}_n$ induces a bijection $\sigma_V : V \to V$ on the set of vertices via

$$\sigma_V \circ v = v \circ \sigma^{-1} \tag{1.1}$$

that is $\sigma_V = v \circ \sigma^{-1} \circ ord$. In other words, $\sigma_V(v_i) = v'_i = v_{\sigma^{-1}(i)}$. And vice–versa given $\Sigma_V : V \to V$ a bijection, it determines an element σ of $\mathbb{S}_k$ via $\sigma = ord \circ \sigma_V^{-1} \circ v$.

[1] In [4] we used v_0 for the root, which would mean that $v_0 = v_1$.

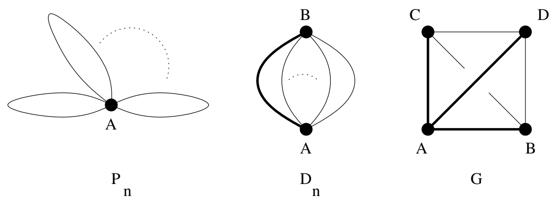

Figure 1. Graphs with spanning trees. The root is A. The petal graphs P_n, the graphs D_n and the graph G

Remark 1.1. Given any order on Γ, one can obtain a map $Aut(V) \to \mathbb{S}_k$. The way the action is set up is that composition in $\mathbb{S}_k$ corresponds to composition of σ_V. In particular, if $v' = \sigma(v)$ and $v'' = \sigma'(v')$ then $v''_i = v'_{(\sigma')^{-1}(i)} = v_{\sigma^{-1}(\sigma')^{-1}(i)} = v_{(\sigma'\sigma)^{-1}}(i)$. Notice that the order used at each step is the induced order.

1.1.2. π_1 of a graph An (edge)-path on a graph is a sequence of oriented edges, $\vec{e}_i$: $i = 1 \dots n$ together with an orientation of each e_i such that for any two consecutive $\vec{e}_i, \vec{e}_{i+1}$: $t(\vec{e}_i) = s(\vec{e}_{i+1})$. An edge path is called reduced, if no two consecutive oriented edges are opposites of each other. Concatenation of paths and inverting a path makes the set of all reduced paths into the so–called path groupoid $\pi_1(\Gamma)$. If in the concatenation of reduced paths there is a pair of adjacent opposite edges, they are simply deleted in the product of π_1. The reduced loops at a fixed vertex $\pi_1(\Gamma, v)$ form a group, the fundamental group at v.[2]

Contracting all of the edges of a spanning tree of Γ leaves a graph with one vertex whose realization is a bouquet of n S^1s, we call such a graph a petal graph P_n. It is well known that $\pi_1(\Gamma) \simeq \pi_1(P_n) = \mathbb{F}_n$, the free group in n generators. Each spanning tree τ gives a different isomorphism. The inverse map is given by associating the following path to a small loop l around a petal of P_n. Let $\vec{e}$ be the unique oriented edge of the preimage of l that is not in the spanning tree and let $s(e) = v, t(e) = w$. Then the inverse image of l is the path $\gamma^\tau_{v_{rt}w} \circ \vec{e} \circ \gamma^\tau_{vv_{rt}}$. Here and below, we use the concatenation of paths $\circ$ in the order of functions, that is $p \circ q$ is: first go through q then through p.

1.2. Labelled graphs, gauging and spanning tree gauge.
A G–labelling for a graph Γ is a morphism $lab : F \to G$ such that $lab(\imath(f)) = lab(f)^{-1}$. Or identifying flags with oriented edges in the notation of §1.1: $lab(\vec{e}) = lab(\overleftarrow{e})^{-1}$. A Γ–labelling lab extends to paths in the following way: let $p = (\vec{e}_1, \dots, \vec{e}_n)$ be a path in Γ, then we define $lab(p) := lab(e_n) \dots lab(e_1)$. Since going back and forth along an edge yields the identity in G, this descends to $lab : \pi(\Gamma) \to G$. A G–labelled graph is a pair of a graph and a G–labelling.

A G–labelled graph with a RST τ is in spanning tree gauge, if $lab^\phi(\vec{e}) = 1$ for all oriented edges/flags $\vec{e}$ in τ. Two labellings lab and lab' are called gauge–equivalent if they differ by a gauging ϕ, i.e. $\exists \phi : lab' = lab^\phi$.

Remark 1.2. Of course by composing a G–labelleing by a group homomorphism $G \to H$ one obtains an H–labelleing and all constructions below will push–forward. The most general

[2] The data of vertices and edges and ∂ define a 1–dimensional CW complex. If Γ is a graph we denote by $|\Gamma|$ its realization, given by gluing together intervals (one for each edges) at their end points if these correspond to shared vertices. An edge path gives an actual path on $|\Gamma|$ starting and ending at a vertex. A path is reduced if the realization does not go back and forth through an edge. It is a fact that $\pi(|G|) = \pi(\Gamma)$ and $\pi_1(|G|, v) = \pi_1(\Gamma, v)$ are the topological fundamental groupoid and group.

labelling one can have in a spanning tree gauge is to label the edges by the elements of a free group. This will then be the free group on $1-\chi(\Gamma)$ generators, where $\chi(\Gamma)$ is the Euler characteristic of Γ. Another useful example for physics is when $G = U(1)$. This is of course also the universal receptacle of characters.

Definition 1.3. A gauge element ϕ is a map $\phi : V \to G$. The gauging of a G-labelling lab by a gauge element ϕ is defined to be the labelling lab^ϕ given by $lab^\phi(\vec{e}) = \phi(t(\vec{e}))lab(\vec{e})\phi(s(\vec{e}))^{-1}$, for any oriented edge $\vec{e}$. The gauge elements form a group, via point–wise multiplication, the gauge group of Γ. It is isomorphic to G^V and after the choice of an order to G^k where $k = |V|$.

Lemma 1.4. *Given a G–labelled graph (Γ, lab) and a rooted spanning tree (τ, v_{rt}), then there is a gauge element ϕ such that (Γ, lab^ϕ) is in spanning tree gauge w.r.t. τ. Explicitly, $\phi(v) = lab(p)$, where p is the shortest edge path from v to v_{rt} along τ, will be such an element and it is the unique such element satisfying he condition $\phi(v_{rt}) = 1 \in G$.*

Proof. If one sets $\phi(v'_{rt}) = 1 \in G$ then one can iteratively solve for ϕ of all the other vertices using the distance from the root as determined by the RST. The result is the above explicit gauge element. □

To make all dependencies clear, we will sometimes write $\phi = \phi^{lab}(\to \tau)$. Given lab, we will write $lab_\tau := lab^{\phi(\to\tau)}$ and write $\phi^{lab_\tau}(\tau \to \tau')$ for the re–gauging of lab_τ into spanning tree gauge w.r.t. τ'.

There is some ambiguity in the choice of ϕ above. One parameter is the initial condition $\phi(v'_{rt}) = 1 \in G$.

Lemma 1.5. *Given any lab in spanning tree gauge, its stabilizer subgroup under gauging will be the* constant functions *$\phi : V \to Z(H)$, where H is the subgroup of G generated by all the $lab(\vec{e})$ and $Z(H)$ is its centralizer in G. If G is Abelian this subgroup is isomorphic to G in the diagonal embedding into G^V and the different choices of ϕ of Lemma 1.4 are distinguished by their value on the root $\phi(v_{rt})$. For any lab its stabilizer subgroup is conjugate (inside G^V) to such a subgroup.*

Proof. Fix $\phi(v_{rt}) = g$ then along any edge of the spanning tree the label is 1, so that invariance of lab means that the function ϕ must be constant $\phi(v) \equiv g$. Furthermore for lab to be invariant, we need that for any non–spanning tree edge $\vec{e}$: $g\, lab(\vec{e})g^{-1} = lab(\vec{e})$, so that indeed $g \in Z(H)$. The other statements follow from this. □

Corollary 1.6. *If the image of lab generates G then $G^V/Z(G)$ acts as the effective gauge group. If G is Abelian, then for any lab its stabilizer group is G and the quotient G^V/G acts effectively.* □

Corollary 1.7. *If lab and lab′ are gauge equivalent, and lab is in spanning tree gauge w.r.t. the RST τ and lab′ is in spanning tree gauge w.r.t. τ', then there is a gauge element ϕ of the form given in Lemma 1.4 such that $lab' = lab^{\phi\circ\psi}$, where ψ is a constant regauging that is unique up to $Z(H)$. In particular if G is Abelian we can choose $\psi = id$.*

Proof. The assumption is that there exists a ϕ, such that $lab' = lab^\phi$. Any other gauging element ϕ' satisfying $lab \circ \sigma_E^{-1} = lab^{\phi'}$ would differ by an element in the stabilizer of lab which by the above lemma is a constant function to G. So, by using such a function, we can (a) normalize the given ϕ to the one of Lemma 1.4 and (b) obtain any other ϕ that solves the problem from this one via elements in the stabilizer of lab. □

Corollary 1.8. *Up to a constant regauging ψ, $lab_\tau^{\phi^{lab_\tau}(\tau\to\tau')} = lab_{\tau'}^\psi$ and if G is Abelian $lab_\tau^{\phi^{lab_\tau}(\tau\to\tau')} = lab_{\tau'}$.*

Proof. Obviously the two labellings are gauge equivalent, so since they are both in spanning tree gauge w.r.t. τ' they agree up to a constant regauging, which acts trivially in the Abelian case. □

1.2.1. Re–gauging matrices For any triple (lab, τ, τ'), with τ, τ' ORSTs and lab in spanning tree gauge with respect to τ, let $\sigma \in \mathbb{S}_k$ be the permutation, such that $ord' = \sigma \circ ord$ or, in other words, $v'_i = v_{\sigma^{-1}(i)}$. Let $\phi = \phi^{lab}(\tau \to \tau')$ be the gauge element of Lemma 1.4 that re–gauges lab for spanning tree gauge τ to spanning tree gauge τ' and set $\Phi = diag(\phi(v'_1), \dots, \phi(v'_k))$. Then define

$$M_{lab_\tau}(\tau \to \tau') := \Phi P_\sigma \tag{1.2}$$

here we added the subscript τ merely as a reminder that lab is in spanning tree gauge w.r.t. τ.

1.2.2. Cocycle and re–gauging groupoid action For any ordered pair of RST, (τ, τ') with roots v_{rt} and v'_{rt}, we define $p(\tau, \tau') := \gamma^{\tau}_{v'_{rt} v_{rt}}$ to be the shortest path from v_{rt} to v'_{rt} in τ' and for any triple of RSTs (τ, τ', τ'') with roots $v_{rt}, v'_{rt}, v''_{rt}$ set:

$$l(\tau, \tau', \tau'') := p^{-1}(\tau, \tau') p(\tau', \tau'') p(\tau, \tau'') = \gamma^{\tau}_{v_{rt} v'_{rt}} \gamma^{\tau'}_{v'_{rt} v''_{rt}} \gamma^{\tau}_{v''_{rt} v_{rt}} \tag{1.3}$$

which is a loop at v_{rt}.

Proposition 1.9. *[4] The matrices above satisfy*

$$M_{lab_{\tau'}}(\tau' \to \tau'') M_{lab_\tau}(\tau \to \tau') = C(\tau, \tau', \tau'') M_{lab_\tau}(\tau \to \tau') \tag{1.4}$$

where $C(\tau, \tau', \tau'') = lab_\tau(l(\tau, \tau', \tau''))$.

1.3. Quantum enhanced symmetries for labelled graphs
A symmetry σ_Γ of a graph Γ as above is a pair (σ_V, σ_F) of a bijection of the set of vertices $\sigma_V : V \to V$ and a bijection of set of flags σ_F which is compatible with ∂, i.e. $\partial(\sigma_F(f)) = \sigma_F(\partial(f))$. These symmetries from the group $Aut(\Gamma)$. The map on flags induces a map on directed edges which we denote by σ_E.

Any element σ_Γ of $Aut(\Gamma)$ pushes forward RSTs. If τ is an RST, then $\tau' = \sigma_V(\tau)$, the image of τ, is again an RST, with root $\sigma_V(v_{rt})$. Symmetries of the graph act on a G–labelling as follows: $\sigma_\Gamma(lab) = lab \circ \sigma_E^{-1}$. On orders they act according to $\sigma_\Gamma(ord) = ord \circ \sigma_V^{-1}$, so that $ord'(v_i) = \sigma(i)$, and $v'_i = v_{\sigma^{-1}(i)}$ give the induced order on τ'.

Remark 1.10. Given an order ord there is a map $Aut(\Gamma) \to \mathbb{S}_k$. The composition of automorphisms on vertices corresponds to the composition of elements of symmetric group as in Remark 1.1. An element of $\mathbb{S}_k$ is liftable if it is in the image of this map. A symmetric graph is a graph where this map is bijective.

A classical symmetry for a labelled graph (Γ, lab) is a symmetry of the graph that also satisfies $\sigma_\Gamma(lab) = lab \circ \sigma_E = lab$. These from the group $Aut(\Gamma, lab)$. If the labelling is constant this is the symmetry group of the graph. In general this will be a subgroup of it.

Definition 1.11. A *quantum enhanced symmetry* of a labelled graph is a symmetry σ_Γ of the underlying graph, such that there is a gauge group element ϕ for which $(lab \circ \sigma_E)^\phi = lab$. If lab is in spanning tree gauge w.r.t. τ, then a quantum enhanced symmetry is strict if the ϕ can be chosen to be $\phi^{lab\circ\sigma}(\sigma^{-1}(\tau) \to \tau)$.[3]

[3] Notice we switched to a right action since spanning trees will push–forward and we will get a representation of these groups in this fashion.

The quantum enhanced symmetry group of the labelled graph will be a subgroup $Aut_q(\Gamma, lab) \subset Aut(\Gamma)$ of the symmetry group of the underlying graph which contains the classical symmetry group of the labelled graph $Aut_q(\Gamma, lab) \supset Aut(\Gamma, lab)$. It usually is strictly bigger, see §3.[4] For a lab in spanning tree gauge w.r.t. the strict quantum enhanced symmetry group $Aut^0_q(\Gamma, lab)$ is a subgroup of $Aut_q(\Gamma, lab)$. Notice that if G is Abelian $Aut^0_q(\Gamma, lab) = Aut_q(\Gamma, lab)$.

Remark 1.12. Notice that if lab is in spanning tree gauge w.r.t. τ, then $\sigma_V(lab)$ is in spanning tree gauge w.r.t. $\sigma_V(\tau)$. Thus to check if σ_Γ is an enhanced quantum symmetry, for a lab in spanning tree gauge w.r.t. a tree τ, by Corollary 1.7, we only have to check if $lab \circ \sigma_E = lab^{\phi\circ\psi}$ for the preferred quantum enhanced symmetry ϕ given in Lemma 1.4 up to a constant ψ that transforms $lab \circ \sigma_V^{-1}$ into spanning tree gauge w.r.t. to τ. If G is Abelian or σ_V is strict, then we can choose $\psi \equiv 1$.

1.4. Projective representation for quantum enhanced symmetries
Fix a G–labelled graph (Γ, lab) and an ORST τ. Assume that lab is in spanning tree gauge with respect to τ. This is without loss of generality, since by Lemma 1.4, for any RST τ there is an invertible gauge element $\phi(\tau)$ such that $lab^{\phi(\tau)}$ is in spanning tree gauge w.r.t. τ. For $\sigma_\Gamma \in Aut_q(\Gamma, lab)$, set

$$\rho_{lab_\tau}(\sigma_\Gamma) := M_{lab_\tau}(\tau \to \sigma(\tau))$$

Theorem 1.13. *[4] Let lab_τ be in spanning tree gauge w.r.t. the ORST τ, then ρ_{lab_τ} is a projective representation of $Aut^0_q(\Gamma, lab)$ with cocycle $C(\sigma'_\Gamma, \sigma_\Gamma) = \rho(l(\sigma'_\Gamma \circ \sigma_\Gamma(\tau_0), \sigma_\Gamma(\tau_0), \tau_0)$.* □

Recall that if G is Abelian, then $Aut^0_q = Aut_q$.

Remark 1.14. The concrete form of the action depends on the choice of the preferred gauge elements. If one would consistently use the normalisation $\phi(v_{rt}) = g$ then one would get an equivalent representation. The fact that this is a projective representation is, however, due exactly to this choice of normalisation. When performing several of these gaugings monodromy can appear from the movement of the root of the spanning trees, this is what is captured by the cocyle.

1.4.1. The action as pre–gauging or re–gauging The relationship between the actions on lab and the spanning trees is as follows. Given the data $(lab, \tau, \sigma_\Gamma)$ with lab in spanning tree gauge w.r.t. τ, and σ_Γ strict, there are the two ways to check whether lab and $\sigma_\Gamma^{-1}(lab) = lab \circ \sigma$ are gauge equivalent, provided by Corollary 1.7.

The first is to act by σ^{-1} sending lab to $lab \circ \sigma$ which is in spanning tree gauge $\sigma^{-1}(\tau)$ and then to re–gauge to τ.

$$lab \overset{\text{act by } \sigma_\Gamma^{-1}}{\longrightarrow} lab \circ \sigma \overset{\text{re–gauge}}{\longrightarrow} (lab \circ \sigma)^{\phi^{lab\circ\sigma}(\sigma^{-1}(\tau)\to\tau)}$$

The second pre–gauges lab from τ to $\sigma(\tau)$, so that then applying σ_Γ will put it back into spanning tree gauge w.r.t. τ:

$$lab \overset{\text{pre–gauge}}{\longrightarrow} lab^{\phi^{lab}(\tau\to\sigma(\tau))} \overset{\text{act by } \sigma_\Gamma^{-1}}{\longrightarrow} lab^{\phi^{lab}(\tau\to\sigma(\tau))} \circ \sigma$$

[4] These quantum enhanced symmetries naturally appear in quantum contexts, where the labelling by is $U(1)$ and the re–gauging is a choice of phase–shifts. The stabilizer is then an overall phase–shift, so that the effective phase shifts are in $U(1)^k/U(1)$.

Lemma 1.15. *There is the following symmetry between pre–gauging and re–gauging:*

$$M_{lab_\tau}(\tau \to \sigma_\Gamma(\tau)) = M_{(lab\circ\sigma)_{\sigma_\Gamma^{-1}(\tau)}}(\sigma_\Gamma^{-1}(\tau) \to \tau) \tag{1.5}$$

Proof. The proof is a tedious but straightforward unraveling of definitions. □

This equation (1.5) states that the two matrices obtained from the two gaugings above coincide. We used the latter version above. In the situation where $\sigma_\Gamma \in Aut_q^0(\Gamma, lab)$, we furthermore know that in both cases the final labelling is again lab due to Corollary 1.7. This is why there is an action. The cocycle stems from the fact that the normalisation of two repeated pre– or re–gaugings need not be 1 on the *original* root v_{rt}.

1.4.2. Liftings and extensions We can actually take any triple of an element $\sigma \in Aut(\Gamma)$, an ORST τ and a labelling lab and associate to it the matrix $\rho_{lab,\tau}(\sigma_\Gamma)$, if σ is not in $Aut_q(\Gamma, lab)$ then the re–gauged $(lab \circ \sigma)^{\phi^{lab\circ\sigma}(\sigma^{-1}(\tau)\to\tau)}$ will not equal lab and there is no obvious group action.

There is however an action of a certain groupoid and if the action of $Aut(\Gamma)$ is liftable there is a fixed point group that acts. These will be described in the next section. In this section, we do however define the relevant matrices and matrix products.

Definition 1.16. $\sigma \in Aut(\Gamma)$ is liftable for lab if there is an element $\Psi = \Psi(\sigma) \in Aut(G)$ such that

$$(lab \circ \sigma)_{\sigma^{-1}(\tau)}^{\phi^{lab\circ\sigma}(\sigma^{-1}(\tau)\to\tau)} = (lab \circ \sigma)_\tau = (\Psi \circ lab)_\tau \tag{1.6}$$

Consider the composition of two re–gaugings induced by $\sigma_\Gamma, \sigma'_\Gamma \in Aut(\Gamma)$.

$$\tau \to \sigma_\Gamma(\tau) \to \sigma\sigma'_\Gamma(\tau)$$

Theorem 1.17. *If all σ are liftable for lab, then $\Psi(\rho_{lab_\tau})(\sigma'_\Gamma)\rho_{lab_\tau}(\sigma_\Gamma) = C(\sigma', \sigma)\rho_{lab_\tau}(\sigma'_\Gamma\sigma_\Gamma)$.*

Proof. There is the following chain of equalities

$$\begin{aligned}\Psi \circ (M_{lab_\tau}(\tau \to \sigma'(\tau))) &= M_{\Psi\circ lab_\tau}(\tau \to \sigma'(\tau)) \\ &= M_{(lab\circ\sigma)_{\sigma^{-1}\tau}^{\phi^{lab\circ\sigma}(\sigma^{-1}(\tau)\to\tau)}}(\tau \to \sigma'(\tau)) \\ &= M_{lab_{\sigma(\tau)}}(\sigma(\tau) \to \sigma'(\sigma(\tau))\end{aligned}$$

where the first equality is by definition, the second equality is due to (1.5) and the last equation is a straightforward, but tedious check along the lines of [1]. The claim now follows from (1.4). □

1.5. Remark on Quivers

Instead of graphs, one can use quivers. A quiver or a directed graph $\vec{\Gamma}$ is a graph Γ in which all edges are oriented, this means that we have two maps $s, t : E \to V$. Given a quiver there is a natural forgetful map which forgets the orientation of the edges. We call two quivers groupoid equivalent, if they have the same underlying graph. We will use the notation $\vec{\Gamma}$ for quivers and Γ for the underlying graph. A (directed–edge) path on a quiver is a sequence of directed edges $e_i : i = 1 \dots n$, such that for two consecutive directed edges e_i, e_{i+1}: $t(e_i) = s(e_{i+1})$.

Fix a monoid G. A G–labelled quiver is a quiver $\vec{\Gamma}$ together with a map $lab : E \to G$. There is an obvious extension of ρ to all directed edge paths.

If G is a group, we can extend the definition of ρ to the full path groupoid as follows:. Given an edge path on the underlying graph $p = (e_1, \dots, e_n)$, we define $\epsilon_i = 1$ if the orientation of

e_i in the path agrees with that of Γ and $\epsilon_i = -1$ if the orientations are opposite. The formula $\hat{lab}(p) = lab(e_n)^{\epsilon_n} \dots lab(e_1)^{\epsilon_1}$ defines a morphism of the path groupoid into G. We call two G–labelled quivers (Γ, lab) and (Γ', lab') groupoid equivalent, if the underlying graphs are equivalent and $lab(e) = lab'(e)$ whenever their orientations agree and $lab(e) = lab^{-1}(e)$ if the orientations differ in the two graphs. A Γ labelled graph is equivalence class of G–labelled quivers under groupoid equvialence.

2. Categorical formulation and extension of the actions and representations

The notions become totally natural in the language of categories and it is possible to generalize the actions by the individual enhanced quantum automorphisms groups of fixed labellings into an action of a bigger groupoid.

2.1. Categories and fundamental groups

A category $\mathcal{C}$ is a collection/class of objects and for each pair of objects X, Y a collection of morphisms $Hom_{\mathcal{C}}(X,Y)$, called morphisms from X to Y, together with an associative composition morphism $Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z) \stackrel{\circ}{\to} Hom({}_{\mathcal{C}}X,Y)$, and identities $id_x \in Hom(X,X)$. Writing $g \circ f := \circ(f,g)$, being an identity means that $id_X \circ f = f, g \circ id_X = g$ for f any morphism into X and g any morphism out of X. A category is small if the objects and all the $Hom_{\mathcal{C}}(X,Y)$ are sets. If f is a map from X to Y, then X is called the source of f and Y is called the target of f, and one writes $X \stackrel{f}{\to} Y$.[5]

Maps between categories are called functors. Functors from a category $\mathcal{C}$ to a category $\mathcal{D}$ again form a category $Fun(\mathcal{C},\mathcal{D})$ whose morphisms are called natural transformations. To keep track for two functors $F, G : \mathcal{C} \to \mathcal{D}$ one writes $Nat(F,G)$ for $Hom_{Fun(\mathcal{C},\mathcal{D})}(F,G)$.

2.2. Groupoids

A groupoid is a category whose morphisms are all isomorphisms. A group correspondes to a groupoid with only one object and vice–versa via the following identification: $Hom(*,*) = G$, where $*$ is the unique object and with composition of morphisms given by group multiplication. For any object x in a groupoid, $Hom(x,x) = End(x) = Aut(x)$ is a group called the automorphisms group of x. A groupoid has a natural (contravariant) auto–equivalence *op* which is the identity on objects and sends any morphism to its inverse.

In this language a group representation becomes a functor $\underline{G} \to Vect_k$ to the category of k-vector spaces.[6] A functor from a groupoid to $\mathcal{V}ect_k$ is given by a collection of vector spaces and isomorphisms between them — one for every morphism in the groupoid. We consider not only functors to $\mathcal{V}ect_k$ but also functors of groupoids from a groupoid Γ to a fixed $\underline{G}$.

A set X determines a groupoid in several ways. One is the discrete groupoid, which just has id_x for $x \in X$ as its morphisms. There is another canonical groupoid, which we call $\mathcal{K}(X)$, the complete groupoid. It has as morphisms pairs of elements (x,y) which we write $x \to y$ with the composition $(y \to z) \circ (x \to y) = x \to z$. This makes $(x,x) = id_x$ and $y \to x = (x \to y)^{-1}$.[7]

A set X with an action of a group G determines a groupoid in the following fashion. The objects are the set X the morphisms are $X \times G$ with the source map s given by $s(x,g) = x$ and

[5] A slick way to give a small category is as a pair of sets, M, O (morphisms and objects) with two maps $M \rightrightarrows_t^s O$, the source and target maps, together with a composition morphism $\circ : M \,{}_s\!\times_t M \to M$ and a section $id : O \to M$ of both s and t such that $id_X = id(X)$ is an identity. Here $M \,{}_s\!\times_t M \to M \subset M \times M$ is the relative product consisting of elements (f,g) with $t(f) = s(g)$ and being an identity means the same thing as above.

[6] A general groupoid can have many components, but is is equivalent as a category to a disjoint union of group categories. In our case the groupoids have one component and hence all of them are equivalent to some $\underline{G}$.

[7] This is the same as using a morphism $X \times X \rightrightarrows_{t=p_2}^{s=p_1} X$ with composition given by the map $(X \times X) \,{}_s\!\times_t (X \times X) \to X \times X \times X \times X \stackrel{p_{14}}{\to} X \times X$ where p_{14} is the projection to the first and fourth component.

target map $t(x,g) = g(x)$. This becomes clear in the notation $x \xrightarrow{g}$ or $x \xrightarrow{g} g(x)$. Composition is defined for $x \xrightarrow{g}$ and $y \xrightarrow{h}$ if $g(x) = y$ then $(x \xrightarrow{g}) \circ (y \xrightarrow{h}) = x \xrightarrow{hg}$, this groupoid is called the action groupoid. It is also a semi–direct product and this is why we write $\underline{X \rtimes G}$ with the underline to stress that it is a groupoid. This groupoid coincides with $\mathcal{K}(X)$ if the action is regular, that is X is a G–torsor.

There is a canonical functor $st : \underline{X \rtimes G} \to \mathcal{K}(X)$ which is identity on objects and on morphisms is defined as $st(x \xrightarrow{g} g(x)) = x \to g(x)$ or in other notation $st(\phi) = (s(\phi), t(\phi))$. The functor st is bijective on objects, but does not need to be surjective on morphisms. It will usually partition X into components $X = X_1 \amalg \cdots \amalg X_n$, such that $st(\underline{X \rtimes G}) = \mathcal{K}(X_1) \amalg \cdots \amalg \mathcal{K}(X_n)$

2.3. The constructions above in category theoretical language

A quiver $\vec{\Gamma}$, determines a category $P(\vec{\Gamma})$, whose objects are vertices and whose morphisms are paths on the quiver. This is also the free category on the morphisms given by $\vec{\Gamma}$. A graph Γ determines a groupoid $\pi(\Gamma)$, its path–groupoid, in the following way. Choose any quiver whose underlying graph is the given graph and consider the category it defines. Now add additional morphisms which are inverses to the morphisms given by the directed edges. This is equivalent to saying that the morphisms are the reduced paths on Γ. A functor on such a groupoid is given by the values on the vertices and on the directed edges of any quiver representing the graph. A G–labelling of a quiver is a functor lab from the quiver $\vec{\Gamma}$ to the groupoid $\underline{G}$. Such a functor, by a universal property, factors through the groupoid $\pi(\Gamma)$. Groupoid equivalence of labellings then means that the induced functors from $\pi(\Gamma)$ to $\underline{G}$ agree.

In other workds, a G–labelling of Γ is equivalent to a functor $lab : \pi(\Gamma) \to \underline{G}$. lab is in spanning tree gauge with respect to τ if $lab|_{\pi(\tau)}$ is trivial, i.e. maps all morphisms to the id_*.

Regauging in this language is a natural transformation from one labelling functor to another. Such a natural transformation is given by a collection of maps: $\phi(v) \in Hom_{\underline{G}}(F(v), F'(v)) = G$, one for each vertex v, which for each directed edge $\vec{e}$ form v to w satisfy $\phi(w)^{-1} \circ F(\vec{e}) = F'(\vec{e}) \circ \phi(v)$. These natural transformations are invertible and yield isomorphisms of functors. In other words, lab and lab' are gauge equivalent if they are representatives of the same isomorphism class of functors $[lab] = [lab']$.

Proposition 2.1. *$Nat(lab, -) = Map(V, G)$ and each such map ϕ gives an invertible natural transformation $lab \to lab^{\phi}$. Thus, $Fun(\pi(\Gamma), \underline{G})$ is a groupoid and re–gaugings $\phi \in Map(V, G)$ act transitively on the elements of a given isomorphism class $[lab]$.*

In the case that the image of lab generates G, the group of automorphisms of lab is G and hence the group G^V/G acts simply transitively on the functors in a given equivalence class lab. □

An auto–equivalence of $\pi(\Gamma)$ is a functor from $\pi(\Gamma) \to \pi(\Gamma)$ that is isomorphic to the identity functor. Any graph symmetry $\sigma \in Aut(\Gamma)$ yields an auto–equivalence on $\pi(\Gamma)$, which we also denote by σ. On objects it is given by $\sigma(v) = \sigma_V(v)$ and on morphisms as follows. Pick $\vec{e}$ with orientation $s(e) = v$ and $t(e) = w$ then define $\sigma(\vec{e}) = \sigma_F(\vec{e})$ which has the orientation given by the source $\sigma_V(v)$ and target $\sigma_V(w)$. This gives an action of $Aut(\Gamma)$ on $Fun(\pi(\Gamma), \underline{\Gamma})$ given by $lab \mapsto lab \circ \sigma^{-1}$. Note if lab is in spanning tree gauge w.r.t. τ then $\sigma(lab) = lab \circ \sigma^{-1}$ is in spanning tree gauge w.r.t. the push–forward $\sigma(\tau)$. To check that $[lab] = [lab \circ \sigma^{-1}]$ we can apply Corollary 1.7.

Lemma 2.2. *A quantum enhanced symmetry of G–labelled graph, that is a functor $lab : \pi(\Gamma) \to \underline{G}$, is an element $\sigma \in Aut(\Gamma)$ which satisfies that $[lab] = [lab \circ \sigma]$, i.e. there exists an equivalence ϕ: $(lab \circ \sigma^{-1})^{\phi} \simeq lab$. If lab is in spanning tree gauge w.r.t. τ then $\sigma \in Aut(\Gamma)$ is a strict quantum enhanced symmetry if $lab = \sigma(lab)^{\phi}$ in the notation above.*

The group of auto–equivalences ψ of $\underline{G}$, which correspond to automorphisms of G, also acts on $Fun(\pi(\Gamma), \underline{G})$ via $lab \to \psi \circ lab$.

Definition 2.3. An element of $\sigma \in Aut(\Gamma)$ is called transferable via lab if there exists a $\psi \in Aut(\underline{G})$ such that $(lab \circ \sigma)^{\phi^{lab\circ\sigma}(\sigma^{-1}(\tau)\to\tau)} = \psi \circ lab$.

There are usually many more auto–equivalences of $\underline{G}$ than those transferred from symmetries of the underlying graph. As an example lets look at the n–petal graph with a functor to the free group $\mathbb{F}_n$ which associates one generator to each (directed) edge. The auto–equivalences correspond to automorphism of the free group, while the graph symmetries only give the permutations of the generators.

2.4. Groupoid extensions and representations
With this language in place, we can also enlarge the setting of Theorem 1.13.

Fix a graph Γ and let X be the set of all ORSTs of Γ. The re–gauging groupoid is by definition $\mathcal{K} = \mathcal{K}(X)$. With this notation we can translate the results of

Theorem 2.4. *Let G be Abelian Fix a G–labelling lab and assume it is in spanning tree gauge with respect to some ORST τ then then the matrices $M_{lab_\tau}(\tau \to \tau')$ of equation (1.2) yield projective representation of $\mathcal{K}$ in $M_{|V|}(G)$ with cocycles $lab(l)$ with l defined in equation (1.3). Moreover this lifts to the groupoid extension of $\mathcal{K}$ by $\pi(\Gamma)$ given by $\mathcal{L} = \amalg_{v\in V}\pi_1(\pi(\Gamma), v)$ defined in Proposition 2.5 below.*

Moreover via st this yields a projective representation of $\underline{X \rtimes Aut(\Gamma)}$, which induce the projective representations of 1.13.

2.4.1. Groupoid extensions Even in the case that G is not Abelian, there is a Groupoid extension, which in the Abelian case lift the projective action to an honest action. There are functors $p^{op} : \mathcal{K}^{op} \to \pi(\Gamma)$ and $l^{op} : \mathcal{K}_1^{op}{}_t \times_s \mathcal{K}_1^{op} \to \mathcal{L}$

$$p^{op}(g^{op}) := \gamma^{\tau}_{v_{rt}v'_{rt}}, \quad l^{op}(g^{op}, h^{op}) := l(h, g) \tag{2.1}$$

Proposition 2.5. *[4] The pair (p^{op}, l^{op}) are an element of $C^2_{\pi(\Gamma)}(\mathcal{K}, \mathcal{L})$ that is a $\pi(\Gamma)$–crossed $\mathcal{K}$ 2–cocycle with values in $\mathcal{L}$. By general theory, [7, 9, 10] the noncommutative cocycle (p, l) gives rise to a groupoid extension (Σ, b) over $\pi(\Gamma)$*

$$\Sigma : 1 \to \mathcal{L} \to \hat{\mathcal{K}} \to \mathcal{K} \to 1 \quad b : \hat{\mathcal{K}} \to \pi(\Gamma) \tag{2.2}$$

2.5. Choice of spanning tree gauges as functors
One can add one more layer of functoriality to the choices of spanning tree gauges to understand how the action of $Aut_q(\Gamma, lab)$ is induced as well as its generalization to $Aut(G)$.

Fix lab, then this defines a functor $gauge : \mathcal{K} \to Fun(\pi(\Gamma), \underline{G})$. On objects it is defined by $gauge(\tau) = lab_\tau$ where lab_τ is the re–gauging of lab into spanning tree gauge w.r.t. τ. On morphisms we set $gauge(\tau \to \tau')$ to be the natural transformation given by re–gauing lab_τ to be in spanning tree gauge w.r.t. τ' using the re–gauging of Lemma 1.4. Even more is true, $gauge$ is a projective functor $gauge : Fun(\pi(\Gamma), \underline{G}) \to Fun(\mathcal{K}, Fun(\pi(\Gamma), \underline{G}))$. Given a natural transformation $\phi : lab \to lab^\phi$ on the left, it maps to id. It is a strict functor if G is Abelian, otherwise there might be a 2–morphism according to Corollary 1.7. The exact behaviour is the content of Theorem 1.13.

Now $Aut(\pi(\Gamma))$ acts on $Fun(\pi(\Gamma), \underline{G})$ and this action gets transferred by $gauge$. The content of Theorem 1.13 is that if one restricts to strict quantum enhanced automorphisms, one again obtains an action by projective functors.

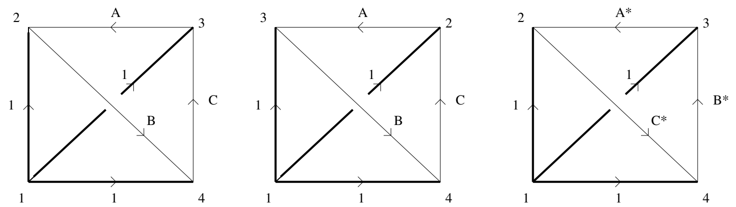

Figure 2. Calculation of the action of (23) on T^3. The original graph, the pushed forward order and the move into the old position to read off the morphism

2.6. Lifts

Finally to explain Theorem 1.17 we notice that there is also an action of $Aut(\underline{G})$ on $Fun(\pi(\Gamma), \underline{G})$. If one takes both actions together, one obtains the bi–groupoid ρ, s, λ : $Aut(\pi(\Gamma)) \ltimes Fun(\pi(\Gamma), \underline{G}) \rtimes Aut(\underline{G}) \to Fun(\pi(\Gamma), \underline{G})$. By restricting to liftable enhanced symmetries and letting (ψ, lab) be the inverse of (lab, σ), we get automorphisms of lab, which are sequences of ψ's and σ which via *gauge* get transferred to projective functors.

3. Representations

We will now give the details for the example of the tetrahedral graph or the full square. It has symmetry group $\mathbb{S}_4$. It acts transitively on all ORST. The subgroup of $\mathbb{S}_3$ acts transitively on all orders for a fixed RST. Fix $G = \mathbb{F}_3$ the free group on three generators. We will denote inverses by $*$ to keep with the application in physics, where we apply a character to $U(1)$. We fix an initial rooted spanning tree and order as in the first picture of Figure 2 and fix the $G = \mathbb{F}_3$–labelling as indicated.

We will use the following graphical calculation technique:

1. Write down the graph together with the initial spanning tree and order.
2. Push–forward the spanning tree and the order. This is given by replacing i by $\sigma(i)$.
3. Write the re–gauging at the vertices. They are given by going along the new spanning tree to the root and multiplying the labels in order.
   (a) Read off the matrix Φ by putting the G label of a vertex onto the diagonal in the place indicated by the label $\sigma(i)$.
   (b) Obtain $M = \Phi P_\sigma$
4. Perform the regauging, by multiplying the label of an ordered edge from the left by the label of its target and on the right by the inverse label of its source.
5. Rearrange the vertices with σ_V. Formally, this is done by writing v_i next to $\sigma(i)$, but one can just use the new numbering to move the graph back into its old position and read off the transformation ψ.

3.1. Lifts of the action

The lift of the action of $\mathbb{S}_4$ on $\mathbb{F}_3$ is fixed once we know the action of the generators $(12), (23)$ and (34).

The action of (23) is graphically calculated in Figure 2, from which one reads off $\Psi((23))(A, B, C) = (A^*, C^*, B^*)$. Here (A, B, C) is the notation for the initially chosen basis of $\mathbb{T}^3$.

A similar calculation shows that $\Psi((34))(A, B, C) = (B^*, A^*, C^*)$. A consequence is that the cycle $(234) = (23)(34)$ acts as $\Psi((234))(A, B, C) = \Psi((23))(B^*, A^*, C^*) = (B, C, A)$ and is the cyclic permutation.

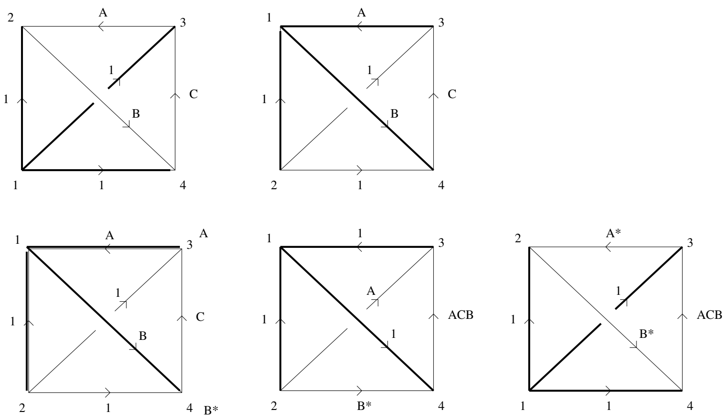

Figure 3. Calculation of the action of (12) on T^3

The action of (12) is more complicated as the root is moved. For this we calculate graphically, see Figure 3, and read off Ψ as: $(A, B, C) \mapsto (A^*, B^*, ACB)$.

This allows us to compute fixed points and stabilizer groups if we take a character. This is equivalent to treating A, B, C as variables in $U(1)$. We will first concentrate on non–Abelian stabilizer groups. There are only two fixed points under the full $\mathbb{S}_4$ action and these are $(1, 1, 1)$ and $(-1, -1, -1)$. The group A_4, the subgroup of all even permutations, is the stabilizer group of the two points (i, i, i) and $(-i, -i, -i)$. One can readily check that these are the only non–Abelian stabilizer groups. The other possibility would be $\mathbb{S}_3$, but a short calculation shows that anything that is stabilized by any $\mathbb{S}_3$ subgroup is stabilized by all of $\mathbb{S}_4$.

3.1.1. Representations We collect together the matrices M needed for further calculation. Again, we fix our initial ordered rooted spanning tree as before.

Using short hand notation, the matrix for the re–gauging induced by the transpositions $(12), (13), (14)$ from the initial spanning tree to the pushed forward one are

$$\rho_{12} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & A & \\ & & & B^* \end{pmatrix}, \rho_{13} = \begin{pmatrix} 0 & & 1 & \\ & A^* & & \\ 1 & & 0 & \\ & & & C \end{pmatrix}, \rho_{14} = \begin{pmatrix} 0 & & & 1 \\ & B & 0 & \\ & 0 & C^* & \\ 1 & & & 0 \end{pmatrix}$$

The calculation for ρ_{12} can be read off from Figure 3. For this we read off the matrix Φ from the re–gauging parameter and the matrix M_σ is given by the permutation we are considering. The other calculations are similar. All other transpositions, viz. those not involving 1, simply yield permutation matrices as there is no re–gauging involved. It is convenient to also have the following matrices as a reference:

$$\rho_{(12)(34)} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & B^* \\ & & A & 0 \end{pmatrix}, \quad \rho_{(14)(23)} = \begin{pmatrix} & & & 1 \\ & & C^* & \\ & B & & \\ 1 & & & \end{pmatrix}$$

and finally

$$\rho_{(132)} = \begin{pmatrix} 0 & 1 & 0 & \\ 0 & 0 & A & \\ 1 & 0 & 0 & \\ & & & B^* \end{pmatrix}$$

3.1.2. The point $(1,1,1)$. At $(1,1,1)$, the matrices $\rho_{12}, \rho_{23}, \rho_{34}$ give the usual representation of $\mathbb{S}_4$ on $\mathbb{C}^4$. As is well known this representation decomposes into the trivial representation and an irreducible 3–dim representation.

3.1.3. The point $(-1,-1,-1)$. In this case, the matrices $\rho_{12}, \rho_{23}, \rho_{34}$ only give a projective representation. As one can check $\rho_{12}\rho_{23}\rho_{12} = -\rho_{13}$ while $\rho_{23}\rho_{12}\rho_{23} = \rho_{13}$ for instance. Define the 1–cocycle λ by $\lambda(\sigma) = (-1)$ if 1 appears in a cycle of length > 1 and 1 else. So that $\lambda((12)) = \lambda((13)) = \lambda((123)) = -1$ while $\lambda((23)) = \lambda((24)) = \lambda((234)) = 1$. Then one calculates that $\tilde{\rho} := \rho \circ \lambda$ has a trivial cocycle c and thus ρ is isomorphic to a true linear representation of $\mathbb{S}_4$. Checking the characters, one sees again that in this case the irreducible components of $\tilde{\rho}$ are again the one–dimensional trivial representation and the 3–dimensional standard representation. The trivial representation is spanned by $(-1,1,1,1)$.

Remark 3.1. We would like to remark that the choice of λ amounts to choosing a different gauge for the root vertex, namely -1 instead of 1.

3.1.4. The points (i,i,i) *and* $(-i,-i,-i)$. These points are similar to each other. We will treat the first one in detail. Again, we have only a projective representation of A_4 aka. the tetrahedral group T. Namely, $\rho_{(12)(34)}\rho_{(13)(24)} = -i\rho_{(14)(23)}$. Again we can scale by a 1–cocycle λ. This time $\lambda(id) = 1$, $\lambda((ij)(kl)) = i$, $\lambda(ijk) = 1$ if $1 \notin \{i,j,k\}$, and $\lambda((ijk)) = i$ if $1 \in \{i,j,k\}$. The resulting representation $\tilde{\rho} = \rho\circ\lambda$ is then still a projective representation, but is it a representation of the unique non–trivial $\mathbb{Z}/2\mathbb{Z}$ extension of A_4, which goes by the names $2T, 2A_4, SL(2,3)$ or the binary tetrahedral group. This group is well known. It is presented by generators s and t with the relations $s^3 = t^3 = (st)^2$. In $SL(2,3)$ (that is the special linear group of 2×2 matrices over the field with three elements $\mathbb{F}_3$), one can choose $s = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ and $t = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$.

For $2A_4$ using a set theoretic section $\wedge$ of the extension sequence

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 2A_4 \underset{\wedge}{\overset{\longrightarrow}{\longleftarrow}} A_4 \longrightarrow 1 \tag{3.1}$$

and z as a generator for $\mathbb{Z}/2\mathbb{Z}$, we can pick $s = z\widehat{(123)}, t = z\widehat{(234)}$ as generators. Now we can check the character table, Table 1, and find that the representation $\tilde{\rho}$ over the complex numbers decomposes as the sum of two irreducible two–dimensional representations $\chi_5 \oplus \chi_6$. In fact, these are the two representations into which the unique real irreducible 4–dimensional representation of complex type splits over $\mathbb{C}$.

The explicit computation for the representation

$$\begin{aligned} \tilde{\rho}(s) &= -\lambda((123))\rho_{(123)} = \begin{pmatrix} 0 & 0 & -i & 0 \\ -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ \tilde{\rho}(t) &= -\lambda((234))\rho_{(234)} = -\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{aligned} \tag{3.2}$$

is as follows. Suppose the $\tilde{\rho} = \bigoplus_{i=1}^{7} a_i\rho_i$, where ρ_i is the irrep with character χ_i. Now $tr(id) = 4, tr(-1) = -4$, using the character table this implies that the coefficients $a_1 = a_2 = a_3 = a_7 = 0$ and furthermore $(*)\, a_4 + a_5 + a_6 = 2$. We furthermore have that $tr(s) = -1$ so that $a_4 + \omega a_5 + \omega^2 a_6 = -1$ which together with (*) implies that $a_4 = 0, a_5 = a_6 = 1$. This fixes

Representative	1	-1	s^3	t^2	s^2	t	s
Elts in Conj. Class	1	1	6	4	4	4	4
Order	1	2	3	3	4	4	6
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ω	ω^2	ω^2	ω
χ_3	1	1	1	ω^2	ω	ω	ω^2
χ_4	2	-2	0	-1	-1	1	1
χ_5	2	-2	0	$-\omega$	$-\omega^2$	ω^2	ω
χ_6	2	-2	0	$-\omega^2$	$-\omega$	ω	ω^2
χ_7	3	3	-1	0	0	0	0

Table 1. Character table of $2 \cdot A_4$ [14], where $\omega = e^{\frac{2\pi i}{3}}$.

a, b, c	Group	Iso class of of extension	type	Dim of Irreps
$(0,0,0)$	$\mathbb{S}_4$	$\mathbb{S}_4$	trivial	1,3
(π,π,π)	$\mathbb{S}_4$	$\mathbb{S}_4$	trivialisable	1,3
$(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ $(\frac{3\pi}{2},\frac{3\pi}{2},\frac{3\pi}{2})$	A_4	$2A_4$	isomorphic extension	2,2

Table 2. Possible choices of parameters (a, b, c) leading to non–Abelian enhanced symmetry groups

the decomposition into irreps. As a double check one can verify that the rest of the equations are also satisfied.

The analysis of the complex conjugate point $(-\pi/2, -\pi/2, -\pi/2)$ is analogous.

We would briefly like to remark that these special points have a meaning in the study of real materials which are in the form of a Gyroid. In these materials, band sticking is forced by the presence of the symmetries and leads to special properties of the material [2].

3.2. Groupoid calculation

Here we calculate one example of Theorem 1.17:

$$\rho_{(12)(34)} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & A \\ & & B^* & 0 \end{pmatrix}, \quad \rho_{(13)(24)} = \begin{pmatrix} & & 1 & 0 \\ & & 0 & A^* \\ 1 & 0 & & \\ 0 & C & & \end{pmatrix}$$

Using the substitution $\rho'_{(13)(24)} = \rho_{(13)(24)}(A \to B, C \to ABC)$, we indeed obtain

$$\rho_{(1234)}\rho'_{(13)(24)} = B^*\rho_{(14)(23)} = B^* \begin{pmatrix} & & & 1 \\ & & B & \\ & C^* & & \\ 1 & & & \end{pmatrix}$$

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