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Orbifold Frobenius Algebras, Cobordisms and Monodromies

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ABSTRACT. We introduce the new algebraic structure replacing Frobenius algebras when one is considering functors from objects with finite group actions to Frobenius algebras, such as cohomology, the local ring of an isolated singularity, etc.. Our new structure called a G–twisted Frobenius algebra reflects the "stringy" geometry of orbifold theories or equivalently theories with finite gauge group. We introduce this structure algebraically and show how it is obtained from geometry and field theory by proving that it parameterizes functors from the suitably rigidified version of the cobordism category of one dimensional closed manifolds with G bundles to linear spaces. We furthermore introduce the notion of special G–twisted Frobenius algebras which are the right setting for studying e.g. quasi–homogeneous singularities with symmetries or symmetric products. We classify the possible extensions of a given linear data to G–Frobenius algebras in this setting in terms of cohomological data.

Introduction

We consider the "stringy" geometry and algebra of global orbifold theories. They arise when one is considering the "stringy" extension of a classical functor which takes values in Frobenius algebras –such as cohomology, local ring, K–theory, etc.– in the presence of a finite group action on the objects. In physics the names of orbifold theories or equivalently theories with a global finite gauge group are associated with this type of question [DHVW, DW,DVVV,IV,V]. A common aspect is a new linear structure given by augmenting the traditional one of G–invariants by twisted sectors. Here one usually obtains a sum over representatives of conjugacy classes of the group in question moded out by the action of the centralizer of these elements. The formula for equivariant K–theory [AS] or the orbifold cohomology of [CR] are an example of this. We note that the product in the former is not "stringy" since it does not mix the twisted sectors. The use of orbifold constructions is also the cornerstone of the original mirror construction [GP]. The orbifolds under study in that context are so–called Landau–Ginzburg orbifold theories, which have so far not been studied mathematically. These correspond to the Frobenius manifolds coming from singularities and are studied in detail in the present paper under the names Special or more precisely Jacobian. Together with the theory of tensor

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product [K3] both operations used in [GP] to obtain mirror symmetry are now mathematical in the realm of Frobenius algebras.

We take a novel approach to this procedure by breaking it into two steps: 1) adding the twisted sectors, *one for each group element* together with a group action of G and *a group graded multiplication.* 2) taking the G–invariants which should again yield a Frobenius algebra.

The novelty of our approach is that we regard a *non–commutative structure before taking the group quotient.* The axiomatics of this structure called G–twisted Frobenius algebras were first presented in [K1]. The present text adds the geometric descriptions in terms of cobordisms of G–bundles and functors to linear spaces to our general theory.

This geometric description should be viewed as an extension of the result of Atiyah, Dubrovin, Dijkgraaf and Segal linking the cobordism description of topological field theory to the algebraic structure of Frobenius algebras.

In order to make the exposition more tractable we shortened the proofs to sketches, the full theory and detailed proofs can be found in [K2], which was made available to the referee.

We furthermore introduce a large class of examples —the so–called special G–Frobenius algebras– that we consider in detail. For these algebras we prove a Reconstruction Theorem which describes the possible extensions of underlying direct sum data to G–twisted Frobenius algebra in terms of cohomological data.

This is part of the general setup of (re–)construction of "stringy" geometry. The paradigm for this is the situation of a geometric object with a group action where one can obtain a collection of Frobenius algebras, one for each group element, by considering the Frobenius algebras associated to each fixed point set. This decomposition carries a classical product which is just the direct sum of the products. One example is again equivariant or orbifold K–theory or Bredon cohomology. In the "stringy" context, however, this multiplication is not the right one, since it should respect the group grading.

The construction of twisted sectors is not merely an auxiliary artifact, but is essential. This is clearly visible in the case of the Frobenius algebra associated to the singularity of type A_n together with a $\mathbf{Z}/(n+1)\mathbf{Z}$ action and the singularity of type A_{2n-3} with $\mathbf{Z}/2\mathbf{Z}$ action, which are worked out in detail in the last paragraph. The particular example of type A_n together with a $\mathbf{Z}/(n+1)\mathbf{Z}$ action exhibits a version of Mirror–Symmetry in which it is self–dual, as is expected. Here the twisted sectors shed some light on this duality, since in reality it is the sum of twisted sectors that is dual to the untwisted one.

In the case that the Frobenius algebra one starts out with comes from a semi–simple Frobenius manifold and the quotient of the twisted sector is not trivial, then there is unique extension to the level of Frobenius manifolds. This is the case in the above example of A_{2n+3} and D_n. There are other situations where the extension to the deformations (i.e. to the Frobenius manifold level) is unique; one being symmetric products (cf. §7 and [K4]).

The paper is organized as follows: After recalling the general facts about cobordisms and Frobenius algebras in §1, we give the algebraic structures in an axiomatic fashion in §2. There is a slight difference to the theory with trivial group in that there are two structures with slightly different G-action to be considered which are nevertheless present. These versions differ by a twist with a character. In the singularity version, on the non–twisted level, they correspond to the different

G–modules on the cohomology and the Milnor ring [W]. In physical terms, these two structures are related by spectral flow. To be even more precise, one structure carries the natural multiplication and the other the natural scalar product.

In §3 we provide a geometric cobordism realization of the theory. There are ramifications of this presentation with [FQ, T], if the above mentioned character is trivial. This is, however, not necessarily the case and moreover in the most interesting examples this is not the case.

In §4 we introduce a large class of examples we introduce the so–called special G–Frobenius algebras. For these we prove a reconstruction Theorem which classifies the possible "stringy" multiplications and G–actions extending the classical direct sum in terms of cohomological data.

Special G–Frobenius algebras have as a subset Jacobian G–Frobenius algebras. This allows us to apply our theory to Landau–Ginzburg models or singularity theory –in a sense the original building blocks for mirror symmetry– which we do in §5. This class encompasses those examples whose origin lies in singularity theory and examples of manifolds whose cohomology ring can be described as a quotient by a Jacobian ideal in the spirit of the Landau–Ginzburg—Calabi–Yau correspondence. Here it is important to note that everything can be done in a super (i.e. $\mathbf{Z}/2\mathbf{Z}$–graded) version. This introduces a new degree of freedom into the construction corresponding to the choice of parity for the twisted sectors.

For special G–Frobenius algebras with grading we give a mirror construction in §6. This construction is based on the idea of spectral flow, which in our situation boils down to using the exponential of the grading operator to shift the group grading. In §7 we explicitly work out several examples including the transition from the singularity A_{2n+3} to D_n via a quotient by $\mathbf{Z}/2\mathbf{Z}$, the self duality of A_n and the case of $*/G$.

Lastly in §8 we comment on recent developments including symmetric group actions on powers of a Frobenius algebra which are related to the Hilbert scheme of smooth projective varieties [LS].

1. Frobenius Algebras and Cobordisms

In this paragraph, we recall the definition of a Frobenius algebra and its relation to the cobordism–category definition of a topological field theory [A,Du,Dij].

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1.1. Frobenius algebras. We begin by recollecting well known facts on Frobenius algebras and their cobordism realization in order to stress the parallel exposition of twisted G Frobenius algebras in the next section.

1.1.1. Definition. A *Frobenius algebra* (FA) over a field k of characteristic 0 is $\langle A, \circ, \eta, 1\rangle$, where

A is a finite dim k–vector space,
$\circ$ is a multiplication on A: $\circ : A \otimes A \to A$,
η is a non-degenerate bilinear form on A, and
1 is a fixed element in A – the unit.
satisfying the following axioms:

a) *Associativity*
$(a \circ b) \circ c = a \circ (b \circ c)$

b) *Commutativity*
$a \circ b = b \circ a$

c) *Unit*:
$\forall \in a : 1 \circ a = a \circ 1 = a$

d) *Invariance*:
$\eta(a, b \circ c) = \eta(a \circ b, c)$

1.1.2. Remark. By using η to identify A and A^* – the dual vector space of A– these objects define a one–form $\epsilon \in A^*$ called the co–unit and a three–tensor $\mu \in A^* \otimes A^* \otimes A^*$.

Using dualization and invariance these data are interchangeable with η and $\circ$ via the following formulas. Explicitly, after fixing a basis $(\Delta_i)_{i\in I}$ of A, setting $\eta_{ij} := \eta(\Delta_i, \Delta_j)$ and denoting the inverse metric by η^{ij}

$\epsilon(a) := \eta(a, 1)$,
$\mu(a, b, c) := \eta(a \circ b, c) = \epsilon(a \circ b \circ c)$,
$a \circ b = \sum_{ij} \mu(a, b, \Delta_i)\eta^{ij}\Delta_j$ and
$\eta(a, b) = \epsilon(a \circ b)$.

1.1.3. Notation. We call $\rho \in A$ the element dual to ϵ. This is the element which is Poincaré dual to 1.

1.1.4. Grading. A graded Frobenius algebra is a Frobenius algebra together with a group grading of the vector space A: $A = \bigoplus_{i\in I} A_i$ where I is a group together with the following compatibility equations: denote the I–degree of an element by deg.

1) 1 is homogeneous; $1 \in A_d$ for some $d \in I$
2) η is homogeneous of degree $d + D$.
   I.e. for homogeneous elements a, b $\eta(a, b) = 0$ unless $\deg(a)+\deg(b) = d+D$.
   This means that ϵ and ρ are of degree D.
3) $\circ$ is of degree d, this means that μ is of degree $2d + D$, where again this means that
   $\deg(a \circ b) = \deg(a) + \deg(b) - d$

1.1.5. Definition. An even derivation $E \in Der(A, A)$ of a Frobenius algebra A is called an *Euler field*, if it is conformal and is natural w.r.t. the multiplication, i.e. for some $d, D \in k$ it satisfies:

$$\eta(Ea, b) + \eta(a, Eb) = D\eta(a, b) \tag{1.1}$$

and

$$E(ab) = Ea\, b + a\, Eb - d\, ab \tag{1.2}$$

Such derivation will define a grading on A by its set of eigenvalues.

1.1.6. Super-grading. For an element a of a super vector space $A = A_0 \oplus A_1$ denote by $\tilde{a}$ its $\mathbf{Z}/2\mathbf{Z}$ degree, i.e. $\tilde{a} = 0$ if it is even ($a \in A_0$) and $\tilde{a} = 1$ if it is odd ($a \in A_1$). This changes the axiom for commutativity into one on super–commutativity.

b)$^\sigma$ *Super–commutativity*

$$a \circ b = (-1)^{\tilde{a}\tilde{b}} b \circ a$$

The grading for super Frobenius algebras carries over verbatim.

1.2. Cobordisms.

1.2.1. Definition. Let $\mathcal{COB}$ be the category whose objects are one–dimensional closed oriented (topological) manifolds considered up to orientation preserving homeomorphism and whose morphisms are cobordisms of these objects.

I.e. $\Sigma \in Hom(S_1, S_2)$ if Σ is an oriented surface with boundary $\partial\Sigma \equiv -S_1 \coprod S_2$. The composition of morphisms is given by glueing along boundaries with respect to orientation reversing homeomorphisms.

1.2.2. Remark. The operation of disjoint union makes this category into a monoidal category with unit $\emptyset$.

1.2.3. Remark. The objects can be chosen to be represented by disjoint unions of the circle with the natural orientation S^1 and the circle with opposite orientation $\bar{S}^1$. Thus a typical object looks like $S = \coprod_{i \in I} S^1 \coprod_{j \in J} \bar{S}^1$. Two standard morphisms are given by the cylinder, and thrice punctured sphere.

1.2.4. Definition. Let $\mathcal{VECT}_k$ be the monoidal category of finite dimensional k–vector spaces with linear morphisms with the tensor product providing a monoidal structure with unit k.

1.2.5. Theorem. (Atiyah, Dijkgraaf, Dubrovin) [A,Dij, Du] There is a 1–1 correspondence between Frobenius algebras over k and isomorphism classes of covariant functors of monoidal categories from $\mathcal{COB}$ to $\mathcal{VECT}_k$, natural with respect to orientation preserving homeomorphisms of cobordisms and whose value on cylinders $S \times I \in Hom(S, S)$ is the identity.

Under this identification, the Frobenius algebra A is the image of S^1, the multiplication or rather μ is the image of a thrice punctured sphere and the metric is the image of an annulus.

2. Orbifold Frobenius algebras

2.1. G–Frobenius algebras. We fix a finite group G and denote its unit element by e.

2.1.1. Definition. A *G–Frobenius algebra* (FA) over a field k of characteristic 0 is $< G, A, \circ, 1, \eta, \varphi, \chi >$, where

G finite group
A finite dim G-graded k–vector space
$A = \oplus_{g\in G} A_g$
A_e is called the untwisted sector and
the A_g for $g \neq e$ are called the twisted sectors.
$\circ$ a multiplication on A which respects the grading:
$\circ : A_g \otimes A_h \to A_{gh}$
1 a fixed element in A_e–the unit
η non-degenerate bilinear form
which respects grading i.e. $g|_{A_g\otimes A_h} = 0$ unless $gh = e$.
φ an action of G on A (which will be by algebra automorphisms),
$\varphi \in \mathrm{Hom}(G, \mathrm{Aut}(A))$, s.t. $\varphi_g(A_h) \subset A_{ghg^{-1}}$
χ a character $\chi \in \mathrm{Hom}(G, k^*)$

Satisfying the following axioms:
NOTATION: We use a subscript on an element of A to signify that it has homogeneous group degree –e.g. a_g means $a_g \in A_g$– and we write $\varphi_g := \varphi(g)$ and $\chi_g := \chi(g)$.

a) *Associativity*
$(a_g \circ a_h) \circ a_k = a_g \circ (a_h \circ a_k)$
b) *Twisted commutativity*
$a_g \circ a_h = \varphi_g(a_h) \circ a_g$
c) *G Invariant Unit*:
$1 \circ a_g = a_g \circ 1 = a_g$
and
$\varphi_g(1) = 1$
d) *Invariance of the metric*:
$\eta(a_g, a_h \circ a_k) = \eta(a_g \circ a_h, a_k)$
i) *Projective self–invariance of the twisted sectors*
$\varphi_g|A_g = \chi_g^{-1} id$
ii) *G–Invariance of the multiplication*
$\varphi_k(a_g \circ a_h) = \varphi_k(a_g) \circ \varphi_k(a_h)$
iii) *Projective G–invariance of the metric*
$\varphi_g^*(\eta) = \chi_g^{-2}\eta$
iv) *Projective trace axiom*
$\forall c \in A_{[g,h]}$ and l_c left multiplication by c:
$\chi_h \mathrm{Tr}(l_c \varphi_h|_{A_g}) = \chi_{g^{-1}} \mathrm{Tr}(\varphi_{g^{-1}} l_c|_{A_h})$

An alternate choice of data is given by a one–form ϵ, the co–unit with $\epsilon \in A_e^*$ and a three–tensor $\mu \in A^* \otimes A^* \otimes A^*$ which is of group degree e, i.e. $\mu|_{A_g\otimes A_h\otimes A_k} = 0$ unless $ghk = e$.

The relations between $\eta, \circ$ and ϵ, μ are analogous to those of 1.1.2.

Again, we denote by $\rho \in A_e$ the element dual to $\epsilon \in A_e^*$ and Poincaré dual to $1 \in A_e$.

2.1.2. Remarks.

1) A_e is central by twisted commutativity and $\langle A_e, \circ, \eta|_{A_e\otimes A_e}, 1\rangle$ is a Frobenius algebra.
2) All A_g are A_e-modules.

3) Notice that χ satisfies the following equation which completely determines it in terms of φ. Setting $h = e, c = 1$ in axiom iv)

$$\dim A_g = \chi_{g^{-1}}\mathrm{Tr}(\varphi_g|_{A_e}) \tag{2.1}$$

by axiom iii) the action of φ on ρ determines χ up to a sign

$$\chi_g^{-2} = \chi_g^{-2}\eta(\rho, 1) = \eta(\varphi_g(\rho), \varphi_g(1)) = \eta(\varphi_g(\rho), 1) \tag{2.2}$$

4) Axiom iv) forces the χ to be group homomorphisms, so it would be enough to assume in the data that they are just maps.

2.1.3. Proposition. The G invariants A^G of a G–Frobenius algebra A form an associative and commutative algebra with unit. This algebra with the induced bilinear form is a Frobenius algebra if and only if $\sum_g \chi_g^{-2} = |G|$, which means that the Schur–Frobenius indicator is one. Embedding $k$in $\mathbf{C}$ yields $\chi \in \mathrm{Hom}(G, U(1))$ this implies $\forall g : \chi_g = \pm 1$.

2.1.4. Definition. A G–Frobenius algebra is called an *orbifold model* if the data $\langle A^G, \circ, 1\rangle$ can be augmented by a compatible metric to yield a Frobenius algebra. In this case, we call the Frobenius algebra A^G a *G–orbifold Frobenius algebra.*

2.2. Super-grading. We can enlarge the framework by considering super–algebras rather than algebras. This will introduce the standard signs.

The action of G as well as the untwisted sector should be even. The axioms that change are

b$^\sigma$) *Twisted super–commutativity*
$a_g \circ a_h = (-1)^{\tilde{a}_g\tilde{a}_h}\varphi_g(a_h) \circ a_g$

iv$^\sigma$) *Projective super–trace axiom*
$\forall c \in A_{[g,h]}$ and l_c left multiplication by c:
$\chi_h \mathrm{STr}(l_c\varphi_h|_{A_g}) = \chi_{g^{-1}}\mathrm{STr}(\varphi_{g^{-1}}l_c|_{A_h})$

where STr is the super–trace.

2.3. Geometric model – spectral flow. The axioms of the G–Frobenius algebra are well suited for taking the quotient, since the multiplication is G–invariant. However, this is not the right framework for a geometric interpretation. In order to accommodate a more natural coboundary description, we need the following definition which corresponds to the physical notion of Ramond ground states:

2.3.1. Definition. The *Ramond space* of a G–Frobenius algebra A is the G–graded vector–space

$$V := \oplus_g V_g := \bigoplus_g A_g \otimes k$$

together with the G-action $\bar{\varphi} := \varphi \otimes \chi$, the induced metric $\bar{g}$ and the induced multiplication $\bar{\circ}$.

This space is essentially the correct free cyclic module generated by A.

All structures of A can be naturally transferred but some of the axioms get twisted. They read:

b') *Projective twisted commutativity*
$v_g \bar{\circ} v_h = \chi_g \bar{\varphi}_g(v_h)\bar{\circ} v_g = \bar{\varphi}_g(v_h \bar{\circ} v_g)$

c') *Projectively invariant unit:*
$v \bar{\circ} v_g = v_g \bar{\circ} v = v_g$
and
$\bar{\varphi}_g(v) = \chi_g v$

1') *Self-invariance of the twisted sectors*
$\bar{\varphi}_g | V_g = id$

2') *Projective G–invariance of multiplication*
$\bar{\varphi}_k(v_g \bar{\circ} v_h) = \chi_k \bar{\varphi}_k(v_g) \bar{\circ} \bar{\varphi}_k(v_h)$

3') *G–Invariance of metric*
$\bar{\varphi}_g^*(\bar{\eta}) = \bar{\eta}$

4') *Trace axiom*
$\forall c \in V_{[g,h]}$ and l_c left multiplication by c:
$\mathrm{Tr}(l_c \circ \bar{\varphi}_g|_{V_h}) = \mathrm{Tr}(\bar{\varphi}_{h^{-1}} \circ l_c|_{V_g})$

2.3.2. Remark. In the theory of singularities, the untwisted sector of the Ramond space corresponds to the forms $H^{n-1}(V_\epsilon, \mathbf{C})$ while the untwisted sector of the G–twisted Frobenius algebra corresponds to the Milnor ring [Wa]. These are naturally isomorphic, but have different G–module structures. In that situation, one takes the invariants of the Ramond sector, while we will be interested in invariants of the G–Frobenius algebra and not only the untwisted sector (cf. [K5] and see also §7).

3. Bundle cobordisms, finite gauge groups, orbifolding and G–Ramond algebras

In this section, we introduce two cobordism categories which correspond to G–orbifold Frobenius algebras and Ramond G–algebras, respectively. Again G is a fixed finite group.

3.1. Bundle cobordisms. In all situations, gluing along boundaries will induce the composition and the disjoint union will provide a monoidal structure.

3.1.1. Definition. Let $\mathcal{GBCOB}$ be the category whose objects are principal G–bundles over one–dimensional closed oriented (topological) manifolds, pointed over each component of the base space, whose morphisms are cobordisms of these objects (i.e. principal G–bundles over oriented surfaces with pointed boundary).

More precisely, $B_\Sigma \in Hom(B_1, B_2)$ if Σ is an oriented surface with boundary $\partial\Sigma = -S_1 \coprod S_2$ and B_Σ is a bundle on Σ which restricts to B_1 and B_2 on the boundary.

The composition of morphisms is given by gluing along boundaries with respect to orientation reversing homeomorphisms on the base and covering bundle isomorphisms which align the base–points.

3.1.2. Remark. The operation of disjoint union makes this category into monoidal category with unit $\emptyset$ formally regarded as a principal G bundle over $\emptyset$.

3.1.3. Remark. Typical objects are bundles B over
$S = \coprod_{i \in I} S^1 \coprod_{j \in J} \bar{S}^1$.
Let (S^1, ν) $\nu \in S^1)$ be a pointed S^1.

3.1.4. Structure Lemma. The space $Bun(S^1, G)$ of G bundles on (S^1, ν) can be described as follows:

$$Bun(S^1, G) = (G \times F)/G$$

where F is a generic fibre regarded as a principal G-space and G acts on itself by conjugation and the quotient is taken by the diagonal action.

3.1.5. Remark. Usually one uses the identification

$$Bun(M, G) = \mathrm{Hom}(\pi_1(M), G)/G,$$

which we also use in the proof. However, for certain aspects of the theory – more precisely to glues and to include non–trivial characters– it is vital to include a point in the bundle and a trivialization rather then just a point in the base.

Proof of the Structure Lemma. Given a pointed principal G bundle (B, S^1, π, F, G), $b \in B$ we set $\nu = \pi(b) \in S^1$. The choice of $b \in F = B_\nu$ gives an identification of F with G, i.e. we let $\beta : G \mapsto F$ be the admissible map in the sense of [St] that satisfies $\beta(e) = b$. We set $g \in G$ to be the element corresponding to the monodromy around the generator of $\pi_1(S^1)$. Notice that since we fixed an admissible map everything is rigid – there are no automorphisms– and the monodromy is given by an element, not a conjugacy class. Thus we associate to (B, b) the tuple (g, b).

Vice versa, given (g, b) we start with the pointed space (S^1, ν) and construct the bundle with fiber G, monodromy g and marked point $b = \beta(e) \in B_\nu$.

The bijectiveness of this construction follows by the classical results quoted above [St].

The choice in this construction corresponds to a choice of a point $b \in B$. Changing b amounts to changing β and the monodromy g. Moreover, moving the point $\nu = \pi(b)$ and moving b inside the fiber by parallel transport keeps everything fixed. Moving b inside the fixed fiber by translation (once ν is fixed) corresponds to translation by the group action in the fiber i.e. the translation action of G on F and simultaneous conjugation of the monodromy.

This observation leads us to the following definition:

3.1.6. Definition. We call a bundle over a closed one–dimensional manifold rigidified if its components of the base space are labeled and the bundle is pointed over each component of the base space. a trivialization around the projection of this point are fixed in each component of the base. We denote such a bundle (B, b) where $b = (b_0, \dots, b_n; \phi_0, \dots, \phi_n)$ is the set of marked points b_i over each component of the base together with the trivializations ϕ_i. Furthermore, if the surface has genus zero we realize it in the plane as a pointed disc with all boundaries being S^1. We label the outside circle by 0. If π is the bundle projection, we set $x_i := \pi(b_i)$, we call b_0 the base–point, the B_{x_i} the special fibers and B_{x_0} the initial fiber.

Furthermore, if a given surface has genus zero we realize it in the plane as a pointed disc with all boundaries being S^1. We label the outside circle by 0.

If π is the bundle projection, we set $x_i := \pi(b_i)$, we call b_0 the base–point, the B_{x_i} the special fibers and B_{x_0} the initial fiber.

We call two rigidified bundles equivalent if there exists a bundle map between them that preserves the monodromy of the marked point and preserves the trivializations as well as the orientation of the base.

3.1.7. Definition. Let $\mathcal{GBCOB}^*$ be the category whose objects are equivalence classes of rigidified principal G–bundles over one–dimensional closed oriented (topological) manifolds considered up to pull–back under orientation preserving homeomorphism of the base respecting the markings and whose morphisms are cobordisms of these objects (i.e. principal G–bundles over oriented surfaces with boundary together with rigidification on the boundary, a choice of base–point $x_0 \in \partial\Sigma$ compatible with the rigidification and a choice of curves Γ_i from x_0 to x_i which identify the trivializations via parallel transport, where we used the notation above.).

I.e. objects are bundles B over $S = \coprod_{i\in I} S^1 \coprod_{j\in J} \bar{S}^1$ with base–points and trivializations over the various components of the base up to equivalence. An element $B_\Sigma \in Hom((B_1, \bar{b}_1), (B_2, \bar{b}_2))$ is an equivalence class of a G bundle over a bounded surface Σ with boundary $\partial\Sigma = \bar{S}_1 \sqcup S_2$ whose boundary $\partial B_\Sigma = \bar{B}_1 \sqcup B_2$ together with rigidification data $\bar{b}_1$and $\bar{b}_2$ over the boundaries and rigidification over the interior given by a choice of paths Γ_i from x_0 to x_i (the images of the marked points in the components of the boundary). Such bundles are regarded as equivalent if there is an orientation preserving homoeomorphism between the underlying surfaces that induces an orientation preserving homoemorphims on the data such that the equivalence class of the boundary data is preserved. We will call S_1 the inputs and S_2 the outputs.

3.1.8. Construction. We define the pointed bundle (g, h) to be the bundle which is obtained by glueing $I \times G$ via the identification $(0, e) \sim (1, h^{-1}gh)$ and marking the point $(0, h)$, where $I = [0, 1]$ the standard interval.

This produces all monoidal generators.

3.1.9. Lemma. The elements of Construction 3.1.8 are up to reversing the orientation a the monoidal basis of objects of $\mathcal{GBCOB}*$. Therefore the monoidal basis can be identified by $G \times G$ up to involution.

Proof. The only thing to show is that each class of rigidified bundles has exactly one element of the above kind.

Given a rigidified bundle over S^1 the trivialization yields a map to $I \times G/\sim$ with $(0, e) \sim (1, k)$ where k is the monodromy given by the trivialization. Under this identification let $(0, h)$ be the image of the point b_0. Let g be the monodromy of this point. Then $k = h^{-1}gh$ and the bundle is in the same class as (g, h). It is clear that there is at most one such element in each class since both the monodromy and the trivialization have to be preserved.

A generating object can thus be depicted by an oriented circle with labels (g, h) where $(g, h) \in G \times G$. We use the notation (g, h) for positively oriented circles and $\overline{(g, h)}$ for negatively oriented circles. We will consider functors V with an involutive property we have that we will have $V(\overline{(g, h)}) \simeq V((g^{-1}, h))^*$ where $*$ denotes the dual. General objects are then just disjoint unions of these, i.e. tuples (g_i, h_i). Homomorphisms on the generators are given by the trivial bundle cylinder with different trivializations on both ends represents the natural diagonal action of G by conjugation and translation described in 3.1.4, so that this diagonal G–action is realized in terms of cobordisms.

The natural $G \times G$ action, however, cannot be realized by these cobordisms and we would like to enrich our situation to this case by adding morphisms corresponding to the G action on the trivializations which is the action of the global gauge group.

3.1.10. Definition. Let $\mathcal{RGBCOB}$ be the category obtained from $\mathcal{GBCOB}*$ by adding the following morphisms. For any n–tuple $(k_1, \dots, k_n) : k_i \in G$ and any object $(g_i, h_i) : i = 1, \dots n$ we set

$$\tau(k_1, \dots, k_n)(g_i, h_i)_{i \in \{1, \dots n\}} := (g_i, k_i h_i)_{i \in \{1, \dots n\}}$$

We call these morphisms of type II and the morphisms coming from $\mathcal{GBCOB}*$ of type I. We also sometimes write II_k for τ_k.

3.1.11. Remark. There is a natural forgetful functor from $\mathcal{RGBCOB}$ to $\mathcal{GBCOB}$.

Given a character $\chi \in Hom(G, k^*)$ we can form the fibre–product $G \times_\chi k$. This gives a functor $k[G]_\chi$ from the category $\mathcal{VECT}_k$ to $k[G] - \mathcal{MOD}$, the category of $k[G]$–modules.

3.1.12. Definition. A G_χ–orbifold theory is a monoidal functor V from $\mathcal{RGBCOB}$ to $k[G] - \mathcal{MOD}$ satisfying the following conditions:

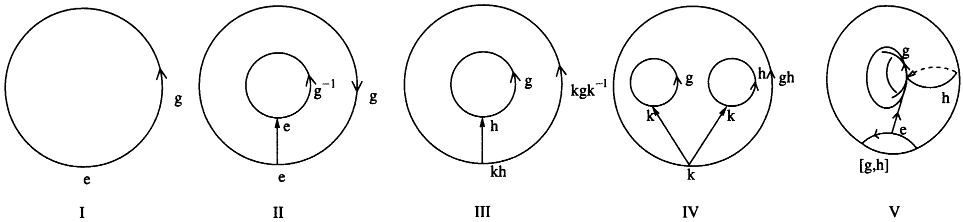
i) The image of V lies in the image of $k[G]_\chi$.
ii) Objects of $\mathcal{RGBCOB}$ which differ by morphisms of type II are mapped to the same object in $k[G] - \mathcal{MOD}$.
iii) The value on morphisms of type I does not depend on the choice of connecting curves and associated choice of trivializations or base–point.
iv) The morphisms of type II are mapped to the G–action by χ.
v) The functor is natural with respect to morphisms of the type $\tau(k, \dots, k)$. I.e. $V(\tau(k, \dots, k) \circ \Sigma) = V(\tau_{out}(k^{-1}, \dots, k^{-1})) \circ V(\Sigma) \circ V(\tau_{in}(k, \dots, k))$; where τ_{in} and τ_{out} operator on the inputs and outputs respectively.
vi) V associates id to cylinders $B \times I, (b, 0) \in B \times 0, (b, 1) \in B \times 1$ considered as cobordisms from (B, b) to itself.
vii) V is involutive: $V(\bar{S}) = V^*(S)$ where $*$ denotes the dual vector space with induced k[G]–module structure. In accordance, the morphism of type II commute with involution, i.e. they are mapped to the G–action by χ^{-1}.
viii) The functor is natural with respect to orientation preserving homeomorphisms of the underlying surface of a cobordism and pull–back of the bundle.

3.1.13. Corollary. G_χ–orbifold theories are homotopy invariant.

3.1.14. Remark. Given a choice of connecting curves, we can identify all fibres over special points. Therefore after fixing one identification of a fibre with G, we can identify the other marked points as translations of points of parallel transport and label them by group elements, which we will do.

The action of $\tau(k, \dots, k)$ corresponds to a change of identification for one point and simultaneous re–gauging of all other points via this translation, i.e. a diagonal gauging. Therefore given a cobordism, we can fix an identification of all fibres with G.

3.1.15. Definition and Notation. We will fix some standard bundle cobordisms pictured below.



I: The *standard disc bundle* is the disc with a trivial bundle and positively oriented boundary considered as a cobordism between $\emptyset$ and (e,e); it will be denoted by D.

II: The *standard g–cylinder bundle* is the cylinder $S^1 \times I$ with the bundle having monodromy g around $((S^1,0))$ considered as a cobordism between $(g,e) \coprod \overline{(g,e)}$ and $\emptyset$; it will be denoted by C_g.

III: The *$(g,h)^k$–cylinder bundle* is the cylinder with a bundle having monodromy g around $(S^1,0)$ considered as a cobordism between (g,h) and (kgk^{-1},kh); it will be denoted by $C^h_{g,k}$.

IV: The *standard $(g,h)^k$–trinion bundle* is the trinion with the bundle having monodromies g around the first S^1 and h around the second S^1 and translations e for τ_{01}, τ_{02} considered as a cobordism between $(g,k),(h,k)$ and (gh,k); it will be denoted by $T^k_{g,h}$.

V: The *(g,h)–torus bundle* is the once–punctured torus with the principal G–bundle having monodromies g and h around the two standard cycles considered as a cobordism between $([g,h],e)$ and $\emptyset$; it will be denoted by $E^k_{g,h}$.

3.1.16. Proposition. To fix a G_χ–orbifold theory on the objects of the type (g,e) and to fix a G_χ–orbifold theory on the morphisms it suffices to fix it on bundles over the standard cylinder bundle C, the (g,h)–cylinder bundles $C_{g,h}$, and on the standard trinion T.

Idea of the proof. Any object is can be obtained from the monoidal generators and the morphisms of type II. Any morphism of type I is given by a surface which can be decomposed into discs, cylinders and trinions.

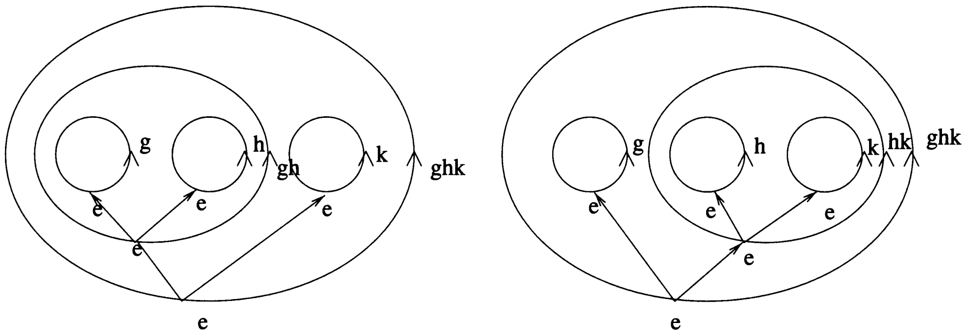
3.1.17. Proposition and Definition. For any G_χ–orbifold theory V set

$$\begin{array}{lcl} V((g,e)) & = & V_g \\ V(T^e_{g,h}) & = & \bar{\circ} : V_g \otimes V_h \to V_{gh} \\ V(D)(1_k) & = & v \\ V(C_g) & = & \bar{\eta}|_{V_g \otimes V_{g^{-1}}} \\ V(C_{g,h}) & = & \bar{\varphi}_h : V_g \mapsto V_{hgh^{-1}} \end{array}$$

then this data together with g and χ form a Ramond–G algebra $\langle G, V, \bar{\circ}, v, \bar{\eta}, \bar{\varphi}, \chi\rangle$ which we call the *associated G–Ramond algebra to V.*

Proof. It is clear by 3.1.16 that given a G_χ orbifold theory it is completely fixed by its associated G–Ramond algebra. The converse is also true:

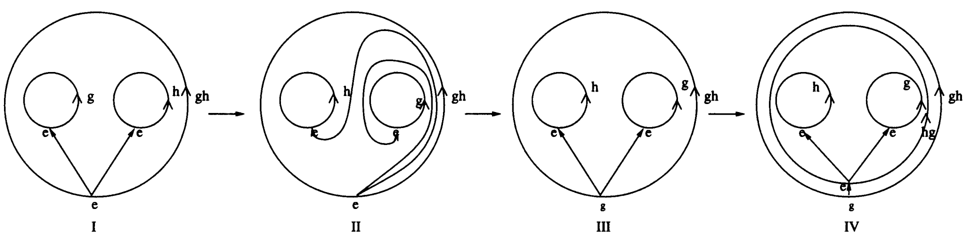
The axiom a) follows from the standard gluing procedures of TFT. I.e. the usual gluing of a disc with 3 holes from two discs with two holes in two different ways.



For axiom b') we regard the following commutative diagrams

$$\begin{array}{ccccccc}
(g,e)\coprod(h,e) & \xrightarrow{\tau_{12}} & (h,e)\coprod(g,e) & & (h,e)\coprod(g,e) & & (g,e)\coprod(h,e) \\
\downarrow T^e_{g,h} & \Rightarrow & \downarrow II & \Rightarrow & \downarrow III & \Rightarrow & \downarrow IV \\
(gh,e) & \xrightarrow{id} & (gh,e) & & (gh,g) & & (gh,g) \\
& & & \Downarrow V & & & \\
V_g\otimes V_h & \xrightarrow{\tau_{12}} & V_h\otimes V_g & \xrightarrow{id} & V_h\otimes V_g & \xrightarrow{id} & V_h\otimes V_g \\
\downarrow \bar{\circ} & & & & & & \downarrow \bar{\varphi}_g\bar{\circ} \\
V_{gh} & \xrightarrow{id} & V_{gh} & \xrightarrow{id} & V_{gh} & \xrightarrow{id} & V_{gh}
\end{array}$$

where we have used axiom viii) for the first move and axiom iii) for the second and gluing for the last one.

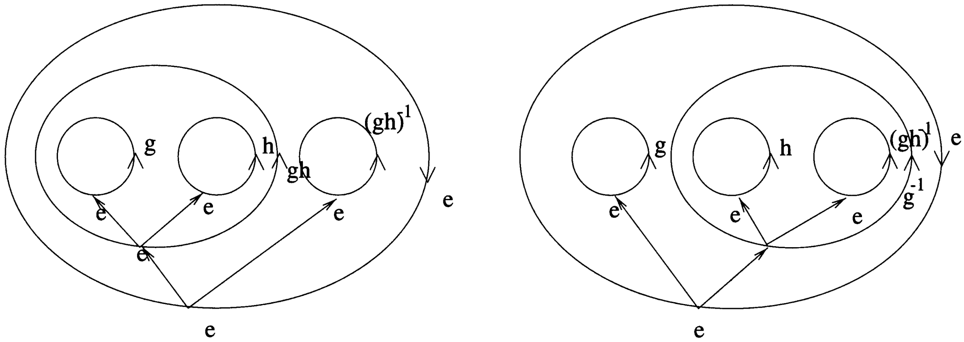


The unit of axiom c) is given by $\bar{D}^e$. The projective invariance follows from

$$\begin{array}{ccccccc}
\emptyset & & & \xrightarrow{D} & & & (e,e) \\
\downarrow \| & & & & & & \downarrow \| \\
\emptyset & \xrightarrow{D} & (e,e) & \xrightarrow{C^k_{(e,e)}} & (e,k) & \xrightarrow{II_{k-1}} & (e,e) \\
& & & \Downarrow V & & & \\
k & \xrightarrow{v} & V_e & \xrightarrow{\bar{\varphi}_k} & V_e & \xrightarrow{\bar{\chi}_{k-1}} & V_e \\
\downarrow \| & & & & & & \downarrow \| \\
k & & & \xrightarrow{1^e} & & & V_e
\end{array}$$

where in the third line $1_k \mapsto v \mapsto \chi_k v \mapsto v$.

The axiom d) of the invariance of the metric is again a standard gluing argument.



I.e. using $\bar{\epsilon}$ and associativity:

$$\bar{\eta}(v_g \bar{\circ} v_h, v_{(gh)^{-1}}) = \bar{\epsilon}((v_g \bar{\circ} v_h) \bar{\circ} v_{(gh)^{-1}}) = \bar{\epsilon}(v_g \bar{\circ}(v_h \bar{\circ} v_{(gh)^{-1}})) = \bar{\eta}(v_g, v_h \bar{\circ} v_{(gh)^{-1}})$$

For axiom 1')

We use the following diagram:

$(g,e) \stackrel{id}{\to} (g,e) \quad (g,e)$

$\downarrow id \Rightarrow \downarrow \tilde{C} \Rightarrow \downarrow C$

$(g,e) \stackrel{id}{\to} (g,e) \quad (g,g)$

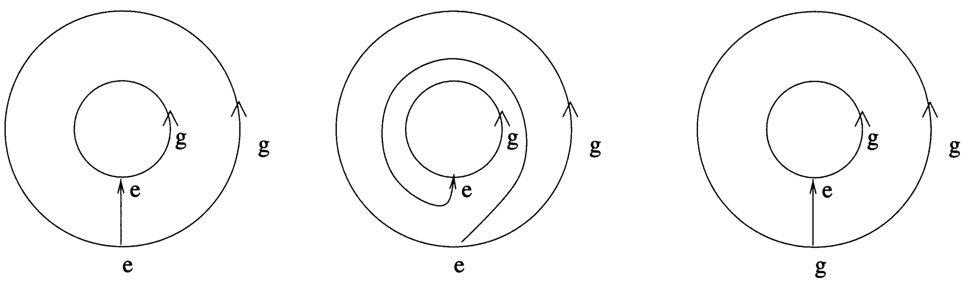
$\Downarrow V$

$V_g \stackrel{id}{\to} V_g \stackrel{id}{\to} V_g$

$\downarrow id \qquad \downarrow \bar{\varphi}_g$

$V_g \stackrel{id}{\to} V_g \stackrel{id}{\to} V_g$

where we used axiom viii) for the first move and axiom iii) for the second.



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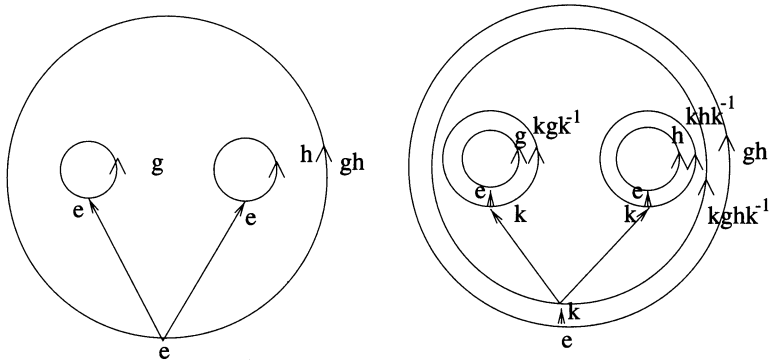
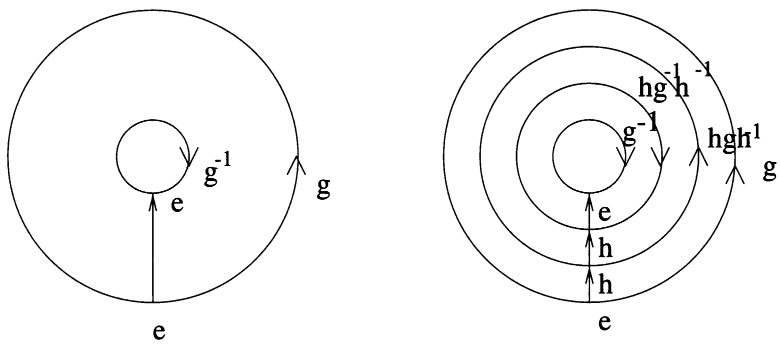
The axiom 2') follows from the diagrams

$$\begin{array}{ccccccc} (g,e)\coprod(h,e) & \stackrel{(C^k_{(g,e)},C^k_{(h,e)})}{\longrightarrow} & (kgk^{-1},k)\coprod(khk^{-1},k) & \stackrel{T^k_{kgk^{-1},khk^{-1}}}{\longrightarrow} & (kghk^{-1},k) & \stackrel{C^{k^{-1}}_{(kghk^{-1},k)}}{\longrightarrow} & (gh,e) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ V_g \otimes V_h & \stackrel{\bar{\varphi}_k \otimes \bar{\varphi}_k}{\longrightarrow} & V_{kgk^{-1}} \otimes V_{khk^{-1}} & \stackrel{\circ^k}{\longrightarrow} & V_{kghk^{-1}} & \stackrel{\bar{\varphi}_{k^{-1}}}{\longrightarrow} & V_{gh} \end{array}$$



$$
\begin{array}{ccccccc}
(g,e)\coprod(h,e) & & & \xrightarrow{T^e_{gh}} & & & (gh,e)\\
\downarrow \| & & & & & & \downarrow \|\\
(g,e)\coprod(h,e) & \xrightarrow{(II_k,II_k)} & (g,k)\coprod(h,k) & \xrightarrow{T^k_{g,h}} & (gh,k) & \xrightarrow{II_{k^{-1}}} & (gh,e)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
V_g\otimes V_h & \xrightarrow{\bar{\chi}_k\otimes\bar{\chi}_k} & V_g\otimes V_h & \xrightarrow{\circ^k} & V_{gh} & \xrightarrow{\bar{\chi}_{k-1}} & V_{gh}\\
\downarrow \| & & & & & & \downarrow \|\\
V_g\otimes V_h & & & \xrightarrow{\bar{\circ}} & & & V_{gh}
\end{array}
$$

The axiom 3') – G–invariance of the metric – follows from the following diagrams



$$
\begin{array}{ccccc}
(g,e)\coprod\overline{(g,e)} & & \xrightarrow{A^e} & & K\\
\downarrow \| & & & & \downarrow \|\\
(g,e)\coprod\overline{(g,e)} & \xrightarrow{(C,\bar{C})} & (kgk^{-1},k)\coprod\overline{(kgk^{-1},k)} & \xrightarrow{A^k} & K\\
\downarrow (id,\bar{\ }) & & \downarrow (id,\bar{\ }) & & \downarrow (id,\bar{\ })\\
(g,e)\coprod(g^{-1},e) & \xrightarrow{(C,C)} & (kgk^{-1},k)\coprod(kg^{-1}k^{-1},k) & \xrightarrow{A^k} & K\\
\downarrow & & \downarrow & & \downarrow\\
V_g\otimes V_{g^{-1}} & \xrightarrow{\bar{\varphi}_k\otimes\bar{\varphi}_k} & V_{kgk^{-1}}\otimes V_{kg^{-1}k^{-1}} & \xrightarrow{\eta^k} & K
\end{array}
$$

and

$$\begin{array}{ccccc}
(g,e)\coprod\overline{(g,e)} & & \xrightarrow{A^e} & & K \\
\downarrow \| & & & & \downarrow \| \\
(g,e)\coprod\overline{(g,e)} & \overset{(II_k,II_k)}{\rightarrow} & (g,,k)\coprod\overline{(g,k)} & \overset{A^k}{\rightarrow} & K \\
\downarrow (id,\bar{}) & & \downarrow (id,\bar{}) & & \downarrow (id,\bar{}) \\
(g,e)\coprod(g^{-1},e) & \overset{(II_k,II_k}{\rightarrow} & (g,,k)\coprod(g^1,k) & \overset{A^k}{\rightarrow} & K \\
\downarrow & & \downarrow & & \downarrow \\
V_g\otimes V_{g^{-1}} & \overset{\bar{\chi}_k\otimes\bar{\chi}_{k-1}}{\rightarrow} & V_g\otimes V_{g^{-1}k} & \overset{\eta^k}{\rightarrow} & K \\
\downarrow \| & & & & \downarrow \| \\
V_g\otimes V_{g^{-1}} & & \xrightarrow{\eta} & & K
\end{array}$$

Lastly axiom 4') comes from gluing a once punctured torus in two different ways.

3.1.18. Proposition. Given a G–Ramond algebra V there is a unique G_χ orbifold theory $\mathcal{V}$ s.t. V is its associated G–Ramond algebra.

Combining the Propositions 3.1.16 and 3.1.18 using standard arguments we can obtain:

3.1.19. Theorem. There is a 1–1 correspondence between G-twisted FA and isomorphism classes of G_χ–orbifold theories as χ runs trough the characters of G.

3.1.20. Corollary. There is a 1–1 correspondence between G-orbifold FA and isomorphism classes of monoidal functors which are identity on cylinders and satisfy the involutive property from $\mathcal{GBCOB}$ to $\mathcal{VECT}$ which lift to $\mathcal{RGBCOB}$.

3.2. Spectral flow. In the previous paragraph, we chose the geometric version to correspond to the Ramond picture, as is suggested by physics, since we are considering the vacuum states in their Hilbert–spaces at the punctures. From physics one expects that by the spectral flow, the vacuum states should bijectively correspond to the chiral algebra. In the current setting the difference only manifests itself in a change of G–action given by a twist resulting from the character χ. We can directly produce this G–action and thereby the G–Frobenius algebra by considering the G–action not by $C^e_{e,k}$, but by $II_{k^{-1}} \circ C^e_{e,k}$. This statement is proven by re-inspection of the commutative diagrams in the proof of 3.1.17.

4. Special G–Frobenius algebras

In this section we restrict ourselves to a subclass of G–twisted Frobenius algebras. This subclass is large enough to contain all G–Frobenius algebras arising from singularities, symmetric products and spaces whose cohomology of the fixed point sets are given by restriction of the comhomology of the ambient space. The restriction allows us to characterize the possible G–Frobenius structures for a given collection of Frobenius algebras as underlying data in terms of cohomological data. The restriction we will impose (cyclicity of the twisted sectors) can easily be generalized to more generators; which will render everything matrix–valued.

4.1. Definition. A *special G–Frobenius algebra* is a G–twisted Frobenius algebra whose components A_g are cyclic A_e–modules, together with two collections of maps $(r_g), (i_g)$ indexed by G where

$r_g : A_e \to A_g$ is the map of A_e–modules induced by multiplication, we write $a_g := r_g(a)$. In particular, if one sets $1_g := r_g(1)$ we obtain $r_g(a) = a1_g$. Notice that it is equivalent to specifying the map r_g or generators 1_g of A_g as cyclic A_e–modules.
$i_g : A_g \mapsto A_e$ are collections of maps s.t. each i_g is an injection which splits the exact sequence of A_e-modules.

$$0 \longrightarrow I_g \longrightarrow A \underset{i_g}{\overset{r_g}{\rightleftarrows}} A_g \longrightarrow 0 \tag{4.1}$$

Here $I_g \subset A_e$ is the annihilator of 1_g and thus of A_g. We denote the concatenation of i_g and r_g by π_g

$$\pi_g = r_g \circ i_g : A_e \to A_e$$

and we take the statement that i_g is a section of A_e modules to mean

$$i_g(a_e b_g) = \pi_g(a_e) i_g(b_g)$$

Furthermore the following holds:

$$\forall g \in G : \varphi_g(1_h) = \varphi_{g,h} 1_{ghg^{-1}}$$

for some $\varphi_{g,h} \in K$ and $\varphi_{g,1} = 1$.

4.1.1. Special super G–Frobenius algebra. The super version of special G–Frobenius algebras is straightforward. Notice that since each A_g is a cyclic A_e–algebra its parity is fixed to be $(-1)^{\tilde{g}} := \widetilde{1_g}$ times that of A_e. I.e. $a_g = i_g(a_g)1_g$ and thus $\widetilde{a_g} = \widetilde{i_g(a_g)}\widetilde{1_g}$. In particular if A_e is purely even, A_g is purely of degree $\tilde{g}$.

4.2. Proposition. Let $\eta_g : i_g(A_g) \otimes i_g(A_g) \to K$ be given by the following formula:

$$\eta_g(a, b) = \eta(r_g(a), r_{g^{-1}}(b)) = \epsilon(\gamma_{g,g^{-1}} \pi_g(ab))$$

Then $i_g(A_g) \subset A_e$ together with η_g and the induced multiplication $a \circ_g b = i_g(a \circ r_g(b)) = \pi_g(a \circ b)$ is a Frobenius sub–algebra. Furthermore η_g establishes an isomorphism between $i_g(A_g)$ and $i_{g^{-1}}(A_{g^{-1}})$.

4.3. Proposition. The following equality holds $\pi_{g^{-1}} \circ \pi_g = \pi_g$ showing that $\pi_{g^{-1}}|_{i_g(A_g)} = id$ identifying $i_g(A_g)$ and $i_{g^{-1}}(A_{g^{-1}})$.

4.4. Definition. Let A be a special G–twisted Frobenius algebra A. We define a G *graded 2–cocycle with values in* A_e to be a map $\gamma : G \times G \to A_e$ which satisfies

$$\gamma_{g,h} := \gamma(g, h) \in i_{gh}(A_{gh}) \tag{4.2}$$

and

$$\pi_{ghk}(\gamma_{g,h}\gamma_{gh,k}) = \pi_{ghk}(\gamma_{h,k}\gamma_{g,hk}) \tag{4.3}$$

We will call such a cocycle *associative* if also $\forall g, h, k \in G$:

$$\pi_{ghk}(\pi_{gh}(i_g(A_g) i_h(A_h)) i_k(A_k) \gamma_{g,h}\gamma_{gh,k}) = \pi_{ghk}(i_g(A_g) \pi_{hk}(i_h(A_h) i_k(A_k)) \gamma_{h,k}\gamma_{g,hk}) \tag{4.4}$$

and we call a cocycle *section independent* if $\forall g, h \in G$

$$(I_g + I_h)\gamma_{g,h} \subset I_{gh} \tag{4.5}$$

4.5. Remark. Notice that if g is section independent then it follows that the multiplication is independent of the choice of section and the cocycle is automatically associative.

4.6. Definition. A *non-abelian G 2-cocycle with values in K^** is a map $\varphi : G \times G \to K$ which satisfies:

$$\varphi_{gh,k} = \varphi_{g,hkh^{-1}}\varphi_{h,k} \tag{4.6}$$

where $\varphi_{g,h} := \varphi(g,h)$ and

$$\varphi_{e,g} = \varphi_{g,e} = 1$$

Notice that in the case of a commutative group G this says that the $\varphi_{g,h}$ form a two cocycle with values in K^*.

Furthermore setting $g = h^{-1}$, we find:

$$\varphi_{g^{-1},ghg^{-1}} = \varphi^{-1}_{g,h}$$

Using the multiplication law $1_g 1_h = \gamma_{g,h} 1_{gh}$ to define $\gamma_{g,h}$ one obtains:

4.7. Proposition. A special G–twisted Frobenius algebra A gives rise to an associative G 2–cocycle γ with values in A and to a non–abelian G 2–cocycle φ with values in K^*, which satisfy the compatibility equations

$$\varphi_{g,h}\gamma_{ghg^{-1},g} = \gamma_{g,h} \tag{4.7}$$

and

$$\varphi_{k,g}\varphi_{k,h}\gamma_{kgk^{-1},khk^{-1}} = \varphi_k(\gamma_{g,h})\varphi_{k,gh} \tag{4.8}$$

4.7.1. Proposition. A special G-Frobenius algebra gives rise to collection of Frobenius–algebras $(A_g, \circ_g, 1_g, \eta_g)_{g\in G}$ together with a G–action on A_e. A graded G 2–cocycle with values in A_e and a compatible non–abelian G 2–cocycle with values in k^*. Furthermore:

i) $\eta_e(\varphi_g(a), \varphi_g(b)) = \chi_g^{-2}\eta_e(a,b)$
ii) $\eta_g(a_g, b_g) = \eta(i_g(a_g)i_g(b_g)\gamma_{g,g^{-1}}, 1)$
iii) The projective trace axiom $\forall c \in A_{[g,h]}$ and l_c left multiplication by c:

$$\chi_h \mathrm{Tr}(l_c\varphi_h|_{A_g}) = \chi_{g^{-1}}\mathrm{Tr}(\varphi_{g^{-1}}l_c|_{A_h}) \tag{4.9}$$

4.8. Theorem. (Reconstruction) Given a Frobenius algebra (A_e, η_e) and a G-action on $A : \varphi : G \times A \to A$ together with the following data

- Frobenius algebras $(A_g, \eta_g | g \in G \setminus \{1\})$.
- Injective algebra homomorphisms $i_g : A_g \to A_e$ s.t. $i_g(A_g) = i_{g^{-1}}(A_{g^{-1}})$
- Restriction maps: $r_g : A \to A_g$ s.t. $r_g \circ i_g = id$.
- A graded associative G 2–cocycle γ with values in A_e and a compatible non–abelian G 2–cocycle φ with values in K^*, s.t. $\eta_g(a,b) = \eta_g(\gamma_{g,g^{-1}1_g}(a), i_g(b))$.
- A group homomorphism $\chi : G \to K^*$

such that i)-iii) of 4.7.1 hold, then there is a unique extension of this set of data to a special G–twisted Frobenius algebra, i.e. there is a unique special G–twisted Frobenius algebra with these underlying data.

Idea of a proof.

Use the multiplication law

$$a_g b_h = r_{gh}(i_g(a_g)i_h(b_h)\gamma_{g,h})1_{g,h} \tag{4.10}$$

and the G–action

$$\varphi_g(a_h) = \varphi_g(i_h(a_h))\varphi_{g,h}1_{ghg^{-1}} \tag{4.11}$$

and check all properties.

4.9. Rescaling. Given a special G–Frobenius algebra, we can rescale the cyclic generators by λ_g, i.e. we take the same underlying G–Frobenius algebra, but rescale the maps r_g to $\tilde{r}_g$ with $\tilde{1}_g = \tilde{r}_g(1) = \lambda_g 1_g$. We also fix $\lambda_e = 1$ to preserve the identity.

This yields an action of $\mathrm{Map}_{\text{pointed spaces}}(G, k^*)$ on the cocycles γ and φ preserving the underlying G–Frobenius algebra structure.

The action is given by:

$$\begin{aligned} \gamma_{g,h} &\mapsto \tilde{\gamma}_{g,h} = \frac{\lambda_g \lambda_h}{\lambda_{gh}} \gamma_{g,h} \\ \varphi_{g,h} &\mapsto \tilde{\varphi}_{g,h} = \frac{\lambda_h}{\lambda_{ghg^{-1}}} \varphi_{g,h} \end{aligned} \tag{4.12}$$

4.9.1. Remark. We can introduce the groups associated with the classes under this scaling and see that the classes of γ correspond to classes in $H^2(G, A)$. We can also identify the non–abelian cocycles φ with one–group cocycles with values in $k^*[G]$ where we treat $k^*[G]$ as an abelian group with diagonal multiplicative composition

$$(\sum_g \lambda_g g) \cdot (\sum_h \mu_h h) := \sum_g \lambda_g \mu_g g \tag{4.13}$$

and G–action given by conjugation:

$$s(g)(\sum_h \lambda_h h) = \sum_h \lambda_h g h g^{-1} \tag{4.14}$$

4.10. Lemma. If $\gamma_{g,h} = 0$ then $\pi_h(\gamma_{g,g^{-1}}) = 0$ and $\pi_g(\gamma_{h,h^{-1}}) = 0$

4.10.1. Graded special G–Frobenius algebras. Consider a set of graded Frobenius algebras satisfying the reconstruction data: $\{(A_g, \eta_g) : g \in G\}$ with degrees $d_g := \deg(\eta_g)$
s.t. $A_g \simeq A_{g^{-1}}$. E.g. in the cohomology of fixed point sets d_g is given by the dimension and for the Jacobian Frobenius manifolds (see the next section) d_g fixed by the degree of $Hess(f_g) = \rho_g$. Furthermore, the reconstructed $\{\eta|_{(A_g \otimes A_{g^{-1}}}, g \in G\}$ have degree $d_g = d_{g^{-1}}$.

For a G–twisted FA the degrees all need to be equal to $d := d_e$. To achieve this, one can shift the grading in each A_g by s_g. This amounts to assigning degree s_g to 1_g. This is the only freedom, since the multiplication should be degree–preserving and all A_g are cyclic.

Set
$s_g^+ := s_g + s_{g^{-1}}$
$s_g^- := s_g - s_{g^{-1}}$
Then $s_g^+ := d - d_g$ for grading reasons, but the shift s^- is more elusive.

4.10.2. Definition. The standard shift for a Jacobian Frobenius algebra (see below) is given by

$$s_g^+ := d - d_g$$

and

$$s_g^- := \frac{1}{2\pi i}\mathrm{tr}(\log(g)) - \mathrm{tr}(\log(g^{-1})) := \frac{1}{2\pi i}(\sum_i \lambda_i(g) - \sum_i \lambda_i(g^{-1}))$$
$$= \sum_{i:\lambda_i\neq 0} 2(\frac{1}{2\pi i}\lambda_i(g) - 1)$$

where the $\lambda_i(g)$ are the logarithms of the eigenvalues of g using the branch with values in $[0, 2\pi)$ i.e. cut along the positive real axis.

In this case we obtain:

$$s_g = \frac{1}{2}(s_g^+ + s_g^-) = \frac{1}{2}(d - d_g) + \sum_i (\frac{1}{2\pi i} i(g) - \frac{1}{2})$$

4.11. Remark–Grading and Monodromies. The grading s^- has a geometric meaning in the case of singularities. Here the original grading is given by the monodromy operator. In the twisted sector the monodromy operator is replaced by the product of the operator and the operator represented by g. This is described in [K5].

4.11.1. Reconstruction for graded special G–Frobenius algebras. In the Reconstruction program the presence of a non–trivial grading can greatly simplify the check of the trace axiom. E.g. if $A_{[g,h]}$ has no element of degree 0 then both sides of this requirement are 0 and if $[g, h] = e$ one needs only to look at the special choices of c with $\deg(c) = 0$ which most often is just $c = 1$, the identity.

4.11.2. Ramond–grading. The grading in the Ramond–sector is by the following definition

$$\deg(v) := -\frac{d}{2}$$

This yields

$$\deg(\bar{\eta}) = 0 \text{ and } \deg(\bar{o}) = \frac{d}{2}$$

5. Jacobian Frobenius algebras

5.1. Definition. A Frobenius algebra A is called *Jacobian*

if it can be represented as the Milnor ring of a function f. I.e. if there is a function $f \in \mathcal{O}_{\mathbf{A}^n_K}$ s.t. $A = \mathcal{O}_{\mathbf{A}^n_K}/J_f$ where J_f is the Jacobian ideal of f. And the bilinear form is given by the residue pairing. This is the form given by the the Hessian of f: $\rho := \mathrm{Hess}_f$.

If we write $\mathcal{O}_{\mathbf{A}^n_k} = k[x_1 \dots x_n]$, J_f is the ideal spanned by the $\frac{\partial f}{\partial x_i}$.

A *realization of a Jacobian Frobenius algebra* is a pair (A, f) of a Jacobian Frobenius algebra and a function f on some affine k space $\mathbf{A}^n_k$, i.e. $f \in \mathcal{O}_{\mathbf{A}^n_k} = k[x_1 \dots x_n]$ s.t. $A = k[x_1 \dots x_n]$ and $\rho := \det(\frac{\partial^2 f}{\partial x_i \partial x_j})$.

A *small realization of a Jacobian Frobenius algebra* is a realization of minimal dimension, i.e. of minimal n.

5.2. Definition. A *natural G action on a realization of a Jacobian Frobenius algebra* (A_e, f) is a linear G action on $\mathbf{A}_k^n$ which leaves f invariant.

Given a natural G action on a realization of a Jacobian Frobenius algebra (A, f) set for each $g \in G$, $\mathcal{O}_g := \mathcal{O}_{\mathrm{Fix}_g(\mathbf{A}_k^n)}$. This is the ring of functions of the fixed point set of g for the G action on $\mathbf{A}_k^n$. These are the functions fixed by g: $\mathcal{O}_g = k[x_1, \dots, x_n]^g$.

Denote by $J_g := J_{f|_{\mathrm{Fix}_g(\mathbf{A}_k^n)}}$ the Jacobian ideal of f restricted to the fixed point set of g.

Define

$$A_g := \mathcal{O}_g / J_g \tag{5.1}$$

The A_g will be called twisted sectors for $g \neq 1$. Notice that each A_g is a Jacobian Frobenius algebra with the natural realization given by $(A_g, f|_{\mathrm{Fix}_g})$. In particular, it comes equipped with an invariant bilinear form $\tilde{\eta}_g$ defined by the element $\mathrm{Hess}(f|_{\mathrm{Fix}_g})$.

For $g = 1$ the definition of A_e is just the realization of the original Frobenius algebra, which we also call the untwisted sector.

Notice there is a restriction morphism $r_g : A_e \to A_g$ given by $a \mapsto a|_{\mathrm{Fix}_g} \bmod J_g$.

Denote $r_g(1)$ by 1_g. This is a non–zero element of A_g since the action was linear. Furthermore it generates A_g as a cyclic A_e module.

The set $\mathrm{Fix}_g \mathbf{A}_k^n$ is a linear subspace. Let I_g be the vanishing ideal of this space. We obtain a sequence

$$0 \to I_g \to A_e \stackrel{r_g}{\to} A_g \to 0$$

Let i_a be any splitting of this sequence induced by the inclusion: $\hat{i}_g : \mathcal{O}_g \to \mathcal{O}_e$ which descends due to the invariance of f.

In coordinates, we have the following description. Let $\mathrm{Fix}_g \mathbf{A}_k^n$ be given by equations $x_i = 0 : i \in N_g$ for some index set N_g.

Choosing complementary generators $x_j : j \in T_g$ we have $\mathcal{O}_g = k[x_j : j \in T_g]$ and $\mathcal{O}_e = k[x_j, x_i : j \in T_g, i \in N_g]$. Then $I_g = (x_i : i \in N_g)_{\mathcal{O}_e}$ the ideal in $\mathcal{O}_e$ generated by the x_i and $\mathcal{O}_e = I_g \oplus i_g(A_g)$ using the splitting i_g coming from the natural inclusion $\hat{i}_g : k[x_j : j \in T_g] \to k[x_j, x_i : j \in T_g, i \in N_g]$. We also define the projections

$$\pi_g : A_1 \to A_g; \pi_g = i_g \circ r_g$$

which in coordinates are given by $f \mapsto f|_{x_j = 0 : j \in N_g}$ Let

$$A := \bigoplus_{g \in G} A_g$$

where the sum is a sum of A_e modules.

5.3. Definition. A discrete Torsion is a map $\epsilon : G \times G \to k$, s.t.

$$\epsilon(g, h) = \epsilon(h^{-1}, g) \quad \epsilon(g, g) = 1 \quad \epsilon(g_1 g_2, h) = \epsilon(g_1, h)\epsilon(g_2, h)$$

5.4. Theorem. Given a natural G action on a realization of a Jacobian Frobenius algebra (A_e, f) with a quasi–homogeneous function f of degree d and type $\mathbf{q} = (q_1, \dots, q_n)$ together with the natural choice of splittings i_g the possible structures of special G twisted Frobenius algebra on the A_e module $A := \bigoplus_{g \in G} A_g$ are in 1–1 correspondence with elements of $\bar{Z}^2(G, A)$, where $\bar{Z}^2$ are G graded cocycles and a compatible non–abelian two cocycle φ with values in k^*, which define a choice of discrete torsion.

6. Mirror construction for special G–Frobenius algebras

6.1. Double grading. We consider Frobenius algebras with grading in some abelian group I.

$$A = \bigoplus_{i \in I} A_i$$

This grading can be trivially extended to a double grading with values in $I \times I$ in two ways

$$A^{cc} = \bigoplus_{i \in I} A_{i,i} \quad \text{and} \quad A^{ac} = \bigoplus_{i \in I} A_{i,-i}$$

corresponding to the diagonal $\Delta : I \to I \times I$ and $(id, -) \circ \Delta : I \to I \times I$. We call bi–graded Frobenius algebras of this form of (c,c)–type and of (a,c)–type, respectively. In the language of Euler fields we consider the field $(E, \bar{E}) = (E, E)$ or $(E, \bar{E}) = (E, -E)$.

These gradings become interesting for special G–Frobenius algebras, since in that case the shifts will produce a possible non–diagonal grading.

6.2. Definition. Given a graded special G–Frobenius algebra we assign the following bi–degrees to 1_g

$$(E, \bar{E})(1_g) := (s_g, s_{g^{-1}})$$

It is clear that A_e is of (c,c)–type. A is however only of (c,c) type if $s_g = s_{g^{-1}}$.

Furthermore for the Ramond–space of A we assign the following bi–degree to v

$$(E, \bar{E})(v) := (-\frac{d}{2}, -\frac{d}{2})$$

6.3. Euler–twist (spectral flow). In this section, we consider a graded special G–Frobenius algebra and construct a new vector–space from it. We denote the grading operator by E:

6.3.1. Definition. The twist–operator j for an Euler–field E is

$$j := exp(2\pi i E)$$

We denote the group generated by j by J.

We call a special G–Frobenius algebra *Euler* if there is a special $\tilde{G}$–Frobenius algebra $\tilde{A}$ of which A is a subalgebra where $\tilde{G}$ is a group that has G and J as subgroups.

6.3.2. Definition. The dual $\check{A}$ to an Euler special G–Frobenius algebra A of (c,c)–type is the vector space

$$\check{A} := \bigoplus_{g \in G} \check{A}_g := \bigoplus_{g \in G} V_{gj^{-1}}$$

with the A_e and G–module structure determined by $\tau_j^g : \check{A}_g \simeq V_{gj^{-1}}$ together with the bi–grading

$$(E, \bar{E})(\check{1}_g) := (s_{gj^{-1}} - d, s_{gj^{-1}})$$

where $\check{1}_g$ denote the generator of $\check{A}_g$ as A_e–module and the bi–linear form

$$\check{\eta} := \tau_j^*(\bar{\eta})$$

where $\tau_j := \oplus_{g\in G}\tau_j^g$ and V and $\bar\eta$ refer to the Ramond–space of A.

7. Explicit Examples

7.1. Self duality of A_n. We consider the example of the Jacobian Frobenius Algebra associated to the function z^{n+1}

$$A_n := \mathbf{C}[z]/(z^n)$$

together with the induced multiplication and the Grothendieck residue. Explicitly:

$$z^i z^j = \begin{cases} z^{i+j} & \text{if } i+j \le n \\ 0 & \text{else} \end{cases}$$

the form

$$\eta(z^i, z^j) = \delta_{i,n-1-j}$$

and the grading:

$$E(z^i) := \frac{i}{n+1}$$

which means $\rho = z^{n-1}$ and $d = \frac{n-1}{n+1}$.

We consider just the group $J \simeq \mathbf{Z}/(n+1)\mathbf{Z}$ with the generator j acting on z by multiplication with $\zeta_{n+1} := \exp(2\pi i\frac{1}{n+1})$. We have

$$Fix_{j^i}\begin{cases} \mathbf{C} & \text{if } i=0 \\ 0 & \text{else} \end{cases}$$

and thus

$$A_{j^i}\begin{cases} A_n & \text{if } i=0 \\ 1_{j^i} & \text{else} \end{cases}$$

Furthermore we have the following grading;

$$(E,\bar E)(1_{j^i}) = \begin{cases} (0,0) & \text{for } i=0 \\ (\frac{i-1}{n+1}, \frac{n-i}{n+1}) & \text{else} \end{cases}$$

which means $\rho_{j^i} = 1_{j^i}$ and $d_{j^i} = 0$.

Using the reconstruction Theorem we have to find a cocycle γ and a compatible action φ. There is no problem for the metric since always $|N_g| = 1$ and if $n+1$ is even $\det(\zeta^{\frac{n+1}{2}}) = -1$. Since the group J is cyclic there is no freedom of choice for ϵ and just two choices of parity are possible corresponding to $j \mapsto \pm 1$.

From the general considerations we know $\gamma_{j^i,j^{n-1-i}} \in A_e$ and $\deg(\gamma_{j^i,j^{n-1-i}}) = d - d_{j^i} = \frac{n-1}{n+1}$ which yields

$$\gamma_{j^i,j^{n-1-i}} = ((-1)^{\tilde j i}\zeta^i)^{1/2}\rho = ((-1)^{\tilde j}\zeta)^{i/2}z^{n-1}$$

for the other γ notice that $\deg(1_{j^i}) + \deg(1_{j^k}) = \frac{i+k-2}{n+1}$ while $\deg(1_{j^{i+k}}) = \frac{i+k-1}{n+1}$ if $i+k \neq n+1$, but there is no element of degree $\frac{1}{n+1}$ in $A_{j^{i+k}}$ for $i+k \neq n+1$.

Hence

$$\gamma_{j^i,j^k} = \begin{cases} ((-1)^{\tilde j}\zeta)^{i/2}z^{n-1} & \text{for } i+j = n+1 \\ 0 & \text{else} \end{cases}$$

this means that

$$\varphi_{j^i,j^k} = (-1)^{\tilde j^i \tilde j^k}\zeta^{-i}$$

Therefore the G–invariants $A^G = A_1$ are generated by the identity 1.

The Ramond grading of this algebra is

$$(E,\bar{E})(1_{j^i}v) = \begin{cases} (0,0) & \text{for } i=0 \\ (\frac{k}{n+1}-\frac{1}{2}, -\frac{k}{n+1}+\frac{1}{2}) & \text{else} \end{cases}$$

Since $j \in G$ the special G–Frobenius algebra is Euler and the dual is defined, moreover $G = J$ so that the vector–space structures of A and $\check{A}$ coincide. The grading is given by:

$$(E,\bar{E})(1_{j^i}v) = \begin{cases} (0,0) & \text{for } i=0 \\ (-\frac{k-(n+1)}{n+1}-\frac{1}{2}, \frac{n+1-k}{n+1}+\frac{1}{2}) & \text{else} \end{cases}$$

The G–action is given by

$$\check{\varphi}_{j^i,j^k} = \begin{cases} (-1)^{\tilde{j}^i\tilde{j}^k}\zeta^i & \text{for } k=1 \\ (-1)^{\tilde{j}^i\tilde{j}^k} & \text{else} \end{cases}$$

This G–action leaves all even sectors $\check{A}_{j^i}$ invariant except for $i = 1$ if $\tilde{j} = 1$. Thus with the choice of all even sectors we have as G–modules

$\check{A} \simeq A_n$

where more explicitly

$\check{1}_{j^i} \mapsto z^{n+1-i} : k = 2, \dots n$ and $\check{1}_0 \mapsto 1$.

Notice that this A_n is of (a,c)–type however. Also since $J = G$ the form $\bar{\eta}$ pulls back and gives a non–degenerate form on $\check{A}^G$, which is the usual form on A_n. Furthermore the usual multiplication on A_n is compatible with everything so that we can say that A_n is self–dual under this operation.

7.2. D_n from a special $\mathbf{Z}/2\mathbf{Z}$–Frobenius algebra based on A_{2n-3}. In this section, we show how to get D_n from a special $\mathbf{Z}/2\mathbf{Z}$–Frobenius algebra based on A_{2n-3}. The function for the Frobenius algebra A_{2n-3} is z^{2n-2}. Since this is an even function, $\mathbf{Z}/2\mathbf{Z}$ acting via $z \mapsto -z$ is a symmetry. There are two sectors, the untwisted and the twisted sector containing the element 1_{-1} with degree 0. The multiplication is fixed by $\deg(\gamma_{-1,-1}) = \frac{2n-4}{2n-2}$ thus

$$\gamma_{-1,-1} = z^{2(n-2)}$$

again the group is cyclic so the G–action only depends on the choice of parity of the -1–sector.

In the untwisted sector we have $A_e^{\mathbf{Z}/2\mathbf{Z}} = \langle 1, z^2, \dots, z^{2(n-1)} \rangle \simeq A_{n-1}$ and the action of -1 on 1_{-1} is given by

$$\varphi_{-1,-1} = (-1)^{\widetilde{-1}\widetilde{-1}+1}$$

so that if $\sigma(-1) = 1$

$$A^{\mathbf{Z}/2\mathbf{Z}}_{2n-3,\mathbf{Z}/2\mathbf{Z}} \simeq D_n$$

and if $\sigma(-1) = 0$ we just obtain the invariants of the untwisted sector which are isomorphic to A_{n-1}.

The untwisted sector is given by the singularity A_{n-1} as expected upon the transformation $u = z^2$. Notice that the invariants of the Ramond sector yield the singularity A_{n-2} as expected from [W]. These are of the form $u^i du$ or $z^{2i+1}dz$ with $i = 0, \dots, n-3$.

7.3. Point mod G. In the theory of Jacobian Frobenius algebras there is the notion of a point played by a Morse singularity $z_1^2 + \cdots + z_n^2$. Any finite subgroup $G \subset O(n, k)$ leaves this point invariant.

The G–twisted algebra after possibly stabilizing is the following.

$$A = \bigoplus_{g \in G} k1_g$$

And the grading is $\deg(1_g) = (\frac{1}{2}s_g^-, \frac{1}{2}s_g^-)$, since $d = d_g = 1$.

Using 4.10 it follows that the γ cannot vanish, thus fixing φ and ϵ, so that the possibilities are enumerated by the graded cocycles. The compatibility equations hold automatically.

SPECIAL CASE.

If we assume that $G \subset O(n, \mathbf{C})$ and that $s_g^- = 0$ (i.e. $\sum_{i:\lambda_i \neq 0} \frac{1}{2\pi i}\lambda_i = \frac{|N_g|}{2} \in \mathbf{N}$), then we the cocycles lie in $H^2(G, k^*)$ and the possible algebra structures are those of twisted group algebras.

7.3.1. Point mod $\mathbf{Z}/n\mathbf{Z}$. By the above analysis we realize $\mathbf{Z}/n\mathbf{Z}$ as the sub–group of rotations of order n in $\mathbf{C}$. We have that $s_g^- = 0$ and thus we can choose the full cocycle making A into A_n, multiplicatively, with trivial grading and trivial G–action if one chooses all even sectors. The metric, however, will not be consistent with A_n. The identity pairs with itself for instance. Dualizing A, we obtain the following space

$$\check{A} = \oplus_{j^i} A_{j^{i-1}}$$

with again a trivial $(E, \bar{E})$ grading. Choosing the generator $\tilde{j} := j^{-1}$ for J, the metric reads

$$\tilde{\eta}(\tilde{1}_{\tilde{j}^i}, \tilde{1}_{\tilde{j}^k}) = \begin{cases} 1 & \text{if } i + k = n - 1 \\ 1 & i = k = n \\ 0 & \text{else} \end{cases}$$

This metric is compatible with the following multiplication:

$$\tilde{1}_{\tilde{j}^i} \circ \tilde{1}_{\tilde{j}^k} = \begin{cases} \tilde{1}_{\tilde{j}^{i+k}} & \text{if } i + k \leq n - 1 \\ \tilde{1}_{\tilde{j}^{n-1}} & i = k = n \\ 0 & \text{else} \end{cases}$$

8. Recent developments

After the first presentation of our axiomatic treatment of the "stringy" geometry of orbifolds at WAGP00 Oct. 2000 in Trieste [K1] two papers appeared taking the same point of view in a special setting: in [LS] the authors were considering the cohomology of the Hilbert scheme of points of a variety and came up with an algebraic structure on the n–th tensor power of a Frobenius algebra. It can be shown (cf. [K4] and see below) that this is the essentially unique structure of special S_n–twisted Frobenius algebra on $A^{\otimes n}$. The second paper taking our point of view is [FG] where the special setting of of Gromov–Witten invariants of a global quotient are considered and shown to satisfy our axioms.

8.1. Symmetric Powers. For the full calculations see [K4] Let A be a Frobenius algebra with degree d. We consider $A^{\otimes n}$ with the permutation action of the symmetric group group $\mathbf{S}_n$.

Regarding first a Jacobian Frobenius algebra A given by the function f, $A^{\otimes n}$ is the Jacobian algebra of the function $f(z_1)+\cdots+f(z_n)$ and action of permuting the variable coincides with the action of permuting the tensor factors. The restriction to the fixed point set $\mathrm{Fix}(\sigma) : \sigma \in \mathbf{S}_n$ is nothing but the contraction μ_σ which is the iterated contraction by multiplication over the indices provided by the partition corresponding to the cycle of σ.

The standard degree shifts in the case of a power of a Jacobian algebra can be calculated to be

$$s^+ = d - d_\sigma = d|\sigma|, \quad s^- = 0$$

where $|\sigma|$ is the minimal number of transposition needed to represent σ.

In the general setting we can use the contraction maps μ_σ as the maps r_σ and keep the grading.

8.2. Theorem. Given a Jacobian Frobenius algebra A there are exactly two $\mathbf{S}_n$ Frobenius algebra structures on $A^{\otimes n}$ with restriction maps r_σ and the natural grading above. These correspond to an all even case and a super–graded case. These cases differ exactly by a choice of algebraic discrete torsion [K4].

Here the notion of algebraic discrete torsion is introduced in [K4] and extends the notion of discrete torsion for Jacobian Frobenius algebras as presented above.

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