http://dx.doi.org/10.1090/conm/403/07596

Contemporary Mathematics
Volume **403**, 2006

Singularities with Symmetries, Orbifold Frobenius Algebras and Mirror Symmetry

Ralph M. Kaufmann

ABSTRACT. Previously, we introduced a duality transformation for Euler G-Frobenius algebras. Using this transformation, we prove that the simple A, D, E singularities and Pham singularities of coprime powers are mirror self-dual, where the mirror duality is implemented by orbifolding with respect to the symmetry group generated by the grading operator and dualizing. We furthermore calculate orbifolds and duals to other G-Frobenius algebras which relate different G-Frobenius algebras for singularities to each other. In particular, using orbifolding and the duality transformation we provide mirror pairs for the simple boundary singularities B_n and F_4. Lastly, we relate our constructions to r spin-curves, classical singularity theory, and foldings of Dynkin diagrams.

Introduction

In [**Ka03**] we introduced a duality transformation for Euler G-Frobenius algebras which are graded Frobenius algebras whose grading operator is realized by the action of a central element. Using this transformation in the setting of isolated singularities with symmetries, we prove that the simple singularities A, D, E and certain Pham singularities are mirror self-dual where the mirror duality is implemented by orbifolding with respect to the symmetry group generated by the grading operator and dualizing. In particular, the invariants of the orbifold are A_1 while the invariants of the dual are the simple singularities of type A, D, E one started out with. Thus orbifolding and dualizing provides a mirror dual pair to the pair (W, A_1) which is naturally associated to W, for W one of the simple singularities A_n, D_n, E_6, E_7, E_8. We also show that the same holds true for Pham singularities of co-prime powers.

Furthermore, we calculate orbifolds and duals to other G-Frobenius algebras which relate different G-Frobenius algebras for singularities to each other. We thereby provide more mirror pairs, notably mirror pairs for the simple boundary singularities. In particular, $((B_n, I_2(4)), (I_2(4), B_n))$ is obtained by orbifolding and dualizing either A_{2n-1} or D_{n+1} by $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$. And $((F_4, I_2(4)), (I_2(4), F_4))$ is obtained by orbifolding E_6 and dualizing with respect to $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

2000 *Mathematics Subject Classification.* Primary: 14B05; Secondary: 14J32, 14N35, 32B30.

The invariants of the G-Frobenius algebras based on the singularities with symmetries are related to the singularities considered on the orbifold of $\mathbb{C}^n$ with respect to the symmetry group, while the duals conjecturally play a role in the analogs of r spin-curves built on quasi-homogeneous polynomials, of which special types have been studied by [**FJR**]. For the exact formulation of these conjectures we refer to §5.1.

The operation of dualizing as defined in [**Ka03**] was inspired by the representation theory of $N = 2$ super-conformal field theory applied to orbifold Landau-Ginzburg models [**IV90**]. Although the background is very elaborate and involves many highly complicated concepts, in the special case we are considering all can be stated in terms of G-Frobenius algebras or $D(k[G])$-modules and algebras, where $D(k[G])$ stands for the Drinfel'd double of the group ring $k[G]$. $D(k[G])$-modules are a special type of G-graded G-modules, namely those where the G-action acts by conjugation on the G-grading, cf. [**Mo93, Ka02b, JKK03**].

We will first review the background for this operation and then comment on its realization on the level of Euler G-Frobenius algebras. The reader not inclined to read about physics can thus skip the following two paragraphs and further comments about physics which can be considered as motivation and continue to the purely algebraic part of the paper.

A so-called $(2,2)$ super-conformal field theory has an $N = 2$ super-conformal symmetry for both the left and the right movers. This implies that there are four finite rings which are closed under the naïve operator product. These rings are called $(c,c),(a,c),(a,a)$ and (c,a), respectively. In terms of representation theory these rings are given by fields which are annihilated by certain operators or equivalently satisfy certain constraints for their eigenvalues with respect to the operators $J_0, \bar{J}_0, L_0, \bar{L}_0$ of the two $N = 2$ super-conformal algebras, which are usually called $q, \bar{q}, h$ and $\bar{h}$, respectively. The left c or a stands for left chiral or anti-chiral and the letter a or c on the right for right chiral or right anti-chiral. An element $|\phi\rangle$ is left chiral if $G^+_{-1/2}|\phi\rangle = 0$ or equivalently $h = \frac{q}{2}$. It is called left anti-chiral if $G^-_{-1/2}|\phi\rangle = 0$ or equivalently $h = -\frac{q}{2}$. Right chiral means that $\bar{G}^+_{-1/2}|\phi\rangle = 0$ or equivalently $\bar{h} = \frac{\bar{q}}{2}$, and finally right anti-chiral means that $\bar{G}^-_{1/2}|\phi\rangle = 0$ or equivalently $\bar{h} = -\frac{\bar{q}}{2}$. It turns out the rings (a,a) and (c,a) can be recovered from (c,c) and (a,c) by charge conjugation. Thus one confines oneself to study the former two rings. Mirror symmetry as it was originally conceived in physics was an operation which takes one conformal field theory T and produces another conformal field theory $\check{T}$ such that the (c,c) ring of T is isomorphic to the (a,c)-ring of $\check{T}$ and vice versa.

One special type of $N = 2$ theory is given by the so-called Landau-Ginzburg theory, which is the conformally invariant fixed point of the Lagrangian

$$\mathcal{L} = \int K(X, \bar{X}) d^2z d^4\theta + \int (f(z_i) + \text{ complex conjugate}) \, d^2z d^2\theta,$$

where f is a quasi-homogeneous function of fractional degree q_i for z_i. This model leads to a trivial (a,c)-ring and a (c,c)-ring which is given by $\mathbb{C}[\mathbf{z}]/J_f$ where $J_f = (f_{z_i})$ is the Jacobian ideal. Moreover, the bi-degree $(q,\bar{q})$ for z_i is given by (q_i, q_i).

The above considerations are the starting point for a purely algebraic consideration. If the function f above has an isolated singularity at zero, the situation is one that has been studied for a long time by mathematicians. The (c,c)-ring is, in this case, just the local, or Milnor, ring of the singularity. The only unusual thing is the bi-grading instead of the grading, but in fact the bi-grading is just a diagonal grading obtained from the usual grading in singularity theory and it contains no additional information. It will however play an important role later on.

In the setup above, the quasi-homogeneity of the function f allows one to consider it as a function on a weighted projective space. In the case that the polynomial describes a Calabi-Yau hypersurface, the claim that these two geometries (singularity/Calabi-Yau) should give the same Frobenius manifolds of field theories is the famous Landau-Ginzburg/Calabi-Yau correspondence. Thus by analyzing this correspondence one is naturally lead to consider the quotient of the theory by a finite symmetry group. In general, one can consider a group $G \subset GL(\mathbb{C}, n)$ which leaves $f(z_1, \ldots, z_n)$ invariant and consider the resulting orbifold. This particular situation and the general setup of global orbifolds was analyzed in [**Ka03**]. It turns out that the algebraic object one is dealing with is an extension of the Milnor ring, which by itself is a Frobenius algebra, to a G-Frobenius algebra in the sense of [**Ka03**]. A G-Frobenius algebra has a G-action and the invariants of this G-Frobenius are expected to form a Frobenius algebra. These will be bi-graded in a natural way. In physics terms this algebra of invariants is the (c, c)-ring of the orbifold model. Now again appealing to physics, the orbifold theory should also have an (a, c)-ring. This ring is what is computed by the duality transformation we gave in [**Ka03**]. To be precise, the ring (a, c) will be equal to the G-invariants of the dual $D(k[G])$ model. In order to define the full dual it is necessary for the group of symmetries to contain the symmetry provided by the exponential grading operator $J = diag(\exp(2\pi i q_1), \ldots \exp(2\pi i q_n))$.

We call the transformation a mirror transformation, since as we show below, the orbifold of the simple singularities of type A, D, E by the symmetry group generated by J has a trivial (c, c) and an (a, c)-ring that is isomorphic to the Milnor ring of the singularity and hence is mirror dual to the original Milnor ring. Thus for these singularities the operation of orbifolding and taking the invariants of the dual implements mirror symmetry. If one would like to phrase mirror symmetry in terms of A-models and B-models, the Landau-Ginzburg model is a B-model. In mathematical terms the B-Model is the Milnor ring with the diagonal bi-grading (q, q). The corresponding mirror model is an A-model (not to be confused with the A-type singularity) which would be given by the Milnor ring but with a grading of $(-q, q)$. This would be a "Landau-Ginzburg A-model." For the B-model the (a, c)-ring is trivial and for the A-model the (c, c)-ring is trivial.

In [**Ka03**], we have made the case that for global orbifolds it is not enough to consider just the invariants of the G-action of the G-Frobenius algebra, but instead one needs to consider the whole G-Frobenius algebra. The fruitfulness of this point of view can be seen for instance in its application to symmetric products [**Ka04**]. Another instance where the relevance of the G-Frobenius algebras is apparent is in the tensor product which exists on the level of G-Frobenius algebras and not their invariants. The philosophy extends beyond the level of Frobenius algebras to their deformations, G-cohomological field theories, as demonstrated in [**JKK03**].

The dualization as we described it in [**Ka03**], and which we will review below, does not always provide a G-Frobenius algebra. In fact, generally the data of the $D(k[G])$ model with metric does not afford a G-Frobenius algebra structure, although it is expected that there is a Frobenius structure on the invariants. This leads us to define the notion of a degenerate G-Frobenius algebra below. Here one adds an additional metric which is equal to the original metric when restricted to the invariants, but is allowed to be degenerate on the non-invariant elements and is invariant with respect to a G-graded multiplication. This multiplication together with the metric descends to a Frobenius algebra on the invariants.

It is this type of structure that arises in the theory of spin curves [**JKV01, PV01, P02**] and the construction of cohomological field theories from certain singularities with fixed Abelian groups H containing the grading symmetry J, which have recently started to be investigated [**FJR**]. We conjecture that the resulting theory is the deformation of the dual of the orbifold of the singularity with respect to the group H. Although the structures coincide on the invariant part, on the degenerate part the matching of non-invariant elements is only almost realized. There are additional elements which can be explained by interpreting the Milnor rings inside degenerate G-Frobenius algebras, as we discuss in §5.1.

In these geometric settings the g-twisted sectors—this is another name for the group degree g part of the Frobenius algebra for $g \neq e$—which have a degenerate metric are related to a certain behavior called Ramond type. In the case of the A_n singularities there is only one such sector and the entire sector is degenerate. In the cases of D and E, the structure is more complicated and there are invariant elements in g-twisted sectors which have a degenerate metric. The appearance of these degenerate elements is stunning and maybe a nuisance from the point of view of spin-curves, but is natural from the G-Frobenius point of view. Moreover, regarding our dualization on the level of G-Frobenius algebras as mirror symmetry, we expect this kind of behavior for the mirror dual "A-model" of a singularity, the construction of which was Witten's original motivation for considering the spin-curve picture [**W91, W92**].

A note of caution about nomenclature. One would be inclined to call the sectors having degenerate pairings in the new metric Ramond sectors. This might however lead to confusion, since the term Ramond already has a meaning in the theory of G-Frobenius algebras [**Ka03**] and orbifold Landau-Ginzburg theory. Therefore we will call them *sectors of Ramond type* and hope to avoid the confusion.

We recall that the Ramond-G-algebra, or state-space, for a G-Frobenius algebra is a cyclic module for the G-Frobenius algebra whose G-action is determined by compatibility and the fact that the generator of the cyclic algebra is the one-dimensional representation of G which is given by the character χ which is part of the data of a G-Frobenius algebra. The component of this space of group degree g would be naturally called the g-twisted Ramond sector. The "Ramond" in this name stands for the Ramond ground states. This Ramond space plays a fundamental role in the theory of singularities as it corresponds as a $D(k[G])$-module to the middle dimensional cohomology of the Milnor fibers in an orbifold model, while the G-Frobenius algebra corresponds as a $D(k[G])$-module to the orbifold Milnor ring or universal deformation space (see the remarks in §5.2 below). For the untwisted sectors, i.e., the subalgebras of group degree e, this statement was first proved in [**Wa80**].

In the spirit of the mirror construction for simple singularities, one expects for a given theory T with a symmetry group G and a subgroup $H \subset G$ of symmetries that $(T/H)^H \simeq (((T/H)^H/(G/H))^\vee)^{(G/H)}$, where the superscript stands for taking the invariants and $\vee$ stands for dualizing. This type of transformation was used by

[GP90] to produce the first mirror pairs. The general statement has to be taken as always *cum grano salis*, but as we show below, it is true in many instances in a suitable interpretation. For us T/H will signify a set of data including an H-Frobenius subalgebra of a G-Frobenius algebra which is derived from a Frobenius algebra T with symmetry group G together with its $\vee$ dual.

Lastly, the untwisted sector of an orbifold associated to a singularity can, under certain conditions, be related to the folding of an associated Dynkin diagram. We emphasize that there are foldings and orbifoldings of diagrams. The $\mathbb{Z}/2\mathbb{Z}$ folding of A_{2n-1} yields B_n, while the $\mathbb{Z}/2\mathbb{Z}$ orbifolding yields D_{n+1}.

In order to understand the operation of folding, we also include section 5.3 in this paper on groups of projective symmetries. This is a new construction for Frobenius algebras which we expect to be able to extend to the respective Frobenius manifolds and to a full theory of G-Frobenius algebras. On the algebra level, we obtain the classical folding results for Coxeter groups identifying the sub-Frobenius algebra with the Coxeter group of the folded diagram [**St01, St02**]. The relation to singularity theory and the G-modules appearing in this setting is also briefly discussed. One could hope to extend the folding to all the diagrams of [**Z98**] and find the corresponding orbifold theory.

We will work in the setting of G-Frobenius algebras over a field k of characteristic zero (or prime to $|G|$) as it was established in [**Ka03**]. To understand the constructions of G-Frobenius algebras it is important to see that they are usually performed in four steps. 1. One constructs a G-graded k-module $A = \bigoplus A_g$ with a non-degenerate pairing between A_g and $A_{g^{-1}}$. A_e is usually called the untwisted sector and A_g is called the g-twisted sector. 2. One constructs a $D(k[G])$-module structure on A. This action has to respect certain axioms, among them the self-invariance for the twisted sectors and the so-called restricted trace condition, which necessitates the introduction of a character $\chi \in \mathrm{Hom}(G, k^*)$. The $D(k[G])$-module property is equivalent to introducing an action of G, usually called φ, such that $\varphi(g)(A_h) \subset A_{ghg^{-1}}$. 3. One enlarges the picture by introducing an A_e-module structure on A which is compatible with the $D(k[G])$-module structure. 4. Lastly one adds a multiplication which is compatible with the G-grading to make the $D(k[G])$-module into a G-Frobenius algebra. Comparing G-Frobenius algebras on different levels of this construction compares to the topological mirror symmetry of dimensions and vector spaces vs. that of full Frobenius manifolds.

These are also the steps of the (re)construction program as explained in [**Ka03, Ka04**]. Here the data for the first step is usually provided by the geometric setup. For the second step there are usually several different choices. This is, however, expected, since there is the phenomenon of discrete torsion for orbifolds. As we demonstrated in [**Ka02b**], for every G-Frobenius algebra there exists a family of G-Frobenius algebras indexed by elements of $\alpha \in Z^2(G, k^*)$ with the same underlying data as mentioned in step 1 (up to a re-scaling of the metrics pairing the twisted sectors). In the last step there is an additional compatibility condition of the pairing, which might force one to again re-scale the pairings between the twisted sectors. For all the conditions, we refer to [**Ka03**]. We will, however, review the construction for singularities with symmetries below.

The dualization is an involution on triples (A, j, χ) of a $D(k[G])$-module A, an element $j \in Z(G)$, the center of G, and a one-dimensional representation of G, also known as a character. For a special type of graded G-Frobenius algebras, χ is part of the data while j corresponds to the grading operator. If one includes the other structures of a G-Frobenius algebra, then the operation ceases to be an involution, as, for instance, the metric will be compatible with the group grading only up to a shift. To compensate for the different behavior of the duals, we introduce the notion of a degenerate G-Frobenius algebra of group degree j for an element $j \in Z(G)$.

The paper is organized as follows: In the first section, we review the construction and basic properties of G-Frobenius algebras and consider special types of graded G-Frobenius algebras called Euler and G-Euler. The second section contains the definition for the dualization for Euler $D(k[G])$-modules. The third section applies the first two sections to the G-Frobenius algebras resulting from quasi-homogeneous polynomials in general. The fourth section contains explicit calculations for a large list of examples. From these examples we obtain the theorem about the mirror-self duality of the simple singularities, i.e., those of ADE type and the Pham singularities for coprime powers. The examples also provide mirror pairs for the simple boundary singularities B_n and F_4 and produce G_2 as the untwisted sector of a D_4 orbifold. In the last section, we connect our calculations to spin-curves, classical results in the theory of singularities and foldings of Dynkin diagrams.

Acknowledgments

We would like to thank the organizers of the AMS special session "Gromov-Witten Theory of Spin Curves and Orbifolds" Tyler Jarvis, Takashi Kimura and Arkady Vaintrob for doing such an excellent job in providing a common platform for all the exciting research revolving around spin curves and orbifolds. Special thanks goes to the referee for his thorough reading of the manuscript and helpful remarks which have helped to strengthen the exposition.

It is also a pleasure to thank Boris Dubrovin, Claus Hertling, Tyler Jarvis, Takashi Kimura, Yongbin Ruan, Ian Strachan, Kyoji Saito, and Jean-Bernard Zuber for discussions which were relevant for various stages of the presented research.

Notation

Throughout this paper, we will fix the ground field k to be $\mathbb{C}$ to avoid switching back and forth. Strictly speaking, this is necessary only for the considerations about singularities. In general, for finite a fixed finite group G, for the concept of G-Frobenius algebras, it is enough to consider fields of characteristic prime to the order G. In most examples, the natural setting would be a field of characteristic zero which contains all $|G|$-th roots of unity or a field k of characteristic zero with a fixed embedding of $\bar{k} \hookrightarrow \mathbb{C}$.

1. Graded G-Frobenius algebras

1.1. G-Frobenius algebras. We would like to recall the definition of a G-Frobenius algebra of [**Ka03**]. Although it has now appeared in many places we think it convenient for the reader to display it here once more.

DEFINITION 1.1. *A G-Frobenius algebra (GFA) over the field k is given by the data $\langle G, A, \circ, 1, \eta, \varphi, \chi\rangle$, with*

G	*a finite group*
A	*a finite dim G-graded k-vector space*
	$A = \oplus_{g \in G} A_g$
	A_e is called the untwisted sector and
	the A_g for $g \neq e$ are called the twisted sectors.
$\circ$	*a multiplication on A which respects the grading:*
	$\circ : A_g \otimes A_h \to A_{gh}$
1	*a fixed element in A_e-the unit*
η	*a non-degenerate bilinear form*
	which respects grading i.e., $g\vert_{A_g \otimes A_h} = 0$ unless $gh = e$.
φ	*an action of G on A (which will be by algebra automorphisms),*
	$\varphi \in \mathrm{Hom}(G, \mathrm{Aut}(A))$, *such that* $\varphi_g(A_h) \subseteq A_{ghg^{-1}}$
χ	*a character* $\chi \in \mathrm{Hom}(G, k^*)$

satisfying the following axioms:
NOTATION: *We use a subscript on an element of A to signify that it has homogeneous group degree, (e.g., a_g means $a_g \in A_g$) and we write $\varphi_g := \varphi(g)$ and $\chi_g := \chi(g)$.*

(i) *Associativity*
$(a_g \circ a_h) \circ a_k = a_g \circ (a_h \circ a_k)$

(ii) *Twisted commutativity*
$a_g \circ a_h = \varphi_g(a_h) \circ a_g$

(iii) *G-Invariant Unit:*
$1 \circ a_g = a_g \circ 1 = a_g$
and
$\varphi_g(1) = 1$

(iv) *Multiplicative invariance of the metric:*
$\eta(a_g, a_h \circ a_k) = \eta(a_g \circ a_h, a_k)$

(v) *Projective self-invariance of the twisted sectors*
$\varphi_g|A_g = \chi_g^{-1} id$

(vi) *G-Invariance of the multiplication*
$\varphi_k(a_g \circ a_h) = \varphi_k(a_g) \circ \varphi_k(a_h)$

(vii) *Projective G-invariance of the metric*
$\varphi_g^*(\eta) = \chi_g^{-2} \eta$

(viii) *Projective trace axiom*
$\forall c \in A_{[g,h]}$ and l_c left multiplication by c:
$\chi_h \mathrm{Tr}(l_c \varphi_h|_{A_g}) = \chi_{g^{-1}} \mathrm{Tr}(\varphi_{g^{-1}} l_c|_{A_h})$

For the examples in §4 it is essential that we consider G-Frobenius algebras with non-trivial characters.

REMARK 1.1. *Instead of using a left action of G on A, one can also use a right action as for instance is done in* [**JKK03**]. *Since if φ is a left action $\psi(g) := \varphi(g^{-1})$ is a right action, it does not matter which choice is made.*

REMARK 1.2. *Another way to characterize the G-grading and G-action is to say that it is a $D(k[G])$-module. This statement is equivalent to saying that A is*

G-graded and the G-action is such that

$$\varphi(g)A_h \subseteq A_{ghg^{-1}} \text{ or equivalently } \psi(g)A_h \subseteq A_{g^{-1}hg} \tag{1.1}$$

see e.g., [**Ka02b**]. *We use the nomenclature of $D(k[G])$-module, rather than G-graded G-module since it includes the condition (1.1).*

The compatibilities of the multiplication with the grading and the G-action can also be rephrased as: A is a $D(k[G])$-module algebra.

1.2. Restriction. The operation of restriction of a G-Frobenius algebra to a H-Frobenius algebra for a subgroup $H \subset G$ is discussed in [**Ka03**] and is given by $res(A)^G_H := \bigoplus_{h \in H} A_h$ and restricting all structures.

By forgetting or omitting the multiplicative structure and considering just the action of the subgroup H we obtain the restriction from a $D(k[G])$ to a $D(k[H])$-module.

1.3. Super-grading. We also need to enlarge the framework by considering super-algebras rather than algebras. This will introduce the standard signs.

DEFINITION 1.2. *A G-twisted Frobenius super-algebra over k is a tuple $\langle G, A, \circ, 1, \eta, \varphi, \chi \rangle$, with*

- G *a finite group*
- A *a finite dimensional $G \times \mathbb{Z}/2\mathbb{Z}$-graded k-vector space* $A = A_0 \oplus A_1 = \oplus_{g \in G}(A_{g,0} \oplus A_{g,1}) = \oplus_{g \in G} A_g$
  A_e is called the untwisted sector and is even.
  The A_g for $g \neq e$ are called the twisted sectors.
- $\circ$ *a multiplication on A which respects both gradings:* $\circ : A_{g,i} \otimes A_{h,j} \to A_{gh,i+j}$
- 1 *a fixed element in A_e—the unit*
- η *a non-degenerate even bilinear form which respects grading i.e., $g|_{A_g \otimes A_h} = 0$ unless $gh = e$*
- φ *an action by even algebra automorphisms of G on A,* $\varphi \in \mathrm{Hom}(G, \mathrm{Aut}(A))$, *such that* $\varphi_g(A_h) \subseteq A_{ghg^{-1}}$
- χ *a character* $\chi \in \mathrm{Hom}(G, k^*)$

satisfy the axioms (i)–(viii) of a G-Frobenius algebra with the following alteration of (ii) and (viii):

(ii)$^\sigma$ *Twisted super-commutativity*

$$a_g \circ a_h = (-1)^{\tilde{a}_g \tilde{a}_h} \varphi_g(a_h) \circ a_g$$

(viii)$^\sigma$ *Projective super-trace axiom*

$\forall c \in A_{[g,h]}$ *and l_c left multiplication by c:*

$$\chi_h \mathrm{STr}(l_c \varphi_h|_{A_g}) = \chi_{g^{-1}} \mathrm{STr}(\varphi_{g^{-1}} l_c|_{A_h})$$

where STr *is the super-trace.*

1.4. Graded G-Frobenius algebras.

DEFINITION 1.3. *We call a (super) G Frobenius algebra A* graded by an additive group *I if it is graded as a (super) algebra by I and the metric is homogeneous of a fixed degree d, i.e., for homogeneous a, b the metric satisfies $\eta(a,b) = 0$ unless $deg(a) + deg(b) = d$, where we denote the I-degree of a homogeneous element $a \in A$*

by deg(a). *If* $I = \mathbb{Q}$, *we simply call* A graded. *We also call* d *the* degree of the Frobenius algebra.

1.5. The grading operator. Given a graded G-Frobenius algebra A, we define the grading operator Q to be given by

$$Q(a) := \deg(a)a \quad \text{if } a \text{ is homogeneous.} \tag{1.2}$$

Sometimes this type of operator is also called E.

In the case that A is graded, we furthermore define the operator

$$J := \exp(2\pi i Q). \tag{1.3}$$

This definition is also possible, if for instance, k is of characteristic 0 and an embedding of $\bar{k} \subset \mathbb{C}$ has been fixed.

DEFINITION 1.4. *We call a graded* $D(k[G])$*-module* $A = \bigoplus_{g\in G} A_g$ Euler *if the operator* $J|_{A_e}$ *is described by the action of a central element* j *of the group* G *on* A_e. *i.e., there exists a* $j \in Z(G)$, *the center of* G, *which satisfies* $\varphi(j)|_{A_e} = J|_{A_e}$.

We call a graded $D(k[G])$*-module* G-Euler *if there exists a* $j \in Z(G)$ *such that* $\varphi(h^{-1}j)|_{A_h} = J|_{A_h}$ *for all* $h \in G$.

We call a graded $D[k[H])$*-module* A quasi-Euler *(or* quasi-G-Euler*) if there is a group* G, *such that* H *is a subgroup of* G $(G \supset H)$ *and there exists an Euler (or* G*-Euler)* $D(k[G])$*-module* B *such that the restriction of the* $D(k[G])$*-module* B *to its* $D[k[H])$*-sub module* $res_H(B)$ *is* A.

An Eulerization *of a quasi-Euler* $D(k[H])$*-module is a fixed choice of* $D(k[G])$*-module* B *as above.*

A G*-Frobenius algebra is called Euler,* G*-Euler, quasi-Euler or quasi-*G*-Euler if its underlying* $D(k[G])$*-module is Euler,* G*-Euler, quasi-Euler or quasi-*G*-Euler, respectively.*

1.6. Bi-Graded G-Frobenius algebras.

DEFINITION 1.5. *We call a (super)* G *Frobenius algebra* A bi-graded by an additive group I *if it is bi-graded as a (super) algebra by* I. *If* $I = \mathbb{Q}$, *we simply call* A *bi-graded.*

1.6.1. Notation. We will usually denote the two grading operators by Q and $\bar{Q}$. Given a bi-homogeneous element a we will denote its degree with respect to Q by $q(a) = Q(a)$ and its degree with respect to $\bar{Q}$ by $\bar{q}(a) = \bar{Q}(a)$. We will also use the notation $(q(a), \bar{q}(a))$ to denote the bi-degree.

DEFINITION 1.6. *Fix a graded Frobenius algebra* A *with grading operator* Q.

We define its (c,c)-realization $A^{(c,c)}$ *to be given by the Frobenius algebra* A *together with the bi-grading* (Q,Q), *i.e.,* $\bar{Q} = Q$.

We define the (a,c)-realization *of* A, *denoted by* $A^{(a,c)}$, *to be given by the Frobenius algebra* A *together with the bi-grading* $(Q,-Q)$, *i.e.,* $\bar{Q} = -Q$.

REMARK 1.3. *The terminology stems from the representation theory of the* $N = 2$ *super-conformal algebra, as explained in the introduction.*

In the case of quasi-homogeneous functions with an isolated singularity at zero, the above realizations will yield an A*- and* B*-model for the singularity as follows.*

Both the A- and the B-model will consist of a pair of Frobenius algebras, called the (c,c)-ring and the (a,c)-ring.

The B-model for a singularity will have as a (c,c)-ring the (c,c)-realization of the Milnor ring and a trivial (a,c)-ring.

The A-model for a singularity will have a non-trivial (a,c)-ring given by the (a,c)-realization of the Milnor ring and a trivial (c,c)-ring.

Notice that in the B-model realization there will only be elements of diagonal bi-grading in the (c,c)-ring and in the A-model there will only be anti-diagonally graded (i.e., of bi-grading $(q,-q)$) elements in the (a,c)-ring.

1.7. Constructing G-Frobenius algebras. When one constructs G-Frobenius algebras from geometric or algebraic data, the different structures are usually introduced one after the other. A good example of this procedure is given by the construction of G-Frobenius algebras from isolated singularities with symmetries reviewed below in §3. Also, some operations like the duality discussed below are given on a certain level of structure. The usual order in which the structures are introduced is as follows.

(1) **The G-graded k-module.** Usually the first structure to be given for any G-Frobenius algebra is its *additive* structure $A := \bigoplus_{g\in G} A_g$.

On this level it is also usual to introduce the non-degenerate pairing η which pairs A_g with $A_{g^{-1}}$.

(2) **The G-graded G-module or $D(k[G])$-module structure.** The next property which is usually introduced is a *G-action* on A, usually denoted by φ for a left action (see e.g., [**Ka03**]) which makes A into a $D(k[G])$-module cf. §1.2.

Further data and conditions:

(a) Along with the G-action, the function $\chi : G \to k^*$ is fixed since χ can be derived from the G-action via the condition of projective self-invariance (axiom (v)).

(b) From the projective G-invariance of the metric (axiom (vii)), it follows that the function χ^2 has to be a character, i.e., a one-dimensional representation.

(3) **The $D(k[G])$-A_e-bi-module**. Usually the untwisted sector A_e is naturally a Frobenius algebra. The next step in constructing a G-Frobenius algebra is then the *A_e-module* structure for each A_g: $A_e \otimes A_g \to A_g$, which will be a part of the algebra multiplication. These operations turn A into an A_e-module. They are usually already present in the geometry by functoriality [**Ka04**]. The A_e-module structure has to be compatible with the G-action so $\varphi_g(a_e b_h) = \varphi_g(a_e)\varphi_g(b_h)$.

The A_e-module structure leads to a second compatibility condition of the G-action with the pairing which is given by restriction of the trace axiom to the case $g = e, c = 1 \in A_e$. This condition is called the *restricted trace axiom or condition.* This condition effectively relates the dimension of the various twisted sectors A_g to the character χ and G-action on the identity sector:

$$\chi_h \mathrm{STr}\varphi(h)|_{A_e} = \mathrm{STr}\, id|_{A_h} = \mathrm{sdim}(A_h).$$

Also, the trace axiom puts constraints on the possible G-actions. The constraints can be quite effective, but they define the action at most up to discrete torsion [**Ka02b**].

(4) **The G-Frobenius algebra.** The last step is to introduce the *stringy multiplications*: $A_g \otimes A_h \to A_{gh}$, i.e., the algebra structure. This structure has to satisfy all axioms of a GFA. This means it has to be compatible with the G-action, the metric, and the full trace axiom.

1.8. The metric and the grading. When constructing a G-Frobenius algebra in the above fashion, the metric and the grading can be introduced at the end, but usually, there is a natural choice in each step, which may be modified in the next step.

1.8.1. The metric. The metric, i.e., non-degenerate even symmetric pairing, is usually introduced in step (1) and may be re-scaled in step (4) by a factor to ensure the compatibility of the metric with the multiplication (multiplicative invariance of the metric, axiom (vi)).

1.8.2. The grading. In step (1) first there is usually a grading $Q^{(1)}$ inherent in the definition of each A_g when introducing the metric, which is usually also inherent in the construction. Each of the pairings $A_g \otimes A_{g^{-1}} \to k$ is usually homogeneous of some fixed degree d_g with $d_g = d_{g^{-1}}$.

The first alteration of the grading is a shift of the grading for each A_g by $\frac{1}{2}s^+(g) := \frac{1}{2}(d_e - d_g)$, i.e., the new grading for an element $a_g \in A_g$ is $Q^{(2)}(a_g) = Q^{(1)}(a_g) + \frac{1}{2}s^+(g)$. This makes the metric homogeneous of degree $d := d_e$ on all of A. Notice that $s^+(g) = s^+(g^{-1})$.

The second alteration appears in step (2). For physically inspired reasons, one often makes an additional shift $\frac{1}{2}s^-(g)$ depending on g, which has to preserve the homogeneity of the metric. This condition then translates to the condition that the second shift is anti-symmetric $s^-(g) = -s^-(g^{-1})$.

Mathematically this shift is usually postulated in an *ad hoc* fashion. Nevertheless, its effectiveness in orbifold cohomology [**CR00, CR02**], for instance, is apparent. In the setting of singularities there is an interpretation in terms of the monodromy operator. This will be discussed elsewhere [**Ka**].

The final grading for an element $a_g \in A_g$ is

$$\mathcal{Q}(a_g) = Q^{(2)}(a_g) + \frac{1}{2}s^-(g) = Q^{(1)}(a_g) + \frac{1}{2}(s^+(g) + s^-(g)).$$

If the G-action of step (2) is induced by a linear G action, there is a standard choice for this shift, given by

DEFINITION 1.7. *The standard grading shift for a G-Frobenius algebra with a choice of linear representation $\rho : G \to GL_n(k)$ is given by*

$$s_g := \frac{1}{2}(s_g^+ + s_g^-) \tag{1.4}$$

with

$$s_g^+ := d - d_g \tag{1.5}$$

and

$$s_g^- := \frac{1}{2\pi i}\left(\mathrm{tr}(\log(g)) - \mathrm{tr}(\log(g^{-1}))\right) := \frac{1}{2\pi i}\left(\sum_k \lambda_k(g) - \sum_k \lambda_k(g^{-1})\right)$$

$$(1.6) \qquad = \sum_{k:\lambda_k\neq 0}\left(\frac{1}{2\pi i}2\lambda_k(g) - 1\right),$$

where the $\lambda_k(g)$ are the logarithms of the eigenvalues of $\rho(g)$ using the arguments in $[0, 2\pi)$.

This means that if $\rho(g) = diag(exp(2\pi i\nu_1), \ldots, exp(2\pi i\nu_n))$ with $0 \leq \nu_k < 1$, then $\lambda_k = 2\pi i\nu_k$.

REMARK 1.4. *Notice that if $\rho(g) = diag(exp(2\pi i\nu_1(g)), \ldots, exp(2\pi i\nu_n(g)))$ with $0 \leq \nu_k(g) < 1$, then*

$$\nu_k(g^{-1}) = \begin{cases} 0 & \text{if } \nu_k(g) = 0 \\ 1 - \nu_k(g) & \text{otherwise} \end{cases}$$

$$\nu_k(gh) = \nu_k(g) + \nu_k(h) - \Theta((\nu_k(g) + \nu_k(h)) - 1),$$

where Θ is the step function

$$\Theta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

REMARK 1.5. *In the case of orbifold cohomology* [**CR00, CR02**], *one starts with an action of G on the manifold M and induces an action on the tangent space M which defines the shift s^- via (1.6), and the shift s_g^+ is defined by $d_g :=$* $\dim(Fix(g) \subset M)$. *For general orbifolds this reasoning is understood locally* [**CR00, CR02**]. *For global orbifolds the expressions can, however, be understood globally.*

If $\rho(g) = diag(\exp(2\pi i\lambda_1), \ldots, \exp(2\pi i\lambda_n))$, then $d_g = \sum_{k:\lambda_k=0} 1$ and $d - d_g = \sum_{k:\lambda_k\neq 0} 1$, so $s_g = \sum_{k:\lambda_k\neq 0}\frac{1}{2\pi i}\lambda_k(g) = \sum_k q_k$, yielding agreement with the definition (1.4) above and the one of [**CR00, CR02**] *in that particular case.*

Notice that for the last expression of equation (1.6), we can use the branch of the logarithm obtained by cutting along $[0, \infty)$.

1.8.3. The super-grading. As for the grading, usually each A_g comes with an intrinsic super-grading. In step (1) one usually allows the freedom to shift the super-grading by a $\mathbb{Z}/2\mathbb{Z}$-valued function. The restrictions on this function come from the existence of an even, non-degenerate, quasi-homogeneous metric and in step (3) from the trace axiom. In step (4) there is a condition of the super-grading which is derived from the fact that χ is a character and the trace condition.

1.8.4. Bi-grading.

DEFINITION 1.8. *Set $\bar{s}_g := \frac{1}{2}(s_g^+ - s_g^-)$. Since $s_g^+ = s_{g^{-1}}^+$ and $s_g^- = -s_{g^{-1}}^-$, it follows that $\bar{s}_g = s_{g^{-1}}$. We define the bi-grading $(\mathcal{Q}, \bar{\mathcal{Q}})$ by*

$$\mathcal{Q}(a_g) := Q(a_g) + s_g \quad \bar{\mathcal{Q}}(a_g) := Q(a_g) + \bar{s}_g \quad \text{for } a_g \in A_g$$

REMARK 1.6. *This grading is physically motivated* [**IV90**] *and basically means that the natural bi-degree of the so-called "twist field" is* $(s_g, \bar{s}_g)$. *As mentioned previously, without the background of representation theory of the* $N = 2$ *super-conformal algebra, this grading seems* ad hoc *from a mathematical standpoint. Furthermore, it has an interpretation in terms of a twisted monodromy operator in the theory of quasi-homogeneous isolated singularities* [**Ka**].

The relevance of diagonally and anti-diagonally graded elements is that only elements of this kind should yield invariants under the G-action in the setting of the orbifold mirror philosophy when it is applied to construct A- and B-models for singularities. The grading condition is thus an internal consistency check on the operations used to yield Theorem 4.1 below.

2. A mirror type transformation

In the following, we will construct an involution for triples $\langle A, j, \chi\rangle$ of $D(k[G])$-modules A, elements j of the center of G and characters $\chi \in \mathrm{Hom}(G, k^*)$.

In the case of an Euler $D(k[G])$-module, we take the element j to be the element defined by the Euler property.

We also extend the operation to include a non-degenerate pairing and a bi-grading.

This involution induces via restriction a dualization on quasi-Euler $D[k[H])$-modules (without pairing) with fixed Eulerization.

In the case that A is an Euler Frobenius algebra or a quasi-Euler Frobenius algebra with a fixed Eulerization, we let Q be the grading operator and $j \in G$, such that $\rho(j) = exp(2\pi Q) = J$. In this case the data (A, j, χ) is fixed by the G-Frobenius algebra and the element j yielding the grading.

Our operation conjecturally acts as a mirror transformation on the underlying Euler G-Frobenius algebras in the sense of the orbifold mirror philosophy, see §2.6.

The additional bi-grading is conjecturally compatible with interchange of the B-model—whose non-trivial ring is the (c, c)-ring and is given by the (c, c)-realization of the Milnor ring—and the A-model—whose non-trivial (a, c)-ring is given by the (a, c)-realization of the Milnor ring—for Landau-Ginzburg theories with respect to mirror symmetry.

In fact, we will prove that the orbifold mirror philosophy is correct in the case of the simple singularities A_n, D_n, E_6, E_7, E_8 and yields mirror pairs for the simple boundary singularities B_n, F_4 in a suitable interpretation.

REMARK 2.1. *The definition of the dual comes from physics* [**IV90, V89**], *where the dual* $D(k[G])$*-module is obtained by using an endofunctor in the category of representations of the* $N = 2$ *super-conformal algebra which translates in our case to an isomorphism of* $D(k[G])$*-modules. This endofunctor is generally known as spectral flow and has a particular realization discussed below in the case of G-Frobenius algebras.*

2.1. The G-graded k-module structure.

DEFINITION 2.1. *Given a G-graded k-module A and an element* $j \in Z(G)$, *we define the dual* $\check{A}$ *to be the G-graded k-module*

$$\check{A}_g := A_{gj^{-1}}, \quad \check{A} := \bigoplus_{g \in G} \check{A}_g. \tag{2.1}$$

REMARK 2.2. *The above formula states that as k-modules A and $\check{A}$ are isomorphic, it is only their G-grading which has changed. We denote the isomorphism by $\Phi : A \to \check{A}$, with $\Phi(A_g) = \check{A}_{gj}$.*

2.2. The metric. With the help of the map Φ^{-1}, we can pull back a given metric η from A to $\check{A}$. We set

$$\check{\eta} = (\Phi^{-1})^* \eta \qquad \check{\eta}(\check{a}, \check{b}) := \eta(\Phi^{-1}(\check{a}), \Phi^{-1}(\check{b})).$$

REMARK 2.3. *Notice if η is homogeneous with respect to the group degree, i.e., pairs A_g with $A_{g^{-1}}$, then $\check{\eta}$ pairs $\check{A}_g$ with $\check{A}_{g^{-1}j^2}$ and thus $\check{\eta}$ is not group degree homogeneous, but of group degree j^2 as a tensor in $\check{A}^* \otimes \check{A}^*$.*

2.3. The G-action or the $D(k[G])$-module structure. Given a triple $\langle A, j, \chi\rangle$ of a $D(k[G])$-module A, an element j of the center of G, and a character $\chi \in \mathrm{Hom}(G, k^*)$, we define $\bar{\varphi} := \varphi \otimes_k \chi$. This is an action of G on the k-module $A \otimes_k k \simeq A$ and thus on the k-module $\check{A}$.

We define the G-action $\check{\varphi}$ on $\check{A}$ to be the induced action of the action on A by $\bar{\varphi}$. That is, for $a \in \check{A}$,

$$\check{\varphi}(h)(a) = \chi(h)\Phi(\varphi(h)(\Phi^{-1}(a))). \tag{2.2}$$

REMARK 2.4. *We see that under this action $\check{\varphi}(h)(\check{A}_g) \subset \check{A}_{hgj^{-1}h^{-1}j} = \check{A}_{hgh^{-1}}$, since we made the assumption that $j \in Z(G)$. Thus we obtain a G-action, which makes $\check{A}$ into a $D(k[G]])$-module.*

REMARK 2.5. *The metric $\check{\eta}$ is G-invariant and hence descends to the G-invariants.*

2.4. The bi-grading of the dual. If A was initially graded by the operator $Q^{(1)}$ and $Q(a_g) = Q^{(1)}(a_g) + s_g$, then set $\check{s}_g := s_{gj^{-1}} - d$ and $\check{\bar{s}}_g := \bar{s}_{gj^{-1}}$, where we recall that d is the degree of the G-Frobenius algebra. We define a bi-grading on $\check{A}$ by

$$\check{\mathcal{Q}}(\check{a}) = Q^{(1)}(a) + \check{s}_g \quad \check{\bar{\mathcal{Q}}} := Q^{(1)}(a) + \check{\bar{s}}_g \quad \text{for } \check{a}_g \in \check{A}_g. \tag{2.3}$$

REMARK 2.6. *An Euler G-Frobenius algebra $A = \langle G, A, \circ, 1, \eta, \varphi, \chi, j\rangle$ naturally gives rise to a triple $\langle A, j, \chi\rangle$ and thus determines a dual $D(k[G])$-module with a non-degenerate pairing and a bi-grading.*

REMARK 2.7. *The motivation for the dual bi-grading again comes from the physical interpretation of GM_f as an orbifold Landau-Ginzburg model and the dualization being implemented by the spectral flow operator $\mathcal{U}_{(1,0)}$* **[IV90]** *which has the natural charge $(d = \hat{c} = \frac{c}{3}, 0)$.*

2.4.1. The involution.

DEFINITION 2.2. *We define the dual of a triple $\langle A, j, \chi\rangle$ of a $D(k[G])$-module A, an element j of the center of G, and a character $\chi \in \mathrm{Hom}(G, k^*)$ to be the triple $\langle \check{A}, j^{-1}, \chi^{-1}\rangle$.*

REMARK 2.8. *Notice that the inclusion of the data j and χ turns the operation on the $D(k[G])$-module into an involution.*

2.4.2. The dual of a quasi-Euler $D[k[H]]$-module with given Eulerization.

DEFINITION 2.3. *We define the dual of a quasi-Euler $D[k[H]]$-module A (or H-Frobenius algebra) with given Eulerization B to be the restriction of $\check{B}$ to H. $\check{A} := res_H(\check{B})$*

REMARK 2.9. *Notice that if $j \notin H$ then we cannot pull back the metric, since if $h \in H$ then the element $h^{-1}j^2$ need not be in H. If, however, for all $h \in H$ also $h^{-1}j^2 \in H$, then we can also pull back the metric.*

2.5. A dual G-Frobenius algebra? We would like to remark that the dualizing process is only a process of dualizing for $D(k[g])$-modules with metric.

One thing which prevents the resulting structure from being a G-Frobenius algebra is that the metric is not G-graded anymore, as remarked above in Remark 2.3. Also the projective self-invariance might not hold. However, there might be choices of G-graded multiplication compatible with the G-action which are in some cases unique.

What is actually expected by physics is that there is a Frobenius algebra structure on the G-invariants of $\check{A}$ with the given metric. It is important to note that physics does not say there should be an algebra isomorphism, and in fact the induced multiplication $\Phi \circ \Phi^{-1}$ will not be G-graded on $\check{A}$ unless $j = e$ and the grading and dualization are trivial.

What we can expect is a Frobenius structure on the invariants, plus a lift of this Frobenius structure to the G-graded equivariant level. This will provide some additional structure. This motivates the following definition.

DEFINITION 2.4. *A degenerate G-Frobenius algebra A of degree $j \in Z(G)$ is given by the data $\langle G, A, \circ, 1, \eta, \eta', \varphi, \chi\rangle$, where $\langle G, A, \circ, 1, \eta, \varphi, \chi\rangle$ are the data of a G-Frobenius algebra, and η' is a second pairing on A. These data satisfy the conditions of a G-Frobenius algebra with the following changes and additions:*

(1) *The non-degenerate paring η and the pairing η' pair $A_{gj^{-1}}$ with $A_{g^{-1}j^{-1}}$.*
(2) $\eta|_{A^G} = \eta'_{A^G}$ *where A^G are the G invariants of A.*
(iv)' *Multiplicative Invariance of the metric η':*
$\eta'(a_g, a_h \circ a_k) = \eta'(a_g \circ a_h, a_k)$.
(v)' *Self-invariance of the twisted sectors*
$\varphi_{gj^{-1}}|A_g = id$.
(vii)' *G-invariance of the metric η*
$\varphi_g^*(\eta) = \eta$.
(viii)' *j twisted trace axiom*
$\forall c \in A_{[g,h]}$ *and l_c left multiplication by c:*
$\mathrm{Tr}(l_c \varphi_{hj^{-1}}|_{A_g}) = \mathrm{Tr}(\varphi_{g^{-1}j} l_c|_{A_h})$.

CONJECTURE 2.1. *We conjecture that there is a degenerate G-Frobenius algebra of degree j on the dual of a G-Euler G-Frobenius algebra.*

In the examples we consider, there is a certain uniqueness in the choice for the multiplication. In order to state this precisely we need the following two definitions.

DEFINITION 2.5. *Fix a $D(k[G])$-, A_e-bi-module $A = \bigoplus A_g$ together with two metrics η, η' and a character χ satisfying all the axioms pertaining to the metrics and the G-actions of a degenerate G-Frobenius algebra of degree j. We call a* *degenerate G-Frobenius algebra A of degree j* maximally non-degenerate *if* $a_g \circ b_h = 0$ *in* A *implies that* $a_g \circ' b_h = 0$ *in any other degenerate G-Frobenius algebra* A' *with the same underlying* $D(k[G])$*-,* A_e*-bi-module* $A' = \bigoplus A_g$*, two metrics* η*,* η'*, and character* χ *.*

We call a maximally non-degenerate G-Frobenius algebra projectively unique *if it agrees with all other maximally non-degenerate G-Frobenius structures when projected to* A/k^**.*

REMARK 2.10. *As demonstrated in* [**Ka02b**]*, twisting by discrete torsion exactly realizes the universal (i.e., applicable to all G-Frobenius algebras) projective rescalings. This means, vice versa, that A and all its twists by discrete torsion are projectively the same.*

2.6. Orbifold mirror philosophy. There is an orbifold mirror philosophy which is motivated by physics (e.g., [**GP90**]) or representation theory which states the following

PHILOSOPHY 2.1. *Let* T *be a* $N = 2$ *theory (which for us at the moment means Frobenius algebra) and let* $H \subset G$ *be symmetry groups with* H *normal in* G*, then*

$$(T/H)^H \simeq (((T/H)^H/(G/H))^\vee)^{(G/H)}.$$

This is too vague to be called a conjecture, since most of the symbols in the statement have no fixed meaning.

For us T/H will mean the pair consisting of a sub-H-Frobenius algebra of a G-Frobenius algebra derived from a Frobenius algebra T which is equipped with a G action and its $^\vee$ dual. Setting $K = G/H$ if H is normal, we will interpret $((T/H)^H/(G/H))$ to be the pair consisting of a K-Frobenius algebra derived from $(T/H)^H$ and its $^\vee$ dual.

In this form we can apply the orbifold mirror philosophy to quasi-homogeneous singularities, where if G is Abelian, the operation of forming the quotient is well defined, up to fixing an additional finite set of data $(\sigma, \epsilon, \gamma)$, see §3 below.

It turns out that even for different actions of H and G this orbifold mirror philosophy holds true and produces mirror pairs. For the precise correspondences derived from the orbifold mirror philosophy in the setting of quasi-homogeneous singularities with an isolated critical point at the origin, see §4.

In order to elucidate the statement, we wish to point out that there is indeed an action of G/H on the H invariants of the sub-H-Frobenius algebra of a G-Frobenius algebra A. For the action to be defined on the restriction $res_H(A)$ we need the additional assumption that H is normal. Then G acts on $res_H(A)$ and the subgroup H acts trivially on $res_H(A)^H$. Thus there is an induced action of G/H on $res_H(A)^H$.

3. Quasi-homogeneous singularities with symmetries

DEFINITION 3.1. *Let* $f : \mathbb{C}^n \to \mathbb{C}$ *be a function which has an isolated singularity at zero. A symmetry of* f *is an element* $S \in GL(n, \mathbb{C})$*, such that* $f(S(\mathbf{z})) = f(\mathbf{z})$*. An isolated singularity with symmetries is a function* $f : \mathbb{C}^n \to \mathbb{C}$ *which has an isolated singularity at zero together with a finite group* G *and a representation* $\rho(G) : G \to GL(n, \mathbb{C})$ *such that* G *acts by symmetries on* f*, i.e.,* $\forall g \in G : g^*(f)(\mathbf{z}) = f(\rho(g)(\mathbf{z})) = f(\mathbf{z})$*.*

We denote by $G_{max} \subset GL(n, \mathbb{C})$ *the maximal group of symmetries of* f*.*

DEFINITION-PROPOSITION 3.1. *For a function $f(\mathbf{z})$ with an isolated singularity at zero, we will denote by M_f the Milnor or local ring of f, which is given by $\mathcal{O}/J_f$, where $\mathcal{O}$ is the ring of germs of holomorphic functions at zero and $J_f = (\frac{\partial f}{\partial z_i})$ is the Jacobian ideal. This ring together with the Grothendieck residue paring η is a graded Frobenius algebra, see e.g.,* [**AGLV93, M99**].

3.1. The graded Frobenius algebra of a quasi-homogeneous function with an isolated singularity at zero. If the function f is also quasi-homogeneous there is a natural grading operator which assigns to each z_i its degree of quasi-homogeneity q_i.

To define the q_i assume that

$$f(\lambda^{Q_1} z_1, \dots, \lambda^{Q_n} z_n) = \lambda^N f(z_1, \dots, z_n)$$

with $Q_i, N \in \mathbb{N}$. Then we set $q_i = \frac{Q_i}{N}$ and define $\deg(z_i) := q_i$, which yields a map $M_f \to \mathbb{Q}$.

The metric for the resulting Frobenius algebra is given by the element which is dual to the identity, and this element is represented by $H := Hess(f)$, the Hessian of f. For a quasi-homogeneous singularity, the degree of the Hessian is the degree of the form η and is denoted by d. By the general theory [**AGLV93**] a formula for d is given by

$$d = \sum_i (1 - 2q_i)$$

and the dimension or the Milnor number of the local algebra is

$$\mu := \dim(M_f) = \prod_i (\frac{1}{q_i} - 1).$$

3.1.1. Examples. In the following examples, we took the liberty to re-scale the Grothendieck residue form, which amounts to multiplying the function f by an overall factor.

(1) The A_n series: $f(z) = z^{n+1}$
$$M_f = \mathbb{C}[z]/(z^n) = \langle 1, z, z^2, \dots, z^{n-1} \rangle \quad \eta(z^i, z^j) = \delta_{i+j,n-1}$$
$q = \frac{1}{n+1}, \mu = n, d = 1 - \frac{2}{n+1} = \frac{n-1}{n+1}$.

(2) The D_n series: The D_{n+1}, $n \geq 3$ singularity is given by the function $f(x, y) = \frac{1}{n}x^n + xy^2$
$$M_f \simeq \mathbb{C}[x, y]/(x^{n-1} + y^2, xy) \simeq \langle 1, x, x^2, \dots, x^{n-1}, y \rangle$$
$\eta(x^i, x^j) = \delta_{i+j,n-1}, \eta(y, y) = -1, \eta(x^i, y) = 0$
$q_x = q_1 = \frac{1}{n}, q_y = q_2 = \frac{n-1}{2n}, \mu = n + 1, d = \frac{n-1}{n}$.

(3) The E_7 singularity: $f(x, y) = \frac{1}{3}x^3 + xy^3$
$$M_f = \mathbb{C}[x, y]/(x^2 + y^3, xy^2) = \langle 1, x, x^2, y, y^2, xy, x^2y \rangle$$
$\eta(x^i y^j, x^k y^l) = \delta_{i+k,2}\delta_{j+l,1}, \eta(y^2, y^2) = -1$
$q_x = q_1 = \frac{1}{3}, q_y = q_2 = \frac{2}{9}, \mu = 7, d = \frac{8}{9}$.

3.1.2. Products. For two functions f and g with an isolated singularity at zero, as shown in [**Ka99, M99**] $M_{f+g} = M_f \otimes M_g$, even on the level of Frobenius manifolds.

3.1.3. Stabilization. Notice that adding squares, an operation known as stabilization, to a function with an isolated singularity ($f \mapsto f + w^2$) leaves the Milnor ring invariant. This fact, which is well known in singularity theory (see e.g., [**AGLV93**]), can also be seen as follows from the point of view of Frobenius algebras.

Since the Frobenius algebra of the singularity $f(w) = w^2$ is given by $M_{w^2} = A_1 = k$ and this is the unit in the monoidal category of Frobenius algebras [**Ka99**], we also find that $M_{f+w^2} \simeq M_f \otimes A_1 \simeq M_f$.

DEFINITION 3.2. *We define* $M_0 := M_{w^2} = A_1$.

REMARK 3.1. *All the following definitions and constructions are invariant under stabilizations, if one extends the group action by the usual embedding of* $GL(n, \mathbb{C})$ *to* $GL(n+1, \mathbb{C})$.

3.2. The G-Frobenius algebra for a singularity with symmetry group G. We would like to recall from [**Ka02, Ka03, Ka04, Ka02a**] that for the data (f, G, ρ), as above, there are several natural G-Frobenius algebras, whose underlying k-module structure and bi-grading are all the same, but whose $D(k[G])$-module structures are in one-to-one correspondence with twists by discrete torsion and whose G-Frobenius structures depend on the choice of a graded compatible cocycle for the quantum multiplication. We will review the construction below following the steps of §1.7.

3.2.1. The G-graded k-module structure. First we show that for the data (f, G, ρ), as above, there is a naturally associated G-graded M_f-module.

Let Fix_g be the fixed point set of g in $\mathbb{C}^n$, in other words the eigenspace to the eigenvalue 1 of $\rho(g)$. Set $f_g := f|_{\mathrm{Fix}_g}$.

We define

$$A_g := M_{f_g} \quad GM_f := \bigoplus_{g \in G} M_{f_g}.$$

REMARK 3.2. *We would like to emphasize the following observations:*

1) *Notice* $A_e = M_f$.
2) *Each of the* $A_g = M_{f_g}$ *is, as a local ring of a quasi-homogeneous function with an isolated singularity at zero, a Frobenius algebra. We denote the metric for the Frobenius algebra* A_g *by* η_g*, its unit by* 1_g*, its degree by* d_g*, and its grading operator by* Q_g.

   The sum of the grading operators Q_g *defines a grading operator* Q *on* A.
3) *We furthermore can use the ring structure of the individual* M_{f_g} *to define a natural* M_f*-module structure by inclusion of function germs. This* A_e*-module structure is compatible with the grading* [**Ka03**].

   In the following examples the multiplication is simply given by:

$$M_f \times M_{f_g} \to M_{f_g} : (a, b) \mapsto a|_{\mathrm{Fix}_g} b.$$

The precise condition for when the A_e*-module structure has the form above is as follows. Let's suppose* g *is diagonal in the basis* $e_i : i \in I$ *corresponding to variables* z_i*, and let* $e_i : i \in I_1$ *correspond to a basis of* $Fix(g)$*, and set* $I_2 := I \setminus I_1$. *The* e_i *with* $i \in I_2$ *form a basis of a complement of the vector space of fixed points.*

Set $Z = (z_i : i \in I_2)$, let $J_g = J_{f_g}$ and $J_{I_1} = (f_{z_i} : i \in I_1)$ be the respective ideals in $\mathcal{O}$. Then the condition is that $J_{I_1} + Z = J_g + Z$.

REMARK 3.3. *All the A_g are cyclic A_e-modules. In the terminology of* [**Ka03**] *GM_f is a special G-Frobenius algebra. Notice that the unit 1_g is a cyclic generator for the A_e-module A_g.*

3.2.2. The grading. The initial grading operator Q from above plays the role of the operator $Q^{(1)}$ of §1.8. The actual grading Q is determined by the degrees of the cyclic generators 1_g.

DEFINITION 3.3. *We define the grading operator $\mathcal{Q}$ on GM_f by*

$$\mathcal{Q}(a_g) = Q(a) + s_g \text{ for } a_g = a1_g,$$

with

$$s_g = \frac{1}{2}(s_g^+ + s_g^-) = \frac{1}{2}(d - d_g) + \sum_{k:\nu_k \neq 0} (\frac{1}{2\pi i}\lambda_k(g) - \frac{1}{2}),$$

where the $\lambda_k(g)$ are the logarithms of the eigenvalues of g using the branch with arguments in $[0, 2\pi)$, i.e., cut along the positive real axis.

This means that

$$s_g = \mathcal{Q}(1_g),$$

and we call s_g the *grading shift.*

3.2.3. Notation. In practice, the choice of logarithms means that in a diagonal form $\rho(g) = diag(exp(2\pi i\nu_1(g)), \dots, exp(2\pi i\nu_n(g))$ and $0 \le \nu_i \le 1$.

For the element j we have by definition $\nu_i(j) = q_i$. Also due to choice of logarithm, $\nu_i(g) = 1 - \nu_i(g^{-1})$. Lastly, $d_g = \sum_{i:\nu_i(g)=0} 1 - 2q_i$ and so

$$s_g^+ = 2\sum_{i:\nu_i(g)\neq 0} (\frac{1}{2} - q_i), \quad s_g^- = 2\sum_{i:\nu_i(g)\neq 0} (\nu_i(g) - \frac{1}{2}) \tag{3.1}$$

$$s_g = \sum_{i:\nu_i(g)\neq 0} (\nu_i(g) - q_i). \tag{3.2}$$

3.2.4. The bi-grading. There is a natural bi-grading for the G-Frobenius algebras of the type GM_f which is given by

$$(\mathcal{Q}, \bar{\mathcal{Q}})(a_g) := (Q(a_g) + s_g, Q(a_g) + \bar{s}_g) \quad \text{for } a_g \in A_g,$$

where we used the notation $\bar{s}_g := \frac{1}{2}(s_g^+ - s_g^-)$.

In the notation above

$$\bar{s}_g = \sum_{i:\nu_i(g)\neq 0} (1 - \nu_i(g) - q_i) = \sum_{i:\nu_i(g^{-1})\neq 0} (\nu_i(g^{-1}) - q_i) = s_{g^{-1}} = d - d_g - s_g. \tag{3.3}$$

LEMMA 3.1. *An element $a_g \in A_g$ has diagonal grading (q, q) if and only if $s_g = s_{g^{-1}}$, i.e., $s_g^- = 0$ or equivalently $\sum_i \nu_i = \frac{1}{2}\mathrm{codim}(\mathrm{Fix(g)})$.*

3.2.5. Super-grading. Recall that A_g are all cyclic A_e-modules and the natural parity for all of A_e is all even. Thus under the assumption that all elements of A_e are even, the possible super-gradings for the M_f module GM_f are given by maps $\sim \in Map(G, \mathbb{Z}/2\mathbb{Z})$. Here $\tilde{1}_g = \tilde{g}$. Here we use $\sim$ both for the grading on A and G, which is justified by the equation above.

3.2.6. The G-actions. The primary choice for a G-action on GM_f would be the induced action via pullback. Since G acts on the collection of fixed point set, $h : \mathrm{Fix}_g \to \mathrm{Fix}_{hgh^{-1}}$, we get a right action ψ on GM_f which coincides with the notation of [**JKK03**]. On the other hand, if we take the associated left action ϕ, we are in line with the definitions of [**Ka03**]—here $\phi(g) := \psi(g^{-1})$.

From the point of view of constructing GFAs, however, all actions of G which preserve the already constructed pieces of the data of a GFA are equally admissible. In particular, notice that each A_g is a cyclic A_e-module and we denote the generator by 1_g. We find that if G is acting via A_e-module automorphisms, then

$$\varphi(g)1_h = \varphi_{g,h}1_{ghg^{-1}} \text{ for some } \varphi_{g,h} \in k^*.$$

From the fact that this is indeed an action of G, we obtain a cocycle condition on the $\varphi_{g,h}$. To be precise, they form a non-Abelian G 2-cocycle with values in k^*, as defined below.

DEFINITION 3.4. *A* non-Abelian G 2-cocycle *with values in k^* is a map $\varphi : G \times G \to k^*$ which satisfies*

$$\varphi_{gh,k} = \varphi_{g,hkh^{-1}}\varphi_{h,k}, \tag{3.4}$$

where $\varphi_{g,h} := \varphi(g,h)$ and

$$\varphi_{e,g} = \varphi_{g,e} = 1.$$

3.2.7. The super-grading. To define the super-grading we make an additional assumption.

Assumption. Keeping the condition that all elements of A_e are even, we furthermore postulate that the G-action is an even action.

This limits the possible super-gradings to functions of $C(G) \to \mathbb{Z}/2\mathbb{Z}$ where $C(G)$ are the conjugacy classes of G.

3.2.8. The conditions from the trace axiom. As demonstrated in [**Ka03**], if we further demand that the restricted trace axiom holds for the above $D(k[G])$-module, this means that the trace axiom holds under the condition $[g,h] = e$ and the choice $c = 1$. Also certain conditions for the character, the super-grading and the cocycle $\varphi_{g,h}$ must hold [**Ka03**]. These are reviewed below.

Recall $\rho : G \to GL(n, \mathbb{C})$ is the linear representation fixed from the beginning.

3.2.9. The trace axiom, the character and the super-grading. From the proof of the Theorem 5.1 [**Ka03**] we extract the following proposition:

PROPOSITION 3.2. *Let GM_f be the $D(k[G])$-module with the G-action given by the cocycle $\varphi_{g,h}$, and fix a super-grading $\sim$ and a character $\chi \in \mathrm{Hom}(G, k^*)$, then the trace axiom for $g = e$ and arbitrary h is satisfied with respect to $\sim$ and χ if and only if χ satisfies*

$$\chi_g = (-1)^{\tilde{g}}(-1)^{n-\dim(\mathrm{Fix}_g)}\det(\rho(g)).$$

REMARK 3.4. *Notice that this entails a condition on the super-grading: Set*

$$\sigma(g) := \tilde{g} + n - \dim(\mathrm{Fix}(g)) \mod 2 \tag{3.5}$$

and call it the sign of g. Then

$$\chi_g = (-1)^{\sigma(g)} \det(g),$$

and therefore $\sigma \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$.

Thus the possible super-gradings $\sim$ *are in 1-1 correspondence with elements* σ *of* $\mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$.

3.2.10. The trace axiom and discrete torsion.

DEFINITION 3.5. *A discrete torsion bi-character for a group* G *is a map from commuting pairs* $(g,h) \in G \times G : [g,h] = e$ *to* k^* *with the properties*

$$\epsilon(g,h) = \epsilon(h^{-1}, g) \quad \epsilon(g,g) = 1 \quad \epsilon(g_1 g_2, h) = \epsilon(g_1, h)\epsilon(g_2, h). \tag{3.6}$$

DEFINITION 3.6. *A non-Abelian 2-cocycle is said to satisfy the condition of discrete torsion with respect to a given* $\sigma \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$ *and a linear representation* $\rho \in \mathrm{Hom}(G, GL(n))$, *if for all elements* $g, h \in G : [g,h] = e$,

$$\epsilon(g,h) := \varphi_{g,h}(-1)^{\sigma(g)\sigma(h)} \det(g)\det(g^{-1}|_{\mathrm{Fix}(h)}) \tag{3.7}$$

is a discrete torsion bi-character.

REMARK 3.5. *Due to the properties of* φ *as a non-Abelian cocycle, the second and third condition of discrete torsion (3.6) are automatically satisfied. If, furthermore,* $\gamma_{g,h} \neq 0$, *then the first condition reduces to*

$$\det(g)\det(g^{-1}|_{\mathrm{Fix}(h)})\det(h)\det(h^{-1}|_{\mathrm{Fix}(g)}) = 1.$$

3.2.11. The action of discrete torsion. In [**Ka02b**], we analyzed the phenomenon of discrete torsion and showed that the different choices φ can by obtained from a fixed $D(k[G])$-module by tensoring with the twisted group algebra $k^\alpha[G]$ with $\alpha \in Z^2(G, k^*)$.

The corresponding discrete torsion bi-character to such an $\alpha \in Z^2(G, k^*)$ is given by

$$\epsilon(g,h) = \frac{\alpha(g,h)}{\alpha(ghg^{-1}, g)}$$

[**Ka02b**].

From the considerations of [**Ka02b**] one obtains

LEMMA 3.2. *For two choices of non-Abelian cocycles* φ *and* φ', *let* $A(\varphi)$ *and* $A(\varphi')$ *be the* $D(k[G])$ *modules based on the k-module* GM_f, *then there is a group cocycle* $\alpha \in Z^2(G, k^*)$, *such that that* $A(\varphi') \simeq A(\varphi) \otimes k^\alpha[G]$.

From the Proof of Theorem 5.1 of [**Ka03**] we can also extract the following:

PROPOSITION 3.3. *The* $D(k[G])$*-module* GM_f *with the G-action given by the cocycle* $\varphi_{g,h}$ *satisfies the super-trace axiom if and only if there is a* $\sigma \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$ *such that* $\varphi_{g,h}$ *satisfies the condition of discrete torsion with respect to a* $\sigma \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$ *and the linear representation* $\rho \in \mathrm{Hom}(G, GL(n))$.

COROLLARY 3.1. *If the group G is Abelian, then specifying a G action by a non-Abelian cocycle* φ *which satisfies the restricted trace axiom for the resulting* $D(k[G])$*-module is equivalent to specifying a discrete torsion bi-character* $\epsilon(g,h)$ *and a group homomorphism* $\sigma \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$

$$\varphi_{g,h} = \epsilon(g,h)(-1)^{\sigma(g)\sigma(h)}\det(g^{-1})\det(g|_{\mathrm{Fix}(h)}) \tag{3.8}$$

3.2.12. The G-Frobenius algebra structures. As explained in [**Ka03**] and [**Ka04**] there is no fixed preferred G Frobenius structure on the M_f-module above in general, but rather a set depending on the choice of a so-called super-sign and a two cocycle. The main result of [**Ka03**] in this respect is

THEOREM 3.1 ([**Ka03**]). *Given a natural G action on a realization of a Jacobian Frobenius algebra (A_e, f) with a quasi-homogeneous function f, let $A := \bigoplus_{g\in G} A_g$ be as defined above, up to an isomorphism of Frobenius algebras on the A_g, then the structures of super G-Frobenius algebra on A are in 1-1 correspondence with triples $(\sigma, \gamma, \varphi)$ where $\sigma \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$, γ is a G-graded, section-independent cocycle compatible with the metric satisfying the condition of supergrading with respect to the natural G action, and φ is a non-Abelian two cocycle with values in k^* which satisfies the condition of discrete torsion with respect to σ and the natural G action, such that (γ, φ) is a compatible pair.*

In many cases the equations for the cocycles allow one to find a unique multiplication up to the twist by discrete torsion. We refer the reader to [**Ka03**] for details.

The cocycle γ is a special type of A_e valued group 2-cocycle which defines multiplication on the cyclic generators via

$$1_g \circ 1_h := \gamma(g,h)1_{gh};$$

the extra conditions ensure that the extension of this multiplication using the cyclic A_e-module structures is well defined. We usually write $\gamma_{g,h}$ for $\gamma(g,h)$.

The function σ is related to the super-sign as follows.

$$\sigma(g) := \tilde{g} + |N_g| \mod 2, \tag{3.9}$$

where $|N_g| := \mathrm{codim}(Fix(g))$ in $\mathbb{C}^n$.

Also,

DEFINITION 3.7. *A cocycle $\gamma \in Z^2(G, A_e)$ is said to satisfy the* condition of supergrading with respect to a given a linear representation $\rho \in \mathrm{Hom}(G, GL(n))$, *if $\gamma_{g,h} = 0$ unless $|N_h| + |N_g| + |N_{gh}| \equiv 0(2)$. Here $|N_g| := \mathrm{codim}(\mathrm{F}ix(\rho(g))$ is the codimension of the fixed point set of g.*

3.2.13. The metric. The metric is constructed in two steps.

In the first step the metric is constructed from the metrics on the individual Frobenius algebras $A_g := M_{f_g}$. First notice that since $\mathrm{Fix}(g) = \mathrm{Fix}(g^{-1})$, we have $A_g = A_{g^{-1}}$. Now A_g has a non-degenerate pairing η_g which we wish to view as a pairing $A_g \otimes A_{g^{-1}} \to k$. We set

$$\eta' := \bigoplus_{g\in G} \eta_g \in A^* \otimes A^*.$$

In order to ensure the compatibility of the metric with the multiplication and the twisted commutativity $\gamma_{g,g^{-1}} = (-1)^{\tilde{g}} \varphi_{g,g^{-1}} \gamma_{g^{-1},g}$ we need to rescale the metric:

$$\eta := \bigoplus_{g\in G} ((-1)^{\tilde{g}} \chi_g)^{1/2} \eta_g.$$

For a discussion of the choice of the square root we refer to [**Ka03**].

3.3. The metric on the invariants. As shown in [**Ka03**] the G-invariants will be a Frobenius algebra with respect to the metric η if and only if $\chi(g) = \pm 1$.

REMARK 3.6. *In physics terms, this means that the spectral flow operator* $\mathcal{U}_{(1,1)}$ *survives in the projection.*

3.4. The dual of a quasi-Euler G-Frobenius algebra for a quasi-homogeneous singularity with symmetries.

3.4.1. The grading operator and the Euler condition. Any non-trivial Frobenius algebra M_f stemming from a quasi-homogeneous singularity has a non-trivial grading operator Q as discussed above, and

$$J := exp(2\pi i Q) = diag(exp(2\pi i q_1), \dots, \exp(2\pi i q_n))$$

generates a non-trivial finite cyclic group $\langle J \rangle \subset GL(n)$ of order $ord(J)$, the order of J. Moreover, fixing J as the generator we can identify this group with $\mathbb{Z}/ord(J)\mathbb{Z}$ with a fixed generator j acting via $\rho(j) = J$.

REMARK 3.7. *This means that G_{max} is non-trivial. Since any symmetry has to preserve the grading j, it is in the center of G_{max} and thus any of the G_{max} Frobenius algebras constructed in §3.2 will be Euler. This will also be true for any subgroup $H \subset G_{max}$ which contains $\langle J \rangle$.*

LEMMA 3.3. *If for all $g : \epsilon(g, j^{-1})(-1)^{\sigma(g)(\sigma(j)+1)} = 1$, then the corresponding G-Frobenius algebras of §3.2 will be G-Euler. This is, for instance, the case if $\forall \epsilon(g, j) \equiv 1$ and $\sigma(j) = 1$ or $\sigma \equiv 0$.*

PROOF.

$$\varphi_{h^{-1}j,h} = \epsilon(hj, h)(-1)^{\sigma(h^{-1}j)\sigma(h)} \exp(2\pi i \sum_{i:\nu_i \neq 0} (\nu_i - q_i)) = \exp(2\pi i s_g)$$

□

Assumption: We will assume that in the data (f, G, ρ), $\langle J \rangle \subset \rho(G)$, when considering duals on the level of $D(k[G])$-modules. Going to the algebra level we postulate that the G-Frobenius algebra structures above are Euler or quasi-Euler with fixed Eulerization.

REMARK 3.8. *The assumption above holds for G_{max}, so if it does not hold for a subgroup $H \subset G_{max}$, then if we are in a quasi-Euler case with fixed Eulerization, we can enlarge H to G_{max} and perform the dualization for the G_{max}-Frobenius algebra and then reduce to the H-Frobenius sub-algebra or the respective $D(k[H])$-module.*

3.4.2. The dual k-module. Given the G-Frobenius algebra GM_f, its dual k-module is defined as

$$\check{A}_g = A_{gj^{-1}} = M_{f|_{\mathrm{Fix}(gj^{-1})}},$$

where j is the group element defining the exponential of the grading operator Q via $\rho(j) = \exp(2\pi i Q)$.

3.4.3. The dual $D(k[G])$-module. The G-module structure is given by pulling back the action and scaling by χ. In the case of a singularity the character χ is determined by a choice of sign function $\sigma \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$ given by $\chi(g) = (-1)^{\sigma(g)} \det(g)$. If we denote the G-action on $\check{A}$ by $\check{\varphi}$, then using the k-module isomorphism $M : A_g \to A_{gj^{-1}}$

$$\check{\varphi}(g)(\check{a}_h) := \chi(g) M \varphi(g) M^{-1}(\check{a}_h) \in \check{A}_{hgh^{-1}}, \quad \text{for } \check{a}_h \in \check{A}_g;$$

or if we denote $M(a) =: \check{a}$ and fix $\sigma \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$, then for $\check{a} \in \check{A}_h$

$$\check{\varphi}(g)(\check{a}) := (-1)^{\sigma(g)} \det(g) (\check{\varphi(g)(a)}) \in \check{A}_{ghg^{-1}}.$$

Using equation (3.8) for $\check{a}_h = M(a1_{hj^{-1}}) \subset \check{A}_h$

$$\begin{aligned} \check{\varphi}(g)(\check{a}_h) &= \check{\varphi}(g) M(a1_{hj^{-1}}) \\ &= \epsilon(g, hj^{-1})(-1)^{\sigma(g)(\sigma(h)+\sigma(j)+1)} \det(g|_{\mathrm{Fix}(hj^{-1})}) M(a1_{ghg^{-1}j^{-1}}). \end{aligned} \tag{3.10}$$

REMARK 3.9. *If $\det(g) = (-1)^{\sigma(g)}$, then χ is trivial. This means that the dual and the G-Frobenius algebra have the same invariants.*

LEMMA 3.4. *$\check{1}_e$ is invariant if and only if GM_f is G-Euler.*

PROOF. Since $Fix(j) = \emptyset$, as f can be chosen to contain no linear terms (the linear terms would actually only add Eigenspaces of Eigenvalue one), the condition

$$\check{\varphi}(g)(\check{1}_e) = \check{1}_e$$

reads

$$\forall g \in G : \epsilon(g, j^{-1})(-1)^{\sigma(g)(\sigma(j)+1)}. \tag{3.11}$$

This is precisely the condition to be G-Euler of Lemma 3.3. □

COROLLARY 3.2. *Unless $\forall g \in G : \epsilon(g, j^{-1})(-1)^{\sigma(g)(\sigma(j)+1)} \det(g|_{Fix(j)}) = 1$, there is no Frobenius structure on the G invariants of $(GM_f)^\vee$ for GM_f with these invariants.*

PROOF. Without this condition there will be no invariant unit for the (a, c)-ring since for non-trivial grading $\check{1}_e$ is the only element of bi-degree $(0, 0)$. □

Assumption: Due to the content of the lemma, we will only consider taking the invariants of the dual of a fixed $D(k[G])$-module structure on GM_f if it is G-Euler.

3.4.4. The bi-grading for the dual. The bi-grading for the dual is given by the general formula (2.3)

$$\check{Q}(\check{a}) = Q(a) + \check{s}_g \quad \check{\bar{Q}} := Q(a_g) + \check{\bar{s}}_g \quad \text{for } \check{a}_g \in \check{A}_g.$$

In the notation of 3.2.3 this reads as
$\check{s}_g := s_{gj^{-1}} - d = \sum_{i:\nu_i(gj^{-1}) \neq 0} (\nu_i(g) + (1 - q_i) - \Theta(\nu_i(g) - q_i) - \sum_i 1 - 2q_i$
and $\check{\bar{s}}_g := \bar{s}_{gj^{-1}} = \sum_{i:\nu_i(gj^{-1}) \neq 0} (1 - (\nu_i(g) + (1 - q_i) - \Theta(\nu_i(g) - q_i) - q_i)$ and thus

$$\check{s}_g = \sum_{\nu_i(gj^{-1}) \neq 0} (\nu_i(g) - \Theta(\nu_i(g) - q_i)) - d_{gj^{-1}} \tag{3.12}$$

$$\check{\bar{s}}_g = \sum_{\nu_i(gj^{-1}) \neq 0} \Theta(\nu_i(g) - q_i) - \nu_i(g). \tag{3.13}$$

REMARK 3.10. *An element in $\check{A}_g$ has anti-diagonal grading $(-q, q)$ if and only if $Q(a_g) = \frac{1}{2}d_{gj^{-1}}$*

3.4.5. The metric. The metric is as in the general case the pulled back metric. It will have group degree j^2 and will be homogeneous of bi-degree $(-d, d)$.

3.4.6. The degenerate G-Frobenius structure. As remarked previously, for the dual $D(k[G])$-module one cannot expect a G-Frobenius structure, but what we called a degenerate G-Frobenius algebra of group degree j, which induces a $C(G)$ graded Frobenius structure on the invariants, in the sense of [**JKK03**].

3.5. Mirror symmetry for singularities. In the framework of mirror symmetry a quasi-homogeneous function f with an isolated singularity is considered as a Landau-Ginzburg B-Model and hence has, as a (c, c) ring the (c, c) realization of M_f, and has a trivial (a, c) ring A_1.

DEFINITION 3.8. *We call a G-Euler G-Frobenius algebra A, together with a degenerate G-Frobenius algebra of degree j on $\check{A}$,* a model for the mirror dual *of a singularity M_f if the G invariants of A are spanned by $1 \in A$ and the G-invariants of $\check{A}$ are the (a, c)-realization. We also just say in short $(\check{A})^G$ is the mirror dual to M_f.*

We call a pair consisting of a G-Euler G-Frobenius algebra A and an H-Euler H-Frobenius algebra B, together with a degenerate G-Frobenius algebra of degree j on $\check{A}$ and a degenerate H-Frobenius algebra of degree j' on $\check{B}$, a mirror dual pair if (i) $A^G = (\check{B})^H$ and (ii) $(\check{A})^G = B^H$. In short, we say A and B are mirror dual.

Constructions for mirror pairs come from the orbifold mirror philosophy.

4. A mirror theorem for simple singularities and other examples

In this section, we calculate the orbifolds and duals in several examples. We will consider the first example in the greatest detail and then leave slightly more details to the reader as we continue to make the text more concise.

The main result of this is the following theorem, whose proof follows from the calculations below which are collected in Table 1.

REMARK 4.1. *In order to explain all the entries in Tables 1 and 2, we recall that according to* [**Du96**] *for a given Coxeter group W from the list $A_n, B_n, D_n, E_6, E_7, E_8, H_3, H_4, F_4, G_2$ and $I_2(k)$, there is a definition for an associated Frobenius manifold. We denote by W the Frobenius algebra for the corresponding Coxeter group.*

THEOREM 4.1. *Let f be one of the simple singularities A_n, D_n, E_6, E_7 and E_8 or a Pham singularity with coprime powers, let J be the exponential grading operator and $\Gamma := \langle j \rangle$ with $\rho(j) = J$. Then there is a projectively unique, maximally non-degenerate, degenerate Γ-Frobenius algebra structure of degree j on $(\Gamma M_f)^\vee$. Moreover the invariants of the Γ-Frobenius algebra ΓM_f are one-dimensional and yield the Frobenius algebra A_1, while the invariants of the $(\Gamma M_f)^\vee$ are isomorphic as a bi-graded Frobenius algebra to $M_f^{(a,c)}$.*

In short, the A, D, and E singularities are mirror self dual, in the sense that $((\Gamma M_g)^\vee)^\Gamma$ is the mirror dual.

It would be tempting to conjecture that if Γ is the group generated by the grading operator, then $((\Gamma M_f)^\vee)^\Gamma$ is the mirror dual. This is, however, not true

M_f	restriction	G	σ	GM_f^G	$((GM_f)^\vee)^G$
A_n		$\mathbb{Z}/(n+1)\mathbb{Z}$	0	A_1	A_n
A_{2n-1}		$\mathbb{Z}/(n+1)\mathbb{Z}$	1	A_1	B_n
A_{2n-1}		$\mathbb{Z}/2\mathbb{Z}$	0	B_n	$I_2(4)$
A_{2n-1}	n odd for dual	$\mathbb{Z}/2\mathbb{Z}$	1	D_{n+1}	A_1
A_{2n-1}		$\mathbb{Z}/n\mathbb{Z}$	0	$I_2(4)$	B_n
D_{n+1}		$\mathbb{Z}/(2n\mathbb{Z})$	0	A_1	A_{2n-1}
D_{n+1}	n even	$\mathbb{Z}/(2n\mathbb{Z})$	1	A_1	D_{n+1}
D_{n+1}	n odd	$\mathbb{Z}/n\mathbb{Z}$	0	A_1	D_{n+1}
D_{n+1}	n even	$\mathbb{Z}/n\mathbb{Z}$	0	$I_2(4)$	B_n
D_{n+1}		$\mathbb{Z}/2\mathbb{Z}$	0	B_n	$I_2(4)$
D_{n+1}	n odd for dual	$\mathbb{Z}/2\mathbb{Z}$	1	A_{2n-1}	$I_2(4)$
$A_{k_1-1}\otimes\dots A_{k_n-1}$	k_i coprime	$\mathbb{Z}/k_1\mathbb{Z}\times\dots\mathbb{Z}/k_n\mathbb{Z}$	0	A_1	$A_{k_1-1}\otimes\dots A_{k_n-1}$
E_6		$\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	0	A_1	E_6
E_7		$\mathbb{Z}/9\mathbb{Z}$	0	A_1	E_7
E_8		$\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/5\mathbb{Z}$	0	A_1	E_8

TABLE 1. Since all groups are cyclic, and $\epsilon\equiv 0$, $\mathrm{Hom}(G,\mathbb{Z}/2\mathbb{Z}) = e$ or $\mathrm{Hom}(G,\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, defining the entry in the column σ. The conditions for the duals are the conditions to be quasi-Euler.

as the example of the elliptic singularity P_8 or a Pham singularity of non-coprime exponents such as $x_0^5+\dots+x_4^5$ below shows. One obstruction is that ΓM_f^Γ is more than one-dimensional. If $\Gamma M_f^\Gamma\simeq A_1$ we would, however, expect that $((\Gamma M_f)^\vee)^\Gamma$ is the dual.

CONJECTURE 4.1. *For an isolated singularity f, let Γ be the group generated by the exponential grading operator J. If $\Gamma M_f^\Gamma\simeq A_1$, then $((\Gamma M_f)^\vee)^\Gamma$ is the mirror dual.*

Also, from the explicit calculations below, we obtain the following.

THEOREM 4.2. *The orbifold mirror philosophy produces mirror pairs for the self-dual cases listed in Table 1, with the group $G=\Gamma$ being the group generated by the exponential grading operator and $H=e$.*

The orbifold mirror philosophy holds exactly for the case of $T=A_{2n-1}, G=\mathbb{Z}/(2n\mathbb{Z}), H=\mathbb{Z}/2\mathbb{Z}$, with n odd, and the choice of $\sigma=1$ for $\mathbb{Z}/(2n\mathbb{Z})$ which restricts to $\sigma=1$ for $\mathbb{Z}/2\mathbb{Z}$ and $\sigma=0$ for $G/H=\mathbb{Z}/n\mathbb{Z}$.

PROOF. The first statement follows from Theorem 4.1. For the second statement, first notice using Table 1 that indeed with the indicated choice of σ for n odd, the pair $((T/H)^H,((T/H)^\vee)^H)$ is given by (D_{n+1},A_1). Furthermore, using the description of the action of $G=\mathbb{Z}/(2n\mathbb{Z})$ on A_{2n-1} of §4.3, we see that the induced action of $G/H=\mathbb{Z}/n\mathbb{Z}$ on the H invariants given by D_n is exactly given by the action of $\mathbb{Z}/n\mathbb{Z}$ used in §4.8. Thus again using Table 1, we obtain that the pair $(T/H)^H/(G/H))^{G/H},((T/H)^H/(G/H)))^\vee)^{G/H}$ is indeed the mirror dual (A_1,D_{n+1}).

□

4.1. Orbifold mirror philosophy for self-dual theories. We would like to extend our orbifold mirror philosophy to include some of the cases involving

T	G	H	$K=G/H$	$\begin{pmatrix}(T/H)^H,\\ ((T/H)^\vee)^H\end{pmatrix}$	$\begin{pmatrix}(T/H)/(K))^K,\\ (((T/H)/K)^\vee)^K)\end{pmatrix}$
A_{2n-1} n odd	$\mathbb{Z}/(2n\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$	(D_{n+1}, A_1)	(A_1, D_{n+1})
A_{2n-1}	$\mathbb{Z}/(2n\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$	$(B_n, I_2(4))$	$(I_2(4), B_n)$
D_{n+1} n even	$\mathbb{Z}/(2n\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$	$(B_n, I_2(4))$	$(I_2(4), B_n)$
E_6	$\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	$e\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$(F_4, I_2(4))$	$(I_2(4), F_4)$

TABLE 2. Mirror pairs from orbifold mirror philosophy

Coxeter groups which are not simply laced. This requires some new definitions, since so far we have only defined how to construct a G-Frobenius algebra based on singularities.

We call a theory T with symmetries G *self-dual* if $T \cong ((T/G)^\vee)^G$, and for these theories we generalize the orbifold mirror philosophy in the following way.

PHILOSOPHY 4.1. *Given a self-dual theory T with symmetries G and subgroups H, K of G such that $G/H \cong K$ there should be actions of H and K such that there is an isomorphism*

$$(T/H)^H \simeq ((T/K)^\vee)^K.$$

This means that in the orbifold mirror philosophy we should substitute T/K for $(T/H)^H/(G/H)$. In view of the self-duality condition, the orbifold mirror philosophy 4.1 amounts to a type of associativity for successive quotients, namely

$$\text{“}(((T/G)^G)/H)^H \cong (T/(G/H))^{G/H} = (T/K)^K\text{”}.$$

Here we put quotation marks around the equation to indicate that unless suitably interpreted, this is not a mathematically strict statement, but rather a background philosophy.

Nevertheless, using the extended orbifold mirror philosophy 4.1 together with the calculations listed in Table 1 and the forming of tensor products (see §4.6.3) we obtain

THEOREM 4.3. *The extended orbifold mirror philosophy holds for the entries in Table 2 and produces the additional mirror pairs $((B_n, I_2(4)), (I_2(4), B_n))$ and $((F_4, I_2(4)), (I_2(4), F_4))$.*

4.2. Boundary singularities. Another approach to the entries of Table 2 is to interpret the orbifold mirror philosophy in the setting of boundary singularities. This requires some new definitions, since so far we have only defined how to construct a G-Frobenius algebra based on singularities and on the boundary singularities.

Recall that a boundary singularity is a function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ which has an isolated singularity at zero, and if x is the first coordinate, the restriction of f to the hypersurface $x = 0$ is again a function with an isolated singularity at zero.

Like for singularities, there is a classification of boundary singularities [**AGV85, AGV88**]. The simple boundary singularities are

$B_n: f(x,y) = x^n + y^2$.
$C_n: f(x,y) = xy + y^n$.
$F_4: f(x,y) = x^2 + y^3$.

The role of the Milnor ring is played by $M_{(f,x)} = \mathcal{O}_{\mathbb{C}^{n+1}}/(xf_x, f_{y_1}, \dots f_{y_n})$, where $x, y_1, \dots, y_n$ are the coordinates of $\mathbb{C}^{n+1}$. Again a quasi-homogeneity of f also determines a grading of $M_{(f,x)}$, and there is again a pairing making $M_{(f,x)}$ into a (graded) Frobenius algebra.

REMARK 4.2. *From the point of view of the Frobenius algebras, the cases B_n and C_n coincide. The difference is subtle and manifests itself only in the intersection pairing on the cohomology* [**A78, AGV85, AGV88**]. *Also, there is an identification $B_2 = C_2$.*

To study the geometry of a boundary singularity one passes to the two-sheeted ramified cover of $\mathbb{C}^{n+1}$ with ramification on the boundary $x = 0$. One then lifts f to $\hat{f}$ [**AGV85, AGV88**], which we will call the ambient singularity. In other words, one studies $\hat{f}(\hat{x}, \hat{y}_1, \dots \hat{y}_n) = f(\hat{x}^2, \hat{y}_1, \dots \hat{y}_n)$.

For the simple boundary singularities the lifts are
$B_n: \hat{f}(\hat{x}, \hat{y}) = \hat{x}^{2n} + \hat{y}^2$ and the ambient singularity is A_{2n-1}.
$C_n: f(\hat{x}, \hat{y}) = \hat{x}^2\hat{y} + \hat{y}^n$ and the ambient singularity is D_{n+1}.
$F_4: f(\hat{x}, \hat{y}) = \hat{x}^4 + \hat{y}^3$ and the ambient singularity is E_6.

In order to deal with orbifolding of the mentioned boundary singularities, we will adopt either one of the following conventions.

CONVENTION 4.1. *Given a group K acting by symmetries on a boundary singularity T, we will take T/K to mean a pair consisting of a K-Frobenius algebra together with its $\vee$ dual for a lift of the action K to the ambient singularity.*

CONVENTION 4.2. *Given a group K acting by symmetries on a boundary singularity T, let $\widehat{T/K}$ be a K-Frobenius algebra together with its $\vee$ dual obtained by forgetting the boundary structure. If $(\widehat{T/K})^K$ is given by a singularity which is again a boundary singularity, we will take $(T/K)^K$ to mean that boundary singularity.*

REMARK 4.3. *Both the conventions are legitimate in terms of unfoldings of singularities and of boundary singularities with symmetries as we will show elsewhere* [**Ka**].

THEOREM 4.4. *With either one of the Conventions 4.1 or 4.2, we obtain mirror pairs from the orbifold mirror philosophy for the cases involving boundary singularities listed in Table 2.*

PROOF. First notice that the last case involving F_4 follows from the previous ones by using the tensor product, see §4.6.3.

For Convention 4.1 we remark that we obtain the boundary singularity of the invariants by substituting $x = z^2$ in the case of A_{n+1} . The induced action on the invariants generated by $x = z^2$ is according to §4.3 by multiplication by ζ_n^i, which lifts to the action of $\mathbb{Z}/n\mathbb{Z}$ on A_{2n-1} as indicated in Table 1. In the case of D_{n+1} the substitution to obtain the boundary singularity is $z_1 = y^2$ and $z_2 = x$. According to §4.7 the induced action of $K = \mathbb{Z}/n\mathbb{Z}$ on $z_2 = x$ which generates the invariants is by ζ_n^i, which again lifts to the action of $\mathbb{Z}/n\mathbb{Z}$ discussed in Table 1.

Lastly, for Convention 4.2, we see that indeed the invariants are given by the boundary singularity B_n. Forgetting the boundary structure leads to treating B_n as

A_{n-1}. Now the claims again follow by inspection of Table 1 and noticing vice-versa that treating A_{n-1} as a boundary singularity results in the boundary singularity B_n. □

REMARK 4.4. *The connection between the boundary singularities and the Coxeter groups is as follows. After making the identifications of Remark 4.2, the Frobenius algebras from the boundary singularities B_n, C_n and F_4 coincide with the Frobenius algebras coming from Coxeter groups* [**Du96**]. *Notice that in this notation $B_2 = C_2 = I_2(4)$.*

REMARK 4.5. *In the third line we would expect the boundary singularity C_n from the point of view of the ambient singularity. In terms of the Frobenius algebras B_n and C_n are of course the same. This distinction can only be made by inspecting the intersection form which we plan to do in* [**Ka**].

REMARK 4.6. *It is interesting to note that for B_n in either of its usual descriptions of folding, A_{2n-1} or D_{n+1}, we obtain a non-trivial (a,c)-ring, which is $I_2(4)$. The same holds true for F_4. This feature seems to distinguish B_n and F_4 as simple* boundary *singularities.*

4.3. The case of A_n. The A_n singularity is given by the function $f := x^{n+1}$. The maximal symmetry group is given by $G := G_{max} = \mathbb{Z}/(n+1)\mathbb{Z}$. Set $\zeta_{n+1} := \exp(2\pi i \frac{1}{n+1})$. The exponential grading operator is $J = \zeta_{n+1} id$ and $G = \langle j \rangle$ with $\rho(j) = J$.

4.3.1. The G-graded k-module GM_f. Since

$$Fix_{j^i} = \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{otherwise,} \end{cases}$$

as k-modules $M_f = A_n$ and $M_{f|_0} = k = A_1$, where A_k denotes the Frobenius algebra of the A_k singularity.

The k-module GM_f is given by

$$GM_f := \bigoplus_{i=0}^{n} M_{f_{j^i}}, \quad M_{f_{j^i}} := \begin{cases} A_n & i = 0 \\ A_1 & i = 1, \ldots, n \end{cases}, \quad GM_f = A_n \oplus A_1 \oplus \cdots \oplus A_1.$$

We denote the generator of the j^i twisted sector $M_{f_{j^i}}$ by 1_{j^i} and have the representation

$$\rho(j^k) = \zeta_{n+1}^k = \exp(2\pi i \frac{k}{n+1}) = \exp(2\pi i \nu(j^k)) \quad \nu(j^k) = \frac{k}{n+1}.$$

4.3.2. The grading. The grading is determined by the shifts

$$s^+_{j^i} = \begin{cases} 0 & \text{if } i = 0 \\ \frac{n-1}{n+1} & \text{if } i \in \{1, \ldots, n\} \end{cases}, \quad s^-_{j^i} = \begin{cases} 0 & \text{if } i = 0 \\ \frac{2i-n-1}{n+1} & \text{if } i \in \{1, \ldots, n\}. \end{cases}$$

Thus

$$s_{j^i} = \begin{cases} 0 & \text{if } i = 0 \\ \frac{i-1}{n+1} & \text{if } i \neq 0. \end{cases}$$

4.3.3. The super-grading. The choices of super-gradings $\tilde{}$ are determined by a choice of $\sigma \in \mathrm{Hom}(\mathbb{Z}/(n+1)\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ by

$$\tilde{1}_g \equiv |N_g| + \sigma(g),$$

where

$$|N_{j^i}| = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } 1 \leq i \leq n. \end{cases}$$

Now if $n = 2m$ is even, $\mathrm{Hom}(\mathbb{Z}/(2m+1)\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = e$, so $\sigma(j^i) \equiv 0$.

If $n = 2m-1$ is odd, $\mathrm{Hom}(\mathbb{Z}/(2m)\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, and there are two choices for σ: either $\sigma(j^i) \equiv 0$ or $\sigma(j^i) \equiv i \mod 2$.

Let us fix $\sigma \in \mathrm{Hom}(\mathbb{Z}/(n+1)\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$.

4.3.4. The bi-grading.

$$(Q, \bar{Q})(1_{j^i}) = \begin{cases} (0,0) & \text{for } i = 0 \\ (\frac{i-1}{n+1}, \frac{n-i}{n+1}) & \text{otherwise} \end{cases}$$

REMARK 4.7. *The elements with a (q,q)-grading are the elements $1, z, \ldots, z^{n-1} \in A_e$, and additionally $1_{\frac{n+1}{2}}$ in the case that $n = 2m-1$ is odd. The latter element is not invariant under the whole group $\mathbb{Z}/(2m)\mathbb{Z}$, but is invariant under the subgroup $\mathbb{Z}/2\mathbb{Z}$, see below.*

4.3.5. The G-action. We have already fixed $\sigma \in \mathrm{Hom}(\mathbb{Z}/(n+1)\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$.

Since $\mathbb{Z}/(n+1)\mathbb{Z}$ is Abelian, its action is determined by a choice of discrete torsion ϵ by the trace axiom

$$\epsilon(g,h) := \varphi_{g,h}(-1)^{\sigma(g)\sigma(h)} \det(g) \det(g^{-1}|_{\mathrm{Fix}(h)});$$

now since $\epsilon(j^i, j^k) = \epsilon(j,j)^{ik} = 1^{ik} = 1$, we find that $\epsilon \equiv 1$, and this implies that

$$\varphi_{j^i,j^k} = \begin{cases} 1 & \text{if } k = 0 \\ (-1)^{\sigma(j^i)\sigma(j^k)}\zeta^{-i} & \text{if } k \in \{1, \ldots, n\}. \end{cases}$$

4.3.6. The metric. After the necessary re-scaling, the metric is given by

$$\eta(z^i, z^k) = \delta_{i+k,n-1} \quad \eta(1_{j^i}, 1_{j^k}) = \delta_{i+k,n+1}(-\zeta^i)^{1/2}.$$

4.3.7. The G-Frobenius structure. Using Theorem 3.1, we have to find a cocycle γ compatible with the action defined by φ above and the grading.

From the general considerations we know $\gamma_{j^i,j^{n-1-i}} \in A_e$ and $\deg(\gamma_{j^i,j^{n-1-i}}) = d - d_{j^i} = \frac{n-1}{n+1}$, which yields

$$\gamma_{j^i,j^{n-1-i}} = (-\zeta^i)^{1/2} z^n - 1 for i \neq 0$$

for the other values of γ, which has been partially defined above. Notice that $\deg(1_{j^i}) + \deg(1_{j^k}) = \frac{i+k-2}{n+1}$, while $\deg(1_{j^{i+j}}) = \frac{i+k-1}{n+1}$ if $i+k \neq n+1$, but there is no element of degree $\frac{1}{n+1}$ in $A_{j^{i+k}}$ for $i+k \neq n+1$, so that the respective multiplication must yield zero if the condition is not met.

Hence

$$\gamma_{j^i,j^k} = \begin{cases} (-\zeta^i)^{1/2} z^n - 1 & \text{for } i+k = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

4.3.8. The G-invariants. Regardless of the choice of σ, the only invariant of $(\mathbb{Z}/(n+1)\mathbb{Z})M_{z^{n+1}}$ is the identity $1 \in A_e$.

PROPOSITION 4.1. *The $\mathbb{Z}/(n+1)\mathbb{Z}$ invariants of the $D(k[G])$-module $(\mathbb{Z}/(n+1)\mathbb{Z})M_{z^{n+1}}$ are spanned by $1 \in A_e$, and thus any $\mathbb{Z}/(n+1)\mathbb{Z}$-Frobenius algebra built on $\mathbb{Z}/(n+1)\mathbb{Z}M_{z^{n+1}}$ has as its invariants the Frobenius algebra A_1.*

REMARK 4.8. *This is the expected result since the dual of (A_n, A_1) is (A_1, A_n) according to* [**V89, IV90**].

4.3.9. The dual G-graded k-module. The dual G-graded k-module

$$(GM_f)^\vee := \bigoplus_{g \in \mathbb{Z}/(n+1)\mathbb{Z}} \check{M}_{f_g}$$

is given by

$$\check{M}_{f_{j^{-i}}} := \begin{cases} A_1 & i \in \{0, \dots, n-1\} \\ A_n & i = n \end{cases} \qquad (GM_f)^\vee = A_1 \oplus \cdots \oplus A_1 \oplus A_n.$$

REMARK 4.9. *Notice that it is convenient to choose the generator j^{-1} for the group $\mathbb{Z}/(n+1)\mathbb{Z}$ instead of j.*

4.3.10. The dual $D(k[G])$-module. The G-action on the $\check{M}_{f_{j^{-k}}}$ is given by

$$\check{\varphi}_{j^{-i}, j^{-k}} = \varphi_{j^{-i}, j^{-(k+1)}} \chi(j^{-i}) = \begin{cases} (-1)^{\sigma(j^{-i})} \zeta^{-i} & \text{for } k = n \equiv -1 \ (n+1) \\ (-1)^{\sigma(j^{-i})(\sigma(j^{-(k+1)})+1)} & \text{otherwise.} \end{cases}$$

4.3.11. The bi-grading. The bi-grading is given by

$$(\check{\mathcal{Q}}, \bar{\check{\mathcal{Q}}})(\check{1}_{j^{-k}}) = \begin{cases} (-\frac{k}{n+1}, \frac{k}{n+1}) & k \in \{0, 1, \dots, n-1\} \\ (-\frac{n-1}{n+1}, 0) & \text{for } k = n. \end{cases}$$

PROPOSITION 4.2. *In the case that $\sigma \equiv 0$ the $\mathbb{Z}/(n+1)\mathbb{Z}$-invariants of $((\mathbb{Z}/(n+1)\mathbb{Z})M_{z^{n+1}})^\vee = ((\mathbb{Z}/(n+1)\mathbb{Z})A_n)^\vee$ form the linear subspace*

$$\langle \check{1}_e, \dots, \check{1}_{j^{-(n-1)}} \rangle.$$

This subspace is isomorphic as a graded k-module to A_n.

In the case that $n = 2m - 1$ is odd and $\sigma(j^i) \equiv i \mod 2$ the $\mathbb{Z}/(n+1)\mathbb{Z}$ invariant subspace is

$$\langle \check{1}_e, \check{1}_{j^{-2}}, \dots, \check{1}_{j^{-2m}} \rangle.$$

This subspace is isomorphic as a graded k-module to the (a, c)-realization of the sub-k-module $B_m \subset A_{2m-1}$.

4.3.12. The metric on the dual $D(k[G])$ algebra. The metric is, after re-scaling the generators by a non-zero factor, given by the formulas

$$\begin{aligned} \check{\eta}(1_{j^{-i}}, 1_{j^{-k}}) &= \delta_{i+k,n-1} \text{ for } i, j \in \{0, \dots, n-1\} \\ \check{\eta}(z^i 1_{j^{-n}}, z^k 1_{j^{-n}}) &= \delta_{i+k,n-1} \\ \check{\eta}(1_{j^{-i}}, z^k 1_{j^{-n}}) &= \check{\eta}(z^k 1_{j^{-n}}, 1_{j^{-i}}) = 0 \text{ for } i \in \{0, \dots, n-1\}. \end{aligned}$$

4.3.13. The degenerate G-Frobenius algebra structure. There is a multiplication compatible with the bi-grading. It is unique up to scaling of the generators and is given by

$$\check{1}_{j-i}\check{1}_{j-k} = \begin{cases} 1_{j-(i+k)} & \text{if } i+k \leq n-1 \\ 0 & \text{if } i+k \geq n \end{cases} \tag{4.1}$$

$$\check{1}_{j-i}\check{1}_{j-n} = 0 \text{ for } i \in \{0,\dots,n-1\} \tag{4.2}$$

$$z^{n-1-i}\check{1}_{j-n}z^{n-1-k}\check{1}_{j-n} = \begin{cases} z^{n-1-(i+k)}1_{j-n} & \text{if } i+k \leq n-1 \\ 0 & \text{if } i+k \geq n. \end{cases} \tag{4.3}$$

The following statement is straightforward.

LEMMA 4.1. *This multiplication renders the metric $\check{\eta}$ invariant, i.e., it satisfies $\check{\eta}(\check{a},\check{b}\check{c}) = \check{\eta}(\check{a},\check{b}\check{c})$. Furthermore, $\check{\eta}$ is the projectively unique non-degenerate pairing compatible with the bi-grading, and the above multiplication is the projectively unique maximally non-degenerate multiplication rendering the metric invariant.*

REMARK 4.10. *The multiplication above is* not *compatible with the grading and group grading. But changing the equation (4.3) to*

$$z^{n-1-i}\check{1}_{j-n}z^{n-1-k}\check{1}_{j-n} = 0 \tag{4.4}$$

yields a multiplication that is (a) compatible with the bi-grading (b) compatible with the group grading and (c) compatible with the G-module structure and thus is compatible with the $D(k[G])$-module. This multiplication does not, however, render the pairing $\check{\eta}$ invariant.

LEMMA 4.2. *The multiplication of Remark 4.10 is the projectively unique maximally non-degenerate multiplication which is compatible with the $D(k[G])$-module structure; in other words, the multiplication turns the $D(k[G])$-module into a $D(k[G])$-module and co-module algebra.*

REMARK 4.11. *The metric $\check{\eta}'$ given by*

$$\check{\eta}'(\check{a},\check{b}) := \begin{cases} \check{\eta}(\check{a},\check{b}) & \textit{for } \check{a} \in A_{j-i}, \check{b} \in A_{j-k} \quad i,k \in \{0,\dots,n-1\} \\ 0 & \textit{otherwise} \end{cases}$$

is invariant with respect to the multiplication of Remark 4.10.

REMARK 4.12. *The multiplication of Remark 4.10 together with the metric of Remark 4.11, which contains degenerate elements and has a non-trivial annihilator of the whole algebra, is reminiscent of the appearance of so-called Ramond sectors in the theory of spin-curves* [**JKV01, PV01, P02**]. *For a discussion, see §5.1 below.*

Collecting the results from above yields

PROPOSITION 4.3. *In the case $\sigma \equiv 0$, the dual $((\mathbb{Z}/(n+1)\mathbb{Z})M_{z^{n+1}})^\vee = ((\mathbb{Z}/(n+1)\mathbb{Z})A_n)^\vee$ together with the multiplication of Remark 4.10 and the metric of Remark 4.11 forms a degenerate G-Frobenius algebra of charge j which is a projectively unique maximally non-degenerate algebra. The Frobenius algebra given by the invariants with the grading $\bar{Q}$ is isomorphic to A_n as graded Frobenius algebras.*

As bi-graded Frobenius algebras, the invariants of the dual are the (a,c)-realization of A_n: $(\mathbb{Z}/(n+1)\mathbb{Z}A_n)^\vee)^{\mathbb{Z}/(n+1)\mathbb{Z}} = A_n^{(a,c)}$.

In this sense, $((\mathbb{Z}/(n+1)\mathbb{Z}A_n)^\vee)^{\mathbb{Z}/(n+1)\mathbb{Z}} = A_n$ *is the mirror and* A_n *is mirror self-dual.*

In the case $n = 2m-1$ *and* $\sigma(j^i) \equiv i \mod 2$ *the dual* $(\mathbb{Z}/(n+1)\mathbb{Z}A_n)^\vee$ *together with the multiplication of Remark 4.10 and the metric of Remark 4.11 forms a degenerate G-Frobenius algebra of degree j which is the projectively unique maximally non-degenerate algebra. This Frobenius algebra of the group invariants is isomorphic to the Frobenius sub-algebra* $B_m \subset A_n$ *as graded Frobenius algebras. In terms of the bi-grading, the invariants of the dual of* $\mathbb{Z}/(n+1)\mathbb{Z}A_n$ *with non-trivial* σ *are* $B_m^{(a,c)}$.

4.4. The case of A_{2n-1} with a $\mathbb{Z}/2\mathbb{Z}$ action. In the case of A_{2n-1}, we can restrict ourselves to the action of the subgroup $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/(2n\mathbb{Z})$, generated by $j^n = -1$ which acts on z by $z \mapsto -z$.

We now consider the singularity A_{2n-1} with the group of symmetries $\mathbb{Z}/2\mathbb{Z}$.

4.4.1. The G-Frobenius algebras. The data of the bi-graded $D(k[G])$-module can be read off by restricting the data of 4.3.

There is a unique twisted sector for the element j^n and the algebra

$$M_f = A_{2n-1},\ M_{f_{-1}} = A_1 = k \quad (\mathbb{Z}/2\mathbb{Z})M_{z^{2n}} = A_{2n-1} \oplus k.$$

The G action is again determined by the fact that $\mathbb{Z}/2\mathbb{Z}$ is cyclic, forcing $\epsilon \equiv 1$ and a choice of $\sigma \in \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$

$$\varphi_{-1,-1} = (-1)^{\sigma(-1)+1}.$$

The bi-grading is given by

$$s^+_{-1} = \frac{n-1}{n}, \quad s^-_{-1} = 0, \quad s_{-1} = \bar{s}_{-1} = \frac{n-1}{2n}.$$

The super-grading is given by $\tilde{1}_{-1} \equiv \sigma(-1) \mod 2$.

The metric is

$$\eta(z^i, z^k) = \delta_{i+k,2n-2}, \quad \eta(1_{-1}, 1_{-1}) = 1, \quad \eta(z^i, 1_{-1}) = \eta(1_{-1}, z^i) = 0.$$

Taking into account the results of [**Ka03, Ka04**], there is a unique $\mathbb{Z}/2\mathbb{Z}$-Frobenius algebra structure with the multiplication

$$z^i \circ z^k = z^{i+k} \text{ for } i+k \le 2n-2, \quad z^i \circ z^k = 0 \text{ for } i+k > 2n-2$$
$$z^i \circ 1_{-1} = \delta_{i,0}1_{-1}, \quad 1_{-1} \circ 1_{-1} = z^{2n-2}.$$

This multiplication and the metric are compatible with the bi-grading and yield a $\mathbb{Z}/2\mathbb{Z}$ Frobenius algebra for both choices of σ.

Notice that since $det(g) = \pm 1$ the metric will make the invariants into a Frobenius algebra.

In the case of $\sigma(j^n) \equiv 1 \mod 2$ we obtain as invariants

$$\langle 1, z^2, \dots z^{2(n-1)}, 1_{j^n} \rangle.$$

The bi-grading of the invariants is diagonal and given by

$$((\frac{1}{n}, \frac{1}{n}), \dots, (\frac{n-1}{n}, \frac{n-1}{n}), (\frac{n-1}{2n}, \frac{n-1}{2n})).$$

In the case $\sigma(j^n) \equiv 0$ the space of invariants is

$$\langle 1, z^2, \dots z^{2(n-1)} \rangle$$

and the multiplication, the metric, and the bi-grading are the restrictions of the ones above.

PROPOSITION 4.4. *In total we obtain*

(1) *The $\mathbb{Z}/2\mathbb{Z}$ invariants of $(\mathbb{Z}/2\mathbb{Z})M_{z^{2n-2}}$ with the choice $\sigma(j^n) \equiv 1 \mod 2$ are isomorphic as a bi-graded Frobenius algebra to the (c,c)-realization of $M_{x^n+xy^2} = D_n$.*
(2) *The $\mathbb{Z}/2\mathbb{Z}$ invariants of $\mathbb{Z}/2\mathbb{Z}M_{z^{2n-2}}$ with the choice $\sigma(j^n) \equiv 0 \mod 2$ are isomorphic as a bi-graded Frobenius algebra to $B_n^{(c,c)}$.*

REMARK 4.13. *The result above, in which the invariants of the untwisted sector of A_{2n-1} yield B_n, is an instance of what is called folding (see* [**Z98**] *and §5.3 below).*

4.4.2. The dual $D(k[G])$-module. For the dual, we obtain two sectors

$$\check{M}_{f_e} = \Phi(M_{f_{j^{-1}}}) \simeq A_1, \quad \check{M}_{f_{j^n}} = \Phi(M_{f_{j^{n-1}}}) \simeq A_1.$$

The dual bi-grading is given by

$$\check{s}_e = \bar{\check{s}}_e = 0, \quad \check{s}_{-1} = -\frac{1}{2}\bar{\check{s}}_{-1} = \frac{1}{2}.$$

REMARK 4.14. *In the case that $\sigma \equiv 0$ or the case that n is odd and $\sigma(j) = \sigma(-1) = -1$, the action is the restriction of the Euler G-Frobenius algebra of §4.3, and is thus quasi-Euler.*

In both these cases the dual action is defined and is given by

$$\check{\varphi}_{-1,1} = (-1)^{\sigma(-1)(\sigma(j)+1)} = 1, \quad \check{\varphi}_{-1,-1} = (-1)^{\sigma(-1)(\sigma(-1)+\sigma(j)+1)} = (-1)^{\sigma(-1)}.$$

Since $(A_{2n-1}, \mathbb{Z}/2\mathbb{Z})$ is not Euler, but only quasi-Euler, we cannot pull back the metric, but due to the grading there is projectively only one compatible homogeneous metric.

PROPOSITION 4.5. *Projectively there is a Frobenius algebra structure on the duals compatible with the group grading. This Frobenius algebra structure is isomorphic, as a bi-graded Frobenius algebra, to the (a,c)-realization of the algebra $I_2(4)$. In the case that $\sigma \equiv 0$ the invariants are $I_2(4)$, and in the case that n is odd and $\sigma(-1) = -1$ the invariants are A_1.*

4.5. The case of A_{2n-1} with symmetry group $\mathbb{Z}/n\mathbb{Z}$. In the case of A_{2n-1}, we can also consider the symmetry group $\mathbb{Z}/n\mathbb{Z} \subset \mathbb{Z}/(2n\mathbb{Z})$ which is generated by j^2.

Again the group is cyclic and $\epsilon \equiv 1$. In the case that n is even, there is only one possible choice of $\sigma \equiv 0$. In the case that n is odd, there are two possible choices $\sigma \equiv 0$ or $\sigma(j^{2k}) \equiv k$. The latter choice is not quasi-Euler, however.

4.5.1. The G-Frobenius algebras. The invariants can be read off from §4.3. For $\sigma \equiv 0$ or for $\sigma(j^{2k}) \equiv k$, there are no invariants in the twisted sector and the invariants in the untwisted sector are

$$\langle 1, z^n \rangle.$$

The bi-degrees are $(0,0), (\frac{1}{2}, \frac{1}{2})$.

PROPOSITION 4.6. *Projectively there is only one Frobenius algebra structure on these invariants compatible with the group grading. The resulting Frobenius algebra is isomorphic as a bi-graded Frobenius algebra to the (c,c)-realization of the algebra for the Coxeter group $I_2(4)$. This is also the restriction of the respective multiplication on the unique $\mathbb{Z}/(2n\mathbb{Z})$ Frobenius algebra $\mathbb{Z}/(2n\mathbb{Z})M_{z^{2n}}$.*

4.5.2. The dual. We can only consider the quasi-Euler choice $\sigma \equiv 0$.

The linear spaces for the dual $D(k[G])$-module are all one dimensional $\check{A}_{j^{2k}} = \langle \check{1}_{j^{2k}} \rangle$, and are all invariant.

The bi-degrees are

$$(\mathcal{Q}, \bar{\mathcal{Q}})(\check{1}_{j^{2k}}) = (-\frac{k}{n}, \frac{k}{n})(\check{1}_{j^{2k}}) \quad k \in \{1, \dots, n-1\}.$$

PROPOSITION 4.7. *The dual $((\mathbb{Z}/n\mathbb{Z})A_{2n-1})^\vee$ affords a projectively unique graded $\mathbb{Z}/n\mathbb{Z}$-Frobenius algebra structure with trivial $\mathbb{Z}/n\mathbb{Z}$ action which is equal to the Frobenius algebra of its invariants and is isomorphic to the (a,c)-realization of B_n.*

4.6. The case of $A_{p-1} \otimes A_{q-1}$, especially E_6 and E_8. We will consider the tensor product $A_{p-1} \otimes A_{q-1}$ for coprime p, q.

The corresponding quasi-homogeneous singularity is given by $f = x^p + y^q$. For these singularities, $q_x = \frac{1}{p}, q_y = \frac{1}{q}, d = \frac{2(pq-p-q)}{pq}, \mu = (p-1)(q-1)$.

Since p and q are coprime, $G_{max} = \mathbb{Z}/(pq\mathbb{Z}) = \mathbb{Z}/pZ \times \mathbb{Z}/q\mathbb{Z}$, which is generated by the grading operator $J = (J_p, J_q)$ in the tensor representation for the symmetry groups of the A_n factors:

$$J = \rho(j) = \begin{pmatrix} \zeta_p & 0 \\ 0 & \zeta_q \end{pmatrix},$$

where as usual $\zeta_p = exp(2\pi i \frac{1}{p})$ and $\zeta_q = exp(2\pi i \frac{1}{q})$.

4.6.1. The G-Frobenius algebras. We have

$$f_{j^i} = \begin{cases} z^p + z^q & \text{if } i = 0 \\ z^p & \text{if } i = rp, r \in \{1, \dots, q-1\} \\ z^q & \text{if } i = rq, r \in \{1, \dots, p-1\} \\ 0 & \textit{otherwise}. \end{cases}$$

In these cases, we get the twisted sectors linearly isomorphic to $A_{p-1} \otimes A_{q-1}$, A_{p-1}, A_{q-1} and A_1. The group is cyclic and hence $\epsilon \equiv 1$. In the case that pq is odd, we only have the trivial choice $\sigma \equiv 0$. In the case that it is even, we also have the possibility of setting $\sigma(j^i) \equiv i$. We will leave the latter case to the reader.

The action is given by

$$\varphi_{j^k, j^i} = \begin{cases} 1 & \text{if } i = 0 \\ \zeta_q^{-k} & \text{if } i = rp, \ r \in \{1, \dots, q-1\} \\ \zeta_p^{-k} & \text{if } i = rq, \ r \in \{1, \dots, p-1\} \\ \zeta_q^{-k}\zeta_p^{-k} & \textit{otherwise}, \end{cases}$$

and we see that only $1 \in A_e$ is invariant.

The grading is given by

$$s_{j^i} = \begin{cases} 0 & \text{if } i = 0 \\ \frac{j-1}{q} & \text{if } i = rp = kq + j,\ r \in \{1, \dots, q-1\} \\ \frac{j-1}{p} & \text{if } i = rq = kp + j,\ r \in \{1, \dots, p-1\} \\ \frac{j-1}{p} + \frac{l-1}{q} & \text{if } i = rp + j = kq + l \end{cases}$$

$$\bar{s}_{j^i} = \begin{cases} 0 & \text{if } i = 0 \\ 1 - \frac{j+1}{q} & \text{if } i = rp = kq + j,\ r \in \{1, \dots, q-1\} \\ 1 - \frac{j+1}{p} & \text{if } i = rq = kp + j,\ r \in \{1, \dots, p-1\} \\ 2 - \frac{j+1}{p} - \frac{l+1}{q} & \text{if } i = rp + j = kq + l. \end{cases}$$

PROPOSITION 4.8. *The $\mathbb{Z}/pq\mathbb{Z}$ invariants of the unique $D(k[G])$-module $\mathbb{Z}/pq\mathbb{Z}(A_{p-1} \otimes A_{q-1})$ are one-dimensional and are thus isomorphic to the Frobenius algebra A_1.*

4.6.2. The dual. The dual action is given by

$$\check{\varphi}_{j^k, j^i} = \begin{cases} \zeta_q^{-k}\zeta_p^{-k} & \text{if } i - 1 = 0 \\ \zeta_p^k & \text{if } i - 1 = rp,\ r \in \{1, \dots, q-1\} \\ \zeta_q^k & \text{if } i - 1 = rq,\ r \in \{1, \dots, p-1\} \\ & \textit{otherwise.} \end{cases}$$

From this we obtain a $pq - p - q + 1 = (p-1)(q-1)$ dimensional space of invariants spanned by

$$\langle \check{1}_{j^i} \rangle \quad i - 1 \not\equiv 0 \mod p \text{ and } i - 1 \not\equiv 0 \mod q.$$

The grading is given by

$$\check{s}_{j^i} = \begin{cases} 0 & \text{if } i = 0 \\ -\frac{2(pq-q-p)}{pq} & \text{if } i = 1 \\ \frac{j-1}{q} + \frac{2}{p} - 2 & \text{if } i - 1 = rp = kq + j,\ r \in \{1, \dots, q-1\} \\ \frac{j-1}{p}\frac{2}{q} - 2 & \text{if } i - 1 = rq = kp + j,\ r \in \{1, \dots, p-1\} \\ \frac{j-1}{p} + \frac{l}{q} - 2 & \text{if } i - 1 = rp + j = kq + l \end{cases}$$

$$\bar{\check{s}}_{j^i} = \begin{cases} 0 & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ 1 - \frac{j-1}{q} & \text{if } i - 1 = rp = kq + j,\ r \in \{1, \dots, q-1\} \\ 1 - \frac{j-1}{p} & \text{if } i - 1 = rq = kp + j,\ r \in \{1, \dots, p-1\} \\ 2 - \frac{j-1}{p} - \frac{l-1}{q} & \text{if } i - 1 = rp + j = kq + l, \end{cases}$$

where we choose $i \in \{0, \dots, pq-1\}$.

By comparing degrees we arrive at

LEMMA 4.3. *Let $i \equiv j \mod p, j \in 2, \dots p$ and $i \equiv k \mod q$, $k \in \{2, \dots, q\}$. The map*

$$\check{1}_{j^i} \mapsto x^{p-j}y^{q-l}$$

induces an isomorphism of graded vector spaces between the $\mathbb{Z}/pq\mathbb{Z}$ invariants of $(\mathbb{Z}/pq\mathbb{Z}M_{x^p+y^q})^\vee = (\mathbb{Z}/pq\mathbb{Z}A_{p-1} \otimes A_{q-1})^\vee$ graded by $\bar{Q}$ and the graded Milnor ring $A_{p-1} \otimes A_{q-1}$.

Moreover, as bi-graded space $((\mathbb{Z}/pq\mathbb{Z}A_{p-1}\otimes A_{q-1}))^{\vee})^{\mathbb{Z}/pq\mathbb{Z}}$ *is the* (a,c)*-realization or the A-model of* $A_{p-1}\otimes A_{q-1}$.

By comparing the degrees and group degrees, one obtains

PROPOSITION 4.9. *There is a projectively unique, maximally non-degenerate, degenerate G-Frobenius structure on* $((\mathbb{Z}/pq\mathbb{Z})(A_{p-1}\otimes A_{q-1}))^{\vee}$ *whose invariants are the mirror dual to* $A_{p-1}\otimes A_{q-1}$.

COROLLARY 4.1. *If we restrict ourselves to the case* $p=3,q=4$, *we obtain the mirror to the* E_6 *singularity, and for* $p=3,q=5$ *the mirror for the* E_8 *singularity.*
In the first case the invariants are

$$\check{1}_{j^i},\ i\in\{0,2,3,6,8,11\}\quad \textit{corresponding to } 1,xy^2,y,y^2,x,xy$$

with bi-degrees

$$(0,0),(-\frac{5}{6},\frac{5}{6}),(-\frac{1}{4},\frac{1}{4}),(-\frac{1}{2},\frac{1}{2}),(-\frac{1}{3},\frac{1}{3}),(-\frac{7}{12},\frac{7}{12}).$$

In the second case the invariants are

$$\check{1}_{j^i},\ i\in\{0,2,3,5,8,9,12,14\}\ \textit{corresponding to } 1,xy^3,y^2,x,xy^2,y,y^3,xy$$

with bi-degrees

$$(0,0),(-\frac{14}{15},\frac{14}{15}),(-\frac{2}{5},\frac{2}{5}),(-\frac{1}{3},\frac{1}{3}),(-\frac{11}{15},\frac{11}{15}),(-\frac{1}{5},\frac{1}{5}),(-\frac{3}{5},\frac{3}{5}),(-\frac{8}{15},\frac{8}{15}).$$

4.6.3. The case of E_6 and the relation to F_4. Using the above calculations we can obtain a mirror pair for F_4 from E_6 via the tensor product. For this we use that $E_6=(A_2\otimes A_3)$ and $G_{max}=\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$.

PROPOSITION 4.10. $(F_4,I_2(4))$ *and* $(I_2(4),F_4)$ *are a mirror dual pair obtained from the orbifold mirror philosophy for* E_6 *with* $G_{max}=\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$, $H=e\times\mathbb{Z}/2\mathbb{Z}$ *and* $G/H=\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, *adopting one of the conventions 4.1–4.2 or the extended orbifold mirror philosophy 4.1.*

PROOF. Using the group $\mathbb{Z}/2\mathbb{Z}$ acting via $e\times\mathbb{Z}/2\mathbb{Z}$ we have $E_6/(\mathbb{Z}/2\mathbb{Z})=(A_2\otimes A_3)/(e\times\mathbb{Z}/2\mathbb{Z})=(A_2/e\otimes A_3/(\mathbb{Z}/2\mathbb{Z}))$. Thus by the previous calculations

$$(((\mathbb{Z}/2\mathbb{Z})E_6)^{\mathbb{Z}/2\mathbb{Z}},(((\mathbb{Z}/2\mathbb{Z})E_6)^{\vee})^{\mathbb{Z}/2\mathbb{Z}})=(A_2\otimes I_2(4),A_1\otimes I_2(4))=(F_4,I_2(4)).$$

For the dual pair with $G_{max}=\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$, $H=e\times\mathbb{Z}/2\mathbb{Z}$ and $K=G/H=\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$,

$$\begin{aligned}&((A_2\otimes A_3/(\mathbb{Z}/2\mathbb{Z}))^{\mathbb{Z}/2\mathbb{Z}}/(\mathbb{Z}/3\mathbb{Z})\times\mathbb{Z}/2\mathbb{Z}))^{K}=\\&\qquad(A_2/(\mathbb{Z}/3\mathbb{Z}))^{\mathbb{Z}/3\mathbb{Z}}\otimes((A_3/(\mathbb{Z}/2\mathbb{Z}))/(\mathbb{Z}/2\mathbb{Z}))^{\mathbb{Z}/2\mathbb{Z}}=A_1\otimes I_2(4)\simeq I_2(4)\end{aligned}$$

and

$$\begin{aligned}&((A_2\otimes A_3/(\mathbb{Z}/2\mathbb{Z}))^{\mathbb{Z}/2\mathbb{Z}}/(\mathbb{Z}/(3\mathbb{Z})\times\mathbb{Z}/2\mathbb{Z}))^{\vee})^{K}\\&=((A_2/(\mathbb{Z}/3\mathbb{Z}))^{\vee})^{\mathbb{Z}/3\mathbb{Z}}\otimes(((A_3/(\mathbb{Z}/2\mathbb{Z}))^{\mathbb{Z}/2\mathbb{Z}}/(\mathbb{Z}/2\mathbb{Z}))^{\vee})^{\mathbb{Z}/2\mathbb{Z}}=A_2\otimes I_2(4))\simeq F_4.\end{aligned}$$

In the last two calculations, we used one of the conventions 4.1 or 4.2.

Alternatively, we could use the extended orbifold mirror philosophy 4.1. For this, notice that $(A_2/(\mathbb{Z}/3\mathbb{Z}))^{\mathbb{Z}/3\mathbb{Z}}=A_1$ and $((A_2/(\mathbb{Z}/3\mathbb{Z}))^{\vee})^{\mathbb{Z}/3\mathbb{Z}}=A_2$. Furthermore, $(A_3/(\mathbb{Z}/2\mathbb{Z}))^{\mathbb{Z}/2\mathbb{Z}}=((A_3/(\mathbb{Z}/2\mathbb{Z}))^{\vee})^{\mathbb{Z}/2\mathbb{Z}}=I_2(4)$ by the previous calculations.

□

4.6.4. Certain Pham singularities. The same reasoning holds true for the Pham singularities of coprime powers:

$$f = x_1^{k_1} + \dots x_n^{k_n} \text{with } k_i \text{ pairwise coprime } M_f = A_{k_1-1} \otimes \dots \otimes A_{k_n-1}.$$

Let Γ be the group generated by the grading operator, so that $\Gamma = \mathbb{Z}/k_1\mathbb{Z} \times \dots \times \mathbb{Z}/k_n\mathbb{Z}$.

PROPOSITION 4.11. *The Γ-invariants of the ΓM_f with the choice of trivial σ are A_1, and there is a degenerate, maximally non-degenerate multiplication on the dual $(\Gamma M_f)^\vee$ which is projectively unique, and the Γ-invariants of the dual are the (a, c)-realization of M_f.*

In other words, $((\Gamma M_f)^\vee)^\Gamma$ is the mirror to M_f.

4.7. The case of D_n. Recall that $D_{n+1} = M_{x^n+y^2z}$ with $q_1 = q_x = \frac{1}{n}, q_2 = q_y = \frac{n-1}{2n}, d = \frac{n-1}{n}$. For $n > 3$ the maximal symmetry group is $G_{max} = \mathbb{Z}/2n\mathbb{Z} = \langle\Lambda\rangle$. The maximal symmetry group in the case of D_4 is larger—$G_{max} = \mathbb{Z}/3\mathbb{Z} \times \mathbb{S}_3$—but also contains the group $\mathbb{Z}/6\mathbb{Z}$ generated by Λ. We will make further comments about the case D_4 below in §4.9.3.

If we fix $\zeta_{2n} := \exp(2\pi i \frac{1}{2n})$, then

$$\Lambda = \begin{pmatrix} \zeta_{2n}^2 & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix} \quad \Lambda^i = \begin{pmatrix} \zeta_{2n}^{2i} & 0 \\ 0 & \zeta_{2n}^{-i} \end{pmatrix}$$

$$\Lambda^{n+1} = \begin{pmatrix} \exp(2\pi i \frac{1}{n}) & 0 \\ 0 & \exp(2\pi i \frac{n-1}{2n}) \end{pmatrix} = \exp(2\pi i Q) = \rho(j) = J.$$

This implies that $G_{max} = \langle J \rangle$ if and only if n is even.

Since e fixes both x and y, Λ^l fixes neither x nor y for $l \neq 0, n$ and Λ^n fixes x but not y, we see that the orbifold data is as follows:

$g \in \mathbb{Z}/(2n\mathbb{Z})$	f_g	M_{f_g}	d_g	$\nu_1(g)$	$\nu_2(g)$	$\frac{1}{2}s_g^+$	$\frac{1}{2}s_g^-$	s_g	$\bar{s}_g$
$g = e = \Lambda^0$	$x^n + xy^2$	D_{n+1}	$\frac{n-1}{n}$	0	0	0	0	0	0
$g = \Lambda^l, 0 < l < n$	0	A_1	0	$\frac{l}{n}$	$\frac{2n-l}{2n}$	$\frac{n-1}{2n}$	$\frac{l}{2n}$	$\frac{l+n-1}{2n}$	$\frac{n-1-l}{2n}$
$g = \Lambda^n$	x^n	A_{n-1}	$\frac{n-2}{n}$	0	$\frac{1}{2}$	$\frac{1}{2n}$	0	$\frac{1}{2n}$	$\frac{1}{2n}$
$g = \Lambda^l, n < l < 2n-1$	0	A_1	0	$\frac{l-n}{n}$	$\frac{2n-l}{2n}$	$\frac{n-1}{2n}$	$\frac{l-2n}{2n}$	$\frac{l-n-1}{2n}$	$\frac{3n-1-l}{2n}$

Comparing the degrees, we arrive at

PROPOSITION 4.12. *The only elements of bi-degree (q, q) of the bi-graded $D(k[G])$-module $(\mathbb{Z}/(2n\mathbb{Z}))D_{n+1}$ are the elements of the untwisted sector and those of the Λ^n-twisted sector.*

4.7.1. The G action. Since $\mathbb{Z}/(2n\mathbb{Z})$ is cyclic, there is only one choice of discrete torsion, which fixes the choice of φ to be

$$\varphi_{\Lambda^k,\Lambda^l} = (-1)^{\sigma(k)\sigma(l)} \overset{-1}{\det}(\Lambda^k) \det(\Lambda^k|_{Fix_{\Lambda^l}}),$$

and so

$$\varphi_{\Lambda^k,\Lambda^l} = \begin{cases} 1 & l = 0 \\ (-1)^{\sigma(k)\sigma(l)}\zeta_{2n}^{-k} & l \neq 0, n \\ (-1)^{\sigma(k)\sigma(n)}\zeta_{2n}^{k} & l = n. \end{cases}$$

4.7.2. The metric. The pairing on the twisted sectors is given by

$$\eta(1_{\Lambda^k}, 1_{\Lambda^l}) = \begin{cases} \delta_{k+l,2n}\exp(2\pi i \frac{k}{4n}) & \text{for } k \le n \\ \delta_{k+l,2n}\exp(2\pi i \frac{2n-k}{4n}) & \text{for } k > n \end{cases}, \quad \eta(x^k 1_{\Lambda^n}, x^k 1_{\Lambda^n}) = \delta_{k+l,n-2};$$

all other pairings are zero, except for the pairing on the untwisted sector, which remains the pairing of D_n.

LEMMA 4.4. *The $\mathbb{Z}/(2n\mathbb{Z})$-invariant subspace is $\langle 1 \rangle$, thus*

$$(D_{n+1}/(\mathbb{Z}/(2n\mathbb{Z})))^{\mathbb{Z}/(2n\mathbb{Z})} = A_1$$

as a graded Frobenius algebra.

4.7.3. The G-Frobenius structure. A straightforward calculation shows that

PROPOSITION 4.13. *There is projectively only one G-Frobenius algebra structure compatible with the bi-grading, which is given by*

$$\gamma_{\Lambda^k,\Lambda^l} = \begin{cases} \delta_{k+l,2n}\exp(2\pi i \frac{k}{4n}) & \textit{for } k < n \\ \delta_{k+l,2n}\exp(2\pi i \frac{2n-k}{4n}) & \textit{for } k > n \\ \delta_{l,n}1 & \textit{for } k = n. \end{cases}$$

4.7.4. The dual $D(k[G])$-module. The bi-grading is given by

$$\check{s}_{\Lambda^{-k}} = \begin{cases} -\frac{k}{2n} & \text{if } k \in \{0,1,2,\ldots,n-2,n,\ldots,2n-1\} \\ -\frac{n-1}{n} & \text{if } k = n-1 \\ \frac{3-2n}{2n} & \text{if } k = 2n-1 \end{cases}$$

$$\bar{\check{s}}_{\Lambda^{-k}} = \begin{cases} \frac{k}{2n} & \text{if } k \in \{0,1,2,\ldots,n-2,n,\ldots,2n-1\} \\ 0 & \text{if } k = n-1 \\ \frac{1}{2n} & \text{if } k = 2n-1. \end{cases}$$

LEMMA 4.5. *In the case that n is even, the only elements of bi-grading $(-q,q)$ of the dual $D(k[G])$-module $(\mathbb{Z}/(2n\mathbb{Z})D_{n+1})^\vee$ are*

$$\langle \check{1}_e, \check{1}_{\Lambda^{-1}}, \ldots, \check{1}_{\Lambda^{-(n-2)}}, y\check{1}_{\Lambda^{-(n-1)}}, \check{1}_{\Lambda^{-n}}, \ldots, \check{1}_{\Lambda^{-(2n-2)}}, x^{\frac{n-2}{2}}\check{1}_{\Lambda^{-(2n-1)}} \rangle,$$

and in the case that n is odd, the elements of degree $(-q,q)$ of the dual $D(k[G])$-module $((\mathbb{Z}/(2n\mathbb{Z}))D_{n+1})^\vee$ are

$$\langle \check{1}_e, \check{1}_{\Lambda^{-1}}, \ldots, \check{1}_{\Lambda^{-(n-2)}}, x^{\frac{n-1}{2}}\check{1}_{\Lambda^{-(n-1)}}, y\check{1}_{\Lambda^{-(n-1)}}, \check{1}_{\Lambda^{-n}}, \ldots, \check{1}_{\Lambda^{-(2n-2)}} \rangle.$$

4.7.5. The dual G-action. The dual G-action is given by

$$\check{\varphi}_{\Lambda^{-k},\Lambda^{-l}} = \begin{cases} (-1)^{\sigma(\Lambda^k)(\sigma(\Lambda^l)+\sigma(\Lambda^{n+1})+1)} & \text{for } l \notin \{n-1, 2n-1\} \\ (-1)^{\sigma(\Lambda^k)}\zeta_{2n}^k & \text{for } l = n-1 \\ (-1)^{\sigma(\Lambda^k)(\sigma(\Lambda)+\sigma(\Lambda^{n+1})+1)}\zeta_n^k & \text{for } l = 2n-1. \end{cases}$$

A longer but straightforward calculation shows

PROPOSITION 4.14. *For the different choices of σ we obtain:*

(1) *In the case $\sigma \equiv 0$, the $\mathbb{Z}/(2n\mathbb{Z})$ invariants of $(\mathbb{Z}/(2n\mathbb{Z})D_{n+1})^\vee$ are*

$$\langle \check{1}_e, \check{1}_{\Lambda^{-1}}, \dots, \check{1}_{\Lambda^{-(n-2)}}, y\check{1}_{\Lambda^{-(n-1)}}, \check{1}_{\Lambda^{-n}}, \dots, \check{1}_{\Lambda^{-(2n-2)}} \rangle.$$

Their bi-degrees are $\check{\mathcal{Q}}(\check{1}_{\Lambda^{-k}}) = -\frac{k}{2n}\check{1}_{\Lambda^{-k}}$, $\bar{\check{\mathcal{Q}}}(\check{1}_{\Lambda^{-k}}) = \frac{k}{2n}\check{1}_{\Lambda^{-k}}$ for $k \in \{0, 1, \dots, 2n-1\} \setminus \{n-1\}$ and $\check{\mathcal{Q}}(y\check{1}_{\Lambda^{-(n-1)}}) = -\frac{n-1}{2n}y\check{1}_{\Lambda^{-(n-1)}}$, $\bar{\check{\mathcal{Q}}}(y\check{1}_{\Lambda^{-(n-1)}}) = \frac{n-1}{2n}y\check{1}_{\Lambda^{-(n-1)}}$.

This is the spectrum of the (a,c)-realization of A_{2n-1}, and there is a projectively unique maximally non-degenerate G-Frobenius structure on the dual whose invariants are the (a,c)-realization of A_{2n-1}.

(2) *In the case that $\sigma(\Lambda^k) \equiv k \mod 2$ and n is even, then $\sigma(j) \equiv 1$ and the invariants are*

$$\langle \check{1}_e, \check{1}_{\Lambda^{-2}}, \dots, \check{1}_{\Lambda^{-(2n-2)}}, x^{\frac{n-2}{2}}\check{1}_{\Lambda^{-(2n-1)}} \rangle.$$

Their bi-degrees are $\check{\mathcal{Q}}(\check{1}_{\Lambda^{-2k}}) = -\frac{k}{n}\check{1}_{\Lambda^{-2k}}$, $\bar{\check{\mathcal{Q}}}(\check{1}_{\Lambda^{-2k}}) = \frac{k}{n}\check{1}_{\Lambda^{-2k}}$ for $k \in \{0, 1, \dots, n-1\}$ and $\check{\mathcal{Q}}(x^{\frac{n-2}{2}}\check{1}_{\Lambda^{2n-2}}) = -\frac{n-1}{2n}x^{\frac{n-2}{2}}\check{1}_{\Lambda^{2n-2}}$, $\bar{\check{\mathcal{Q}}}(x^{\frac{n-2}{2}}\check{1}_{\Lambda^{-(2n-1)}}) = \frac{n-1}{2n}x^{\frac{n-2}{2}}\check{1}_{\Lambda^{-(2n-1)}}$.

This is the spectrum of D_{n+1}. Furthermore, there is a unique maximally non-degenerate $\mathbb{Z}/(2n\mathbb{Z})$-Frobenius algebra structure of charge j on $((\mathbb{Z}/(2n\mathbb{Z}))D_{n+1})^\vee$ which has as invariants the (a,c)-realization of D_{n+1}. So for n even, D_{n+1} is self-dual with the choice of non-trivial σ.

(3) *In the case that $\sigma(\Lambda^k) \equiv k \mod 2$ and n is odd, then $\sigma(j) \equiv 0$ and the invariants are*

$$\langle \check{1}_{\Lambda^{-1}}, \check{1}_{\Lambda^{-3}}, \dots, \check{1}_{\Lambda^{-(2n-3)}}, x^{\frac{n-1}{2}}\check{1}_{\Lambda^{-(n-1)}} \rangle.$$

Their bi-degrees are $\check{\mathcal{Q}}(\check{1}_{\Lambda^{-(2k+1)}}) = -\frac{2k+1}{2n}\check{1}_{\Lambda^{-(2k+1)}}$, $\bar{\check{\mathcal{Q}}}(\check{1}_{\Lambda^{-2k}}) = \frac{2k+1}{n}\check{1}_{\Lambda^{-(2k+1)}}$ for $k \in \{0, 1, \dots, n-1\}$, $\check{\mathcal{Q}}(x^{\frac{n-1}{2}}\check{1}_{\Lambda^{n-2}}) = -\frac{n-1}{2n}x^{\frac{n-1}{2}}\check{1}_{\Lambda^{n-2}}$ and $\bar{\check{\mathcal{Q}}}(x^{\frac{n-2}{2}}\check{1}_{\Lambda^{2n-2}}) = \frac{n-1}{2n}$.

This case is non G-Euler, and we see that there is no Frobenius algebra structure on the invariants, since the unit is missing. This means that the prospective unit $\check{1}_e$ is not invariant and there is not even a degenerate G-Frobenius algebra structure on the dual.

4.8. The case of D_{n+1} with the symmetry group $\mathbb{Z}/n\mathbb{Z}$. Let $\mathbb{Z}/n\mathbb{Z} \subset \mathbb{Z}/(2n\mathbb{Z})$ be the subgroup of even powers $\mathbb{Z}/n\mathbb{Z} = \langle \Lambda^{2k} \rangle$.

REMARK 4.15. *Notice that this subgroup is Euler if and only if n is odd. Also in this case $G_{max} \neq \langle j \rangle$ and $\mathbb{Z}/n\mathbb{Z} \simeq \langle j \rangle$.*

Since most calculations are obtained via restriction from those of the previous section, we handle both the G-Frobenius algebras and the duals at the same time.

4.8.1. The bi-gradings. The calculations above for the bi-grading for the G-Frobenius algebra and its dual just restrict to sectors corresponding to the subgroup $\mathbb{Z}/n\mathbb{Z}$.

4.8.2. The $D(k[G])$-modules. Since $\mathbb{Z}/n\mathbb{Z}$ is a cyclic group the discrete-torsion bi-character is trivial: $\epsilon \equiv 1$. In the case that n is odd, there is only one choice for σ: $\sigma \equiv 0$. In the case that n is even, there are two choices for σ: $\sigma \equiv 0$ and $\sigma(\Lambda^{2k}) \equiv k \mod 2$. In the first case the resulting structure is quasi-Euler, while in the second case it is not.

PROPOSITION 4.15. *For n even and any choice of σ the invariants of the resulting $D(k[G])$-module on $(\mathbb{Z}/n\mathbb{Z})D_{n+1}$ are two-dimensional and are generated by $\langle 1_e, x^{\frac{n}{2}}\rangle$. There is a projectively unique Frobenius structure for the invariants which is the structure of the Frobenius algebra $I_2(4)$.*

If n is odd, the invariants of $(\mathbb{Z}/n\mathbb{Z})D_{n+1}$ are one-dimensional and generated by 1_e. Hence they are isomorphic to A_1 as a Frobenius algebra.

In the case that n is even, for the choice of $\sigma \equiv 0$ the resulting $D(k[G])$-module structure on the dual $(\mathbb{Z}/n\mathbb{Z}D_{n+1})^\vee$ has invariants

$$\langle \check{1}_e, \check{1}_{\Lambda^{-2}}, \dots, \check{1}_{\Lambda^{-(2n-2)}}\rangle.$$

Their bi-grading is consistent with the (a,c)-realization of B_n, and the respective Frobenius algebra structure is compatible with the group grading. Also, there is a degenerate G-Frobenius structure of charge j on the dual $(\mathbb{Z}/n\mathbb{Z}D_{n+1})^\vee$ which has as invariants precisely the (a,c)-realization of B_n.

In the case that n is odd, the invariants are

$$\langle \check{1}_e, \check{1}_{\Lambda^{-2}}, \dots, \check{1}_{\Lambda^{-(n-3)}}, x^{\frac{n-1}{2}}\check{1}_{\Lambda^{-(n-1)}}, y\check{1}_{\Lambda^{-(n-1)}}, \check{1}_{\Lambda^{-(n+1)}}, \dots, \check{1}_{\Lambda^{-(2n-2)}}\rangle,$$

with bi-degrees matching the (a,c)-realization of D_{n+1}, and there a degenerate G-Frobenius structure of charge j whose invariants are precisely the (a,c)-realization of D_{n+1}.

So in the case of n odd, D_{n+1} is mirror self dual, with respect to the orbifolding by the symmetry group generated by the grading operator.

4.9. D_n with the symmetry group $\mathbb{Z}/2\mathbb{Z}$. In this subsection, we restrict the action of $G_{max} = \mathbb{Z}/(2n\mathbb{Z})$ to the subgroup $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/(2n\mathbb{Z})$ generated by $\Lambda^n = -1$.

4.9.1. The algebras $(\mathbb{Z}/2\mathbb{Z})D_{n+1}$. There are two twisted sectors which as k-modules, are

$$M_{f_e} = D_{n+1}, M_{f_{-1}} = A_{n-1}$$

$$\phi_{-1,-1} = (-1)^{\sigma(-1)+1}.$$

There are two choices for σ: $\sigma(-1) \equiv 0$ or $\sigma(1) \equiv 1$. The first choice always yields a quasi-Euler $\mathbb{Z}/2\mathbb{Z}$-Frobenius algebra, while the latter choice is quasi-Euler only in the case of n odd.

The bi-grading and $\mathbb{Z}/2\mathbb{Z}$-action can be read off from the tables in the previous section.

After fixing σ there is a unique $\mathbb{Z}/2\mathbb{Z}$-Frobenius algebra structure [**Ka03, Ka04**] which is given by

$$1_{-1} \circ 1_{-1} = x.$$

4.9.2. The duals. For the dual both $\check{M}_{f_e}$ and $\check{M}_{f_{-1}}$ are one-dimensional and have degrees $(0,0), (-1/2, 1/2)$. Since this is at most a quasi-Euler we cannot pull back the metric, but there is projectively only one metric compatible with the group grading.

The action is given by

$$\check{\varphi}_{-1,1} = 1 \quad \check{\varphi}_{-1,1} = (-1)^{\sigma(-1)}.$$

PROPOSITION 4.16. *In the case that $\sigma(-1) \equiv 0$, the invariants are given by*

$$\langle 1, x, \dots, x^{n-1} \rangle.$$

The bi-grading and metric and multiplication are the same as those of the (c,c)-realization of B_n. The dual algebra has a projectively unique Frobenius algebra structure compatible with the bi-grading, that is isomorphic to $I_2(4)$, and the invariants are A_1.

In the case that n is odd and $\sigma(-1) = -1$, the invariants are

$$\langle 1, x, \dots, x^{n-1}, 1_{-1}, x1_{-1}, \dots, x^{n-1}1_{-1} \rangle.$$

The algebra of invariants is isomorphic to the (c,c)-realization of A_{2n-1} as a bi-graded Frobenius algebra.

The dual algebra affords the structure of the (a,c)-realization of $I_2(4)$ with trivial $\mathbb{Z}/2\mathbb{Z}$-action.

4.9.3. The case of D_4 and the relation to G_2. In the case of D_4 the maximal symmetry group is $\langle \Lambda, \frac{1}{2}\begin{pmatrix} -1 & i \\ -3i & 1 \end{pmatrix} \rangle \subset GL(2, \mathbb{C})$.

Let

$$j = \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3 \end{pmatrix} a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, b = \frac{1}{2}\begin{pmatrix} -1 & i \\ -3i & 1 \end{pmatrix}.$$

Then $a^2 = b^2 = id, aba = bab$ and $\langle a, b \rangle \simeq \mathbb{S}_3$ the symmetric group on three elements. Also $\Lambda = aj$ and $\langle \Lambda \rangle = \mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Finally $G_{max} = \mathbb{Z}/3\mathbb{Z} \times \mathbb{S}_3$.

We do not want to present the full calculation, which is quite involved, but note that the G-Frobenius algebra for $D_4/\mathbb{S}_3$ is given as a k-module by $M_{f_e} = D_4$, and $M_{f_a} = M_{f_b} = M_{f_{aba}} = A_2$, and $M_{f_{ab}} = M_{f_{ba}} = A_1$. There are three conjugacy classes, and the invariants are $1, x^2, 1_{ab} \pm 1_{ba}$ where the sign is $+$ if one uses $\sigma \equiv 0$ or $-$ if $\sigma(g) \equiv length(g) \mod 2$.

For the invariance of x^2 notice that in the Milnor ring, without using an isomorphism, $y^2 = -3x^2$ and thus $(\frac{1}{2}(-1+iy))^2 = \frac{1}{4}x^2 - \frac{1}{4}y^2 + \frac{1}{2}ixy \equiv \frac{1}{4}x^2 - \frac{-3}{4}y^2 = x^2$.

In the case of the group $\mathbb{Z}/3\mathbb{Z}$ generated by ab, the k-module is given by $M_f = D_4$, and $M_{f_{ab}} = M_{f_{ba}} = A_1$. The invariants are $1, x^2, 1_{ab}, 1_{ba}$.

LEMMA 4.6. *The invariants in the untwisted sector of $D_4/(\mathbb{Z}/3\mathbb{Z})$ and $D_4/\mathbb{S}_3$ are isomorphic to G_2 as graded Frobenius algebras.*

4.10. The case E_7. Recall that for $E_7 : x^3 + xy^3$, we have the following degrees $q_1 = q_x = \frac{1}{3}, q_2 = q_y = \frac{2}{9}, d = \frac{8}{9}$.

Fix $\zeta_9 := \exp(2\pi i \frac{1}{9})$, then the E_7 singularity has the exponential grading operator $J = \exp(2\pi i Q)$

$$J = \begin{pmatrix} \zeta_9^3 & 0 \\ 0 & \zeta_9^2 \end{pmatrix}.$$

This operator generates a subgroup $\langle J \rangle \subset GL(n, \mathbb{C})$, which is isomorphic to $\mathbb{Z}/9\mathbb{Z}$. We fix a generator j of $\mathbb{Z}/9\mathbb{Z}$ and regard the representation $\rho : \mathbb{Z}/9\mathbb{Z} \to GL(n, \mathbb{C})$ as given by $\rho(j) = J$.

This is also the maximal symmetry group $G_{max} = \langle \Lambda \rangle$

$$\Lambda = \begin{pmatrix} \zeta_9^3 & 0 \\ 0 & \zeta_9^{-1} \end{pmatrix}$$

and $J = \Lambda^7$.

4.10.1. The $\mathbb{Z}/9\mathbb{Z}$-graded k-module $(\mathbb{Z}/9\mathbb{Z})M_f$. The representation is given by

$$\rho(j^i) = \begin{pmatrix} \zeta_9^{3i} & 0 \\ 0 & \zeta_9^{2i} \end{pmatrix}.$$

$g \in \mathbb{Z}/9\mathbb{Z}$	f_g	M_{f_g}	d_g	$\nu_1(g)$	$\nu_2(g)$	s_g^+	s_g^-	s_g	$\bar{s}_g$
$e = j^0$	$x^3 + xy^3$	E_7	$\frac{8}{9}$	0	0	0	0	0	0
j^1	0	A_1	0	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{8}{9}$	$-\frac{8}{9}$	0	$\frac{8}{9}$
j^2	0	A_1	0	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{9}$	$\frac{2}{9}$	$\frac{5}{9}$	$\frac{1}{3}$
j^3	x^3	A_2	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{5}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{1}{9}$
j^4	0	A_1	0	$\frac{1}{3}$	$\frac{8}{9}$	$\frac{8}{9}$	$\frac{4}{9}$	$\frac{2}{3}$	$\frac{2}{9}$
j^5	0	A_1	0	$\frac{2}{3}$	$\frac{1}{9}$	$\frac{8}{9}$	$-\frac{4}{9}$	$\frac{2}{9}$	$\frac{2}{3}$
j^6	x^3	A_2	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{5}{9}$	$-\frac{1}{3}$	$\frac{1}{9}$	$\frac{4}{9}$
j^7	0	A_1	0	$\frac{1}{3}$	$\frac{5}{9}$	$\frac{8}{9}$	$-\frac{2}{9}$	$\frac{1}{3}$	$\frac{5}{9}$
j^8	0	A_1	0	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{8}{9}$	$\frac{8}{9}$	0

LEMMA 4.7. *The elements of bi-degree (q, q) of $(\mathbb{Z}/9\mathbb{Z})E_7$ are exactly the elements in the untwisted sector A_e.*

4.10.2. The G-action. For $\mathbb{Z}/9\mathbb{Z}$ we have $\epsilon \equiv 1$ and $\sigma \equiv 0$, so the G-action is given by

$$\varphi_{j^i,j^k} = \begin{cases} 1 & \text{if } k = 0 \\ \zeta_9^{-2i} & \text{if } k \in \{3,6\} \\ \zeta_9^{-5i} & \text{otherwise} \end{cases}$$

and the character is

$$\chi(j^i) = \zeta_9^{5i}.$$

LEMMA 4.8. *The $\mathbb{Z}/9\mathbb{Z}$-invariants of the only compatible $D(k[\mathbb{Z}/9\mathbb{Z}])$-module structure are given by the unit 1_e.*

4.10.3. The dual bi-grading. The dual grading is given by

	0	1	2	3	4	5	6	7	8
$\check{s}_{j^i}$	0	$-\frac{8}{9}$	$-\frac{8}{9}$	$-\frac{1}{3}$	$-\frac{4}{9}$	$-\frac{2}{9}$	$-\frac{2}{3}$	$-\frac{7}{9}$	$-\frac{5}{9}$
$\bar{\check{s}}_{j^i}$	0	0	$\frac{8}{9}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{5}{9}$

The elements of bi-degree $(-q, q)$ are

$$\langle \check{1}_e, y^2\check{1}_j, \check{1}_{j^2}, \check{1}_{j^3}, \check{1}_{j^5}, \check{1}_{j^6}, \check{1}_{j^8} \rangle.$$

4.10.4. The dual $\mathbb{Z}/9\mathbb{Z}$-action. The dual $\mathbb{Z}/9\mathbb{Z}$-action is given by

$$\check{\varphi}_{j^i,j^k} = \begin{cases} \zeta_9^{5i} & \text{if } k = 1 \\ \zeta_9^{3i} & \text{if } k \in \{4,7\} \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 4.9. *The $\mathbb{Z}/9\mathbb{Z}$-invariants of the dual $((\mathbb{Z}/9\mathbb{Z})E_7)^\vee$ are given by*

$$\langle \check{1}_e, y^2\check{1}_j, \check{1}_{j^2}, \check{1}_{j^3}, \check{1}_{j^5}, \check{1}_{j^6}, \check{1}_{j^8}\rangle;$$

they are all of diagonal bi-degree, and their degrees are

$$(0,0), (-\frac{4}{9}, \frac{4}{9}), (\frac{-8}{9}, \frac{8}{9}), (-\frac{1}{3}, \frac{1}{3}), (-\frac{2}{9}, \frac{2}{9}), (-\frac{2}{3}, \frac{2}{3}), (-\frac{5}{9}, \frac{5}{9}).$$

The pairing, the bi-grading, and the group grading are the same as those of the anti-chiral realization of E_7 under the association $\check{1}_e \mapsto 1, \check{1}_j \mapsto y^2, \check{1}_{j^2} \mapsto x^2y, \check{1}_{j^3} \mapsto x, \check{1}_{j^5} \mapsto y, \check{1}_{j^6} \mapsto x^2, \check{1}_{j^8} \mapsto xy$, so that E_7 is self dual.

Again by inspecting the grading and group grading we have

PROPOSITION 4.17. *There is a unique maximally degenerate G-Frobenius structure of charge j on $((\mathbb{Z}/9\mathbb{Z})E_7)^\vee$ whose invariants form the (a,c)-realization of E_7. Hence $(((\mathbb{Z}/9\mathbb{Z})E_7)^\vee)^{\mathbb{Z}/9\mathbb{Z}}$ is the mirror dual to E_7.*

4.11. The case P_8 or $\tilde{E}_7$. We briefly digress to singularities of higher modularity. The first singularity of this type is $P_8 = x^3 + y^3 + z^3 - axyz$ with $a^3 + 27 \neq 0$ which is also known as $\tilde{E}_7$. The Milnor ring of this singularity is generated by $\langle 1, x, y, z, xy, yz, xz, xyz\rangle$. It is quasi-homogeneous of degrees $q_x = q_y = q_z = \frac{1}{3}$ and $d = 1$.

This singularity is not self-dual. Moreover, in the case that $a \neq 0$ there is no symmetry group which has only A_1 as invariants of the G-Frobenius algebra, since the term xyz always has to remain invariant. So it is impossible for a G-Frobenius algebra built from this singularity to be mirror-dual for any orbifolding group to another singularity. Also the invariants cease to have the diagonal (q,q) or anti-diagonal $(-q,q)$ grading.

Let us calculate P_8/Γ for the group Γ generated by the grading operator $J = diag(\zeta_3, \zeta_3, \zeta_3)$. There are two one-dimensional twisted sectors.

The shifts for the twisted sectors $i = 1, 2$ are $s_J = 0, \bar{s}_J = 1; s_{J^2} = 1, \bar{s}_{J^2} = 0$. Since $det(J^i) = 1$ and necessarily $\sigma \equiv 0, \epsilon \equiv 1$, all elements in the twisted sector are invariant. In total, the invariant elements are

$$1, xyz, 1_J, 1_{J^2} \text{ of degrees } (0,0), (1,1), (1,0) \text{ and } (0,1).$$

For the dual, the action does not change since $\sigma \equiv 0$ and hence $\chi \equiv 1$, and we obtain the same invariants, only with a shifted group grading.

REMARK 4.16. *Notice that the spectrum is such that it looks like the Hodge diamond of a manifold.*

Although the singularity P_8 is not mirror dual to any other singularity, its orbifold version $(\mathbb{Z}/3\mathbb{Z})P_8$ yields a self-dual mirror pair $((\mathbb{Z}/3\mathbb{Z}P_8)^{\mathbb{Z}/3Z}, ((\mathbb{Z}/3\mathbb{Z}P_8)^\vee)^{\mathbb{Z}/3Z})$.

PROPOSITION 4.18. *The G-Euler G-Frobenius algebra $(\mathbb{Z}/3\mathbb{Z})P_8$ is mirror self-dual: $(\mathbb{Z}/3\mathbb{Z}P_8)^{\mathbb{Z}/3Z} \simeq ((\mathbb{Z}/3\mathbb{Z}P_8)^\vee)^{\mathbb{Z}/3Z}$.*

5. Remarks on the relation to spin curves, the geometry of singularities and folding

5.1. Remarks on the relation to r-spin curves and A-models for quasi-homogeneous polynomials. The r-spin curve picture was conceived by Witten as an A-model or σ-Model counterpart for the A_{r-1} Landau-Ginzburg B-model [**W91, W92**]. In his construction and the mathematical constructions of [**JKV01,**

PV01, P02] this was achieved. It turns out, however, that in the formulation there are two types of behaviors at given marked points called Ramond or Neveu-Schwarz. The appearance of the Ramond case introduces an additional element in the state space, which is $n+1$ dimensional in the A_n case. If this element appears in a correlation function the value of the correlation function becomes zero. So the algebra is what we called a degenerate Frobenius algebra of degree j if one assigns the group degree j^{-1} to z and identifies the Ramond element with z^{-n}.

This is the projectively unique maximally degenerate G-Frobenius algebra one obtains from $(pt/(\mathbb{Z}/(n+1)\mathbb{Z}))^{\vee}$ [**Ka03**]. If one considers A_1 as the (a,c)-ring of A_n, then by self duality of A_n one could expect that $((A_1 = pt)/\mathbb{Z}/(n+1)\mathbb{Z})^{\vee})^{\mathbb{Z}/(n+1)\mathbb{Z}} = A_n$ (cf. [**Ka03**]), which is indeed the structure found above. In this interpretation the bi-grading is, however, not straightforward, although the grading could be recovered from the q_i and ν_i by considering the action of $\mathbb{Z}/(n+1)\mathbb{Z}$ on $\mathbb{C}$ by roots of unity.

It would be desirable to consider not only this altered version of our duality applied to the (a,c)-ring, but to see it directly on the (c,c) side.

For this, we would like to give another interpretation of our previous remark on the A_{r-1} model. In the spin-curve picture one considers the equation

$$\mathcal{L}^{\otimes r} \simeq \omega(twisted), \tag{5.1}$$

where $\omega(twisted)$ is a suitably twisted version of the canonical line bundle on the curve, see [**JKV01, JKV00, W91, W92**]. In terms of the singularity and the forming of the Milnor ring, this equation (5.1) is mimicked by setting $z^r = 0$, that is by passing to the quotient $\mathbb{C}[z]/(z^r)$. In this ring, z^{r-1} is not zero and corresponds to the Ramond sector of the spin curves. But any element of the Ramond sector produces a zero value in all the correlation functions. In other words it appears only as a degenerate state. As we project to the invariants the equation $z^{r-1} = 0$ is implemented, which in the spin-curve picture is enforced by the virtual fundamental class. In algebraic terms for A_{r-1}, first one considers $R := \mathbb{C}[z]/(z^r)$ and then $R/(z^{r-1})$. The first quotient also manifests itself in the spin picture as the periodicity obtained from $\omega(twisted) \simeq \mathcal{L}^{\otimes r}$.

Now our degenerate Ramond sector is in fact n-dimensional for A_n and not one-dimensional. Here one should remark that for the construction of an operad in the Ramond case, one would actually have to fix a choice of isomorphism of the line bundle with $\omega(twisted)$. The space of choices for this isomorphism is a principal $\mathbb{Z}/(n+1)\mathbb{Z}$ space and thus if one includes this data in the moduli problem, the state space for the Ramond sector becomes $(n+1)$ dimensional for A_n[**J**]. So indeed the Ramond sector seems to be intrinsically higher-dimensional. The fact that the dimension is not n, but $n+1$, can be understood by the reasoning above. In our description, the singularity in this sector is the singularity A_n. In the Milnor ring interpretation, this produces a Frobenius algebra which has n states. In the spin-representation as discussed above one would expect $n+1$ states, one of which is degenerate.

The musings on this subject are at the moment only on the level of the undeformed algebra, but we hope to make them into more solid statements.

There is a straightforward way to build a spin-curve like picture for any quasi-homogeneous polynomial f. For this one considers a line bundle $\mathcal{L}_i$ for each of the variables z_i and imposes the equations obtained by substituting the line bundles $\mathcal{L}_i$ into the of monomials of the polynomials f and equating these expression to

$\omega(twisted)$. This defines the moduli problem. This approach is being seriously discussed by [**FJR**]. When the polynomial f is such that its maximal symmetry group G_{max} is Abelian and each variable appears by itself, the corresponding virtual fundamental class is constructed in [**FJR**]. The hope is to be able to lift these conditions [**J**]. We would like to point out that the condition on the variables appearing alone in a monomial ensures that $\mathbb{C}[[z_i]]/(m_j)$ is finite dimensional. Here the m_j are the monomials of f.

Further evidence for our interpretation of "Landau-Ginzburg A-models" arises from these constructions. For each element $g \in G_{max}$ there are again two types of behaviors at the marked point which are either of Ramond or of Neveu-Schwarz type. The Ramond means that the isotropy at a marked point is not the full symmetry group while in the Neveu-Schwarz case it is.

Again to turn the resulting moduli spaces into operads it is necessary to include additional data for the Ramond case which is isomorphic to the reduced symmetry group of f_g [**J**].

CONJECTURE 5.1. *We conjecture that the Neveu-Schwarz sectors are in 1-1 correspondence with the one-dimensional twisted sectors and the Ramond sectors are in one-one correspondence with the sectors that are more than one-dimensional, i.e.,* $GM_{f_g} \neq \mathbb{C}$.

This conjecture has been checked against the preliminary results of [**FJR**].

CONJECTURE 5.2. *We expect that the non-degenerate part of the cohomological field theory described by a quasi-homogeneous polynomial is the deformation of the Frobenius algebra of the invariants of* $(G_{max}M_f)^\vee$. *Moreover, we expect that the behavior of the correlation functions is modeled by the deformations of a degenerate G-Frobenius algebra of charge j given by* $(G_{max}M_f)^\vee$, *possibly adding more degenerate elements. More precisely, let* m_j *be the monomials of* f *and* $q_i = \frac{1}{n_i}$ *be the quasi-homogeneous degrees of the* z_i. *In the case that the ring* $\hat{M}_{f_g} := \mathcal{O}/(m_{f_g,j})$ *is finite dimensional, the extra elements should correspond to the extension of basis from* M_{f_g} *to* $\hat{M}_{f_g}$ *for each higher-dimensional sector.*

Our calculations predict that this procedure yields the right result in the case of Pham singularities with coprime powers, such as E_6 and E_8, and indeed this is true by taking tensor products of spin-curves [**JKV00**].

5.2. Orbifolding and the geometry of singularities with symmetries. There is a relationship between our constructions of G-Frobenius algebras for a singularity f as well as the the Ramond state space of [**Ka03**] (not to be confused with the Ramond notation for spin-curves) and classic singularity theory.

For a singularity there are classically two objects which are studied; one is the Milnor ring M_f, which also provides a basis for the miniversal unfolding which can be written as

$$F : (\mathbb{C}^{n+1} \times M_f, 0) \to (\mathbb{C}, 0).$$

This fact affords an extension by the choice of a primitive form [**S82, S83**] to a construction of a Frobenius manifold on the flat space M_f [**Du96**].

The other object of interest is obtained from the fibration which is given by

$$f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0) \tag{5.2}$$

restricted to $X := \{\mathbf{z} \in \mathbb{C}^{n+1} : ||\mathbf{z}|| \leq \epsilon, 0 < f(\mathbf{z}) \leq \delta\}$ for sufficiently small ϵ and $\delta \ll \epsilon$, so that the restriction is a smooth fibration with fibers $X_t = X \cap f^{-1}(t)$ [**AGV85, AGV88**]. The fibers are homotopic to bouquets of spheres and the Betti number of the middle dimensional cohomology of the fibers is the Milnor number μ. The isomorphism between M_f and $H := H^n(F_*, \mathbb{C})$ can be given by a choice of primitive form. Here F_* denotes a generic fiber.

Now suppose $G \subset GL(n+1, \mathbb{C})$ is a group of symmetries. This will act on the total space of the fibration (5.2) and trivially on the base, and thus there is an induced action on H.

Let *det* be the one-dimensional representation of G given by the determinant. The main result of [**Wa80**] is

THEOREM 5.1 ([**Wa80**]). *In the situation described above the* $\mathbb{C}[G]$*-modules* H *and* $M_f \otimes det$ *are isomorphic.*

This implies that while the untwisted sector of the G-Frobenius algebra is isomorphic as a $\mathbb{C}[G]$-module to M_f, the untwisted sector of the Ramond state space is isomorphic as a $\mathbb{C}[G]$-module to H. This untwisted Ramond sector corresponds to the j-twisted sector of the dual.

In exactly the case that the symmetries generate a Coxeter group G, the quotient of $\mathbb{C}^{n+1}$ by G is smooth: $\mathbb{C}^{n+1}/G \simeq \mathbb{C}^{n+1}$. In this situation, one can regard the germ f_G on the quotient. Let μ_G denote the Milnor number of f_G and μ_g those of $f_g := f|_{Fix(G)}$. Here we need to assume that this restriction is again an isolated singularity, which is automatic in the quasi-homogeneous case. Also, fix $d_g = codim(Fix(g))$.

The results of [**Wa80**] are

$$\mu_G = \frac{1}{|G|} \sum_{g \in G} (-1)^{d_g} \mu_g.$$

Furthermore, in [**Wa80**] the equivariant Euler characteristic of the $\mathbb{C}[G]$-modules M_f and H is used to compute the Milnor numbers μ_g. Let $M = H_n(F_*, \mathbb{C})$ and consider its class $[M]$ in the representation ring of G. We can identify this with the ring of class functions and evaluate at elements g.

The formula is [**Wa80**]

$$\mu_g = (-1)^{d_g} [M](g).$$

This gives a way to compute the invariants of the untwisted Ramond state space, which is isomorphic as a G-module to the j twisted sector of $(GM_f)^\vee$. It is interesting to note that the twisted sectors contribute to this calculation through the equivariant Euler characteristic.

One could adapt these techniques to the restrictions of the singularity to the various fixed point sets and obtain formulas for the dimension of the whole space $(M_f)^\vee$.

5.3. Folding. For Dynkin diagrams of the simple singularities, and more generally for the generalized Dynkin diagrams of [**Z98**], there is an operation known as *folding.*

In this section we show that the folding can be described as a non-stringy orbifolding with respect to a group of projective symmetries.

Diagram/group	Folded diagram/group	Folding group	representation
A_n	$I_2(n+1)$	$\mathbb{Z}/(n-1)\mathbb{Z}$	$z \mapsto \zeta_{n-1}z$
A_{2n-1}	B_n	$\mathbb{Z}/2\mathbb{Z}$	$z \mapsto -z$
D_{n+1}	B_n	$\mathbb{Z}/2\mathbb{Z}$	$(x,y) \mapsto (x,-y)$
D_4	G_2	$\mathbb{Z}/2\mathbb{Z}$ [1)]	$(x,y) \mapsto (-x,-y)$
D_6	H_3	$\mathbb{Z}/2\mathbb{Z}$	$(x,y) \mapsto (-x,-y)$
E_6	F_4	$\mathbb{Z}/2\mathbb{Z}$	$(x,y) \mapsto (x,-y)$
E_8	H_4	$e \times \mathbb{Z}/3\mathbb{Z}$	$(x,y) \mapsto (x,\zeta_3 y)$

[1)] This is the simplest group. Other folding groups are $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{S}_3$ as discussed in §4.9.3.

TABLE 3. The Foldings and their Projective Symmetry Groups.

DEFINITION 5.1. *A projective symmetry for a singularity $f : \mathbb{C}^n \to \mathbb{C}$ with an isolated critical point at zero is an element $S \in GL(n,\mathbb{C})$, such that $f(S(\mathbf{z})) = \lambda f(\mathbf{z})$ for some $\lambda \in \mathbb{C}^*$.*

A projective folding group for a quasi-homogeneous singularity f is group G together with a representation of G in $GL(n,\mathbb{C})$ which acts by projective symmetries with the same fixed λ and preserves the unique (up to scalar multiples) element of highest degree.

These type of symmetries act on the Milnor ring, since the local ring $f(\mathbf{z}) = 0$ is equal to that of $\lambda f(\mathbf{z}) = 0$.

REMARK 5.1. *For a sum of two singularities $f+g$, the product of two projective symmetry groups for f and g, respectively, also acts on the Milnor ring $M_{f+g} = M_f \otimes M_g$.*

THEOREM 5.2. *For each of the classical foldings for Coxeter groups there is a group of projective symmetries or a product of two groups of projective symmetries which has as its invariants the Frobenius algebra of the folded graph. The foldings and groups are contained in table 5.3.*

REMARK 5.2. *The utilization of projective symmetries is necessary, since not all foldings can be realized with $\lambda = 1$; in particular, the element of highest degree transforms in the representation $\det(\rho(g))^{-2}$ (see e.g., [**Ka03**]) so that the only folding groups with $\lambda = 1$ will be those whose determinants lie in ± 1. In particular, the $\mathbb{Z}/2\mathbb{Z}$-foldings of A_{2n-1} and D_{n+1}, yielding B_n, as discussed above, and also E_6 to F_4, can be realized by orbifolding. For G_2 the folding can only be obtained via orbifolding by restricting to the classical level, i.e., disregarding the twisted sectors.*

REMARK 5.3. *The folding of E_6 and E_8 can also be understood as the tensor products of the folding on the factors: $A_2 \otimes I_2(4) = F_4$ and $A_2 \otimes I_2(5)$.*

REMARK 5.4. *Unlike in the case of the operation of symmetries, the group action of projective symmetries does not act on the fibration (5.2) fiberwise and hence not obviously on the vanishing cohomology bundle. But on the other hand, it leaves the central fiber invariant and furthermore acts by homothety on the base via $f(z) = t \mapsto f(z) = \frac{1}{\lambda}t$, so we obtain an equivariant action.*

REMARK 5.5. *The relation of folding to the miniversal unfolding space is known and is given in* [**St01, St02**]. *In fact the foldings provide submanifolds of Frobenius manifolds or so-called $\mathcal{F}$-manifolds.*

REMARK 5.6. *It would be desirable to extend the theory of G-Frobenius algebras to these quotients. In fact it seems to be straightforward to generalize some of the construction of* [**Ka03**] *for Jacobian Frobenius algebras with symmetries to those with projective symmetries. Here the twisted sectors would again just be obtained from the function by restriction to the fixed subspace. There is also no obstruction to keeping the grading shifts and dualization processes. One would expect to be able to apply this type of orbifolding to the calculations and definition of* [**Z98**]. *We leave the more careful analysis of this possibility for the future.*

References

[A78] V. I. Arnol'd, *Critical points of functions on a manifold with boundary, the simple Lie groups B_k, C_k, F_4 and singularities of evolutes,* Uspekhi Mat. Nauk **33** (1978), no. 5(203), 91–105.

[AGLV93] V. I. Arnol'd et al., *Singularity theory. I.*, translated from the 1988 Russian original by A. Iacob, Springer, Berlin, 1993

[AGV85] V. I. Arnol'd, S. M. Guseĭn-Zade and A. N. Varchenko, *Singularities of differentiable maps,* Vol. I, Birkhäuser Boston, Inc., Boston, MA, 1985.

[AGV88] V. I. Arnol'd, S. M. Guseĭn-Zade and A. N. Varchenko, *Singularities of differentiable maps,* Vol II, Birkhäuser Boston, Inc., Boston, MA, 1988.

[CR00] W. Chen and Y. Ruan, *A new cohomology theory for orbifold*, preprint 2000, Commun. Math. Phys. to appear.

[CR02] W. Chen and Y. Ruan, *Orbifold Gromov-Witten theory,* Orbifolds in Mathematics and Physics (Madison, WI 2001) *Contemp. Math. 310*, Amer. Math. Soc., Providence, RI., 2002, pp. 25–85.

[Du96] B. Dubrovin, *Geometry of 2D topological field theories*, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math., 1620, Springer, Berlin, 1996, pp. 120–348.

[FJR] H. Fan, T. Jarvis and Y. Ruan, *A Generalization of Spin Orbifold Quantum Cohomology Arising From Quasi-Homogeneous Polynomials*, in preparation.

[GP90] B. R. Greene, M. R. Plesser, *Duality in Calabi-Yau moduli space*, Nuclear Phys. B **338** (1990), 15–37.

[IV90] K. Intriligator and C. Vafa, Landau-Ginzburg orbifolds, *Nuclear Phys. B* **339** (1990), 95–120

[J] T. Jarvis. Private communication. 2003.

[JKK03] T. Jarvis, R. Kaufmann and T. Kimura, *Pointed Admissible G-Covers and G-equivariant Cohomological Field Theories*, preprint math.AG/0302316.

[JKV00] T. Jarvis, T. Kimura and A. Vaintrob, *Tensor products of Frobenius manifolds and moduli spaces of higher spin curves,* Conférence Moshé Flato 1999, Vol. II (Dijon 1999) Math. Phys. Stud., 22, Kluwer Acad. Publ., Dordrecht, 2000, pp. 145–166.

[JKV01] T. Jarvis, T. Kimura and A. Vaintrob, *Moduli spaces of higher spin curves and integrable hierarchies*, Compositio Math. **126** (2001), no. 2, 157–212.

[Ka] R. M. Kaufmann, *Unfolding orbifolded singularities*, in preparation.

[Ka99] R. M. Kaufmann, *The tensor Product in the Theory of Frobenius manifolds*, Int. J. of Math. **10** (1999) 159–206.

[Ka02] R. M. Kaufmann, *Orbifold Frobenius algebras, cobordisms, and monodromies*, Orbifolds in Mathematics and Physics (Madison, WI. 2001), Contemp. Math. 310, Amer. Math. Soc., Providence, RI., 2002, pp. 135–162.

[Ka02a] R. M. Kaufmann, *Discrete torsion, symmetric products and the Hilbert scheme.*, to appear in the conference proceedings of the conference in honor of Yuri Ivanovich Manin's 65th birthday, preprint 2002.

[Ka02b] R. M. Kaufmann, *The algebra of discrete torsion*, preprint, math.AG/0208081.

[Ka03] R. M. Kaufmann, *Orbifolding Frobenius algebras*, Int. J. of Math. **14** (2003), 573-619.

[Ka04] R. M. Kaufmann, *Second quantized Frobenius algebras.* Commun. Math. Phys. (online) DOI: 10.1007/s00220-004-1090-y, to appear in print.

[M99] Yu. Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces*, Colloquium Publ., 47, Amer. Math. Soc., Providence, RI, 1999.

[Mo93] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, American Mathematical Society, Providence, RI, 1993.

[P02] A. Polishchuk, *Witten's top Chern class on the moduli space of higher spin curves*, preprint math.AG/0208112.

[PV01] A. Polishchuk and A. Vaintrob, *Algebraic construction of Witten's top Chern class.* Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), Contemp. Math., 276, Amer. Math. Soc., Providence, RI, 2001, pp. 229–249.

[S82] K. Saito, *Primitive forms for a universal unfolding of a function with an isolated critical point,* Journ. Fac. Sci. Univ. Tokyo, Sec. IA. **28** (1982), 775–792.

[S83] K. Saito, *Period mapping associated to a primitive form,* Publ. Res. Inst. Math. Sci. Kyoto Univ., **19** (1983), 1131–1264.

[St01] I.A.B. Strachan, *Frobenius submanifolds.* J. Geom. Phys. **38** (2001), no. 3–4, 285–307.

[St02] I.A.B. Strachan, *Frobenius manifolds and bi-Hamiltonian structures on discriminant hypersurfaces,* Integrable systems, topology, and physics (Tokyo, 2000), Contemp. Math., 309, Amer. Math. Soc., Providence, RI, 2002, pp. 251–265.

[V89] C. Vafa, String vacua and orbifoldized LG models, *Modern Phys. Lett. A*4 (1989), 1169–1185.

[Wa80] C. T. C. Wall, A note on symmetry of singularities, *Bull. London Math. Soc.*12 (1980), no. 3, 169–175.

[W91] E. Witten, *Algebraic geometry associated with matrix models of two- dimensional gravity,* Topological models in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX (1993), pp. 235–269.

[W92] E. Witten, *The N-matrix model and gauged WZW models.* Nucl. Phys. B **371** (1992), 191–245.

[Z98] J.-B. Zuber, *Generalized Dynkin diagrams and root systems and their folding*, Topological field theory, primitive forms and related topics (Kyoto, 1996), Progr. Math., 160, Birkhäuser Boston, Boston, MA, 1998, pp. 453–493.

E-mail address: `kaufmann@math.okstate.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, 196 AUDITORIUM RD., STORRS, CT 06268