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ORBIFOLDING FROBENIUS ALGEBRAS

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We study the general theory of Frobenius algebras with group actions. These structures arise when one is studying the algebraic structures associated to a geometry stemming from a physical theory with a global finite gauge group, i.e. orbifold theories. In this context, we introduce and axiomatize these algebras. Furthermore, we define geometric cobordism categories whose functors to the category of vector spaces are parameterized by these algebras. The theory is also extended to the graded and super-graded cases. As an application, we consider Frobenius algebras having some additional properties making them more tractable. These properties are present in Frobenius algebras arising as quotients of Jacobian ideal, such as those having their origin in quasi-homogeneous singularities and their symmetries.

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0. Introduction

The subject of this exposition is the general theory of Frobenius algebras with group actions. These structures arise when one is studying the algebraic structures stemming from a geometry associated to a physical theory with a global finite gauge group [6, 5, 11, 22]. The most prominent example of this type in mathematics is the Gromov–Witten theory of orbifolds [2], which are global quotients. The use of orbifold constructions is the cornerstone of the original mirror construction [9]. The orbifolds under study in that context are so-called Landau–Ginzburg orbifold theories, which have so far not been studied mathematically. These correspond to the Frobenius manifolds stemming from singularities and are studied as examples in detail in the present paper.

A common aspect of the physical treatment of quotients by group actions is the appearance of so called twisted sectors. This roughly means that if one wishes to take the quantum version of quotient by a group action, one first has to construct an object for each element in the group together with a group action on this object and in a second step take invariants in all of these components. Whereas classically one considers only the G-invariants of the original object which constitutes the sector associated to the identity, the untwisted sector.

We give the first complete axiomatic treatment, together with a natural geometric interpretation for this procedure, which provides a common basis for applications such as singularities with symmetries or Laundau–Ginzburg orbifolds, orbifold cohomology and quantum cohomology of global quotients and in a sense all other string-orbifold versions of classical theories.

Our treatment shows that the construction of twisted sectors is not merely an auxiliary artifact, but is essential. This is clearly visible in the case of the Frobenius algebra associated to the singularity of type A_n together with a $\mathbf{Z}/(n+1)\mathbf{Z}$ action and the singularity of type A_{2n-3} with $\mathbf{Z}/2\mathbf{Z}$ action, which are worked out in detail in the last paragraph. In particular, the former example exhibits a version of mirror symmetry in which it is self-dual. The twisted sectors are the key in this mirror duality, since it is the sum of twisted sectors that is dual to the untwisted one.

In the present work, we develop the theory of orbifold Frobenius algebras along the now classical lines of Atiyah, Dijkgraaf, Dubrovin and Segal. That is, we start by introducing the algebraic structures in an axiomatic fashion. There is an important difference to the theory with trivial group in that there are two structures with slightly different G-action to be considered which are nevertheless present. These versions differ by a twist with a character. In the singularity version, on the non-twisted level, they correspond to the different G-module structures on the cohomology and the Milnor ring [24]. In physical terms, these two structures are related by spectral flow. To be even more precise, one structure carries the natural multiplication and the other the natural scalar product. The next step is a cobordism realization of the theory. In the case of a trivial character our cobordism theory reduces to that of [21], where the same structure independently appeared from a homotopy rather than an orbifold point of view. For another approach see also [8]. The triviality of the character is, however, not necessary and in the case of the most interesting examples worked out in the last paragraph, this is not the case.

The key structure here is a non-commutative multiplication on the sum of all the twisted sectors before taking invariants which after taking G invariants is commutative. This is a novel approach to global quotients which we first presented at WAGP 2000.

In order to apply our theory to Landau–Ginzburg models or singularity theory — in a sense the original building blocks for mirror symmetry — we introduce a large class of examples, so-called special G-Frobenius algebras.

This class contains the class of Jacobian G-Frobenius algebras which in turn encompass the singularity examples and those of manifolds whose cohomology ring can be described as a quotient by a Jacobian ideal. Here it is important to note that everything can be done in a super (i.e. $\mathbf{Z}/2\mathbf{Z}$-graded) version. This introduces a new degree of freedom into the construction corresponding to the choice of parity for the twisted sectors. Lastly, we explicitly work out several examples including the transition from the singularity A_{2n+3} to D_n via a quotient by $\mathbf{Z}/2\mathbf{Z}$, this is the first purely mathematical version of this correspondence avoiding path integrals. In this situation the presence of the Ramond algebra explains the results obtained by [24] in the situation of singularities with symmetries as studied by Arnold. We furthermore show that $*/G$ leads to (twisted) group algebras.

For Jacobian Frobenius algebras, we also introduce a duality transformation which allows us to show that orbifolding plays the role of mirror symmetry. We show that the pair (A_n, A_1) is mirror dual to (A_1, A_n) via orbifolding by $\mathbf{Z}/(n+1)\mathbf{Z}$. In this way, we find the underlying Frobenius algebra structure for the A-model realization of A_n [12, 23].

In the case that the Frobenius algebra one starts out with comes from a semi-simple Frobenius manifold and the quotient of the twisted sector is not trivial, there is unique extension to the level of Frobenius manifolds. This is the case in the above example of A_{2n+3} and D_n.

The general theory presented here applies to the orbifold cohomology of global quotients. Indeed our postulated non-commutative structure, which is discussed here and had first been presented in detail at WAGP2000, has been found by [7]. Moreover, we have recently given an interpretation of this multiplication in terms of moduli spaces of maps of pointed admissible G-covers [13] where we consider the G-CohFT extension of non-projective G-Frobenius algebras. This can be viewed as the G-equivariant counterpart of the correspondence between Frobenius algebras, CohFT and Gromov–Witten invariants.

Furthermore the theory of special G-twisted Frobenius algebras in the case where G is the symmetric group sheds new light on the construction of [19], adding a new uniqueness result and simplifying the general structure of the multiplication [16]. Recently, we uncovered the structure of discrete torsion and explained its role in [17]. It can be realized via the forming of tensor products — in the sense introduced below — with twisted group rings. The synthesis of these results and an application to the Hilbert scheme can be found in [18]. The discrete torsion also comes in a super-version which explains the freedom to choose a sign in the reconstruction for Jacobian Frobenius algebras discussed in Sec. 5.

1. Frobenius Algebras and Cobordisms

In this section, we recall the definition of a Frobenius algebra and its relation to the cobordism-category definition of a topological field theory [1, 3, 4].

1.1. *Frobenius algebras*

Definition 1.1. A Frobenius algebra (FA) over a field K of characteristic 0 is $\langle A, \circ, \eta, 1\rangle$, where

A is a finite dim K-vector space,
$\circ$ is a multiplication on $A : \circ : A \otimes A \to A$,
η is a non-degenerate bilinear form on A, and
1 is a fixed element in A — the unit

satisfying the following axioms:

(a) Associativity:

$$(a \circ b) \circ c = a \circ (b \circ c)\,.$$

(b) Commutativity:

$$a \circ b = b \circ a\,.$$

(c) Unit:

$$\forall \in a : 1 \circ a = a \circ 1 = a\,.$$

(d) Invariance:

$$\eta(a, b \circ c) = \eta(a \circ b, c)\,.$$

Remark 1.1. By using η to identify A and A^* — the dual vector space of A — these objects define a one-form $\epsilon \in A^*$ called the co-unit and a three-tensor $\mu \in A^* \otimes A^* \otimes A^*$.

Using dualization and invariance these data are interchangeable with η and $\circ$ via the following formulas. Explicitly, after fixing a basis $(\Delta_i)_{i\in I}$ of A, setting $\eta_{ij} := \eta(\Delta_i, \Delta_j)$ and denoting the inverse metric by η^{ij},

$$\begin{aligned} \epsilon(a) &:= \eta(a, 1)\,, \\ \mu(a,b,c) &:= \eta(a \circ b, c) = \epsilon(a \circ b \circ c)\,, \\ a \circ b &= \sum_{ij} \mu(a, b, \Delta_i)\eta^{ij}\Delta_j \quad \text{and} \\ \eta(a,b) &= \epsilon(a \circ b)\,. \end{aligned}$$

We call $\rho \in A$ the element dual to ϵ. This is the element which is Poincaré dual to 1.

Definition 1.2. We call two Frobenius algebras $\langle A, \circ, \eta, 1\rangle$ and $\langle A', \circ', \eta, 1'\rangle$ isomorphic if there is an isomorphism ψ of unital algebras between A and A' and $\psi^*\eta' = \lambda\eta$ for some $\lambda \in K^*$. We call two Frobenius algebras strictly isomorphic if $\lambda = 1$.

1.1.1. *Grading*

A graded Frobenius algebra is a Frobenius algebra together with a group grading of the vector space $A : A = \oplus_{i\in I} A_i$ where I is a group together with the following compatibility equations: denote the I-degree of an element by deg.

(1) 1 is homogeneous; $1 \in A_d$ for some $d \in I$.
(2) η is homogeneous of degree $d + D$, i.e. for homogeneous elements $a, b\, \eta(a, b) = 0$ unless $\deg(a) + \deg(b) = d + D$.

This means that ϵ and ρ are of degree D.

(3) $\circ$ is of degree d, this means that μ is of degree $2d + D$, where again this means that

$$\deg(a \circ b) = \deg(a) + \deg(b) - d\,.$$

Definition 1.3. An even derivation $E \in \mathrm{Der}(A, A)$ of a G twisted Frobenius algebra A is called an Euler field, if it is conformal and is natural w.r.t. the multiplication, i.e. for some d, $D \in K$ it satisfies:

$$\eta(Ea, b) + \eta(a, Eb) = D\eta(a, b) \tag{1.1}$$

and

$$E(ab) = Ea\,b + a\,Eb - d\,ab\,. \tag{1.2}$$

Such a derivation defines a grading on A by its set of eigenvalues.

Remark 1.2. For this type of grading, we will use the group $\mathbf{Q}$. There are two more versions of grading: (1) a $\mathbf{Z}/2\mathbf{Z}$ super-grading, which will be discussed in Sec. 1.1.6 and (2) a grading by a finite group G, which is the content of Sec. 2.1.

Definition 1.4. Given an I-graded Frobenius algebra A, we define its characteristic series as

$$\chi_A(t) := \sum_{i \in I} \dim(A_i) t^i\,. \tag{1.3}$$

We refer to the set $\{d, D; i : \dim A_i \neq 0\}$ as the (I-)spectrum of A.

1.1.2. *Scaling*

If the group indexing the grading has the structure of a Λ-module, where Λ is a ring, we can scale the grading by an element $\lambda \in \Lambda$. We denote the scaled Frobenius algebra by $\lambda A := \oplus_{i \in I} A_{\lambda i}$ where $(\lambda A)_i = A_{\lambda i}$

$$\chi_{\lambda A}(t) := \sum_{i \in I} \dim(\lambda A)_i t^i = t^\lambda \chi_A(t)\,. \tag{1.4}$$

It is sometimes — but not always — convenient to normalize in such a way that $\deg(1) = 0$ where 0 is the unit in I. In the case that the grading is given by $\mathbf{Q}$, this means $\deg_{\mathbf{Q}}(1) = 0$ and in the finite group case $\deg_G(1) = e$ where now e is the unit element of G.

1.1.3. *Operations*

There are two natural operations on Frobenius algebras, the direct sum and the tensor product. Both of these operation extend to the level of Frobenius manifolds, while the generalization of the direct sum is straightforward, the generalization of the tensor product to the level of Frobenius manifolds is quite intricate [14].

Consider two Frobenius algebras $\mathcal{A}' = \langle A', \circ', \eta', 1' \rangle$ and $\mathcal{A}'' = \langle A'', \circ'', \eta'', 1'' \rangle$.

1.1.4. *Direct sum*

Set $\mathcal{A}' \oplus \mathcal{A}'' := \langle A, \circ, \eta, 1\rangle = \langle A' \oplus A'', \circ' \oplus \circ'', \eta' \oplus \eta'', 1' \oplus 1''\rangle$, for example, $(a', a'') \circ (b', b'') = (a' \circ' b', a'' \circ'' b'')$. The unit is $1 = 1' \oplus 1''$ and the co-unit is $\epsilon = (\epsilon', \epsilon'')$.

Lemma 1.1. *If both Frobenius algebras are graded by the same I then their direct sum inherits a natural grading if and only if the gradings can be scaled such that*

$$D' + d' = D'' + d'' := D + d \tag{1.5}$$

where

$$D = D' = D'' \tag{1.6}$$

in this scaling.

Furthermore, the unit will have degree $d' = d'' = d$.

Proof. Equation (1.5) ensures that the three tensor μ is homogeneous of degree $D+2d$. The homogeneity of η yields the second condition: for η to be homogeneous, it is necessary that after scaling ρ' and ρ'' are homogeneous of the same degree $D' = D'' = D$. The two equations together imply the homogeneity of $1 = (1', 1'')$ of degree $d = d' = d''$. □

1.1.5. *Tensor product*

Set $\mathcal{A}' \otimes \mathcal{A}'' := \langle A, \circ, \eta, 1\rangle = \langle A' \otimes A'', \circ' \otimes \circ'', \eta' \otimes \eta'', 1' \otimes 1''\rangle$, for example, $(a', a'') \circ (b', b'') = (a' \circ' b', a'' \circ'' b'')$. The unit is $1 = 1' \otimes 1''$ and the co-unit is $\epsilon = \epsilon' \otimes \epsilon''$.

There are no conditions for grading, i.e. if both Frobenius algebras are I-graded there is a natural induced I-grading on their tensor product. The unit is of degree $d = d' + d''$, the co-unit has degree $D' + D''$ and the multiplication is homogeneous of degree $d = d' + d''$.

1.1.6. *Super-grading*

For an element a of a super vector space $A = A_0 \oplus A_1$ denote by $\tilde{a}$ its $\mathbf{Z}/2\mathbf{Z}$ degree, i.e. $\tilde{a} = 0$ if it is even ($a \in A_0$) and $\tilde{a} = 1$ if it is odd ($a \in A_1$).

Definition 1.5. A super Frobenius algebra over a field K of characteristic 0 is $\langle A, \circ, \eta, 1\rangle$, where

A is a finite dim K-super vector space,
$\circ$ is a multiplication on $A : \circ : A \otimes A \to A$,
which preserves the $\mathbf{Z}/2\mathbf{Z}$-grading
η is a non-degenerate even bilinear form on A, and
1 is a fixed even element in A_0 — the unit,

satisfying the following axioms:

(a) Associativity:

$$(a \circ b) \circ c = a \circ (b \circ c)\,.$$

(b) Super-commutativity:

$$a \circ b = (-1)^{\tilde{a}\tilde{b}} b \circ a\,.$$

(c) Unit:

$$\forall \in a : 1 \circ a = a \circ 1 = a\,.$$

(d) Invariance:

$$\eta(a, b \circ c) = \eta(a \circ b, c)\,.$$

The grading for super Frobenius algebras carries over verbatim.

1.2. *Cobordisms*

Definition 1.6. Let $\mathcal{COB}$ be the category whose objects are one-dimensional closed oriented (topological) manifolds considered up to orientation preserving homeomorphism and whose morphisms are cobordisms of these objects, i.e. $\Sigma \in \mathrm{Hom}(S_1, S_2)$ if Σ is an oriented surface with boundary $\partial\Sigma \equiv -S_1 \amalg S_2$.

The composition of morphisms is given by gluing along boundaries with respect to orientation reversing homeomorphisms.

Remark 1.3. The operation of disjoint union makes this category into a monoidal category with unit $\emptyset$.

Remark 1.4. The objects can be chosen to be represented by disjoint unions of the circle with the natural orientation S^1 and the circle with opposite orientation $\bar{S}^1$. Thus a typical object looks like $S = \amalg_{i \in I} S^1 \amalg_{j \in J} \bar{S}^1$. Two standard morphisms are given by the cylinder, and thrice punctured sphere.

Definition 1.7. Let $\mathcal{VECT}_K$ be the monoidal category of finite dimensional K-vector spaces with linear morphisms with the tensor product providing a monoidal structure with unit K.

Theorem 1.1. (Atiyah, Dijkgraaf, Dubrovin [1, 3, 4]). *There is a* 1–1 *correspondence between Frobenius algebras over* K *and isomorphism classes of covariant functors of monoidal categories from* $\mathcal{COB}$ *to* $\mathcal{VECT}_K$, *natural with respect to orientation preserving homeomorphisms of cobordisms and whose value on cylinders* $S \times I \in \mathrm{Hom}(S, S)$ *is the identity.*

Under this identification, the Frobenius algebra A is the image of S^1, the multiplication or rather μ is the image of a thrice punctured sphere and the metric is the image of an annulus.

2. Orbifold Frobenius Algebras

2.1. *G-Frobenius algebras*

We fix a finite group G and denote its unit element by e.

Definition 2.1. A G-twisted Frobenius algebra (FA) over a field K of characteristic 0 is $\langle G, A, \circ, 1, \eta, \varphi, \chi\rangle$, where

- G finite group;
- A finite dim G-graded K-vector space, $A = \oplus_{g\in G} A_g$, A_e is called the untwisted sector and the A_g for $g \neq e$ are called the twisted sectors;
- $\circ$ a multiplication on A which respects the grading: $\circ : A_g \otimes A_h \to A_{gh}$;
- 1 a fixed element in A_e-the unit;
- η non-degenerate bilinear form, which respects grading i.e. $g|_{A_g \otimes A_h} = 0$ unless $gh = e$;
- φ an action of G on A (which will be by algebra automorphisms), $\varphi \in \mathrm{Hom}(G, \mathrm{Aut}(A))$, s.t. $\varphi_g(A_h) \subset A_{ghg^{-1}}$;
- χ a character $\chi \in \mathrm{Hom}(G, K^*)$;

satisfying the following axioms:

We use a subscript on an element of A to signify that it has homogeneous group degree — e.g. a_g means $a_g \in A_g$ — and we write $\varphi_g := \varphi(g)$ and $\chi_g := \chi(g)$,

(a) Associativity:

$$(a_g \circ a_h) \circ a_k = a_g \circ (a_h \circ a_k)\,.$$

(b) Twisted commutativity:

$$a_g \circ a_h = \varphi_g(a_h) \circ a_g\,.$$

(c) G-invariant unit:

$$1 \circ a_g = a_g \circ 1 = a_g \text{ and } \varphi_g(1) = 1\,.$$

(d) Invariance of the metric:

$$\eta(a_g, a_h \circ a_k) = \eta(a_g \circ a_h, a_k)\,.$$

(i) Projective self-invariance of the twisted sectors

$$\varphi_g|A_g = \chi_g^{-1}\mathrm{id}\,.$$

(ii) G-invariance of the multiplication:

$$\varphi_k(a_g \circ a_h) = \varphi_k(a_g) \circ \varphi_k(a_h)\,.$$

(iii) Projective G-invariance of the metric:

$$\varphi_g^*(\eta) = \chi_g^{-2}\eta\,.$$

(iv) Projective trace axiom:

$$\forall\, c \in A_{[g,h]} \text{ and } l_c \text{ left multiplication by } c\text{:}$$
$$\chi_h \operatorname{Tr}(l_c\varphi_h|_{A_g}) = \chi_{g^{-1}} \operatorname{Tr}(\varphi_{g^{-1}} l_c|_{A_h})\,.$$

An alternate choice of data is given by a one-form ϵ, the co-unit with $\epsilon \in A_e^*$ and a three-tensor $\mu \in A^* \otimes A^* \otimes A^*$ which is of group degree e, i.e. $\mu|_{A_g \otimes A_h \otimes A_k} = 0$ unless $ghk = e$.

The relations between η, $\circ$ and ϵ, μ are analogous to those of the Remark 1.1.

Again, we denote by $\rho \in A_e$ the element dual to $\epsilon \in A_e^*$ and Poincaré dual to $1 \in A_e$.

Remark 2.1. (1) A_e is central by twisted commutativity and $\langle A_e, \circ, \eta|_{A_e \otimes A_e}, 1\rangle$ is a Frobenius algebra.

(2) All A_g are A_e-modules.

(3) Notice that χ satisfies the following equation which completely determines it in terms of φ. Setting $h = e$, $c = 1$ in Axiom (iv)

$$\dim A_g = \chi_{g^{-1}} \operatorname{Tr}(\varphi_g|_{A_e}) \tag{2.1}$$

by Axiom (iii) the action of φ on ρ determines χ up to a sign

$$\chi_g^{-2} = \chi_g^{-2}\eta(\rho, 1) = \eta(\varphi_g(\rho), \varphi_g(1)) = \eta(\varphi_g(\rho), 1)\,. \tag{2.2}$$

(4) Axiom (iv) forces the χ to be group homomorphisms, so it would be enough to assume in the data that they are just maps.

Proposition 2.1. *The G invariants A^G of a G-Frobenius algebra A form an associative and commutative algebra with unit. This algebra with the induced bilinear form is a Frobenius algebra if and only if $\sum_g \chi_g^{-2} = |G|$. If $K = \mathbf{C}$ and $\chi \in \operatorname{Hom}(G, U(1))$ this implies $\forall\, g : \chi_g = \pm 1$.*

Proof. Due to Axiom (ii), the algebra is associative and commutative. And since 1 is G invariant, the algebra has a unit.

Now suppose $\sum_g \chi_g^{-2} = |G|$. Then $\eta|_{A^G \otimes A^G}$ is non-degenerate: let $a \in A^G$ and choose $b \in A$ such that $\eta(a, b) \neq 0$. Set $\tilde{b} = \frac{1}{|G|}\sum_{g \in G} \varphi_g(b) \in A^G$. Then

$$\eta(a, \tilde{b}) = \frac{1}{|G|}\sum_{g \in G} \eta(a, \varphi_g(b)) = \frac{1}{|G|}\sum_{g \in G} \eta(\varphi_g(a)\varphi_g(b))$$
$$= \frac{1}{|G|}\sum_{g \in G} \chi_g^{-2}\eta(a, b) = \eta(a, b) \neq 0\,.$$

On the other hand if $\eta|_{A^G \otimes A^G}$ is non-degenerate then let $a, b \in A^G$ be such that $\eta(a, b) = 1$. It follows:

$$1 = \eta(a,b) = \frac{1}{|G|}\sum_{g\in G}\eta(\varphi_g(a),\varphi_g(b)) = \frac{1}{|G|}\sum_{g\in G}\chi_g^{-2}\eta(a,b) = \frac{1}{|G|}\sum_{g\in G}\chi_g^{-2}$$

so that $\sum_g \chi_g^{-2} = |G|$.

The last statement follows from the simple fact that since $\forall\, g \in G : |\chi_g| = 1$ and if $\sum_g \chi_g^{-2} = |G|$ then $\chi_g^{-2} = 1$ and hence $\chi_g \in \{-1, 1\}$. □

Definition 2.2. A G-Frobenius algebra is called an orbifold model if the data $\langle A^G, \circ, 1\rangle$ can be augmented by a compatible metric to yield a Frobenius algebra. In this case, we call the Frobenius algebra A^G a G-orbifold Frobenius algebra.

2.2. *Super-grading*

We can enlarge the framework by considering super-algebras rather than algebras. This will introduce the standard signs.

Definition 2.3. A G-twisted Frobenius super-algebra over a field K of characteristic 0 is $\langle G, A, \circ, 1, \eta, \varphi, \chi\rangle$, where

- G finite group;
- A finite dimensional $\mathbf{Z}/2\mathbf{Z} \times G$-graded K-vector space;
  $A = A_0 \oplus A_1 = \oplus_{g\in G}(A_{g,0} \oplus A_{g,1}) = \oplus_{g\in G} A_g$,
  A_e is called the untwisted sector and is even,
  the A_g for $g \neq e$ are called the twisted sectors;
- $\circ$ a multiplication on A which respects both gradings:
  $\circ : A_{g,i} \otimes A_{h,j} \to A_{gh,i+j}$;
- 1 a fixed element in A_e — the unit;
- η non-degenerate even bilinear form, which respects grading
  i.e. $g|_{A_g \otimes A_h} = 0$ unless $gh = e$;
- φ an action by even algebra automorphisms of G on A,
  $\varphi \in \mathrm{Hom}_{K-\mathrm{alg}}(G, A)$, such that $\varphi_g(A_h) \subset A_{ghg^{-1}}$;
- χ a character $\chi \in \mathrm{Hom}(G, K^*)$ or if $K = \mathbf{C}$, $\chi \in \mathrm{Hom}(G, U(1))$;

satisfying Axioms (a)–(d) and (i)–(iii) of a G-Frobenius algebra with the following alteration:

(b$^\sigma$) Twisted super-commutativity:

$$a_g \circ a_h = (-1)^{\tilde{a}_g \tilde{a}_h}\varphi_g(a_h) \circ a_g\,.$$

(iv$^\sigma$) Projective super-trace axiom:

$$\forall\, c \in A_{[g,h]} \text{ and } l_c \text{ left multiplication by } c\text{:}$$

$$\chi_h\,\mathrm{STr}(l_c\varphi_h|_{A_g}) = \chi_{g^{-1}}\,\mathrm{STr}(\varphi_{g^{-1}} l_c|_{A_h})\,,$$

where STr is the super-trace.

2.2.1. *Operations*

Restriction: If $H \subset G$ and $A = \oplus_{g\in G} A_g$ then $\tilde{A} := \oplus_{h\in H} A_h$ is naturally a H-Frobenius algebra.

Direct Sum: Given a G-Frobenius algebra A and an H-Frobenius algebra B then $A \oplus B$ is naturally a $G \times H$-Frobenius algebra with the graded pieces $(A \oplus B)_{(g,h)} = A_g \oplus B_h$.

We define the direct sum of two G-Frobenius algebras to be the G-Frobenius subalgebra corresponding to the diagonal $\Delta : G \to G \times G$ in $A \oplus A$.

Tensor product: Given a G-Frobenius algebra A and an H-Frobenius algebra B then $\oplus_{(g,h)}(A_g \otimes B_h)$ is naturally a $G \times H$-Frobenius algebra $(A \otimes B)_{(g,h)} = A_g \otimes B_h$.

We define the tensor product of two G-Frobenius algebras to be the G-Frobenius subalgebra corresponding to the diagonal $G \to G \times G$ in $A \otimes A$.

Braided Tensor product: If A and B are two G-Frobenius algebras with the same character χ, we can define a braided tensor product structure on $A \otimes B$ by setting $(A \otimes B)_g := \oplus_{k\in G} A_k \otimes B_{k^{-1}g}$. For the multiplication we use the sequence

$$A_k \otimes B_{k^{-1}g} \otimes A_l \otimes B_{l^{-1}h} \xrightarrow{(\mathrm{id}\otimes\mathrm{id}\otimes\varphi_{k^{-1}g}\otimes\mathrm{id})\circ\tau_{2,3}} A_k \otimes A_{k^{-1}glg^{-1}k} \otimes B_{k^{-1}g} \otimes B_{l^{-1}h}$$
$$\stackrel{\circ\otimes\circ}{\longrightarrow} A_{glg^{-1}k} \otimes B_{k^{-1}gl^{-1}h} \tag{2.3}$$

and $\oplus_k(\varphi_k \otimes \varphi_{kh^{-1}})$ for the action of h on $\oplus_k A_k \otimes B_{k^{-1}g}$.

Remark 2.2. If one thinks in terms of cohomology of spaces the direct sum corresponds to the disjoint union and the tensor product corresponds to the Cartesian product. The origin of the braided tensor product, however, is not clear yet.

2.3. *Geometric model — spectral flow*

The axioms of the G-Frobenius algebra are well suited for taking the quotient, since the multiplication is G-invariant. However, this is not the right framework for a geometric interpretation. In order to accommodate a more natural co-boundary description, we need the following definition which corresponds to the physical notion of Ramond ground states:

Definition 2.4. A Ramond G-algebra over a field K of characteristic 0 is $\langle G, V, \bar{\circ}, v, \bar{\eta}, \bar{\varphi}, \chi\rangle$,

G finite group;
V finite dim G-graded K-vector space,
$V = \oplus_{g\in G} V_g$,
V_e is called the untwisted sector and the
V_g for $g \neq e$ are called the twisted sectors;

$\bar{\circ}$ a multiplication on V which respects the grading:
$\bar{\circ} : V_g \otimes V_h \to V_{gh}$;
v a fixed element in V_e — the unit;
$\bar{\eta}$ non-degenerate bilinear form, which respects grading
i.e. $\bar{\eta}|_{V_g \otimes V_h} = 0$ unless $gh = e$;
$\bar{\varphi}$ an action by of G on V,
$\bar{\varphi} \in \mathrm{Hom}(G, \mathrm{Aut}(V))$, such that $\bar{\varphi}_g(V_h) \subset V_{ghg^{-1}}$;
χ a character $\chi \in \mathrm{Hom}(G, K^*)$;

satisfying the following axioms:

We use a subscript on an element of V to signify that it has homogeneous group degree — e.g. v_g means $v_g \in V_g$ — and we write $\bar{\varphi}_g := \bar{\varphi}(g)$ and $\chi_g := \chi(g)$,

(a) Associativity:

$$(v_g \bar{\circ} v_h) \bar{\circ} v_k = v_g \bar{\circ} (v_h \bar{\circ} v_k)\,.$$

(b′) Projective twisted commutativity:

$$v_g \bar{\circ} v_h = \chi_g \bar{\varphi}_g(v_h) \bar{\circ} v_g = \bar{\varphi}_g(v_h \bar{\circ} v_g)\,.$$

(c′) Projectively invariant unit:

$$v \bar{\circ} v_g = v_g \bar{\circ} v = v_g \quad \text{and} \quad \bar{\varphi}_g(v) = \chi_g v\,.$$

(d) Invariance of the metric:

$$\eta(v_g, v_h \bar{\circ} v_k) = \eta(v_g \bar{\circ} v_h, v_k)\,.$$

(1′) Self-invariance of the twisted sectors:

$$\bar{\varphi}_g | V_g = \mathrm{id}\,.$$

(2′) Projective G-invariance of multiplication:

$$\bar{\varphi}_k(v_g \bar{\circ} v_h) = \chi_k \bar{\varphi}_k(v_g) \bar{\circ} \bar{\varphi}_k(v_h)\,.$$

(3′) G-Invariance of metric:

$$\bar{\varphi}_g^*(\bar{\eta}) = \bar{\eta}\,.$$

(4′) Trace axiom:

$$\forall\, c \in V_{[g,h]} \text{ and } l_c \text{ left multiplication by } c\text{:}$$

$$\mathrm{Tr}(l_c \circ \bar{\varphi}_g|_{V_h}) = \mathrm{Tr}(\bar{\varphi}_{h^{-1}} \circ l_c|_{V_g})\,.$$

Definition 2.5. A state-space for a G-Frobenius algebra A is a quadruple $\langle V, v, \bar{\eta}, \bar{\varphi}\rangle$, where

V is a G-graded free rank one A-module: $V = \oplus_{g \in G} V_g$,
v is a fixed generator of V — called the vacuum,

$\bar{\eta}$ is non-degenerate bilinear form on V,
$\bar{\varphi}$ is a linear G-action on V fixing v projectively,
i.e. $\bar{\varphi}(g)(\text{span}(v)) \subset \text{span}(v)$,

such that these structures are compatible with those of A (we denote $\bar{\varphi}_g := \bar{\varphi}(g)$)

(a) The action of A respects the grading: $A_g V_h \subset V_{gh}$.
(b) V_h is a rank one free A_h-module and $V_h = A_h v$.
(c) $\bar{\varphi}_g(av) = \varphi_g(a)\bar{\varphi}(v) : \forall\, a \in A,\ v \in V$.
(d) For $a, b \in A : \bar{\eta}(av, bv) = \eta(a, b)$.
(e) $\forall\, g,\, h \in G,\ c \in A_{[g,h]}\ \forall\, c \in V_{[g,h]}$ and l_c left multiplication by c: $\text{Tr}(l_c \circ \bar{\varphi}_g|_{V_h}) = \text{Tr}(\bar{\varphi}_{h^{-1}} \circ l_c|_{V_g})$.

Definition 2.6. We call two state spaces isomorphic, if there is an A-module isomorphism between the two.

Since state spaces are free rank one A-modules, it is clear that all automorphisms are re-scalings of v.

Proposition 2.2. *Given a G-twisted Frobenius algebra A there is a unique state space up to isomorphism and the form $\bar{\eta}$ is G-invariant (i.e. $\bar{\varphi}_g^*(\bar{\eta}) = \bar{\eta}$).*

Proof. We start with a free rank one A-module V and reconstruct all other data. The G-grading on V is uniquely determined from that of A by Axiom (b) and this grading satisfies Axiom (a). Up to isomorphism, we may assume a generator $v \in V$ is fixed, then Axiom (d) determines $\bar{\eta}$ from η. We denote the Eigenvalue of $\bar{\varphi}_g$ on v by λ_g: $\bar{\varphi}_g v = \lambda_g v$. Notice that due to (c) $\bar{\varphi}$ is determined by λ_g. Using Axioms (b), (c) and (e), we find that $\lambda_g = \chi_g$, thus fixing the G-action $\bar{\varphi}$.

Namely with $c = 1$ and $h = e$ in (e):

$$\text{Tr}(\bar{\varphi}_g|_{V_e}) = \lambda_g \,\text{Tr}(\varphi_g|_{A_e}) = \lambda_g \chi_g^{-1} \,\text{Tr}(\varphi_e|_{A_g}) = \lambda_g \chi_g^{-1} \,\text{Tr}(\varphi_e|_{V_g})\,.$$

The equality $\bar{\varphi}_g = \varphi_g \lambda_g$ implies that $\bar{\eta}$ is $\bar{\varphi}$ invariant:

$$\bar{\eta}(\bar{\varphi}_g(av), \bar{\varphi}_g(bv)) = \lambda_g^2 \eta(\varphi_g(a), \varphi_g(b)) = \lambda_g^2 \chi_g^{-2} \eta(a, b) = \bar{\eta}(av, bv)\,.$$

In general,

$$\begin{aligned}
\text{Tr}(l_c \circ \bar{\varphi}_g|_{V_h}) &= \lambda_g \,\text{Tr}(l_c \circ \varphi_g|_{A_h}) \\
&= \lambda_g \chi_g^{-1} \chi_{h^{-1}} \,\text{Tr}(\varphi_{h^{-1}} \circ l_c|_{A_g}) \\
&= \lambda_g \chi_g^{-1} \chi_{h^{-1}} \lambda_{h^{-1}}^{-1} \,\text{Tr}(\bar{\varphi}_{h^{-1}} \circ l_c|_{V_g}) \\
&= \text{Tr}(\bar{\varphi}_{h^{-1}} \circ l_c|_{V_g})\,,
\end{aligned}$$

so that with this choice of $\bar{\varphi}$, Axiom (e) is satisfied. □

Remark 2.3. A state space inherits an associative multiplication $\bar{\circ}$ with unit from A via

$$\forall\, a, b \in A : (av)\bar{\circ}(bv) := (a \circ b)v\,. \tag{2.4}$$

This multiplication makes it into a G-Ramond algebra.

This fact leads us to the following definitions:

Definition 2.7. The Ramond space of a G-Frobenius algebra A is the state-space given by the G-graded vector-space

$$V := \bigoplus_g V_g := \bigoplus_g A_g \otimes K$$

together with the G-action $\bar{\varphi} := \varphi \otimes \chi$, the induced metric $\bar{g}$ and the induced multiplication $\bar{\circ}$ and fixed element $v = 1 \otimes 1$.

Theorem 2.1. *There is a one-to-one correspondence between isomorphism classes of G-Ramond algebras and G-Frobenius algebras.*

Proof. The association of a Ramond space to a G-Frobenius algebra provides the correspondence. The inverse being the obvious reverse twist by χ. □

Remark 2.4. In the theory of singularities, the untwisted sector of the Ramond space corresponds to the forms $H^{n-1}(V_\epsilon, \mathbf{C})$ while the untwisted sector of the G-twisted Frobenius algebra corresponds to the Milnor ring [24]. These are naturally isomorphic, but have different G-module structures. In that situation, one takes the invariants of the Ramond sector, while we will be interested in invariants of the G-Frobenius algebra and not only the untwisted sector (cf. [18] and see also Sec. 7).

3. Bundle Cobordisms, Finite Gauge Groups, Orbifolding and *G*-Ramond Algebras

In this section, we introduce two cobordism categories which correspond to G-orbifold Frobenius algebras and Ramond G-algebras, respectively. Again G is a fixed finite group.

3.1. *Bundle cobordisms*

In all situations, gluing along boundaries will induce the composition and the disjoint union will provide a monoidal structure.

Definition 3.1. Let $\mathcal{GBCOB}$ be the category whose objects are principal G-bundles over one-dimensional closed oriented (topological) manifolds, pointed over each component of the base space, whose morphisms are cobordisms of these objects (i.e. principal G-bundles over oriented surfaces with pointed boundary).

More precisely, $B_\Sigma \in \mathrm{Hom}(B_1, B_2)$ if Σ is an oriented surface with boundary $\partial\Sigma = -S_1 \amalg S_2$ and B_Σ is a bundle on Σ which restricts to B_1 and B_2 on the boundary.

The composition of morphisms is given by gluing along boundaries with respect to orientation reversing homeomorphisms on the base and covering bundle isomorphisms which align the base-points.

Remark 3.1. The operation of disjoint union makes this category into a monoidal category with unit $\emptyset$ formally regarded as a principal G bundle over $\emptyset$.

Remark 3.2. Typical objects are bundles B over $S = \amalg_{i\in I} S^1 \amalg_{j\in J} \bar{S}^1$.

Let $((S^1, \nu)\ \nu \in S^1)$ be a pointed S^1.

Lemma 3.1 (Structure Lemma). *The space* $\mathrm{Bun}(S^1, G)$ *of* G *bundles on* (S^1, ν) *can be described as follows*:

$$\mathrm{Bun}(S^1, G) = (G \times F)/G$$

where F *is a generic fibre regarded as a principal* G*-space and* G *acts on itself by conjugation and the quotient is taken by the diagonal action. The space* $\mathrm{Bun}^*(S^1, G)$ *of isomorphism classes of pointed* G *bundles on* S^1 *is in a bijective correspondence with* G, *where the map is given by evaluating the monodromy. In terms of* $\mathrm{Bun}(S^1, G) \times F$ *the monodromy map gives a projection of onto* G *whose fibre over* $g \in G$ *are the centralizers* $Z(g)$, *which are exactly the isomorphisms in* $\mathrm{Bun}^*(S^1, G)$.

Remark 3.3. Usually one uses the identification

$$\mathrm{Bun}(M, G) = \mathrm{Hom}(\pi_1(M), G)/G\,,$$

which we also use in the proof. However, for certain aspects of the theory — more precisely to glue and to include non-trivial characters — it is vital to include a point in the bundle and a trivialization rather than just a point in the base.

Proof of Lemma 3.1. Given a pointed principal G bundle (B, S^1, π, F, G), $b \in B$ we set $\nu = \pi(b) \in S^1$. The choice of $b \in F = B_\nu$ gives an identification of F with G, i.e. we let $\beta : G \mapsto F$ be the admissible map in the sense of [20] that satisfies $\beta(e) = b$. We set $g \in G$ to be the element corresponding to the monodromy around the generator of $\pi_1(S^1)$. Notice that since we fixed an admissible map everything is rigid — there are no automorphisms — and the monodromy is given by an element, not a conjugacy class. Thus we associate to (B, b) the tuple (g, b).

Vice-versa, given (g, b) we start with the pointed space (S^1, ν) and construct the bundle with fibre G, monodromy g and marked point $b = \beta(e) \in B_\nu$.

That this construction is bijective follows by the classical results quoted above [20].

The choice in this construction corresponds to a choice of a point $b \in B$. Changing b amounts to changing β and the monodromy g. Moreover, moving the point $\nu = \pi(b)$ and moving b inside the fibre by parallel transport keeps everything fixed. Moving b inside the fixed fibre by translation (once ν is fixed) corresponds to translation by the group action in the fibre i.e. the translation action of G on F and simultaneous conjugation of the monodromy. Hence, the first claim follows. The second claim follows from the third which in turn follows from the observation that evaluating the monodromy choosing two different points of F bundles yields the same result if and only if the underlying bundles are isomorphic and the chosen points are related by a shift with an element in the centralizer of the monodromy. □

This observation leads us to the following definition:

Definition 3.2. We call a bundle over a closed one-dimensional manifold rigidified if its components are labeled and the bundle is pointed above each component of the base and trivializations around the projection of the marked points to the base are fixed. We denote such a bundle (B, b) where $b \subset B : b = (b_0, \dots, b_n)$ is the set of base-points for each component.

Furthermore, if a given surface has genus zero we realize it in the plane as a pointed disc with all boundaries being S^1. We label the outside circle by 0.

If π is the bundle projection, we set $x_i := \pi(b_i)$, we call b_0 the base-point, the B_{x_i} the special fibres and B_{x_0} the initial fibre.

3.2. *Rigidified bundle cobordisms and G-Frobenius algebras*

Definition 3.3. Let $\mathcal{GBCOB}^*$ be the category whose objects are rigidified principal G-bundles over one-dimensional closed oriented (topological) manifolds considered up to pull-back under orientation preserving homeomorphism of the base respecting the markings and whose morphisms are cobordisms of these objects (i.e. principal G-bundles over oriented surfaces with boundary together with rigidification on the boundary, a choice of base-point $x_0 \in \partial\Sigma$ compatible with the rigidification and a choice of curves Γ_i from x_0 to x_i which identify the trivializations via parallel transport, where we used the notation above).

That is to say objects are bundles B over $S = \amalg_{i\in I} S^1 \amalg_{j\in J} \bar{S}^1$ with base-points on the various components $\bar{b}_1 = (b_i \in B|_{S^1_i} : i \in I)$, $\bar{b}_2 = (b_j \in B|_{\bar{S}^1_j} : j \in J)$ and $B_\Sigma \in \mathrm{Hom}((B_1, \bar{b}_1), (B_2, \bar{b}_2))$ if Σ is an oriented surface with boundary considered up to orientation preserving homeomorphism with boundary $\partial\Sigma = -S_1 \amalg S_2$ — again up to homeomorphism — and B_Σ is a bundle on Σ which restricts to B_1 and B_2 on the boundary together with rigidification data for the boundary i.e. $(\bar{b}_1, \bar{b}_2)$ with $\bar{b}_1 \subset B_1$ and $\bar{b}_2 \subset B_2$. We will call S_1 the inputs and S_2 the outputs.

The extra structure of curves and base-point allows us to identify the special fibres with the initial fibre via parallel transport. Thus we can describe the rigidification data in terms of b_0, and the group elements g_i defined via $\Gamma_i(b_0) = g_i b_i$.

The composition of morphisms is given by gluing along boundaries with respect to orientation reversing homeomorphisms of the base and covering bundle isomorphisms identifying the base-points.

Remark 3.4. The operation of disjoint union makes this category into a monoidal category with unit G regarded as a bundle over $\emptyset$ with base point e.

Construction 3.1. We define the pointed bundle (g, h) to be the bundle which is obtained by gluing $I \times G$ via the identification $(0, e) \sim (1, g)$ and marking the point $(0, h)$, where $I = [0, 1]$ the standard interval.

This produces all monoidal generators.

Remark 3.5. By the Construction 3.1 up to reversing the orientation, we can produce the monoidal generators of objects of $\mathcal{GBCOB}^*$ with the objects coming from $G \times G$.

A generating object can thus be depicted by an oriented circle with labels (g, h) where $(g, h) \in G \times G$. We use the notation (g, h) for positively oriented circles and $\overline{(g, h)}$ for negatively oriented circles. We will consider functors V with an involutive property. In this case $V(\overline{(g, h)}) \simeq V((g^{-1}, h))^*$ where $*$ denotes the dual. General objects are then just disjoint unions of these, i.e. tuples (g_i, h_i). The homomorphisms on the generators which are given by the trivial bundle cylinder with different trivializations on both ends represent the natural diagonal action of G by conjugation and translation described in Lemma 3.1, so that this diagonal G-action is realized in terms of cobordisms.

The natural $G \times G$ action, however, cannot be realized by these cobordisms and we would like to enrich our situation to this case by adding morphisms corresponding to the G action on the trivializations.

Definition 3.4. Let $\mathcal{RGBCOB}$ be the category obtained from $\mathcal{GBCOB}^*$ by adding the following morphisms. For any n-tuple $(k_1, \ldots, k_n) : k_i \in G$ and any object $(g_i, h_i) : i = 1, \ldots n$ we set

$$\tau(k_1, \ldots, k_n)(g_i, h_i)_{i \in \{1, \ldots n\}} := (g_i, k_i h_i)_{i \in \{1, \ldots n\}} .$$

We call these morphisms of type II and the morphisms coming from $\mathcal{GBCOB}^*$ of type I. We also sometimes write II_k for τ_k.

Remark 3.6. There is a natural forgetful functor from $\mathcal{RGBCOB}$ to $\mathcal{GBCOB}$.

Given a character $\chi \in \mathrm{Hom}(G, K^*)$ we can form the fibre-product $G \times_\chi K$. This gives a functor $K[G]_\chi$ from the category $\mathcal{VECT}_K$ to $K[G]$-$\mathcal{MOD}$, the category of $K[G]$-modules.

Definition 3.5. A G_χ-orbifold theory is a monoidal functor V from $\mathcal{RGBCOB}$ to $K[G]$-$\mathcal{MOD}$ satisfying the following conditions:

(i) The image of V lies in the image of $K[G]_\chi$.
(ii) Objects of $\mathcal{RGBCOB}$ which differ by morphisms of type II are mapped to the same object in $K[G]$-$\mathcal{MOD}$.
(iii) The value on morphisms of type I does not depend on the choice of connecting curves and associated choice of trivializations or base-point.

(iv) The morphisms of type II are mapped to the G-action by χ.
(v) The functor is natural with respect to morphisms of the type $\tau(k,\dots,k)$. That is, $V(\tau(k,\dots,k)\circ\Sigma) = V(\tau_{\text{out}}(k^{-1},\dots,k^{-1}))\circ V(\Sigma)\circ V(\tau_{\text{in}}(k,\dots,k))$; where τ_{in} and τ_{out} operate on the inputs and outputs respectively.
(vi) V associates id to cylinders $B\times I$, $(b,0)\in B\times 0$, $(b,1)\in B\times 1$ considered as cobordisms from (B,b) to itself.
(vii) V is involutive: $V(\bar{S}) = V^*(S)$ where $*$ denotes the dual vector space with induced $K[G]$-module structure. In accordance, the morphism of type II commute with involution, i.e. they are mapped to the G-action by χ^{-1}.
(viii) The functor is natural with respect to orientation preserving homeomorphisms of the underlying surface of a cobordism and pull-back of the bundle.

Corollary 3.1. *G_χ-orbifold theories are homotopy invariant.*

Proof. By standard arguments using naturality and (vi) a homotopy of an object S induces an identity on its image. More precisely given a homotopy of objects $f_t : S \mapsto S$, it induces a map $F : S\times I \mapsto S\times I$ and the following diagram is commutative:

$$\begin{array}{ccc} V(S) & \xrightarrow{V(S\times I)=\text{id}} & V(S) \\ V(f_0)\downarrow & & \downarrow V(f_1) \\ V(S) & \xrightarrow{V(S\times I)=\text{id}} & V(S)\,. \end{array}$$

In particular, if $V(f_0)=\text{id}$ then $V(f_1)=\text{id}$. □

Remark 3.7. Given a choice of connecting curves, we can identify all fibres over special points. Therefore after fixing one identification of a fibre with G, we can identify the other marked points as translations of points of parallel transport and label them by group elements, which we will do.

The action of $\tau(k,\dots,k)$ corresponds to a change of identification for one point and simultaneous re-gauging of all other points via this translation, i.e. a diagonal gauging. Therefore given a cobordism, we can fix an identification of all fibres with G.

Notation 3.1. We will fix some standard bundle cobordisms pictured in Fig. 1.

I: The *standard disc bundle* is the disc with a trivial bundle and positively oriented boundary considered as a cobordism between $\emptyset$ and (e,e); it will be denoted by D.

II: The *standard g-cylinder bundle* is the cylinder $S^1\times I$ with the bundle having monodromy g around $((S^1,0))$ considered as a cobordism between $(g,e)\amalg\overline{(g,e)}$ and $\emptyset$; it will be denoted by C_g.

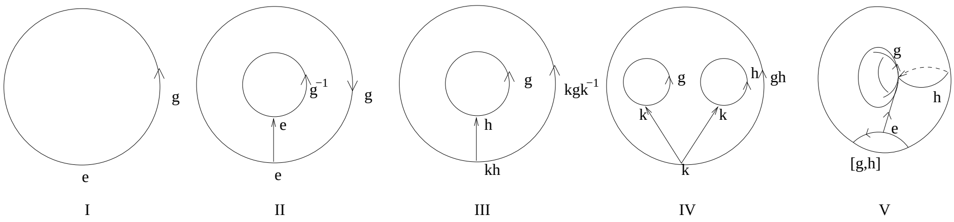

Fig. 1. Standard bundle cobordisms.

III: The $(g,h)^k$-*cylinder bundle* is the cylinder with a bundle having monodromy g around $(S^1, 0)$ considered as a cobordism between (g,h) and (kgk^{-1}, kh); it will be denoted by $C^h_{g,k}$.

IV: The *standard* $(g,h)^k$-*trinion bundle* is the trinion with the bundle having monodromies g around the first S^1 and h around the second S^1 and translations e for τ_{01}, τ_{02} considered as a cobordism between (g,k), (h,k) and (gh,k); it will be denoted by $T^k_{g,h}$.

V: The (g,h)-*torus bundle* is the once-punctured torus with the principal G-bundle having monodromies g and h around the two standard cycles considered as a cobordism between $([g,h],e)$ and $\emptyset$; it will be denoted by $E^k_{g,h}$.

Lemma 3.2. *The bundles over a cylinder that form cobordisms between* (g,e) *and* (h,k) *are parameterized by* G; *it is necessary that* $g = khk^{-1}$. *These cobordisms are given by the* $C^e_{g,h}$.

Proof. Given such a bundle over the cylinder $\Sigma_0 := S^1 \times I$ the translation from $B_{\nu,0}$ to $B_{\nu,1}$ along $\gamma(t) := (\nu, t)$ is a complete invariant. We fix this element $k \in G$. Since in $\pi_1(\Sigma_0, (\nu, 0))$ $C_1 = \gamma C_2 \gamma^{-1}$, it follows that $g = khk^{-1}$. □

Proposition 3.1. *To fix a* G_χ*-orbifold theory on the objects of the type* (g,e) *and to fix a* G_χ*-orbifold theory on the morphisms it suffices to fix it on bundles over the standard cylinder bundle* C, *the* (g,h)*-cylinder bundles* $C_{g,h}$, *and on the standard trinion* T.

Proof. The first claim follows from the condition (ii) and the monoidal structure. For the second claim, first notice that, due to the homotopy Lemma 3.1, V is fixed on D. Also, any bundle over the cylinder is trivial, furthermore by (v), we may regard the cylinder as a cobordism from $B|_{S^1}$ to itself and after applying a morphism of type II we can assume that the boundary objects are of the type (g,e), (h,k). Therefore we know the functor V on all bundles over cylinders. Dualizing and gluing on cylinders, we find that once the functor is defined on the standard trinion, it is defined on all bundles over all trinions. Lastly, given any surface, we can choose a decomposition by a marking into discs, cylinders and trinions, then gluing determines the value of V on this surface. □

Proposition 3.2. *For any G_χ-orbifold theory V set,*

$$V((g,e)) = V_g\,,$$
$$V(T^e_{g,h}) = \bar{\circ} : V_g \otimes V_h \to V_{gh}\,,$$
$$V(D)(1_k) = v\,,$$
$$V(C_g) = \bar{\eta}|_{V_g \otimes V_{g^{-1}}}\,,$$
$$V(C_{g,h}) = \bar{\varphi}_h : V_g \mapsto V_{hgh^{-1}}\,,$$

then this data together with g and χ form a Ramond-G algebra $\langle G, V, \bar{\circ}, v, \bar{\eta}, \bar{\varphi}, \chi\rangle$ which we call the associated G-Ramond algebra to V.

Proof. It is clear by Proposition 3.1 that given a G_χ orbifold theory it is completely fixed by its associated G-Ramond algebra. The converse is also true:

The Axiom (a) follows from the standard gluing procedures of TFT. That is, the usual gluing of a disc with 3 holes from two discs with two holes in two different ways, as shown in Fig. 2.

For Axiom (b′), we regard the following commutative diagrams

$$\begin{array}{ccccccc}
(g,e) \amalg (h,e) & \xrightarrow{\tau_{12}} & (h,e) \amalg (g,e) & & (h,e) \amalg (g,e) & & (g,e) \amalg (h,e) \\
\downarrow T^e_{g,h} & \Rightarrow & \downarrow II & \Rightarrow & \downarrow III & \Rightarrow & \downarrow IV \\
(gh,e) & \xrightarrow{\text{id}} & (gh,e) & & (gh,g) & & (gh,g) \\
& & & \Downarrow V & & & \\
V_g \otimes V_h & \xrightarrow{\tau_{12}} & V_h \otimes V_g & \xrightarrow{\text{id}} & V_h \otimes V_g & \xrightarrow{\text{id}} & V_h \otimes V_g \\
\downarrow \bar{\circ} & & & & & & \downarrow \bar{\varphi}_g \bar{\circ} \\
V_{gh} & \xrightarrow{\text{id}} & V_{gh} & \xrightarrow{\text{id}} & V_{gh} & \xrightarrow{\text{id}} & V_{gh}
\end{array}$$

where we have used Axiom (viii) for the first move and Axiom (iii) for the second and gluing for the last one of the moves depicted in Fig. 3.

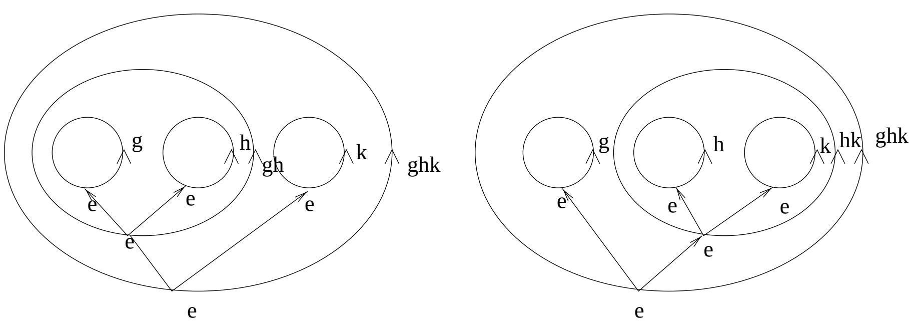

Fig. 2. Associativity.

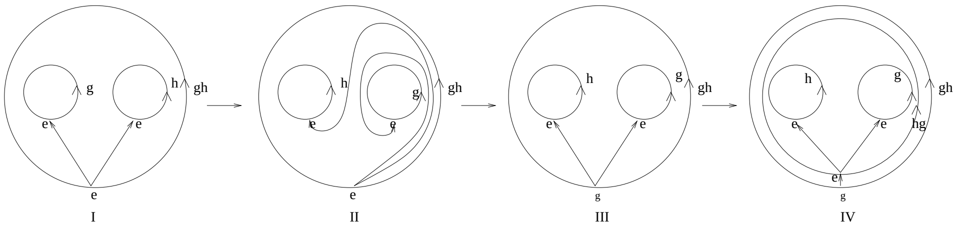

Fig. 3. Twisted commutativity.

The unit of Axiom (c) is given by $\bar{D}^e$. The projective invariance follows from

$$\begin{array}{ccccccc}
\emptyset & & & \overset{D}{\to} & & & (e,e) \\
\downarrow \| & & & & & & \downarrow \| \\
\emptyset & \overset{D}{\to} & (e,e) & \xrightarrow{C^k_{(e,e)}} & (e,k) & \xrightarrow{II_{k^{-1}}} & (e,e) \\
& & & \Downarrow V & & & \\
k & \overset{v}{\to} & V_e & \xrightarrow{\bar{\varphi}_k} & V_e & \xrightarrow{\bar{\chi}_{k^{-1}}} & V_e \\
\downarrow \| & & & & & & \downarrow \| \\
k & & & \xrightarrow{1^e} & & & V_e
\end{array}$$

where in the third line $1_k \mapsto v \mapsto \chi_k v \mapsto v$.

Axiom (d) of the invariance of the metric is again a standard gluing argument, shown in Fig. 4. That is, using $\bar{\epsilon}$ and associativity:

$$\bar{\eta}(v_g \bar{\circ} v_h, v_{(gh)^{-1}}) = \bar{\epsilon}((v_g \bar{\circ} v_h) \bar{\circ} v_{(gh)^{-1}}) = \bar{\epsilon}(v_g \bar{\circ} (v_h \bar{\circ} v_{(gh)^{-1}}))$$

$$= \bar{\eta}(v_g, v_h \bar{\circ} v_{(gh)^{-1}}) \,.$$

For Axiom $(1')$, we use the following diagram:

$$\begin{array}{ccccc}
(g,e) & \overset{\mathrm{id}}{\to} & (g,e) & & (g,e) \\
\downarrow \mathrm{id} & \Rightarrow & \downarrow \tilde{C} & \Rightarrow & \downarrow C \\
(g,e) & \overset{\mathrm{id}}{\to} & (g,e) & & (g,g) \\
& & \Downarrow V & & \\
V_g & \overset{\mathrm{id}}{\to} & V_g & \overset{\mathrm{id}}{\to} & V_g \\
\downarrow \mathrm{id} & & & & \downarrow \bar{\varphi}_g \\
V_g & \overset{\mathrm{id}}{\to} & V_g & \overset{\mathrm{id}}{\to} & V_g
\end{array}$$

where we used Axiom (viii) for the first move and Axiom (iii) for the second move depicted in Fig. 5.

Axiom $(2')$ follows from the diagrams below which correspond to Fig. 6.

$$\begin{array}{ccccccc}
(g,e) \amalg (h,e) & \xrightarrow{(C^k_{(g,e)}, C^k_{(h,e)})} & (kgk^{-1},k) \amalg (khk^{-1},k) & \xrightarrow{T^k_{kgk^{-1},khk^{-1}}} & (kghk^{-1},k) & \xrightarrow{C^{k^{-1}}_{(kghk^{-1},k)}} & (gh,e) \\
& & \Downarrow V & & & & \\
V_g \otimes V_h & \xrightarrow{\bar{\varphi}_k \otimes \bar{\varphi}_k} & V_{kgk^{-1}} \otimes V_{khk^{-1}} & \xrightarrow{\circ^k} & V_{kghk^{-1}} & \xrightarrow{\bar{\varphi}_{k^{-1}}} & V_{gh}
\end{array}$$

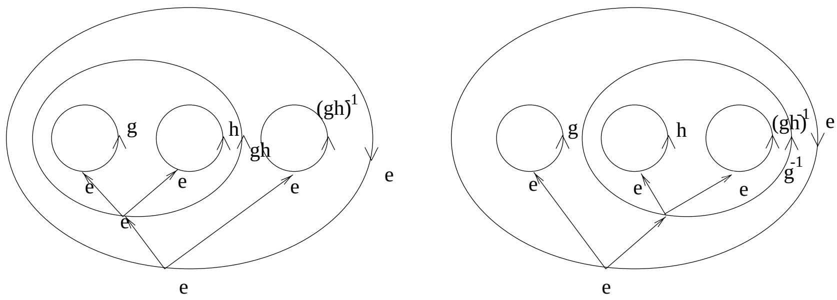

Fig. 4. Invariance of the metric.

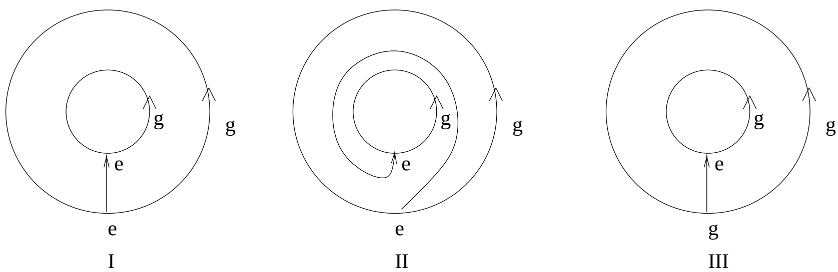

Fig. 5. Self-Invariance of the twisted sectors.

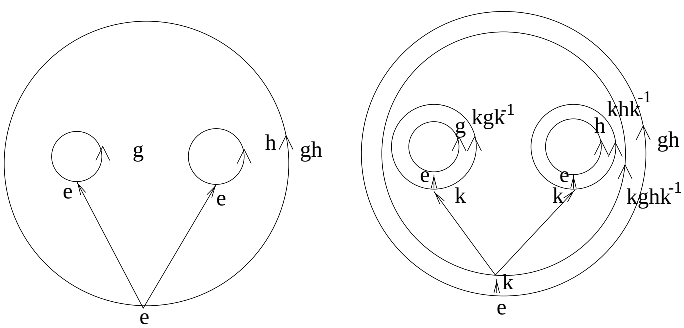

Fig. 6. Projective G-invariance of the multiplication.

and

$$\begin{array}{ccccccc}
(g,e)\amalg(h,e) & & & \xrightarrow{T^{e}_{gh}} & & & (gh,e) \\
\downarrow\| & & & & & & \downarrow\| \\
(g,e)\amalg(h,e) & \xrightarrow{(II_k,II_k)} & (g,k)\amalg(h,k) & \xrightarrow{T^{k}_{g,h}} & (gh,k) & \xrightarrow{II_{k-1}} & (gh,e) \\
 & & \Downarrow V & & & & \\
V_g\otimes V_h & \xrightarrow{\bar{\chi}_k\otimes\bar{\chi}_k} & V_g\otimes V_h & \xrightarrow{\circ^k} & V_{gh} & \xrightarrow{\bar{\chi}_{k-1}} & V_{gh} \\
\downarrow\| & & & & & & \downarrow\| \\
V_g\otimes V_h & & & \xrightarrow{\bar{\circ}} & & & V_{gh}\,.
\end{array}$$

Axiom (3′) — G-invariance of the metric — follows from the following diagrams below which correspond to Fig. 7:

$$
\begin{array}{ccccc}
(g,e)\amalg\overline{(g,e)} & & \overset{A^e}{\rightarrow} & & K \\
\downarrow\, \| & & & & \downarrow\, \| \\
(g,e)\amalg\overline{(g,e)} & \xrightarrow{(C,\bar{C})} & (kgk^{-1},k)\amalg\overline{(kgk^{-1},k)} & \overset{A^k}{\rightarrow} & K \\
\downarrow (\mathrm{id},^{-}) & & \downarrow (\mathrm{id},^{-}) & & \downarrow (\mathrm{id},^{-}) \\
(g,e)\amalg(g^{-1},e) & \xrightarrow{(C,C)} & (kgk^{-1},k)\amalg(kg^{-1}k^{-1},k) & \overset{A^k}{\rightarrow} & K \\
 & & \Downarrow V & & \\
V_g\otimes V_{g^{-1}} & \xrightarrow{\bar{\varphi}_k\otimes\bar{\varphi}_k} & V_{kgk^{-1}}\otimes V_{kg^{-1}k^{-1}} & \overset{\eta^k}{\rightarrow} & K
\end{array}
$$

and

$$
\begin{array}{ccccc}
(g,e)\amalg\overline{(g,e)} & & \overset{A^e}{\rightarrow} & & K \\
\downarrow\, \| & & & & \downarrow\, \| \\
(g,e)\amalg\overline{(g,e)} & \xrightarrow{(II_k,II_k)} & (g,k)\amalg\overline{(g,k)} & \overset{A^k}{\rightarrow} & K \\
\downarrow (\mathrm{id},^{-}) & & \downarrow (\mathrm{id},^{-}) & & \downarrow (\mathrm{id},^{-}) \\
(g,e)\amalg(g^{-1},e) & \xrightarrow{(II_k,II_k)} & (g,k)\amalg(g^{-1},k) & \overset{A^k}{\rightarrow} & K \\
 & & \Downarrow V & & \\
V_g\otimes V_{g^{-1}} & \xrightarrow{\bar{\chi}_k\otimes\bar{\chi}_{k-1}} & V_g\otimes V_{g^{-1}k} & \overset{\eta^k}{\rightarrow} & K \\
\downarrow\, \| & & & & \downarrow\, \| \\
V_g\otimes V_{g^{-1}} & & \overset{\eta}{\rightarrow} & & K\,.
\end{array}
$$

Lastly, Axiom (4′) comes from gluing a once punctured torus in two different ways. □

Proposition 3.3. *Given a G-Ramond algebra V there is a unique G_χ orbifold theory $\mathcal{V}$ such that V is its associated G-Ramond algebra.*

Proof. We need to show that the data is sufficient to reconstruct the functor. For the objects this is clear, due to the monoidal structure. For discs, annuli and

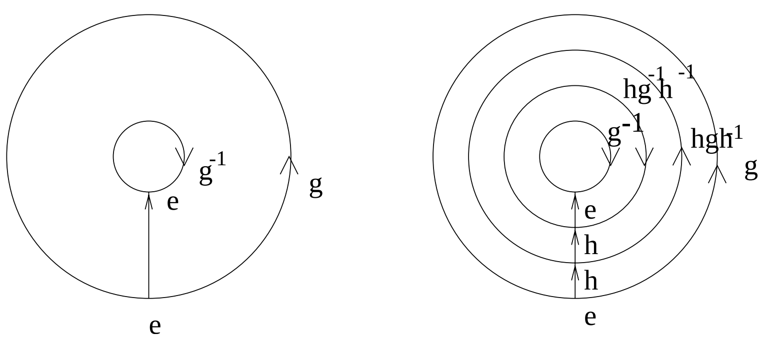

Fig. 7. G-invariance of the metric.

trinions the functor is defined by its basic ingredients, gluing annuli $C_{g,h}$ to the basic trinion, and gauging with morphisms of type II. For other morphisms of type I we notice that we can always decompose a surface into trinions, annuli and discs. For each decomposition there are three choices. The choice of a marking for the decomposition, a choice of orientation and a choice of pairs of points over the marked curves. The second and the third choice can be seen to be irrelevant by inserting two annuli $C_{e,k}$ and $A_{k,e}$ with suitable orientation in a normal neighborhood of the curve in question. The first choice is unique up to two operations [10] which correspond to associativity and the trace axiom, and is thus also irrelevant. □

Combining the Propositions 3.1 and 3.3 with Proposition 3.2, we obtain:

Theorem 3.1. *There is a* 1-1 *correspondence between isomorphism classes of G-twisted Frobenius algebras and isomorphism classes of G_χ-orbifold theories as χ runs through the characters of G.*

Proof. In the standard way, we make the G-twisted Frobenius algebras and the G_χ-orbifold theories into categories by introducing the following morphisms. For G-twisted Frobenius algebras we use algebra homomorphisms respecting all the additional structures and for G_χ-orbifold theories we use natural transformations among functors. The map of associating a G-twisted FA to a G_χ-orbifold theory then turns into a full and faithful functor which is by reconstruction surjective on the objects. □

An immediate consequence of this is:

Corollary 3.2. *There is a* 1-1 *correspondence between G-orbifold FA and isomorphism classes of monoidal functors which are identity on cylinders and satisfy the involutive property from $\mathcal{GBCOB}$ to $\mathcal{VECT}$ which lift to $\mathcal{RGBCOB}$.*

3.3. *Spectral flow*

In the previous paragraph, we chose the geometric version to correspond to the Ramond picture, as is suggested by physics, since we are considering the vacuum states in their Hilbert-spaces at the punctures. From physics one expects that by the spectral flow, the vacuum states should bijectively correspond to the chiral algebra. In the current setting the difference only manifests itself in a change of G-action given by a twist resulting from the character χ. We can directly produce this G-action and thereby the G-Frobenius algebra by considering the G-action not by $C^e_{e,k}$, but by $II_{k^{-1}} \circ C^e_{e,k}$. This statement is proved by re-inspection of the commutative diagrams in the proof of Proposition 3.2.

4. Special *G*-Frobenius Algebras

In this section, we restrict ourselves to a subclass of G-twisted Frobenius algebras. This subclass is large enough to contain all G-Frobenius algebras arising from singularities, symmetric products and spaces whose cohomology of the fixed point sets are given by restriction of the cohomology of the ambient space. The restriction allows us to characterize the possible G-Frobenius structures for a given collection of Frobenius algebras as underlying data in terms of cohomological data. The restriction we will impose (cyclicity of the twisted sectors) can easily be generalized to more generators; which will render everything matrix-valued.

Definition 4.1. We call a G-Frobenius algebra special if all A_g are cyclic A_e modules via the multiplication $A_e \otimes A_g \to A_g$ and there exists cyclic generators 1_g of A_g such that $\varphi_g(1_h) = \varphi_{g,h} 1_{ghg^{-1}}$ with $\varphi_{g,h} \in K^*$.

The last condition is automatic, if the Frobenius algebra A_e only has K^* as invertibles, as is the case for cohomology algebras of connected compact manifolds and Milnor rings of quasi-homogeneous functions with an isolated critical point at zero.

Fixing the generators 1_g, we obtain maps of A_e modules: $r_g : A_e \to A_g$ by setting $r_g(a_e) = a_e 1_g$. This yields a short exact sequence

$$0 \to I_g \to A_e \stackrel{r_g}{\to} A_g \to 0\,. \tag{4.1}$$

It is furthermore useful to fix sections i_g of r_g.

We denote their concatenation by $\pi_g := i_g \circ r_g : A_e \to A_e$.

Lemma 4.1. *$I_g = I_{g^{-1}}$ an thus $A_g \simeq A_{g^{-1}}$ as A_e modules, where the isomorphism can be given by $\psi : r_{g^{-1}} \circ i_g$ for any choice of sections i_g.*

Proof. Suppose $a \in I_g$ then $a1_{g^{-1}} \neq 0$ and $a1_g = 0$. This implies that $\forall\, b \in A_e : \eta(a1_{g^{-1}}, b1_g) = \epsilon(b1_{g^{-1}}a1_g) = 0$ and therefore by the non-degeneracy of η $a1_{g^{-1}} = 0$ and thus $a \in I_{g^{-1}}$. This already implies the isomorphism on the level of vector spaces. For the isomorphism of A_e modules notice that, since $I_g = I_{g^{-1}} = \mathrm{Ann}(1_g) = \mathrm{Ann}(1_{g^{-1}})$ and $i_g(1_g) \equiv 1 \bmod I_g$, $\psi(a1_g) = r_{g^{-1}} \circ i_g(a1_g) = i_g(a1_g)1_{g^{-1}} = \pi_g(ai_g(1_g))1_{g^{-1}} = ai_g(1_g)1_{g^{-1}} = a1_{g^{-1}}$. □

4.1. *Special super G-Frobenius algebra*

The super version of special G-Frobenius algebras is straightforward. Notice that since each A_g is a cyclic A_e-algebra its parity is fixed by $(-1)^{\tilde{g}} := \widetilde{1_g}$. That is $a_g = i_g(a_g)1_g$ and thus $\widetilde{a_g} = \widetilde{i_g(a_g)}\widetilde{1_g}$. In particular if A_e is purely even, A_g is purely of degree $\tilde{g}$.

Proposition 4.1. *Given a special G-Frobenius algebra A and sections i_g. Let $\circ_g : A_g \otimes A_g$ be defined by*

$$a_g \circ_g b_g := r_g(i_g(a_g)i_g(b_g)) = i_g(a_g)i_g(b_g)1_g$$

and define $\eta_g : A_g \otimes A_g \to K$ via

$$\eta_g(a_g, b_g) = \eta(i_g(a_g)1_g, i_g(b_g)1_{g^{-1}}) = \epsilon(i_g(a_g)i_g(b_g)1_g 1_{g^{-1}})\,.$$

Then $(A_g, \circ_g, 1_g)$ is a G-Frobenius algebra. The definitions above are actually independent of the choice of sections i_g. Furthermore, for any section i_g of $r_g : \psi_g := r_{g^{-1}} \circ i_g : A_g \to A_{g^{-1}}$ is an isomorphism of G-Frobenius algebras and A_e-modules.

Proof. That the multiplication is commutative and associative follows from the definition, since A_e is commutative and associative. Since $i_g(1_g) \equiv 1 \bmod I_g$ 1_g is a unit.

That η_g is a symmetric and invariant form can be seen from its expressions in terms of ϵ.

The non-degeneracy follows from the non-degeneracy of η: fix $a_g \in A_g$ and $b_{g^{-1}} \in A_{g^{-1}}$ such that $\eta(a_g, b_{g^{-1}}) \neq 0$. Set $\tilde{b}_g := r_g(i_{g^{-1}}(b_{g^{-1}}))$.

$$\begin{aligned}
\eta_g(a_g, \tilde{b}_g) &= \epsilon(i_g(a_g) i_g(r_g(i_{g^{-1}}(b_{g^{-1}}))) 1_g 1_{g^{-1}}) \\
&= \epsilon(i_g(a_g) r_g(i_g(r_g(i_{g^{-1}}(b_{g^{-1}})))) 1_{g^{-1}}) \\
&= \epsilon(i_g(a_g) i_{g^{-1}}(b_{g^{-1}}) 1_g 1_{g^{-1}}) \\
&= \eta(i_g(a_g)(i_{g^{-1}}(b_{g^{-1}}), \gamma_{g,g^{-1}}) \\
&= \eta(a_g, b_{g^{-1}}) \neq 0\,.
\end{aligned}$$

For the independence of the metric and multiplication on the choice of sections i_g, we remark that for any two sections i_g and i'_g: $i_g(a_g) \equiv i'_g(a_g) \bmod I_g$ and $I_g 1_g = 0$.

For the statements about the isomorphisms of algebras we calculate

$$\begin{aligned}
\psi(a_g) \circ_g \psi(b_g) &= i_{g^{-1}}(r_{g^{-1}}(i_g(a_g))) i_{g^{-1}}(r_{g^{-1}}(i_g(a_g))) 1_g 1_{g^{-1}} \\
&= i_g(a_g) i_g(b_g) 1_{g^{-1}} = \psi(a_g \circ_g b_g)\,,
\end{aligned}$$

and

$$\begin{aligned}
\eta_{g^{-1}}(\psi(a_g), \psi(b_g)) &= \epsilon(i_g(a_g) i_g(b_g) 1_{g^{-1}} 1_g) \\
&= \chi_g \epsilon(i_g(a_g) i_g(b_g) 1_g 1_{g^{-1}}) = \chi_g \eta_g(a_g, b_g)\,. \qquad \square
\end{aligned}$$

For the A_e module isomorphism, notice that $\psi_g(ab_g) = r_{g^{-1}}(i_g(ab_g)) = r_{g^{-1}}(\pi_g(a i_g(b_g))) = r_{g^{-1}}(a i_g(b_g)) = a r_{g^{-1}}(b_g)$.

Remark 4.1. We can also pull back η_g and $\circ_g$ to $i_g(A_g)$ which will make $i_g(A_g)$ into a sub-Frobenius algebra.

Definition 4.2. A special reconstruction datum is a collection of Frobenius algebras $(A_g, \eta_g, 1_g) : g \in G$ together with an action of G by algebra automorphisms on A_e and the structure of a cyclic A_e module algebra on each A_g with generator 1_g such that A_g and A_g^{-1} are isomorphic as of A_e modules algebras.

We can fix the isomorphism $\psi_g : A_g \to A_{g^{-1}}$ via $\psi_g(1_g) = 1_g^{-1}$. This makes ψ into an isomorphism of G-Frobenius algebras.

Definition 4.3. Let A be a special G-twisted Frobenius algebra A. We define a G-graded 2-cocycle with values in A_e to be a map $\gamma : G \times G \to A_e$ which satisfies

$$\gamma_{g,h}\gamma_{gh,k} \equiv \gamma_{h,k}\gamma_{g,hk} \bmod I_{ghk} \,. \tag{4.2}$$

We call a cocycle section independent if $\forall\, g, h \in G$,

$$(I_g + I_h)\gamma_{g,h} \subset I_{gh} \,. \tag{4.3}$$

Two such cocycles are considered to be the same if $\gamma_{g,h} \equiv \gamma'_{g,h} \bmod I_{gh}$ and isomorphic, if they are related by the usual scaling for group cocycles.

Given non-degenerate parings η_g on the A_g, a cocycle is said to be compatible with the metric, if

$$\check{r}_g(1_g) = \gamma(g, g^{-1})$$

where $\check{r}$ is the dual in the sense of vector spaces with non-degenerate metric.

Definition 4.4. A non-abelian G 2-cocycle with values in K^* is a map $\varphi : G \times G \to K^*$ which satisfies:

$$\varphi_{gh,k} = \varphi_{g,hkh^{-1}}\varphi_{h,k} \tag{4.4}$$

where $\varphi_{g,h} := \varphi(g,h)$ and

$$\varphi_{e,g} = \varphi_{g,e} = 1 \,.$$

Notice that in the case of a commutative group G this says that the $\varphi_{g,h}$ form a two cocycle with values in K^*.

Furthermore setting $g = h^{-1}$, we find

$$\varphi_{g^{-1},ghg^{-1}} = \varphi_{g,h}^{-1} \,.$$

Proposition 4.2. *A special G-twisted Frobenius algebra A gives rise to section independent graded G 2-cocycle γ with values in A_e which is compatible with the metric and to a non-abelian G 2-cocycle φ with values in K^*. The following compatibility equations are furthermore satisfied by these cocycles*

$$\varphi_{g,h}\gamma_{ghg^{-1},g} = \gamma_{g,h} \tag{4.5}$$

and

$$\varphi_{k,g}\varphi_{k,h}\gamma_{kgk^{-1},khk^{-1}} = \varphi_k(\gamma_{g,h})\varphi_{k,gh} \,. \tag{4.6}$$

Proof. Given a special G twisted Frobenius structure on A, we fix cyclic generators 1_g of A_g and define $\gamma_{g,h} \in A_{gh}$ by

$$1_g 1_h = \gamma_{g,h} 1_{gh}$$

and $\varphi_{g,h} \in K^*$ by:

$$\varphi_g(1_h) = \varphi_{g,h} 1_{ghg^{-1}} \,.$$

By associativity of the multiplication, we find that the $\gamma_{g,h}$ define a graded 2-cocycle with values in A.

$$\begin{aligned}\gamma_{g,h}\gamma_{gh,k}1_{ghk} &= \gamma_{g,h}1_{gh}1_k = (1_g 1_h)1_k \\ &= 1_g(1_h 1_k) = 1_g\gamma_{h,k}1_{hk} = \gamma_{h,k}\gamma_{g,hk}1_{ghk}\,. \end{aligned} \tag{4.7}$$

So that

$$\pi_{ghk}(\gamma_{g,h}\gamma_{gh,k}) = \pi_{ghk}(\gamma_{h,k}\gamma_{g,hk})\,.$$

Furthermore, since

$$(I_g + I_h)\gamma_{g,h}1_{gh} = (I_g + I_h)1_g 1_h = 0\,,$$

the cocycle is also section independent.

Also, $\forall\, a \in A_e$,

$$\eta(a, \check{r}_g(1_g)) = \eta_g(r_g(a), 1_g) = \eta(\pi_g(a), \gamma_{g,g^{-1}}) = \eta(a, \gamma_{g,g^{-1}})$$

which shows the compatibility with the metric.

φ is a group homomorphism so that

$$\begin{aligned}\varphi_{gh,k}1_{ghkh^{-1}g^{-1}} &= \varphi_{gh}(1_k) = \varphi_g(\varphi_h(1_k)) = \varphi_g(\varphi_{h,k}1_{hkh^{-1}}) \\ &= \varphi_{g,hkh^{-1}}\varphi_{h,k}1_{ghkh^{-1}g^{-1}}\end{aligned}$$

which yields

$$\varphi_{gh,k} = \varphi_{g,hkh^{-1}}\varphi_{h,k}\,. \tag{4.8}$$

By G-twisted commutativity

$$\gamma_{g,h}1_{gh} = 1_g 1_h = \varphi_g(1_h)1_g = \varphi_{g,h}1_{ghg^{-1}}1_g = \varphi_{g,h}\gamma_{ghg^{-1},g}1_{gh}\,.$$

So $\gamma_{g,h}$ and the $\varphi_{g,h}$ satisfy

$$\varphi_{g,h}\pi_{gh}(\gamma_{ghg^{-1},g}) = \pi_{gh}\gamma_{g,h}\,. \tag{4.9}$$

φ is also an algebra automorphism:

$$\varphi_k(1_g)\varphi_k(1_h) = \varphi_k(1_g 1_h)\,.$$

Expressed in the φ's and γ's:

$$\varphi_{k,g}\varphi_{k,h}\gamma_{kgk^{-1},khk^{-1}} = \varphi_k(\gamma_{g,h})\varphi_{k,gh} \tag{4.10}$$

which gives a formula for the action of φ on the γ's. □

Definition 4.5. We call a pair of a section independent cocycle and a non-abelian cocycle compatible if they satisfy Eqs. (4.5) and (4.6).

Corollary 4.1. *A special G-Frobenius algebra gives rise to a collection of Frobenius-algebras $(A_g, \circ_g, 1_g, \eta_g)_{g\in G}$ together with a G-action on A_e, and a compatible pair of a graded, section independent G 2-cocycle with values in A_e that is* *compatible with the metric and a non-abelian G 2-cocycle with values in K^*. All the A_g are cyclic A_e module algebras generated by 1_g and there are A_e module algebra isomorphisms $A_g \simeq A_{g^{-1}}$ of Frobenius algebras. Furthermore the following conditions are satisfied*:

(i) $\varphi_{g,g} = \chi_g^{-1}$.
(ii) $\eta_e(\varphi_g(a), \varphi_g(b)) = \chi_g^{-2}\eta_e(a, b)$.
(iii) *The projective trace axiom* $\forall\, c \in A_{[g,h]}$ *and* l_c *left multiplication by* c:

$$\chi_h \operatorname{Tr}(l_c\varphi_h|_{A_g}) = \chi_{g^{-1}} \operatorname{Tr}(\varphi_{g^{-1}} l_c|_{A_h})\,. \tag{4.11}$$

Theorem 4.1 (Reconstruction). *Given a special reconstruction datum the structures of special G-Frobenius algebras are in* 1-1 *correspondence compatible pairs of a graded, section independent G 2-cocycle with values in A_e that is compatible with the metric and a non-abelian G 2-cocycle with values in K^*. Satisfying the following conditions*:

(i) $\varphi_{g,g} = \chi_g^{-1}$.
(ii) $\eta_e(\varphi_g(a), \varphi_g(b)) = \chi_g^{-2}\eta_e(a, b)$.
(iii) *The projective trace axiom* $\forall\, c \in A_{[g,h]}$ *and* l_c *left multiplication by* c:

$$\chi_h \operatorname{Tr}(l_c\varphi_h|_{A_g}) = \chi_{g^{-1}} \operatorname{Tr}(\varphi_{g^{-1}} l_c|_{A_h})\,. \tag{4.12}$$

Proof. The cyclic A_e-module structure of A_g gives rise to exact sequences (4.1). Let i_g sections of the r_g of (4.1).

Define a multiplication on $A := \oplus_{g\in G} A_g$ via

$$a_g b_h := i_g(a_g) i_h(b_h)\gamma_{g,h} 1_{gh}\,.$$

This multiplication is associative as can be seen by using the section independence of the cocycle. The section independence also guarantees the independence of the choice of sections i_g.

Now we use the φ's to define a G-action by: $\varphi_g(b_h) := \varphi_g(i_h(b_h))\varphi_{g,h} 1_{ghg^{-1}}$.

The compatibility and Eq. (4.6) guarantee that this is a representation and the G action is indeed an action by algebra automorphisms.

The projective self-invariance of the twisted sectors follows from the condition (i).

The G-twisted commutativity

$$a_g b_h = \varphi_g(b_h) a_g$$

also follows from the compatibility.

The form η is defined the following way:

$$\eta(a_g, h_{g^{-1}}) := \eta_e(i_g(a_g) i_{g^{-1}}(b_{a^{-1}})\gamma_{gg^{-1}}, 1)$$

and

$$\eta(a_g, h_h) := 0 \text{ if } gh \neq 1\,.$$

Notice that since the ψ_g is an isomorphism of cyclic A_e modules geberated by 1_g and $1_{g^{-1}}$ it follows that $\psi_g = r_{g^{-1}} \circ i_g$.

Due to the compatibility with the metric, we obtain the following representation:

$$\begin{aligned} \eta(a_g, b_{g^{-1}}) &= \eta_e(i_g(a_g) i_{g^{-1}}(b_{g^{-1}}) \gamma_{g,g^{-1}}, 1) \\ &= \eta_e(i_g(a_g), \check{r}_g(i_{g_{-1}}(b_{g^{-1}}) 1_g)) \\ &= \eta_g(a_g, i_{g^{-1}}(b_{a^{-1}}) 1_g) \\ &= \eta_g(a_g), \psi_{g^{-1}}(b_{g^{-1}}) \,. \end{aligned}$$

The invariance then follows using $I_{gh}\gamma_{gh,(gh)^{-1}} = 0$ via:

$$\begin{aligned} \eta(a_g b_h, d_{(gh)^{-1}}) &= \eta_e(i_g(a_g) i_h(b_h) i_{h^{-1}g^{-1}}(d_{h^{-1}g^{-1}}) \gamma_{g,h} \gamma_{gh,h^{-1}g^{-1}}, 1) \\ &= \eta_e(i_g(a_g) i_h(b_h) i_{h^{-1}g^{-1}}(d_{h^{-1}g^{-1}}) \gamma_{h,h^{-1}g^{-1}} \gamma_{g,g^{-1}}, 1) \\ &= \eta(a_g, b_h d_{h^{-1}g^{-1}}) \,. \end{aligned}$$

□

Remark 4.2. Changing the cyclic generators by elements of K^* leads to isomorphic G-Frobenius algebras and to cohomologous cocycles γ, φ in $Z^2(G, A_e)$ and $Z^2(G, K^*[G])$.

Remark 4.3. By straightforward calculation it can be shown the projective trace axiom is equivalent to

$$\forall\, g, h \in G; c1_{ghg^{-1}h^{-1}} \in A_{[g,h]}, c \in i_{hgg^{-1}h^{-1}}(A_{[gh]}) \,,$$

$$\begin{aligned} &\chi_h \varphi_{h,g} \operatorname{Tr}(l_{\gamma_{ghg^{-1}h^{-1},hgh^{-1}}c} \varphi_h|_{i_g(A_g)}) \\ &\quad = \chi_{g^{-1}} \varphi_{g^{-1},ghg^{-1}} \operatorname{Tr}(\varphi_{g^{-1}} l_{\gamma_{hg^{-1}h^{-1}g,g^{-1}hg}} c|_{i_h(A_h)}) \,. \end{aligned} \tag{4.13}$$

Notice that in the graded case (see below) this condition only needs to be checked for $\deg(\gamma_{ghg^{-1}h^{-1},hgh^{-1}}c) \neq 0$ and $\deg(\gamma_{hg^{-1}h^{-1}g,g^{-1}hg}c) \neq 0$.

Furthermore if $[g,h] = e$ then $\gamma_{ghg^{-1}h^{-1},hgh^{-1}} = \gamma_{hg^{-1}h^{-1}g,g^{-1}hg} = 1_e$ and $\varphi_{g^{-1},ghg^{-1}} = \varphi_{g,h}^{-1} = \varphi_{g^{-1},h}$.

Proposition 4.3. (1) *If $A_g A_h \neq 0$, the compatibility condition of Eq.* (4.5) *already determines the $\varphi_{g,h} \in K^*$.*

(2) *In particular: $\gamma_{g,g} = 0$ unless $\chi_g = 1$ and if $[g,h] = e$ it follows that $\varphi_{g,h}\varphi_{h,g} = 1$ or $\gamma_{g,h} = \gamma_{h,g} = 0$ holds.*

(3) *If also $A_g A_h A_k \neq 0$, the elements defined by Eq.* (4.5) *automatically satisfy the conditions of non-abelian 2-cocycles and the condition of Eq.* (4.6) *is automatically satisfied.*

Proof. Without loss of generality, we may assume that $1_g 1_h 1_k \neq 0$. Then, due to the condition in (4.5),

$$\begin{aligned} 1_g 1_h 1_k = (1_g 1_h) 1_k &= \varphi_{gh}(1_k)(1_g 1_h) \\ &= (\varphi_{gh}(1_k) 1_g) 1_h = \varphi_{gh,k} 1_{ghkh^{-1}g^{-1}} 1_g 1_h \,, \end{aligned}$$

and using associativity, we similarly obtain

$$1_g 1_h 1_k = 1_g(1_h 1_k) = 1_g \varphi_h(1_k) 1_h$$
$$= \varphi_g(\varphi_{b,k} 1_{hkh^{-1}}) 1_g 1_h = \varphi_{g,hkh^{-1}} \varphi_{h,k} 1_{ghkh^{-1}g^{-1}} 1_g 1_h \, .$$

For the first statement in (2) one just needs to plug $g = h$ into Eq. (4.5) and for the second one has to apply the formula twice. For (3), notice that $1_k 1_g 1_h = 1_k(1_g 1_h) = \varphi_k(1_g 1_h) 1_k$ and on the other hand, $1_k 1_g 1_h = \varphi_k(1_g) 1_k 1_h = \varphi_k(1_g) \varphi_k(1_h) 1_k$. □

A useful technical lemma to show that $\gamma_{g,h} \neq 0$ is the following

Lemma 4.2. *If* $\gamma_{g,h} = 0$, *then* $\pi_h(\gamma_{g,g^{-1}}) = 0$ *and* $\pi_g(\gamma_{h,h^{-1}}) = 0$.

Furthermore if $\gamma_{g,h} = \gamma_{h^{-1},g^{-1}} = 1$, *then*

$$\gamma_{gh,(gh)^{-1}} = \gamma_{g,g^{-1}} \gamma_{h,h^{-1}} \, .$$

Proof. If $\gamma_{g,h} = 0$, then

$$0 = \pi_h(\gamma_{g^{-1},gh} \gamma_{g,h}) = \pi_h(\gamma_{g^{-1},g} \gamma_{e,h}) = \pi_h(\gamma_{g^{-1},g}) = \pi_h(\gamma_{g,g^{-1}})$$

and also

$$0 = \pi_g(\gamma_{g,h} \gamma_{gh,h^{-1}}) = \pi_g(\gamma_{g,e} \gamma_{h^{-1},h}) = \pi_g(\gamma_{h,h^{-1}}) \, .$$

Furthermore

$$\gamma_{gh,(gh)^{-1}} = \gamma_{g,h} \gamma_{gh,(gh)^{-1}} = \gamma_{g,g^{-1}} \gamma_{h,h^{-1}g^{-1}} = \gamma_{g,g^{-1}} \gamma_{h,h^{-1}} \, ,$$

since

$$\pi_{g^{-1}}(\gamma_{h,h^{-1}g^{-1}}) = \pi_{g^{-1}}(\gamma_{h^{-1},g^{-1}} \gamma_{h,h^{-1}g^{-1}}) = \pi_{g^{-1}}(\gamma_{h,h^{-1}} \gamma_{e,g^{-1}})$$

and $I_{g^{-1}} \gamma_{g,g^{-1}} = 0$. □

4.1.1. *Graded special G–Frobenius algebras*

Now we consider a set of graded Frobenius algebras satisfying the reconstruction data: $\{(A_g, \eta_g) : g \in G\}$ with degrees $d_g := \deg(\eta_g)$ such that $A_g \simeq A_{g^{-1}}$. For example, in the cohomology of fixed point sets d_g is given by the dimension and for the Jacobian Frobenius manifolds (see the next section) d_g fixed by the degree of $\mathrm{Hess}(f_g) = \rho_g$. Furthermore, the reconstructed $\{\eta|_{(A_g \otimes A_{g^{-1}}}, g \in G\}$ have degree $d_g = d_{g^{-1}}$.

For a G-twisted FA the degrees all need to be equal to $d := d_e$. To achieve this, one can shift the grading in each A_g by s_g. This amounts to assigning degree s_g to 1_g. This is the only freedom, since the multiplication should be degree-preserving and all A_g are cyclic.

Set $s_g^+ := s_g + s_{g^{-1}}$; $s_g^- := s_g - s_{g^{-1}}$. Then $s_g^+ := d - d_g$ for grading reasons, but the shift s^- is more elusive.

Definition 4.6. The standard shift for a Jacobian Frobenius algebra is given by

$$s_g^+ := d - d_g$$

and

$$s_g^- := \frac{1}{2\pi i}\mathrm{tr}(\log(g)) - \mathrm{tr}(\log(g^{-1})) := \frac{1}{2\pi i}\left(\sum_i \lambda_i(g) - \sum_i \lambda_i(g^{-1})\right)$$
$$= \sum_{i:\lambda_i \neq 0} \left(\frac{1}{2\pi i}2\lambda_i(g) - 1\right)$$

where the $\lambda_i(g)$ are the logarithms of the eigenvalues of g using the branch with arguments in $[0, 2\pi)$, i.e. cut along the positive real axis.

In this case we obtain

$$s_g = \frac{1}{2}(s_g^+ + s_g^-) = \frac{1}{2}(d - d_g) + \sum_i \left(\frac{1}{2\pi i}\lambda_i(g) - \frac{1}{2}\right).$$

Remark 4.4. The shift s_g^- is canonical in the case of quasi-homogeneous singularities upon replacing the classical monodromy operator J by Jg. This will be discussed elsewhere [15]. In general it is possible to define the shift s^- if one is additionally given a linear representation of G, such as in orbifold cohomology [2].

The degree of $\gamma_{g,g^{-1}}$ is s_g^+ from comparing degrees in the equation $1_g 1_{g^{-1}} = \gamma_{g,g^{-1}} 1_e$.

Lemma 4.3. *Let A and A_g be a graded Frobenius algebras with the top degree of A_g being d_g then for a section independent cocycle $\gamma_{g,g^{-1}} \subset L \subset A_e$ with* $\dim(L) = \dim(A_g^{d_g})$, *where the superscript denotes a fixed degree.*

Proof. By section independence

$$I_g \gamma_{g,g^{-1}} = 0\,.$$

Thus

$$\gamma_{g,g^{-1}} \in (i_g(A_g)^*)^{d - s_g^+}$$

where * is the dual w.r.t. the form η and we use the splitting induced by the sections i (N.B. if η is also positive definite, we could use an orthogonal splitting)

$$A^k = I_g^k \oplus (i_g(A_g))^k \tag{4.14}$$

and superscripts denote fixed degree. Furthermore

$$\dim((i_g(A_g)^*)^{d_g}) = \dim(i_g(A_g)^{d_g}) = \dim(A^{d_g}) - \dim(I_g)$$
$$= \dim(A^{d_g}) - \dim(\mathrm{Ker}(r_g)|_{A^{d_g}}) = \dim(\mathrm{Im}(r_g)|_{A^{d_g}}) = \dim(A_g^{d_g})$$

where we used the non-shifted grading on A_g. Thus $\gamma_{g,g^{-1}}$ is fixed up to a constant. □

If $\dim A_g = 1$, then $\gamma_{g,g^{-1}}$ is fixed up to normalization by the condition of section independence. The freedom to scale $\gamma_{g,g^{-1}}$ is the same freedom one has in general for choosing a metric for an irreducible Frobenius algebra. Recall that in this case the space of invariant metrics is one dimensional.

In the Reconstruction program the presence of a non-trivial grading can greatly simplify the check of the trace axiom. For example, if $A_{[g,h]}$ has no element of degree 0, then both sides of this requirement are 0 and if $[g,h] = e$, one needs only to look at the special choices of c with $\deg(c) = 0$ which most often is just $c = 1$, the identity.

The grading in the Ramond-sector is by the following definition

$$\deg(v) := -\frac{d}{2}\,.$$

This yields

$$\deg(\bar{\eta}) = 0 \text{ and } \deg(\bar{\circ}) = \frac{d}{2}\,.$$

5. Jacobian Frobenius Algebras

Definition 5.1. A Frobenius algebra A is called Jacobian if it can be represented as the Milnor ring of a function f. That is, if there is a function $f \in \mathcal{O}_{\mathbf{A}^n_K}$ such that $A = \mathcal{O}_{\mathbf{A}^n_K}/J_f$ where J_f is the Jacobian ideal of f. And the bilinear form is given by the residue pairing. This is the form given by the the Hessian of f: $\rho = \mathrm{Hess}_f$.

If we write $\mathcal{O}_{\mathbf{A}^n_K} = K[x_1 \cdots x_n]$, J_f is the ideal spanned by the $\frac{\partial f}{\partial x_i}$.

Definition 5.2. A realization of a Jacobian Frobenius algebra is a pair (A, f) of a Jacobian Frobenius algebra and a function f on some affine K space $\mathbf{A}^n_K$, i.e. $f \in \mathcal{O}_{\mathbf{A}^n_K} = K[x_1 \cdots x_n]$ such that $A = K[x_1 \cdots x_n]$ and $\rho := \det(\frac{\partial^2 f}{\partial x_i \partial x_j})$.

A small realization of a Jacobian Frobenius algebra is a realization of minimal dimension, i.e. of minimal n.

Definition 5.3. A natural G action on a realization of a Jacobian Frobenius algebra (A_e, f) is a linear G action on $\mathbf{A}^n_K$ which leaves f invariant.

5.1. *Special reconstruction data based on a Jacobian Frobenius algebra with symmetries*

Given a natural G action on a realization of a Jacobian Frobenius algebra (A, f) set for each $g \in G$, $\mathcal{O}_g := \mathcal{O}_{\mathrm{Fix}_g(\mathbf{A}^n_K)}$. This is the ring of functions of the fixed point set of g for the G action on $\mathbf{A}^n_K$. These are the functions fixed by g: $\mathcal{O}_g = K[x_1, \ldots, x_n]^g$.

Denote by $J_g := J_{f|_{\mathrm{Fix}_g(\mathbf{A}^n_K)}}$ the Jacobian ideal of f restricted to the fixed point set of g.

Define

$$A_g := \mathcal{O}_g / J_g\,. \tag{5.1}$$

The A_g will be called twisted sectors for $g \neq 1$. Notice that each A_g is a Jacobian Frobenius algebra with the natural realization given by $(A_g, f|_{\mathrm{Fix}_g})$. In particular, it comes equipped with an invariant bilinear form $\tilde{\eta}_g$ defined by the element $\mathrm{Hess}(f|_{\mathrm{Fix}_g})$.

For $g = e$, the definition of A_e is just the realization of the original Frobenius algebra, which we also call the untwisted sector.

Notice there is a restriction morphism $r_g : A_e \to A_g$ given by $a \mapsto a|_{\mathrm{Fix}_g} \bmod J_g$.

Denote $r_g(1)$ by 1_g. This is a non-zero element of A_g since the action was linear. Furthermore it generates A_g as a cyclic A_e module.

We obtain a sequence

$$0 \to I_g \to A_e \stackrel{r_g}{\to} A_g \to 0\,.$$

Let i_a be any splitting of this sequence induced by the inclusion: $\hat{i}_g : \mathcal{O}_g \to \mathcal{O}_e$ which descends due to the invariance of f.

In coordinates, we have the following description. Let $\mathrm{Fix}_g \mathbf{A}^n_K$ be given by equations $x_i = 0 : i \in N_g$ for some index set N_g.

Choosing complementary generators $x_j : j \in T_g$ we have $\mathcal{O}_g = K[x_j : j \in T_g]$ and $\mathcal{O}_e = K[x_j, x_i : j \in T_g, i \in N_g]$. Then $I_g = (x_i : i \in N_g)_{\mathcal{O}_e}$ the ideal in $\mathcal{O}_e$ generated by the x_i and $\mathcal{O}_e = I_g \oplus i_g(A_g)$ using the splitting i_g coming from the natural inclusion $\hat{i}_g : K[x_j : j \in T_g] \to K[x_j, x_i : j \in T_g, i \in N_g]$. We also define the projections

$$\pi_g : A_1 \to A_g; \pi_g = i_g \circ r_g$$

which in coordinates are given by $f \mapsto f|_{x_j = 0 : j \in N_g}$.

Let

$$A := \bigoplus_{g \in G} A_g\,.$$

Definition 5.4. A discrete torsion for a group G is a map from commuting pairs $(g, h) \in G \times G : [g, h] = e$ to K^* with the properties:

$$\epsilon(g, h) = \epsilon(h^{-1}, g)\,, \quad \epsilon(g, g) = 1\,, \quad \epsilon(g_1 g_2, h) = \epsilon(g_1, h)\epsilon(g_2, h)\,. \tag{5.2}$$

Definition 5.5. A non-abelian 2-cocycle is said to satisfy the condition of discrete torsion with respect to a given $\sigma \in \mathrm{Hom}(G, \mathbf{Z}/2\mathbf{Z})$ and a linear representation $\rho \in \mathrm{Hom}(G, GL(n))$, if for all elements $g, h \in G : [g, h] = e$:

$$\epsilon(g, h) := \varphi_{g,h} (-1)^{\sigma(g)\sigma(h)} \det(g) \det(g^{-1}|_{\mathrm{Fix}(h)})$$

is a discrete torsion.

Remark 5.1. Due to the properties of φ as a non-abelian cocycle, φ the second and third condition of discrete torsion (5.2) are automatically satisfied. If furthermore $\gamma_{g,h} \neq 0$, then the first condition reduces to

$$\det(g) \det(g^{-1}|_{\mathrm{Fix}(h)}) \det(h) \det(h^{-1}|_{\mathrm{Fix}(g)}) = 1\,.$$

5.1.1. *Reconstruction for graded special G–Frobenius algebras*

Definition 5.6. A cocycle $\gamma \in Z^2(G, A_e)$ is said to satisfy the condition of supergrading with respect to a given a linear representation $\rho \in \mathrm{Hom}(G, GL(n))$, if $\gamma_{g,h} = 0$ unless $|N_h| + |N_g| + |N_{gh}| \equiv 0(2)$. Here $|N_g| := \mathrm{codim}(\mathrm{Fix}(\rho(g))$ is the codimension of the fixed point set of g.

Theorem 5.1. *Given a natural G action on a realization of a Jacobian Frobenius algebra (A_e, f) with a quasi-homogeneous function f of degree d and type $\mathbf{q} = (q_1, \ldots, q_n)$, let $A := \oplus_{g \in G} A_g$ be as defined previously up to an isomorphism of Frobenius algebras on the A_g then the structures of super G-Frobenius algebra on A are in 1-1 correspondence with triples $(\sigma, \gamma, \varphi)$ where $\sigma \in \mathrm{Hom}(G, \mathbf{Z}/2\mathbf{Z})$ γ is G-graded, section independent cocycle compatible with the metric satisfying the condition of supergrading with respect to the natural G action, φ is a non-abelian two cocycle with values in K^* which satisfies the condition of discrete torsion with respect to σ and the natural G action, such that (γ, φ) is a compatible pair.*

Proof. By the Reconstruction theorem the structures of a G-Frobenius algebra are in 1-1 correspondence with compatible pairs (γ, φ), where γ is compatible with the metric and section independent, subject to the conditions

(i) $\varphi_{g,g} = \chi_g^{-1}$.
(ii) $\eta_e(\varphi_g(a), \varphi_g(b)) = \chi_g^{-2} \eta_e(a, b)$.
(iii) The projective trace axiom $\forall\, c \in A_{[g,h]}$ and l_c left multiplication by c:

$$\chi_h \,\mathrm{STr}(l_c \varphi_h|_{A_g}) = \chi_{g^{-1}} \,\mathrm{STr}(\varphi_{g^{-1}} l_c|_{A_h}) \tag{5.3}$$

where we can reduce to the case that $d_{[g,h]} = 0$, since the algebra is non-trivially graded and the degree zero part of A is just k^*.

Let $\tilde{} \in \mathrm{Map}(G, \mathbf{Z}/2\mathbf{Z})$ be a supergrading in spe. Recall that since A_e is even and the A_g are cyclic, to give a supergrading is equivalent to specifying the superdegrees of 1_g; $\tilde{1}_g := (-1)^{\tilde{g}}$.

Then χ_g are fixed by the equation

$$(-1)^{\tilde{g}} \dim(A_g) = \mathrm{STr}(\mathrm{id}|_{A_{g^{-1}}}) = \chi_g \,\mathrm{STr}(\varphi_g|_{A_e}) = \chi_g \,\mathrm{Tr}(\varphi_g|_{A_e})\,.$$

To calculate the trace on the RHS, we use the character function for a morphism g of degree 0 on graded module $V = \oplus_n V_n$:

$$\chi_{V_n}(g, z) := \sum_{n,\mu} \mu \dim(V_{\mu,n}) z^n$$

where $V_{\mu,n}$ is the eigenspace of Eigenvalue μ on the space V_n. We will use the grading induced by the quasi-homogeneity. That is, let N be such that $q_i = Q_i/N$ with $Q_i \in \mathbf{N}$ and N such that $|G||N$. Then a monomial has degree n if its quasi homogeneous degree is n/N. This is the natural grading for the quasi-homogeneous map $\mathrm{grad}(f)$. Notice that since g commutes with f it preserves the grading. It is clear that this character behaves multiplicatively under concatenations of quasi-homogeneous functions. Therefore by applying Arnold's method, we can pass to a cover of K^n with the projection map $T : T(x_1, \ldots, x_n) = (x^{q_1}, \ldots, x^{q_n})$ and calculate the character for T and for $\text{grad}(f) \circ T$. Then repeating the argument in a simultaneous Eigenbasis of g and the grading of [1], we obtain:

$$\chi_{A_e}(g, z) = \prod_{i=1}^{n} \frac{(\tilde{\mu}_i z)^{N-Q_i} - 1}{(\tilde{\mu}_i z)^{Q_i} - 1}$$

where the $\tilde{\mu}_i$ are the Eigenvalues of some lift of the action of g i.e. $\tilde{\mu}_i^{Q_i} = \mu_i$. Notice that since $|G||N$ $\tilde{\mu}_i^N = 1$, so that in the limit of $z \to 1$, we obtain:

$$\text{Tr}(\varphi_g|_{A_e}) = \prod_{i:\mu_i \neq 1} -\mu_i^{-1} \prod_{i:\mu_i=1} \frac{1}{q_i} - 1 = (-1)^{|N_g|} \det(g)^{-1} \dim(A_g)\,,$$

so that

$$\chi_g = (-1)^{\tilde{g}}(-1)^{|N_g|} \det(g)\,.$$

We set

$$\sigma(g) := \tilde{g} + |N_g| \mod 2 \tag{5.4}$$

and call it the sign of g. Thus we obtain

$$\chi_g = (-1)^{\sigma(g)} \det(g)\,.$$

Notice that $\sigma \in \text{Hom}(G, \mathbf{Z}/2\mathbf{Z})$ since both det and χ are characters. Also notice that a choice of sign corresponds to a choice of parity and vice-versa.

For $\tilde{}$ to be a supergrading, it has to satisfy the condition that if $\gamma_{g,h} \neq 0$, then $\tilde{g} + \tilde{h} \equiv \tilde{gh}(2)$. In view of (5.4), this is equivalent to the condition of supergrading of γ with respect to the natural G action.

This ensures condition (ii). Since for $\rho = \text{Hess}(f)$:

$$\varphi_g(\rho) = \det(g)^{-2}\rho\,.$$

Now consider the direct sum

$$A := \bigoplus_{g \in G} A_g\,.$$

On this direct sum we impose the metric $\eta := \oplus \tilde{\eta}_g$ where $\tilde{\eta}_g$ is the scaled metric

$$\tilde{\eta}_g := ((-1)^{\tilde{g}}\chi_g)^{1/2}\eta_g$$

and η_g is the metric of A_g as a Jacobian Frobenius algebra.

In order to define the root, we choose to cut the plane along the negative real axis. This uniquely defines a square root unless $(-1)^{\tilde{g}}\chi_g = -1$. In the case that $g^2 \neq e$, we can choose roots i and $-i$ for g respectively g^{-1}. The only case that has no solution would be the case of $g^2 = e$ and $(-1)^{\tilde{g}}\chi_g = -1$, but this means that either $\chi_g = -1$ and $\tilde{g} = 1$ or $\chi_g = -1$ and $\tilde{g} = 0$ which cannot happen, since in this case $(-1)^{|N_g|} = \det(g)$ and $(-1)^{\tilde{g}}(-1)^{|N_g|}\det(g) = \chi_g$.

Since $\mathrm{Fix}_g = \mathrm{Fix}_{g^{-1}}$ we have $A_g = A_{g^{-1}}$ and after the shift of metrics the algebras are still isomorphic as Frobenius algebras.

Thus the collection $(A_g, \tilde{\eta}_g, 1_g)$ is a special reconstruction datum which satisfies condition (i) due to the compatibiltity of the cocycles and the compatibilty of γ with the metric.

Finally, we need to check the validity of (iii). Notice that since the multiplication is graded the traces are 0 unless $\deg(\gamma_c) = 0$, so that we can assume that $\gamma_c = 1$. In this case, we have to show:

$$\chi_h \operatorname{STr}(\varphi_h|_{A_g}) = \chi_{g^{-1}} \operatorname{STr}(\varphi_{g^{-1}}|_{A_h})\,.$$

Let x_i be a basis of A_e in which g is diagonal. Then we have to compute the trace of the action of h on the sub-algebra generated by the x_i with eigenvalue 1 under the action of g. This is just the truncated version of the calculation above, so diagonalizing h on $K[x_i : i \in T_g]$ we find using the same characteristic functions and rationale as before:

$$\begin{aligned}
\chi_h \operatorname{STr}(\varphi_h|_{A_g}) &= \chi_h \varphi_{h,g}(-1)^{\tilde{g}} \operatorname{Tr}(\varphi_h|_{i_g(A_g)}) \\
&= \chi_h \varphi_{h,g}(-1)^{\tilde{g}} \prod_{j:\nu_j \neq 1} -\nu_j^{-1} \prod_{j:\nu_j = 1} \frac{1}{q_j} - 1 \\
&= \chi_h \varphi_{h,g}(-1)^{\tilde{g}}(-1)^{|T_g|}(-1)^{|T_g \cap T_h|} \det(h|_{T_g})^{-1} \dim(i_g(A_g) \cap i_h(A_h)) \\
&= \varphi_{h,g}(-1)^{\sigma(g)}(-1)^{\sigma(h)} \det(h) \det(h|_{T_g})^{-1}(-1)^N(-1)^{|T_g \cap T_h|} \\
&\quad \times \dim(i_g(A_g) \cap i_h(A_h)) \\
&= \varphi_{h,g}(-1)^{\sigma(hg)} \det(h|_{N_g})(-1)^{|T_g \cap T_h| + N} \dim(i_g(A_g) \cap i_h(A_h)) \\
&= \epsilon(h,g) T(h,g)
\end{aligned}$$

where ν_j are the eigenvalues of h on $i_g(A_g)$ and

$$\epsilon(g,h) = \varphi_{g,h}(-1)^{\sigma(g)\sigma(h)} \det(g|_{N_h})$$

and we set $\det(g|_{N_h}) := \det(g) \det^{-1}(g|_{T_h})$ if $[g,h] \neq e$.

$$T(h,g) = (-1)^{\sigma(g)\sigma(h)}(-1)^{\sigma(g)+\sigma(h)}(-1)^{|T_g \cap T_h| + N} \dim(i_g(A_g) \cap i_h(A_h))\,.$$

It follows that

$$T(h,g) = T(g,h) = T(g^{-1},h)\,.$$

Notice also that if $[g,h] = e$,

$$\epsilon(gh,k) = \epsilon(g,k)\epsilon(h,k)$$

and

$$T(gh,h) = T(hg,h) = T(g,h)\,.$$

On the other hand

$$\begin{aligned}\mathrm{STr}(\varphi_{g^{-1}}|_{A_h}) &= \epsilon(g^{-1},h)(-1)^{\sigma(g^{-1}h)}(-1)^{|T_g\cap T_h|+N}\dim(i_g(A_g)\cap i_h(A_h))\\ &= \varphi_{h,g}(-1)^{\sigma(g^{-1}h)}\det(g^{-1}|_{N_h})(-1)^{\tilde{g}\tilde{h}}(-1)^{|T_g\cap T_h|}\dim(i_g(A_g)\cap i_h(A_h))\\ &= \epsilon(g^{-1},h)T(g^{-1},h)\end{aligned}$$

where μ_j are the Eigenvalues of g^{-1} on $i_h(A_h)$.

Finally we see that the φ determine the $\epsilon(g,h)$ which have to satisfy the equations

$$\epsilon(g,h)=\epsilon(h^{-1},g)\,,\quad \epsilon(g,g)=1\,,\quad \epsilon(g_1g_2,h)=\epsilon(g_1,h)\epsilon(g_2,h)$$

which are equivalent to the trace axiom.

This relates the freedom of choosing projective factors for the G-action to choices of discrete torsion. □

Lemma 5.1. *The following statements hold*:

(1) *If* $\forall\, g,h,\in G: J_h=J_g$ *any choice of cocycle* γ *with values in* K^* *will give a special* G*-twisted FA. There is only one choice of compatible non-abelian cocycle* φ*. This is the case if* G *acts trivially.*

(2) *By Proposition* 4.3 *if* $[g,h]=e$ *then* $\gamma_{g,h}=0$ *or* $\varphi_{g,h}\varphi_{h,g}=1$*. This implies that if* $\gamma_{g,h}\neq 0$

$$\det(g|_{N_h})\det(h)|_{N_g}=1 \tag{5.5}$$

must hold since $\epsilon(g,h)=\epsilon(h^{-1},g)=\epsilon(h,g)^{-1}$*. Vice-versa*: *if* (5.5) *does not hold for commuting* g,h*, then necessarily* $\gamma_{g,h}=0$*.*

Proof. For (1), notice that all $\pi_h(\gamma_{g,g^{-1}})\neq 0$. Thus by Lemma 4.2, the $\gamma_{g,h}\neq 0$ and furthermore since $r_g=\mathrm{id}$, we see that the $\gamma_{g,h}\in K^*$.

For (2), we use the compatibility twice. If $[g,h]=e:\gamma_{g,h}=\varphi_{g,h}\gamma_{h,g}=\varphi_{g,h}\varphi_{h,g}\gamma_{g,h}$.

Now since by assumption $\gamma_{g,h}\neq 0$, we get the desired result using that if $[g,h]=e:\varphi_{h^{-1},g}=\varphi^{-1}_{h,g}$. □

Remark 5.2. There is a universal action of tensoring with twisted (super) group rings, which allows to recover all possible choices of σ and ϵ. This is discussed in detail in [17].

6. Mirror Construction for Special *G*-Frobenius Algebras

6.1. *Double grading*

We consider Frobenius algebras with grading in some abelian group I.

$$A=\bigoplus_{i\in I}A_i\,.$$

This grading can be trivially extended to a double grading with values in $I \times I$ in two ways

$$A^{cc} = \bigoplus_{i \in I} A_{i,i}$$

and

$$A^{ac} = \bigoplus_{i \in I} A_{i,-i}$$

corresponding to the diagonal $\Delta : I \to I \times I$ and $(\mathrm{id}, -) \circ \Delta : I \to I \times I$.

Definition 6.1. We call bi-graded Frobenius algebras of this form of (c, c)-type and of (a, c)-type, respectively. In the language of Euler fields this means that the Euler field is a pair $(E, \bar{E})$ that satisfies $(E, \bar{E}) = (E, E)$ or $(E, \bar{E}) = (E, -E)$.

These gradings become interesting for special G-Frobenius algebras, since in that case the shifts will produce a possible non-diagonal grading.

Definition 6.2. Given a graded special G-Frobenius algebra, we assign the following bi-degrees to 1_g,

$$(E, \bar{E})(1_g) := (s_g, \bar{s}_g)$$

where $\bar{s}_g := s_{g^{-1}} = \frac{1}{2}(s_g^+ - s_g^-)$.

Furthermore for the Ramond-space of A we assign the following bi-degree to v,

$$(E, \bar{E})(v) := \left(-\frac{d}{2}, -\frac{d}{2}\right) .$$

It is clear that A_e is of (c, c)-type. All of A is however only of (c, c) type if $s_g = \bar{s}_g$.

6.2. *Euler-twist (spectral flow)*

In this section, we consider a graded special G-Frobenius algebra and construct a new vector-space from it. We denote the grading operator by E.

Definition 6.3. The twist-operator j for an Euler-field E is

$$j := \exp(2\pi i E) .$$

We denote the group generated by j by J.

We call a special G-Frobenius Euler if there is a special $\tilde{G}$-Frobenius algebra $\tilde{A}$ of which A is a subalgebra where $\tilde{G}$ is a group that has G and J as subgroups.

Definition 6.4. The dual $\check{A}$ to an Euler special G-Frobenius algebra A of (c, c)-type is the vector space

$$\check{A} := \bigoplus_{g \in G} \check{A}_g := \bigoplus_{g \in G} V_{gj^{-1}}$$

with the A_e and G-module structure determined by $\tau_j^g : \check{A}_g \simeq V_{gj^{-1}}$ together with the bi-grading

$$(E, \bar{E})(\check{1}_g) := (s_{gj^{-1}} - d, \bar{s}_{gj^{-1}}) =: (\check{s}_g, \bar{\check{s}}_g)$$

where $\check{1}_g$ denote the generator of $\check{A}_g$ as A_e-module and the bi-linear form

$$\check{\eta} := \tau_j^*(\bar{\eta})$$

where $\tau_j := \oplus_{g \in G} \tau_j^g$ and V and $\bar{\eta}$ refer to the Ramond-space of A.

7. Explicit Examples

7.1. *Self duality of A_n*

We consider the example of the Jacobian Frobenius Algebra associated to the function z^{n+1}

$$A_n := \mathbf{C}[z]/(z^n)$$

together with the induced multiplication and the Grothendieck residue. Explicitly:

$$z^i z^j = \begin{cases} z^{i+j} & \text{if } i + j \leq n \\ 0 & \text{else} \end{cases}$$

with the form

$$\eta(z^i, z^j) = \delta_{i,n-1-j}$$

and the grading:

$$E(z^i) := \frac{i}{n+1}$$

which means $\rho = z^{n-1}$ and $d = \frac{n-1}{n+1}$.

We consider just the group $J \simeq \mathbf{Z}/(n+1)\mathbf{Z}$ with the generator j acting on z by multiplication with $\zeta_{n+1} := \exp(2\pi i \frac{1}{n+1})$. Now

$$\mathrm{Fix}_{j^i} = \begin{cases} \mathbf{C} & \text{if } i = 0\,, \\ 0 & \text{else}\,, \end{cases}$$

and thus

$$A_{j^i} = \begin{cases} A_n & \text{if } i = 0\,, \\ 1_{j^i} K & \text{else}\,. \end{cases}$$

Furthermore we have the following grading:

$$(E, \bar{E})(1_{j^k}) = \begin{cases} (0, 0) & \text{for } i = 0\,, \\ \left(\dfrac{k-1}{n+1}, \dfrac{n-k}{n+1} \right) & \text{else}\,, \end{cases}$$

and $\rho_{j^i} = 1_{j^i}$ and $d_{j^i} = 0$.

Using the reconstruction theorem we have to find a cocycle γ and a compatible action φ. There is no problem for the metric, since always $|N_g| = 1$ and if $n+1$ is even $\det(\zeta^{\frac{n+1}{2}}) = -1$. Since the group J is cyclic there is no freedom of choice for ϵ and if $n+1$ is even there are two choices of parity are possible corresponding to $\sigma(j) \pm 1$.

From the general considerations we know $\gamma_{j^i,j^{n-1-i}} \in A_e$ and $\deg(\gamma_{j^i,j^{n-1-i}}) = d - d_{j^i} = \frac{n-1}{n+1}$ which yields

$$\gamma_{j^i,j^{n-1-i}} = ((-1)^{\tilde{j}i}\zeta^i)^{1/2}\rho = ((-1)^{\tilde{j}}\zeta)^{i/2} z^{n-1}$$

for the other γ notice that $\deg(1_{j^i}) + \deg(1_{j^k}) = \frac{i+k-2}{n+1}$ while $\deg(1_{j^{i+j}}) = \frac{i+k-1}{n+1}$ if $i+k \neq n+1$, but there is no element of degree $\frac{1}{n+1}$ in $A_{j^{i+k}}$ for $i+k \neq n+1$.

Hence

$$\gamma_{j^i,j^k} = \begin{cases} ((-1)^{\tilde{j}}\zeta)^{i/2} z^{n-1} & \text{for } i+k = n+1\,, \\ 0 & \text{else}\,. \end{cases}$$

And since $\mathbf{Z}/(n+1)\mathbf{Z}$ is Abelian and furthermore there is no non-trivial class of discrete torsion:

$$\varphi_{j^i,j^k} = (-1)^{\sigma(j^i)\sigma(j^k)}\zeta^{-i}\,.$$

Therefore the G-invariants $A^G = A_1$ are generated by the identity 1.

The Ramond grading of this algebra is

$$(E,\bar{E})(1_{j^k}v) = \begin{cases} \left(-\frac{n-1}{2(n+1)}, -\frac{n-1}{2(n+1)}\right) & \text{for } i = 0\,, \\ \left(\frac{k}{n+1} - \frac{1}{2}, -\frac{k}{n+1} + \frac{1}{2}\right) & \text{else}\,, \end{cases}$$

and the action of of G is given by:

$$\bar{\varphi}_{j^i,j^k} = \begin{cases} (-1)^{\sigma(j^i)}\zeta^i & \text{for } k = 0\,, \\ (-1)^{\sigma(j^i)(\sigma(j^k)+1)} & \text{else}\,. \end{cases}$$

Since $j \in G$ the special G-Frobenius algebra is Euler and the dual is defined, moreover $G = J$ so that the vector-space structures of A and $\check{A}$ coincide. The grading is given by:

$$(E,\bar{E})(\check{1}_{j^k}) = \begin{cases} (0,0) & \text{for } k = 0\,, \\ \left(\frac{n-1}{n+1}, 0\right) & \text{for } k = 1\,, \\ \left(\frac{k-(n+1)}{n+1}, \frac{n+1-k}{n+1}\right) & \text{else}\,. \end{cases}$$

The G-action on the $\check{A}_{j^k}$ keeping in mind the shifted group grading is given by:

$$\check{\varphi}_{j^i,j^k} = \begin{cases} (-1)^{\sigma(j^i)}\zeta^i & \text{for } k = 1\,, \\ (-1)^{\sigma(j^i)(\sigma(j^{k-1})+1)} & \text{else}\,. \end{cases}$$

If $\sigma \equiv 0(2)$ all sectors with $k \neq 1$ are invariant. These correspond to the original twisted sectors. There are no invariants in the sector $\check{A}_{j^k}$ which corresponds to the original untwisted sector.

In the case of $n+1$ even and $\sigma(j^k) \equiv k(2)$, we obtain as invariants only those twisted sectors belonging to even powers of j.

Taking into account the grading we find for trivial σ:

$$(A_n/(\mathbf{Z}/\check{(}n+1)\mathbf{Z}))^{\mathbf{Z}/(n+1)\mathbf{Z}} \simeq A_n$$

as a small realization of the Jacobian Frobenius algebra, where more explicitly

$$\check{1}_{j^k} \mapsto z^{n+1-k} : k = 2, \dots n \text{ and } \check{1}_{j^0} \mapsto 1\,.$$

Notice that this A_n is of (a,c)-type, i.e. the bi-degree of z^k is $(-\frac{k}{n+1}, \frac{k}{n+1})$.

Also since $J = G$ the form $\bar{\eta}$ pulls back and gives a non-degenerate form on $\check{A}^G$, which is the usual form on A_n. Furthermore the usual multiplication on A_n is compatible with everything so that we can say that A_n is self-dual under this operation. We would like to point out that in the process of dualization we project out one spurious sector, whose grading is such that there is no Frobenius structure on all of $\check{A}$ since we cannot find an invariant metric — although there is the pulled back non-degenerate metric. All products with fields from the sector $k = 1$ (with the exception of the identity $(\check{1}_e)$ have to vanish for degree reasons. Also in this sector there neither an (a,c) nor a (c,c) type grading.

The structure of this algebra can be seen as analogous to the one found in the r-spin curve of the A_n-model [12, 23], where also a spurious sector dubbed "Ramond" appears. Both our and the r-spin picture are in mirror symmetry parlance the A-model (as opposed to the usual Landau–Ginzburg B-model) version of the A_n model. From our point of view the appearance of the sector $k = 1$ is, however, not mysterious at all, but moreover necessary, since it is the dual of the untwisted sector. This can be seen as an answer to the question of its origin posed in [23].

7.2. D_n from a special $\mathbf{Z}/2\mathbf{Z}$-Frobenius algebra based on A_{2n-3}

In this section, we show how to get D_n from a special $\mathbf{Z}/2\mathbf{Z}$-Frobenius algebra based on A_{2n-3}. The function for the Frobenius algebra A_{2n-3} is z^{2n-2}. Since this is an even function, $\mathbf{Z}/2\mathbf{Z}$ acting via $z \mapsto -z$ is a symmetry. There are two sectors, the untwisted and the twisted sector containing the element 1_{-1} with degree 0. The multiplication is fixed by $\deg(\gamma_{-1,-1}) = \frac{2n-4}{2n-2}$ thus

$$\gamma_{-1,-1} = z^{2(n-2)}$$

again the group is cyclic so the $\mathbf{Z}/2\mathbf{Z}$-action only depends on the choice of parity of the -1-sector.

In the untwisted sector we have $A_e^{\mathbf{Z}/2\mathbf{Z}} = \langle 1, z^2, \dots, z^{2(n-1)} \rangle \simeq A_{n-1}$, the action of -1 on 1_{-1} is given by

$$\varphi_{-1,-1} = (-1)^{\sigma(-1)\sigma(-1)+1}$$

so that if $\sigma(-1) = 1$,

$$(A_{2n-3}/(\mathbf{Z}/2\mathbf{Z}))^{\mathbf{Z}/2\mathbf{Z}} \simeq D_n$$

again as a realization of the Jacobian Frobenius algebra. Notice that in this case $\tilde{1}_{-1} = 0$ so that the algebra is purely even.

If $\sigma(-1) = 0$, we just obtain the invariants of the untwisted sector which are isomorphic to A_{n-1}.

The untwisted sector is given by the singularity A_{n-1} as expected upon the transformation $u = z^2$. Notice that the invariants of the Ramond sector yield the singularity A_{n-2} as expected from [24]. These are of the form $u^i du$ or $z^{2i+1}dz$ with $i = 0, \dots, n-3$.

7.3. *Point* mod *G*

In the theory of Jacobian Frobenius algebras there is the notion of a point played by a Morse singularity $z_1^2 + \cdots + z_n^2$. Any finite subgroup $G \subset O(n, K)$ leaves this point invariant.

The G-twisted algebra after possibly stabilizing is the following.

$$A = \bigoplus_{g \in G} K 1_g\,.$$

And the grading is $\deg(1_g) = (\frac{1}{2}s_g^-, \frac{1}{2}s_g^-)$, since $d = d_g = 1$.

Using Lemma 4.2, it follows that the γ cannot vanish, thus fixing φ and ϵ, so that the possibilities are enumerated by the graded cocycles. The compatibility equations hold automatically.

If we assume that $G \subset O(n, \mathbf{C})$ and that $s_g^- = 0$ (i.e. $\sum_{i:\lambda_i \neq 0} \frac{1}{2\pi i}\lambda_i = \frac{|N_g|}{2} \in \mathbf{N}$), then we the cocycles lie in $H^2(G, K^*)$ and the possible algebra structures are those of twisted group algebras.

7.3.1. *Point* mod $\mathbf{Z}/n\mathbf{Z}$ *and its dual*

By the above analysis we realize $\mathbf{Z}/n\mathbf{Z}$ as the sub-group of rotations of order n in $\mathbf{C}$. We have that $s_g^- = 0$ and thus we can choose the full cocycle making A into A_n, multiplicatively, with trivial grading and trivial G-action if one chooses all even sectors. The metric, however, will not be consistent with A_n. The identity pairs with itself for instance. Dualizing A, we obtain the following space

$$\check{A} = \bigoplus_i A_{j^{i-1}}$$

with again a trivial $(E, \bar{E})$ grading. Choosing the generator $\check{j} := j^{-1}$ for J, the metric reads

$$\check{\eta}(\check{1}_{\check{j}^i}, \check{1}_{\check{j}^k}) = \begin{cases} 1 & \text{if } i + j = n - 1\,, \\ 1 & i = j = n\,, \\ 0 & \text{else}\,. \end{cases}$$

This metric is compatible with the following multiplication:

$$\check{1}_{\check{j}^i} \circ \check{1}_{\check{j}^k} = \begin{cases} \check{1}_{\check{j}^{i+k}} & \text{if } i+j \leq n-1\,, \\ \check{1}_{\check{j}^n} & i=j=n\,, \\ 0 & \text{else}\,. \end{cases}$$

This can be interpreted as $A_{n-1} \oplus A_1$. This is allowed since we do not have any grading restrictions. Again by our mirror symmetry we see that we obtain this A-model by orbifolding the Landau–Ginzburg B-model pair (A_n, A_1) above by $\mathbf{Z}/(n+1)\mathbf{Z}$. We previously did this by regarding the left side of the pair and the above calculation is for the right side of the pair. Thus we again see that the appearance of the extra basis element can be seen as natural from this point of view.

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References

1. M. Atiyah, Topological quantum field theories, *Inst. Hautes Études Sci. Publ. Math.* **68** (1988), 175–186.
2. W. Chen and Y. Ruan, Orbifold quantum cohomology, preprint. math.AG/0005198
3. R. Dijkgraaf, A geometrical approach to two-dimensional conformal field theory, Ph.D. Thesis, Utrecht, 1989.
4. B. Dubrovin, Geometry of 2D topological field theories, in *Integrable Systems and Quantum Groups* (*Montecatini Terme*, *1993*), Lecture Notes in Mathematics 1620, Springer, Berlin, 1996, pp. 120–348.
5. R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, The operator algebra of orbifold models, *Commun Math. Phys.* **123** (1989), 485–526.
6. R. Dijkgraaf and E. Witten, Topological gauge theories and group cohomology, *Commun. Math. Phys.* **129** (1990), 393–429.
7. B. Fantechi and L. Goettsche, Orbifold cohomology for global quotients, preprint. math.AG/0104207

8. D. Freed and F. Quinn, Chern–Simons theory with finite gauge group, *Commun. Math. Phys.* **156** (1993), 435–472.
9. B. R. Greene and M. R. Plesser, Duality in Calabi–Yau moduli space, *Nucl. Phys.* **B338** (1990), 15–37.
10. A. Hatcher and W. Thurston, A presentation for the mapping class group of a closed orientable surface, *Topology* **19** (1980), 221–237.
11. K. Intriligator and C. Vafa, Landau–Ginzburg orbifolds, *Nucl. Phys.* **B339** (1990), 95–120.
12. T. Jarvis, T. Kimura and A. Vaintrob, Moduli spaces of higher spin curves and integrable hierarchies, *Compositio Math.* **126** (2001).
13. T. Jarvis, R. Kaufmann and T. Kimura, Pointed admissible G-covers and G-equivariant cohomological field theories, preprint. math.AG/0302316
14. R. Kaufmann, The tensor product in the theory of Frobenius manifolds, *Int. J. Math.* **10** (1999), 159–206.
15. R. Kaufmann, Singularities and orbifold Frobenius algebras, in preparation.
16. R. Kaufmann, Second quantized Frobenius algebras, preprint. math.AG/0206137
17. R. Kaufmann, The algebra of discrete torsion, preprint. math.AG/0208081
18. R. Kaufmann, Discrete torsion, symmetric products and the Hilbert scheme, preprint.
19. M. Lehn and C. Sorger, The cup product of the Hilbert scheme for K3 surfaces, preprint. math.AG/0012166
20. N. Steenrod, *The Topology of Fibre Bundles*, Princeton Mathematical Series 14, Princeton University Press, Princeton, NJ, 1951.
21. V. Turaev, Homotopy field theory in dimension 2 and group-algebras, preprint. math.QA/9910010
22. C. Vafa, String vacua and orbifoldized LG models, *Modern Phys. Lett.* **A4** (1989), 1169–1185.
23. E. Witten, Algebraic geometry associated with matrix models of two-dimensional gravity, in *Topological Models in Modern Mathematics* (Stony Brook, New York, 1991), Publish or Perish, Houston, TX (1993), pp. 235–269.
   — The N-matrix model and gauged WZW models, *Nucl. Phys.* **B371** (1992), 191–245.
24. C. T. C. Wall, A note on symmetry of singularities, *Bull. London Math. Soc.* **12** (1980), 169–175.