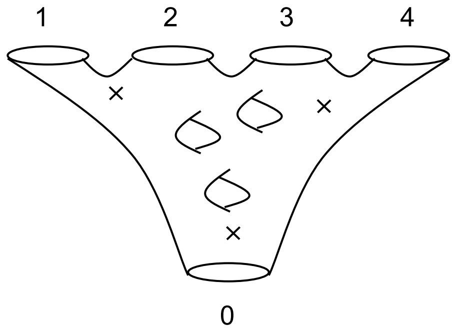
Operads, Strings and Deligne's conjecture

1. Operads
   Fun with algebra and geometry
2. Deligne's conjecture
   Connecting algebra and geometry
3. TFT/Strings
   Getting Physics into the picture
4. Moduli spaces and Arcs
   Basic constructions
5. Operations
   Moduli space actions and String Topology
6. Outlook

Operads:Two indicative examples

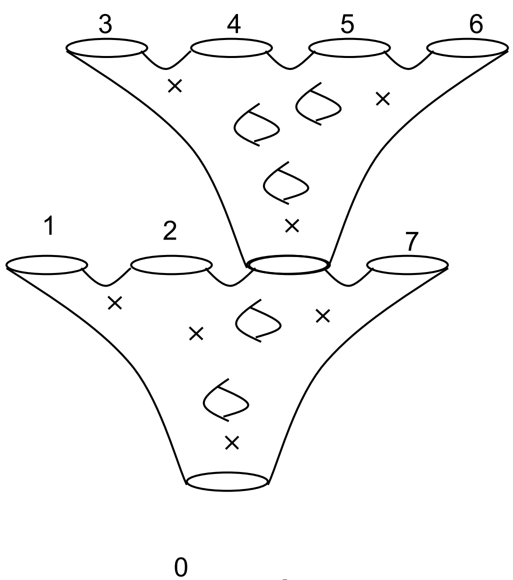


Fix $F_{g,r}^{s}$

Fix $F_{g,n+1}^{s}$ $F_{g',n'+1}^{s'}$ and $1 \leq i \leq n$

Can glue boundary i to boundary 0

i=3
g=2
s=4
g' =3
s' =3
n=4
n' =4



Get $F_{g,n+1}^{s} \circ_i F_{g',n'+1}^{s'} = F_{g+g',n+n'}^{s+s'}$

Let A be an associative, commutative algebra with unit

$$CH^{p}(A,A) := Hom(A^{\otimes p}, A)$$

Fix f^n in CH^n, $g^{n'}$ in $CH^{n'}$ and $1 \leq i \leq r$

Can insert the function g into f at i.

$$\circ_i : CH^m(A,A) \otimes CH^n(A,A) \to CH^{m+n-1}(A,A)$$
$$f \circ_i g(x_1, \ldots, x_{m+n-1}) :=$$
$$f(x_1, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{m+n-1})$$

Notice (1) the gluing/inserting is associative and (2) the symmetric groups S_n act compatibly by permutations of the variables or the labels

Operads the definition

Abstracting from the examples

Definition: An operad is a collection O(n) n in **N** together with operations

$\circ_i: O(n)\otimes O(m)\rightarrow O(m+n-1)$

Which are

1. Associative
2. *S_n*-equivariant.

Clarification:

O(n) are objects in a symmetric monoidal category

And each O(n) has an action of the symmetric group S_n.

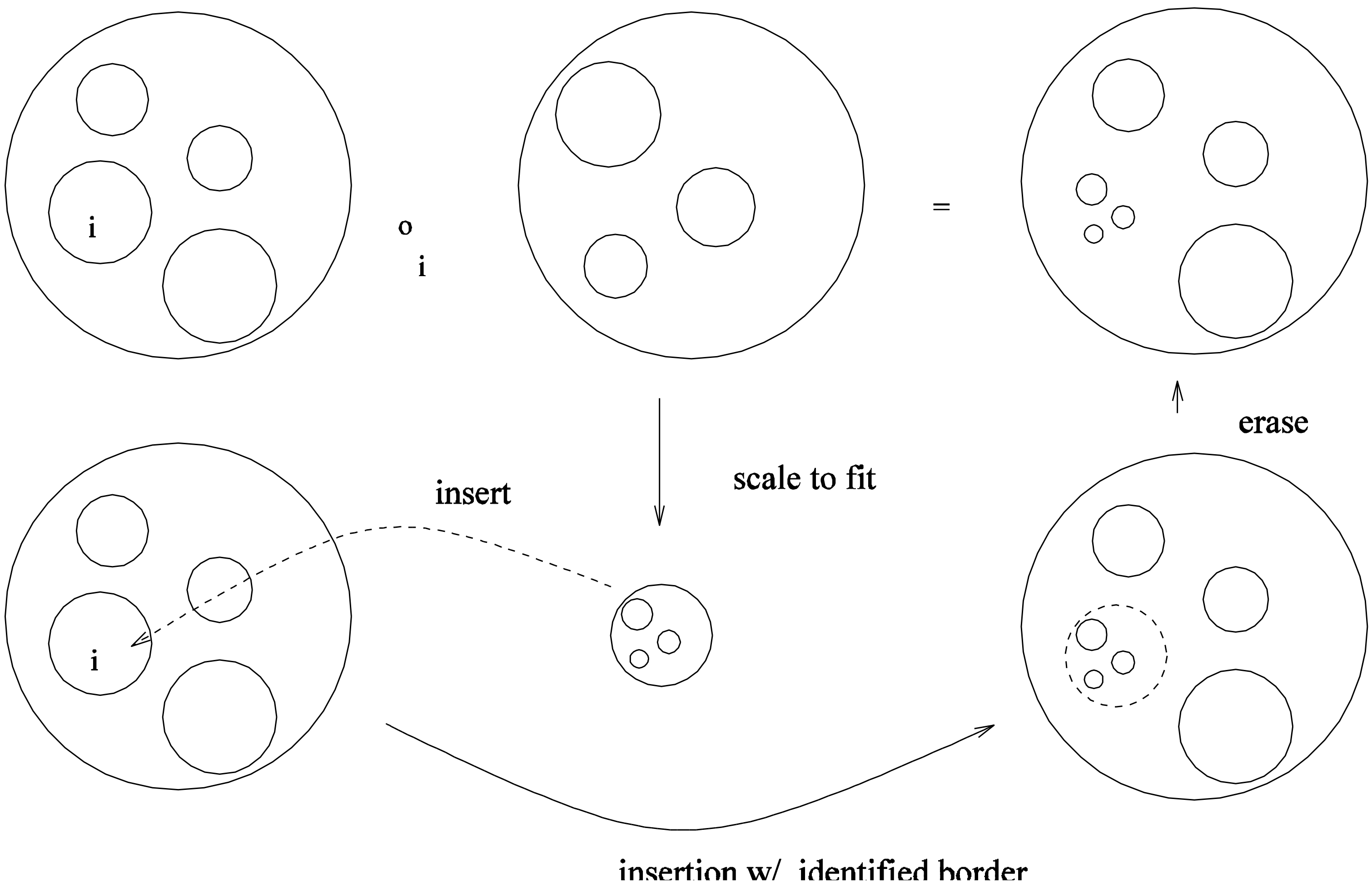
Favorite categories are Finite Sets, Linear Spaces/ Complexes and Topological spaces.

Question: Is there a relation between the first and second example?

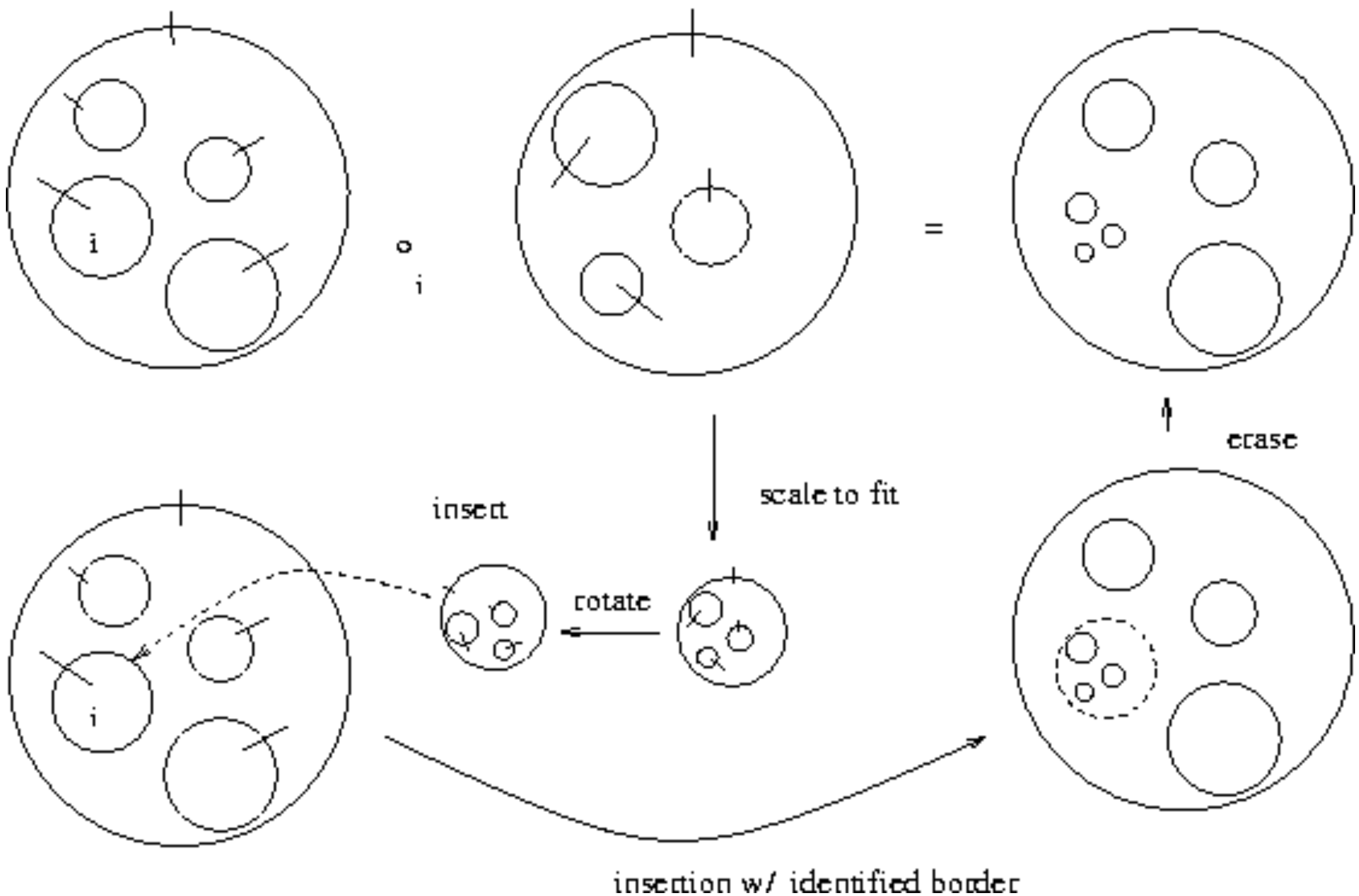
Answer: Yes. A very simple one if one keeps things as they are. But also yes a very interesting one if one "beefs up" the first example.

This is the context of Deligne's conjecture,
string topology, the arc operad and its suboperads.

Example 1: The Little Discs Operad D_2

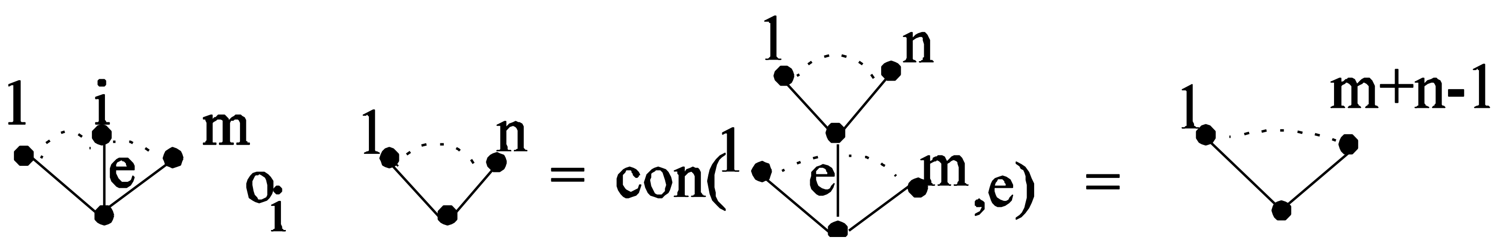


Example 2: Framed Little Discs Operad fD_2



Example 3: Rooted trees

- Comm(n):
  - trivial representation of S_n
  - n-Corolla
- Assoc(n)
  - Permutation representation of S_n
  - Planar n-corolla



Example 4: The homomorphism Operad & Algebras over operads

The Homomorphism Operad Hom_v

- Let V be a vector space.
- $Hom_V(n) := Hom(V^{\otimes n}, V)$
- $f \in Hom_V(n)$, $g \in Hom_V(m)$
- $f \circ_i g(v_1, \ldots, v_{n+m-1}) =$
- $f(v_1, \ldots, v_{i-1}, g(v_i, \ldots v_{i+m-1}), v_{i+m}, \ldots, v_{n+m+1})$
- Usual permutation action

Definition: An Algebra over an operad O is a vector space V together with a map of operads*

$$O \rightarrow Hom_v$$

Think of each element of O(n) as a n-multilinear map.

*All the structures (operations & S_n actions) are preserved.

Algebra meets geometry

Operad	Algebra
Assoc	Associative
Comm	Commutative
$H_*(D_2)$	Gersten-haber (Boadman-Vogt)
$H_*(fD_2)$	Batalin-Vilkovisky (Getzler)

Note: H_* is a functor

Definition: A Gerstenhaber (G) algebra A is an associative commutative graded algebra such with a bracket { • } that satisfies:

$$
\begin{aligned}
(x \cdot y) &= (-1)^{|x||y|} y \cdot x \\
x \cdot (y \cdot z) &= (x \cdot y) \cdot z \\
\{x \bullet y\} &= -(-1)^{|sx||sy|}\{y \bullet x\} \\
\{x \bullet \{y \bullet z\}\} &= \{\{x \bullet y\} \bullet z\} + (-1)^{|sx||sy|}\{y \bullet \{x \bullet z\}\} \\
\{x \bullet y \cdot z\} &= \{x \bullet y\} \cdot z + (-1)^{|sx||y|} y \cdot \{x \bullet z\}.
\end{aligned}
$$

Definition: A Batalin-Vilkovisky (BV) algebra is an associative commutative graded algebra with and operator Δ which satisfies

$$
\begin{aligned}
\Delta^2 &= 0 \\
\Delta(abc) &= \Delta(ab)c + (-1)^{|a|} a\Delta(bc) + (-1)^{|sa||b|} b\Delta(ac) - \Delta(a)bc \\
&\quad -(-1)^{|a|} a\Delta(b)c - (-1)^{|a|+|b|} ab\Delta(c)
\end{aligned}
$$

The Hochschild Complex

- Let A be an associative algebra over a field k.
- The Hochschild cochains are

$$CH^p(A,A) := Hom(A^{\otimes p}, A)$$

- Elementary operations

$$\bullet : CH^m(A,A) \otimes CH^m(A,A) \to CH^{m+n}(A,A)$$

$$f \bullet g(x_{1,} \ldots, x_{m+n}) = f(x_1, \ldots, x_n) g(x_1, \ldots, x_m)$$

$$\circ_i : CH^m(A,A) \otimes CH^n(A,A) \to CH^{m+n-1}(A,A)$$

$$f \circ_i g(x_1, \ldots, x_{m+n-1}) :=$$

$$f(x_1, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{m+n-1})$$

- Differential

$$\partial : CH^n(A,A) \to CH^{n+1}(A,A)$$

$$\partial f(x_1, \ldots, x_{n+1}) = x_1 f(x_2, \ldots, x_{n+1}) - f(x_1 x_2, \ldots, x_{n+1}) + \ldots$$

$$+ (-1)^n f(x_1, \ldots, x_n x_{n+1}) + (-1)^{n+1} f(x_1, \ldots, x_n) x_{n+1}$$

- Gerstenhaber introduced the operations

$$\circ, \{\,,\} : CH^p(A,A) \otimes CH^q(A,A)$$

$$\to CH^{p+q-1}(A,A)$$

$$f \circ g := \sum_{i=1}^{p} (-1)^{(q-1)(i-1)} f \circ_i g$$

$$\{f, g\} := f \circ g - (-1)^{(p+1)(q+1)} g \circ f$$

- Higher brackets and multiplications:

$$\bullet(f_1, \ldots, f_n) := f_1 \bullet \cdots \bullet f_n$$

$$f \circ \{g_1, \ldots, g_n\} :=$$

$$\sum \pm f(x_1, \ldots, x_{i_1-1}, g_1(x_{i_1}, \ldots, x_{i_1+q_1-1}), x_{i_1+q_1}, \ldots,$$

$$x_{i_n-1}, g_n(x_{i_n}, \ldots, x_{i_n+q_n-1}), x_{i_1+p_n} \ldots, x_{q+p_1+\cdots+p_n-n})$$

Any concatenation of these operations can be given by a tree flow chart with black and white vertices.

The Hochschild Complex

- Let *A* be an associative algebra over a field *k*.
- The Hochschild cochains are

$$CH^{p}(A,A) := Hom(A^{\otimes p}, A)$$

- There are two natural operations:

$$\bullet : CH^{m}(A,A) \otimes CH^{m}(A,A) \to CH^{m+n}(A,A)$$
$$f \bullet g(x_{1,},\dots,x_{m+n}) = f(x_1,\dots,x_n)g(x_1,\dots,x_m)$$

$$\circ_i : CH^{m}(A,A) \otimes CH^{n}(A,A) \to CH^{m+n-1}(A,A)$$
$$f \circ_i g(x_1,\ldots,x_{m+n-1}) :=$$
$$f(x_1,\ldots,x_{i-1}, g(x_i,\ldots,x_{i+n-1}), x_{i+n},\ldots,x_{m+n-1})$$

- The Hochschild complex also has a differential which is also derived from the algebra structure.

$$\partial : CH^{n}(A,A) \to CH^{n+1}(A,A)$$
$$\partial f(x_1,\ldots,x_{n+1}) = x_1 f(x_2,\ldots,x_{n+1}) - f(x_1 x_2,\ldots,x_{n+1}) + \ldots$$
$$+ (-1)^n f(x_1,\ldots,x_n x_{n+1}) + (-1)^{n+1} f(x_1,\ldots,x_n)x_{n+1}$$

- **Definition:** The Hochschild complex of A is (*CH**(A,A), ∂), its cohomology is called the Hochschild cohomology and denoted by *HH*(A,A)*.

Higher Order Operations

- Gerstenhaber introduced the operations

$$\circ, \{\ ,\ \} : CH^p(A,A) \otimes CH^q(A,A) \to CH^{p+q-1}(A,A)$$

$$f \circ g := \sum_{i=1}^{p} (-1)^{(q-1)(i-1)} f \circ_i g$$

$$\{f, g\} := f \circ g - (-1)^{(p+1)(q+1)} g \circ f$$

- **Theorem** (Gerstenhaber): HH*(A,A) together with the multiplication • and the bracket { , } is a Gerstenhaber algebra.

Higher brackets and multiplications:

- Iterations of • and { , } give operations

$$\bullet(f_1, \dots, f_n) := f_1 \bullet \dots \bullet f_n$$

$$f \circ \{g_1, \dots, g_n\} :=$$

$$\sum \pm f(x_1, \dots, x_{i_1-1}, g_1(x_{i_1}, \dots, x_{i_1+q_1-1}), x_{i_1+q_1}, \dots,$$

$$x_{i_n-1}, g_n(x_{i_n}, \dots, x_{i_n+q_n-1}), x_{i_1+p_n} \dots, x_{q+p_1+\dots+p_n-n})$$

- Any concatenation of operations is given by a flow chart with black and white vertices whose vertices correspond to the operations above.

Deligne's conjecture and Theorems I

Deligne's conjecture: There is a chain model of the little discs operad and an operation of it on the Hochschild Cochains which lifts the action of the homology of the little discs operad on the Hochschild cohomology.

Theorem 4 [Kont., Tam., V, MS, KontS, BF,K]:

Deligne's conjecture holds in char 0 and in char p.

Importance:

Theorem [Kont., Tam.]: Deligne's conjecture and the formality of the little discs operad imply deformation quantization.

(Fields medal)

Generalizations

Theorem 5 [KontS,K,KSchw] The conjecture also holds in the A_∞ case.

There is even an operad of CW complexes [KSch] whose chains solve the problem.

Theorem 6 [K] (Cyclic Deligne conjecture). The Hochschild Cochains of a Frobenius algebra carry and action of a chain model of the framed little discs operad.

Other Applications: String topology type operations. Use cyclic cohomology [??,Jones]

or use on deRham model [Merk].

Chain Motivation for D's Conjecture

	Operad	"Algebra"
Topological Level	D_2 or equivalent operad	
Chain Level	$\mathrm{Chain}(D_2)$ some chain model eg. $CC_*(Cact^1)$	$CH^*(A,A)$, { , } Homotopy G-alg
Homology level	$H_*(D_2)$ $H_*(Cact^1) \cong H_*(Cact^1)$ $\cong H_*(D_2)$	HH^*, { , } G-alg

Physics Motivation for D's conjecture and cyclic D's conjecture

A little stream of consciousness (encouragement/motivation)

- Little discs are surfaces with boundary. These appear in string theory, CFT and TFT
- TFT is connected to Frobenius algebras
- Gerstenhaber structures and BV structures exist in CFT [Getzler, Lian-Zuckerman???]
- D-branes: Closed strings give deformations on the open strings → Each Riemann surface should give an operation on HH*???better

A little stream of consciousness (worries)(answers)

- Little discs have flat structure
  Go to equivalent model to only "see insertion"
- TFT is cyclic
  The cyclic D conjecture holds for Frobenius Algebras.
- Punctures or boundaries?
  Actually marked boundary =punctured + tangent vector
- What about higher genus and conformal structure?
  There is a moduli space action.

Remaining question: Relation to Gromov-Witten invariants.
Or, how to go to the boundary? (DM or Penner)

Theorems

Theorem: Deligne's conjecture (classic, cyclic, A_∞)

Theorem: There is a (rational) operad structure on the collection of the moduli spaces $M^n_{g,n}$ of surfaces of genus g with n with marked points and tangent vectors at the marked points .

Theorem: There is a chain operad induced by the above which acts on the Hochschild cochains of a Frobenius algebra extending the action of D's conjecture.

Theorem: There is a quasi-PROP of Sullivan chord diagrams.

Note: This contains some of moduli space and some of Penner's compactification

Theorem: There is a CW cell model for this PROP whose cellular chains act in a dg PROP fashion on the Hochschild Cochains of a Frobenius algebra extending the action of D's conjecture.

Two roads to take in the closed case: boundary/puncture

Boundary -> topology

F is a surface with boundary and marked points on the boundary.

Possible interpretations:

1. Cobordisms (forget details of the trajectory)
2. Surfaces with arcs.
3. Operations on the free loop space of a manifold

Algebraic Applications:

1. Action on algebras
2. Cell realization of algebras

Note: all roads lead to Rome, where Rome is the moduli space of

Riemann surfaces

Puncture+cpx structure -> Algebraic geometry.

Remove the boundary so that F is a punctured surface

and

endow F with a complex structure.

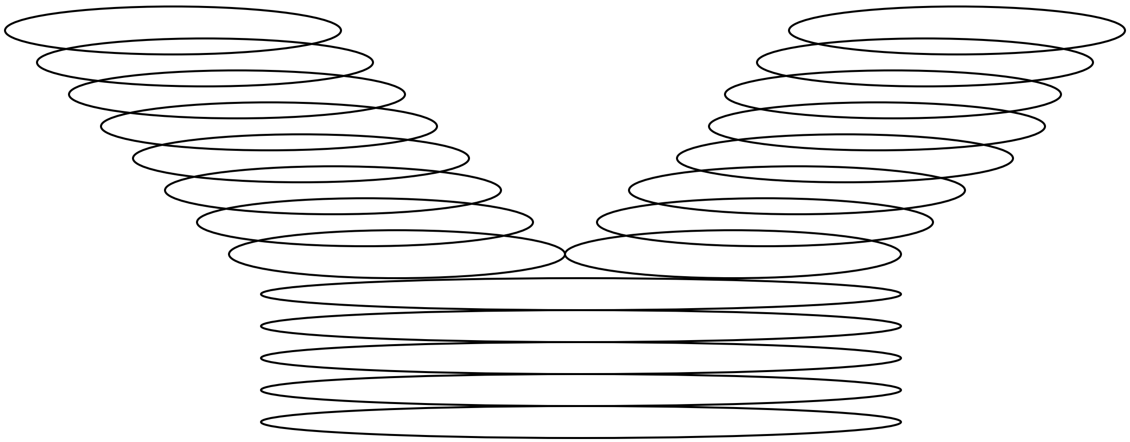
Interpretations:

Gromov-Witten invariants, viz moduli spaces of stable maps

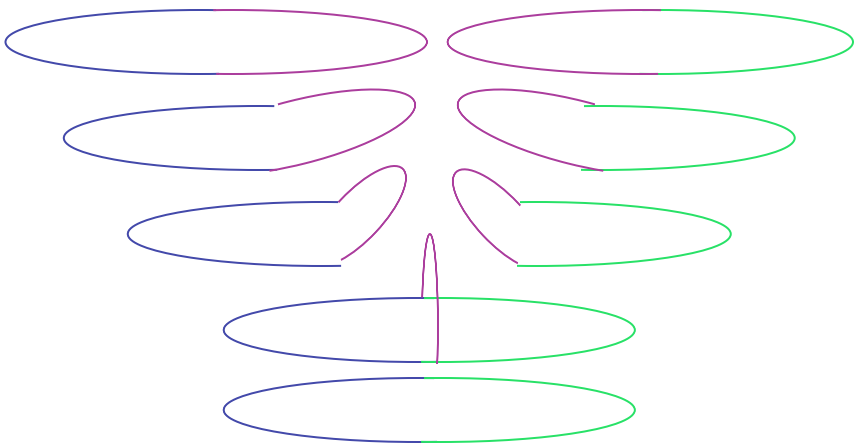
Applications

1. Enumerative problems
2. Deformations of algebras

Naïve Strings Moving & Joining

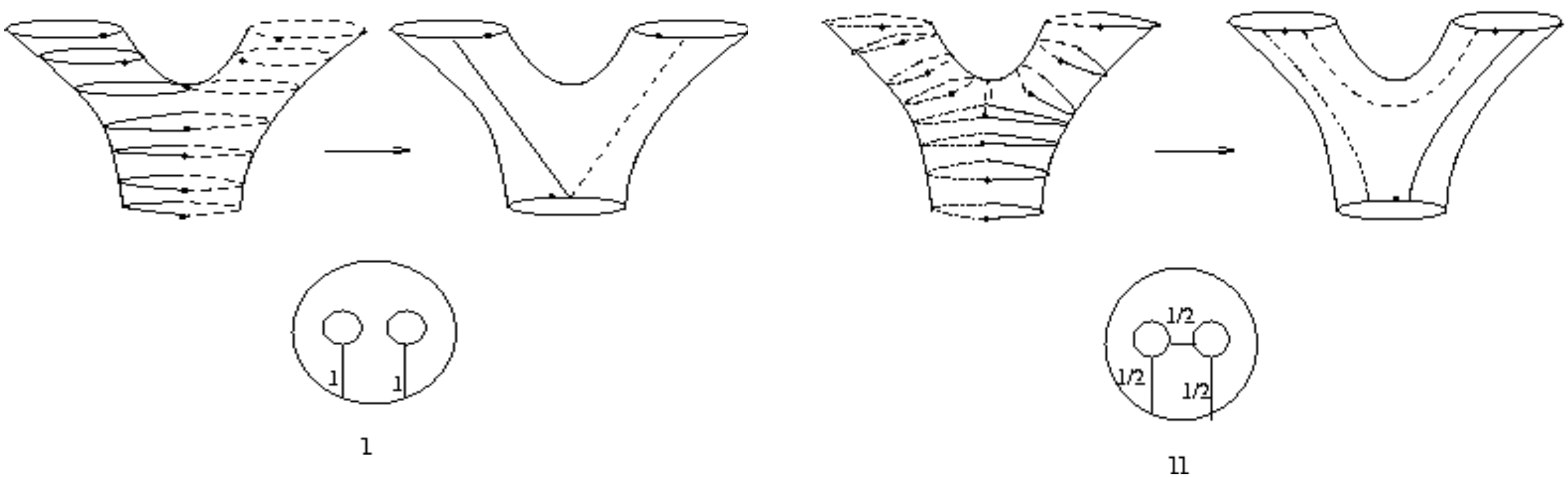


Moving & Annihilating



Moving Strings as Arcs

- **String Interpretation**
- Think of strings moving from boundaries to boundaries on a surface. They may break up and recombine (keep track of length) → bands/ arcs (with weights)



The Arc Spaces

Fix $F^{s}_{g,n+1}$

Fix a window on each boundary component.

Consider arcs running from window to window.

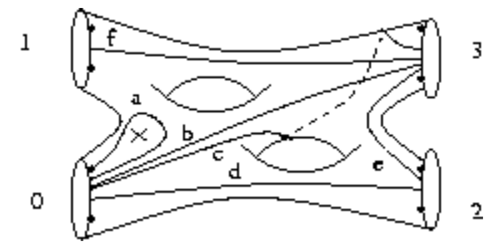
An *essential* arc is an arc which is not homotopic to a part of the boundary.

$S^{s}_{g,r}$:={Homotopy classes of non-empty collections of mutually non-intersecting & non-parallel essential arcs}

$A^{s}_{g,r}$:= (The simplicial realization of the poset $S_{g,s,r}$ whose partial order is given by inclusion) /$\mathcal{PMC}$

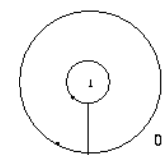
$A^{s}_{g,r}$ is a cell complex.

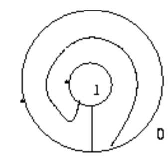
An element of $A^{s}_{g,r}$ is a surface with projectively weighted arcs or bands.



A surface of genus 2 with 1 puncture
4 boundaries and 5 arcs

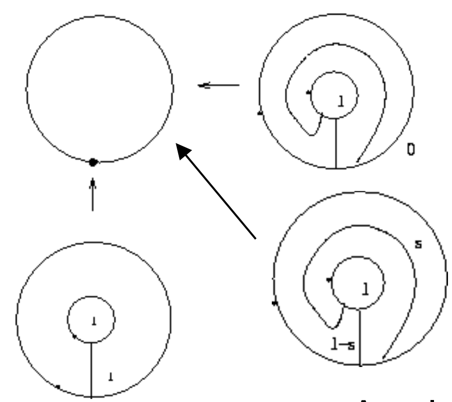
A vertex





A 1-simplex

The space $A^{0}_{0,2} = S^{1}$



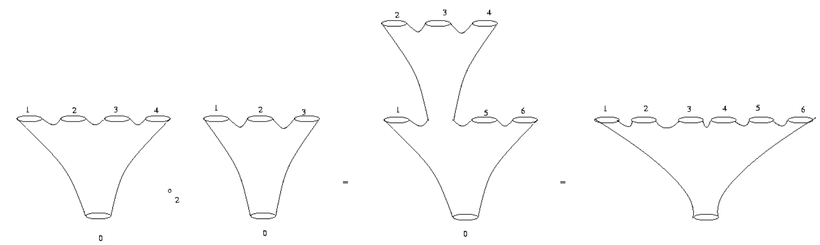
An element of $A^{2}_{0,0}$

The Arc Operad

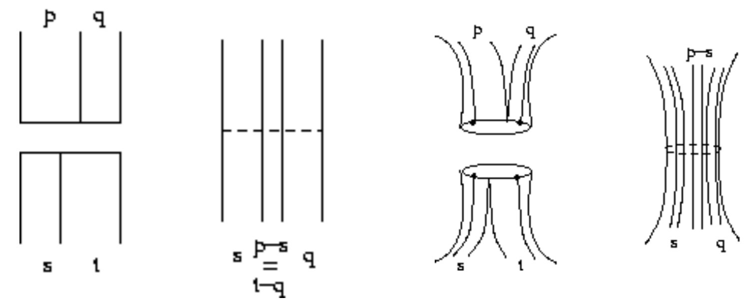
Definition: We define $Arc^s_g(n)$ to be subspaces of $A^s_{g,n+1}$ which consist of all families that hit all boundary components and write Arc(n) for the disjoint union over g and s of the spaces $Arc^s_g(n)$.

Fix two elements of Arc

1. Scale such that the weights at i and 0 agree.
2. Glue the surfaces.
3. Cut and glue the bands according to their least common partition.



Gluing surfaces



Gluing bands of equal total weight

Theorem 1 [KLP]. The gluings above together with the permutation action on the labels endow the collection Arc(n) with the structure of a operad. This operad is also cyclic.

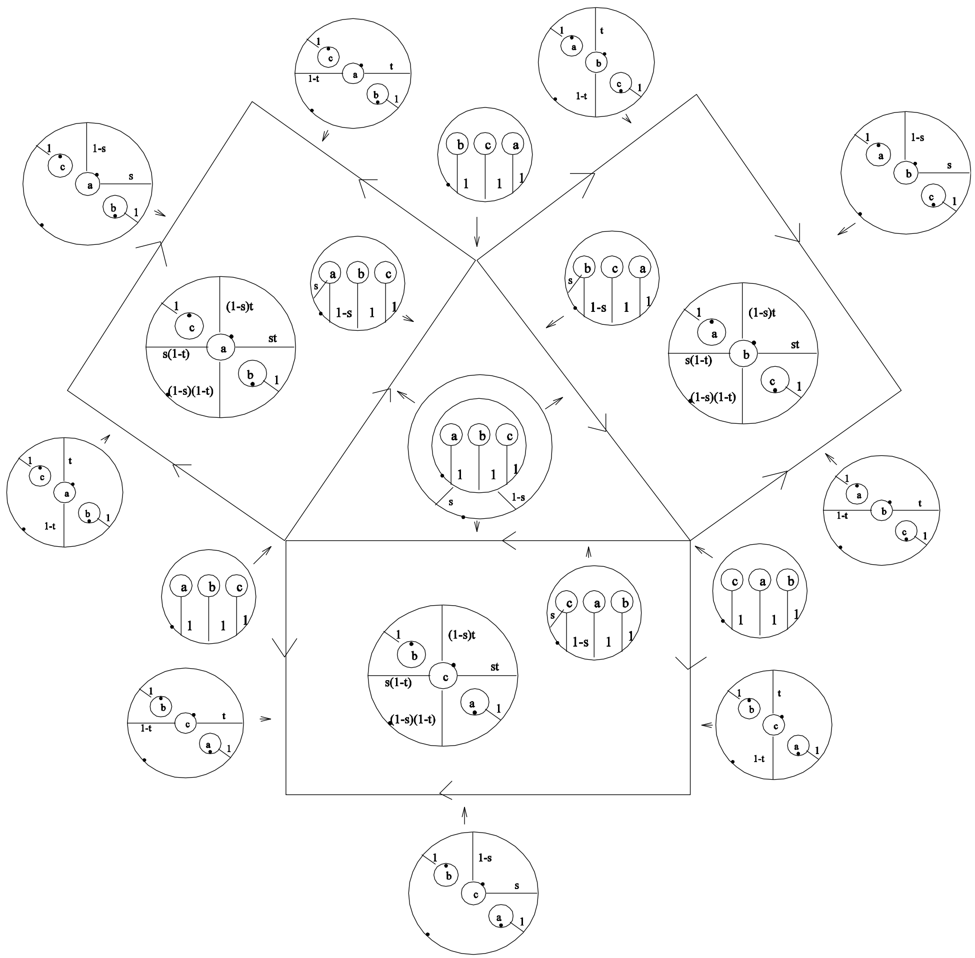
Suboperads of the Arc Operad

Suboperad	Condition
$Arc_{\#}$	All complementary regions are polygons or once punctured polygons
$Arc^0_{\#}$	Same as $Arc_{\#}$ and s=0.
GTree	Only arcs running from 0 to i.
Tree	Same as GTree and g=s=0
LinTree	Same as Tree and linear orders on the arc are compatible

Theorem 2

Suboperad	is homotopy equivalent to
$Arc_{\#}$	[Penner] Decorated Moduli space $M^{dec}{}_{g,n}{}^{s}$
$Arc^0_{\#}$	[K] $M^n{}_{g,n}$
Tree	[K] Voronov's Cacti [K,V] Framed little discs
LinTree	[K] Spineless cacti [K] Little discs

The BV relation



Arcs and Ribbon Graphs and Moduli space

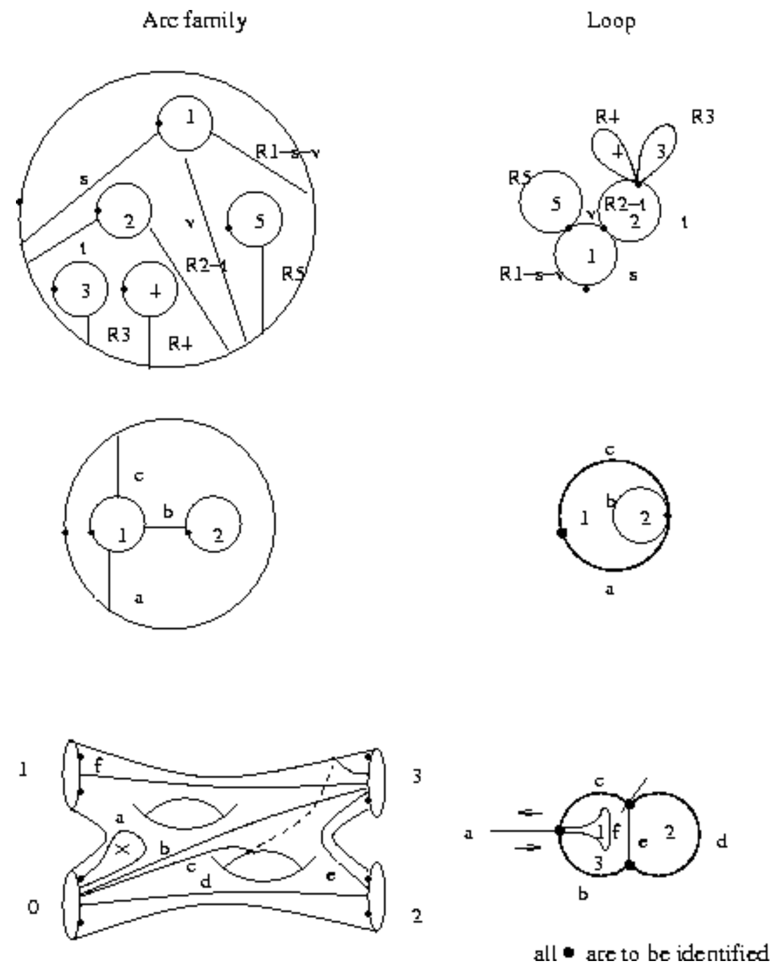
Proposition [K]: The dual graph gives a map from $Arc^0_\#$ to ribbon graphs with a projective metric and a marked point on each boundary cycle.

Corollary: $Arc_\#{}^0{}_{g,n} = M^n{}_{g,n}$

Remarks:

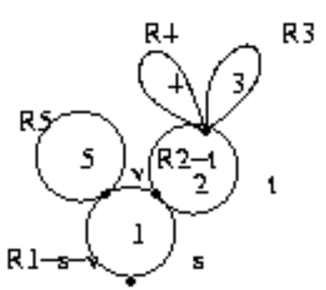
1.This is a specialization of the map Loop of [KLP],defined for Arc.

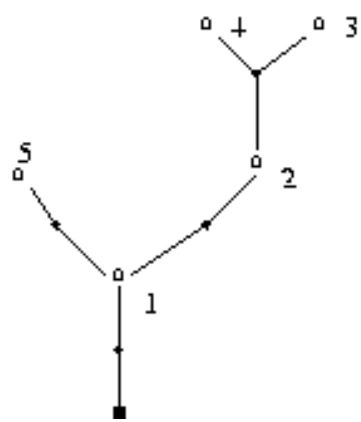
2.From this map we obtain the usual pictures of cacti and spineless cacti up to an overall projective scaling.



A proof of D’s conjecture via the cellular chains of Cact^1

- **Definition**: The incidence graph of a cactus is the graph which has
  - a black vertex for each intersection point and the global marked point,
  - a white vertex for each lobe
  - edges between black and white vertices if the point of the black vertex lies on the lobe of the white vertex.
- **Definition**: The topological type of a cactus is the planar, planted bi-partite tree given by its incidence graph.





- Let $\mathrm{Cact}^1(n)$ be the spineless cacti whose lobes all have size 1 (normalized).
- **Proposition**: The normalized spineless cacti of a given topological type form a cell and these cells form a cellular decomposition of $\mathrm{Cact}^1(n)$.
- **Theorem 7** [K]: The cellular chains $CC_*(\mathrm{Cact}^1(n))$ form an operad and a cell model for Cact and thus for the little discs operad.

Proof of Deligne conjecture [K]: Using the cellular chains $CC_*(\mathrm{Cact})$ regard the tree indexing the cell as a flow chart to obtain the action on Hochschild cochains.
  - black vertex → multiplication
  - white vertex → brace operation

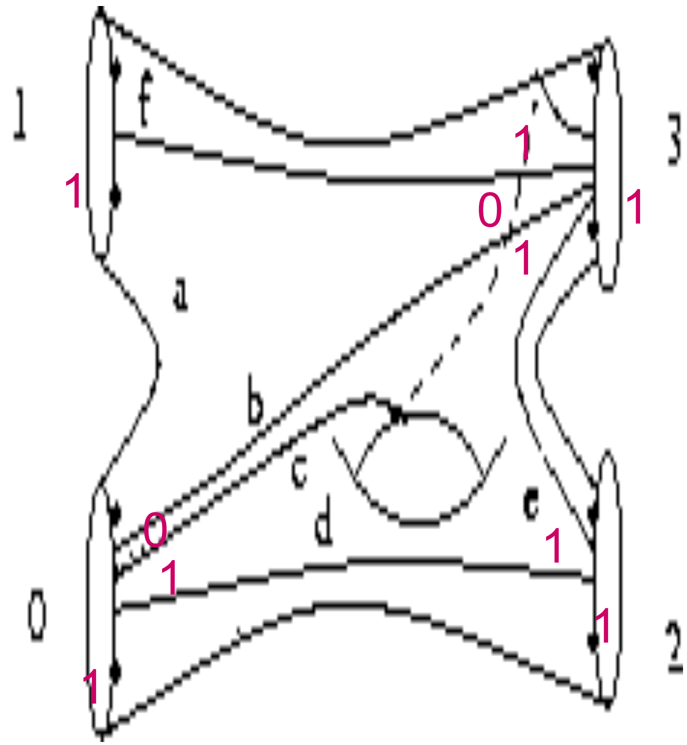
Moduli Space actions

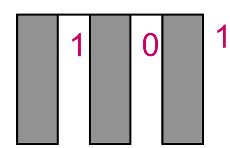
The operations are defined for surfaces with Z/2Z marked angles:

- There are Operad, PROP and dg versions of this operation
- These lift the action of the homology of the little discs operad on the Hochschild cohomology either directly of indirectly. They use the isomorphisms

$CH^n(A,A)\cong A^{\otimes n}\otimes A^* \cong A^{\otimes n+1}\cong CC^n(A)$

Since A is Frobenius these identifications are dg.





Decorate the intra-spaces or angles of the graph by 0 or 1
Usually, we decorate the outside angle by 1

Definition of the action

Using the above isomorphisms the operations are in

$\mathrm{Hom}(CH^*(A,A)^{\otimes k}, CH^*(A,A)^{\otimes l}) \cong \bigoplus_{n,m} \mathrm{Hom}(A^{\otimes n1} \otimes \dots \otimes A^{\otimes nk}, A^{\otimes m1} \otimes \dots \otimes A^{\otimes mk}) \cong A^{*\otimes |n|+|m|}$

We will give the homogenous components corresponding to a surface.

We have to be careful however that these identifications are *not dg* for the Hom complex. (more on this later)

The operations:

1. For a surface with arcs S with k+l boundaries. Fix n1, …,nk,m1,…,ml
2. Duplicate the arcs on the i-th boundary, such that there are n_i respectively m_{i+k} angles with decoration 1. The new angles are all decorated by 1. If this is not possible the operation is 0.
3. The complementary of the arcs are surfaces with boundary.
4. Decorate the angles marked by 1 by the elements of A.

$$(a1,\dots,aN)_S \rightarrow \prod_{\text{complementary regions } P} \int \prod_{I=\text{angles decorated by } 1} a_i e^{-\chi(p)+1}$$

Here $\int a = \langle a,1 \rangle$

The Operations

Moduli space.

1. Fix $S \in Arc^0_{\#g}(n)$. Complementary of the regions surface with arcs are polygons
2. Decorate all angles by 1.

Integrate around the polygon.

Theorem: These are operadic correlation functions, that is they induce the structure of a cyclic operad.

Note:

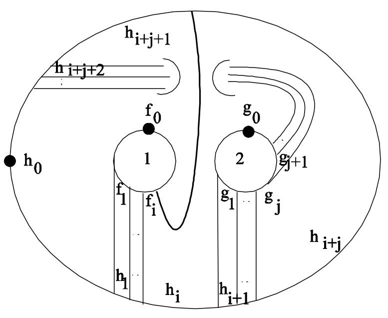
1. Can lift to Z(A,d).
2. The correlators are Feynman rules for the dual graph, and they differ from Kontsevich's CFT correlators for A_∞ algebras.

Sullivan chord diagrams

1. Fix S, with boundaries divided into Ins and Outs, and arcs only running from In to Out. Moreover all In boundaries are hit.
2. Decorate all inner In angles by 1, all inner Out angles by 0 and all outer angles by 0.
3. Extend the PROP structure to the cells of these graphs.

Theorem: If A is commutative Frobenius algebra then the correlators yield a dg-PROP action on the reduced Hochschild co-chains.

The corresponding pictures



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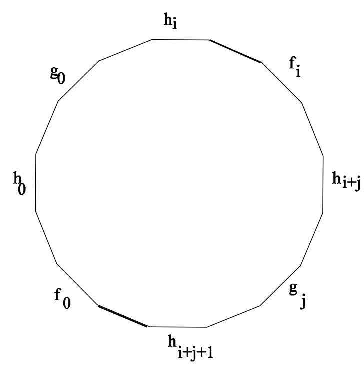

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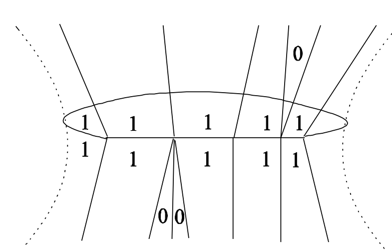
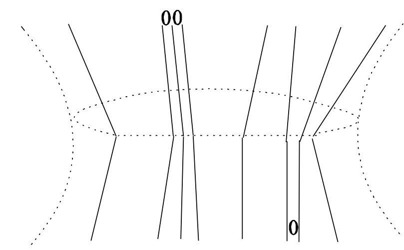
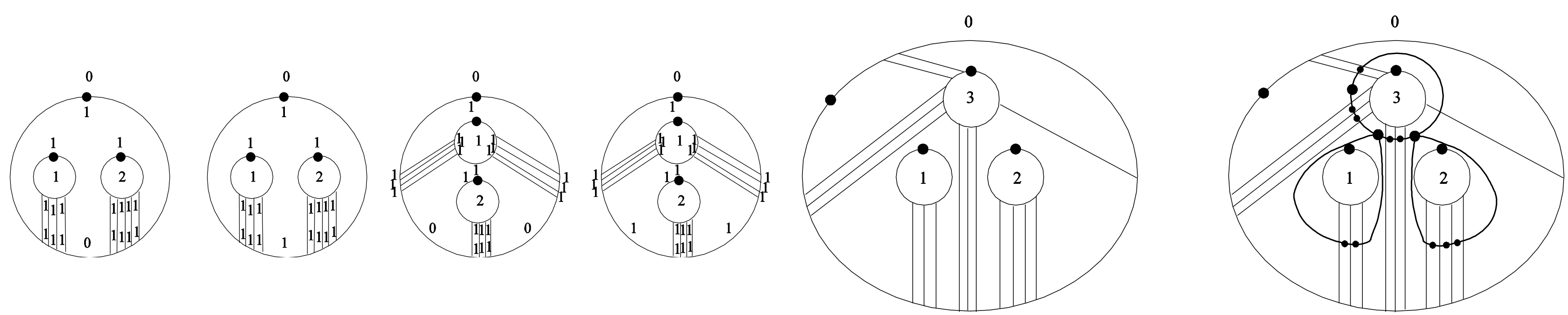
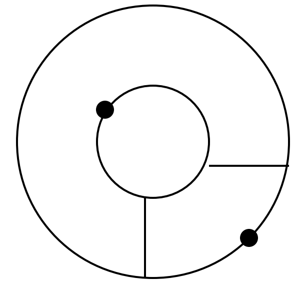


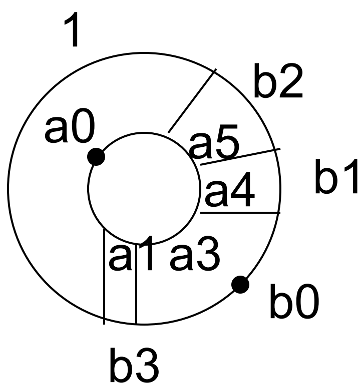




Example:

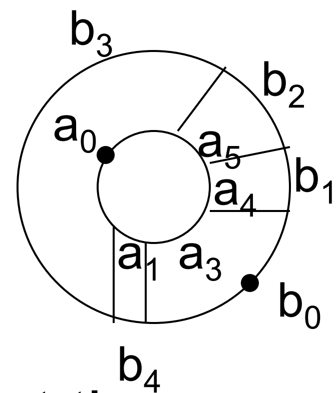


- PROP setting:
  Summands of the type



- For reduced chains get Connes' operator B with $B^2=0$.

- Moduli Setting



- Get the operator N, that is the $1+t+t^2+\ldots$
- This corrseponds to the action of the functor operad Ξ_2 of McClure and Smith.

Moduli Space actions

Proposition. There is a natural filtration on a suboperad of the endomorphism operad of the Hochschild co-chains such that the operad structure is compatible and descends to the associated graded.

Remark: If one is careful about the signs, actually Delinge's conjecture is only true for a suboperad Brace whose degrees are suitably shifted.

Theorem: The operadic correlations functions make the associated graded into an operad on the chain level over the moduli space chain-complex.

Use different decoartion. Decorate half edges not angles

Theorem: There are also actions on a Vector space with a perfect pairing, or a dg-vector space with a perfect parining on (co)-homology.

Outlook

- **Operation of cacti on the cyclic cohomolgy:** Treat *S^1* as a co-simplicial, cyclic object.
- **The A_∞ analog of Deligne's conjecture:** Use a cell decomposition in terms of associahdra and cyclohedra [KSchw].
- **Little k-cubes, k-fold loop spaces and higher genus:** Realizing sequences of fixed complexity on surfaces of higher genus gives a criterion for Tot(X) to be a k-fold loop space.
- **Open/Closed case.** Have a description for the open/closed CFT with Penner.

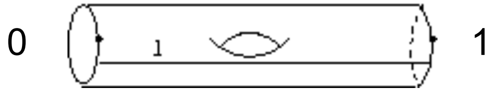
- **Koszul dual Gromov-Witten invariants:** Operation of $Arc_\#$ on the free loop space as an extension of string topology.
- **Relations to Polylogs:** Using the relationship $M_{g,n}(\mathbf{R})$ lift the combinatorial correspondence of the *Arc* operad to algebras describing polylogs to the motivic level.
- **Dwyer-Lashof-Cohen operations:** in both loop space and Hochschild.
- **Rankin-Cohen brackets:** Find a moduli/surface interpretation of these operations on modular forms which can also be described by the geometry of foliations and a tree Hopf algebra.

Chinese Trees and Infinite loop spaces

Definition: The elements in the complement of *$Arc_{\#}$* are called non-effective.

Let *Arc^{ctd}* be the suboperad of connected arc families.

Example: The genus operator Op_g is the arc family



Definition: We define

$StArc_0(n) := \varinjlim (Arc_{g,0}^{ctd}(n))$

where the limit is taken with respect to the system

$$\alpha \to \alpha \circ_i Op_g, \quad \alpha \to Op_g \circ_i \alpha$$

where $Op_g \in Arc^{ctd}_{1,0}(2)$ is non-effektive.

Theorem 9 [K]**:** The spaces $StArc_0(n)$ form an operad.

Theorem 10 [K]**:** The operad $StArc_0(n)$ detects infinite loop spaces, i.e. if X admits an operadic action of $StArc_0(n)$ then it has the homotopy type of an infinite loop space.

Corollary [K]**:** $StArc_0(n)$ has the homotopy type of an infinite loop space.

This can be to be compared to the theorems of Tillmann and Madsen on infinite loop spaces.

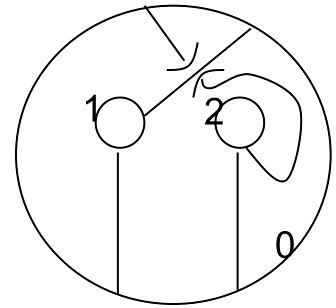
Arcs, Little k-Cubes and operation

Theorem 11: The suboperad of stabilized linear chinese trees has an operadic filtration $StGTree^g$ in terms of effective genus.

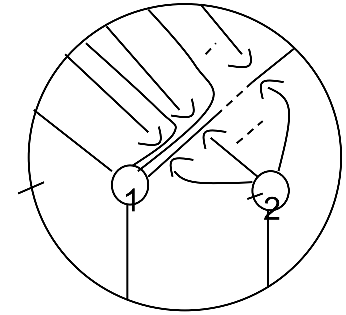
- The operad linear $StGTree^g$ is isomorphic to the little 2g cubes operad.
- Get cells for the $\cup_i$-operations
- A finer filtration gives all k-cubes

Theorem 12: There is a simple description of the cells giving the Dyer-Lashof-Cohen (Araki-Kudo) operations in terms of cells of $CC_*(Cact^1)$ which correspond to the operations found by Tourtchin and Westerland.

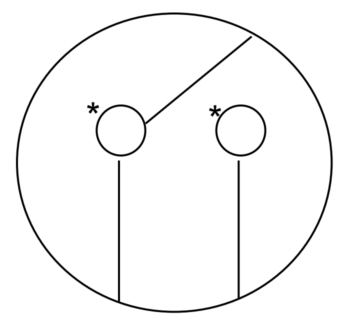
Conjecture: Can also find representatives for the higher operations.



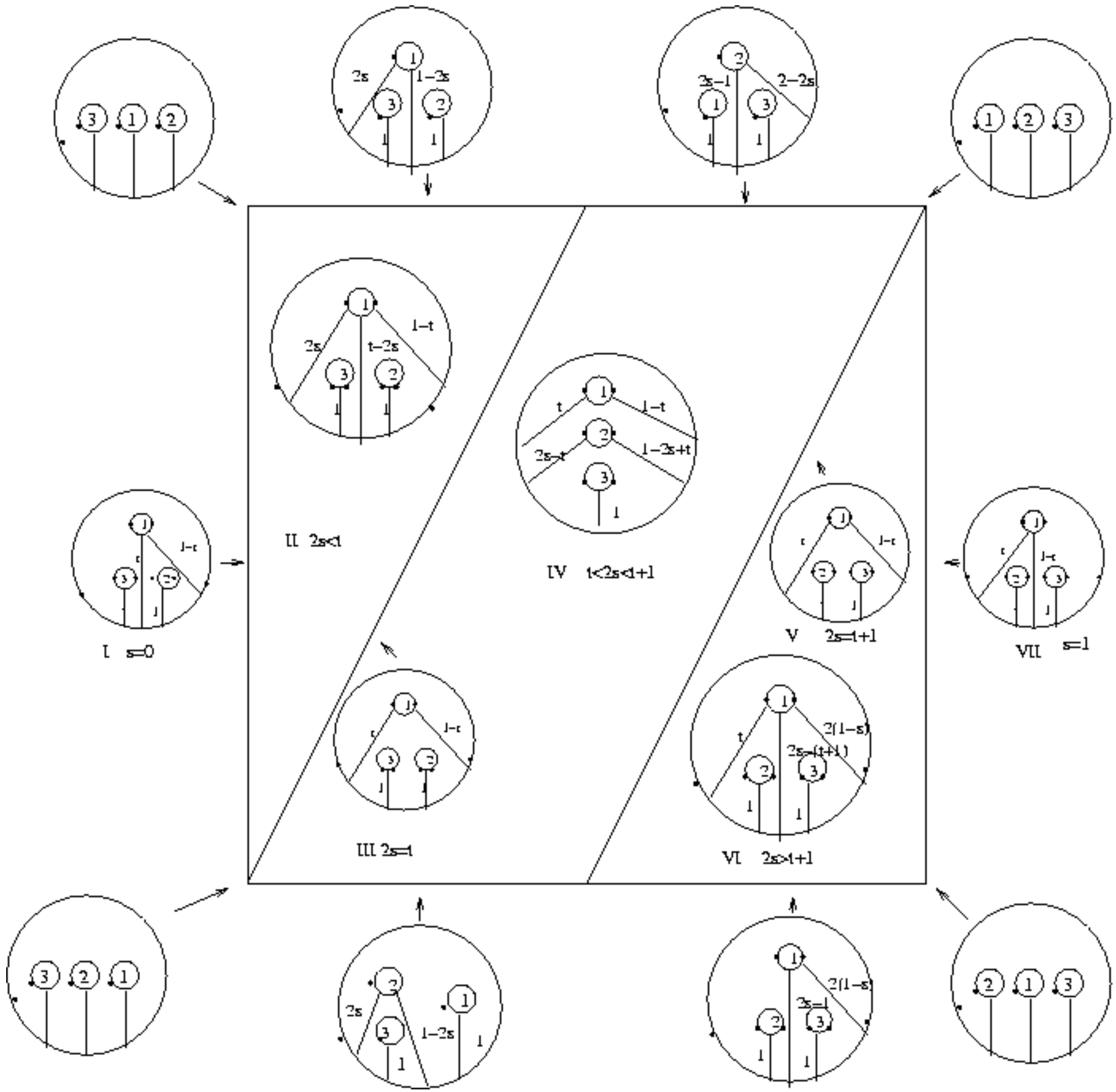
The $\cup_2$-operation



The $\cup_i$-operation



The first Araki-Kudo operation



The first Dyer-Lashof-Cohen operation

A cell interpretation of Hopf algebra of Connes and Kreimer

Connes and Kreimer defined a Hopf algebra based on trees whose antipode describes the addition and subtraction of counterterms in renormalization.

It can also be described as

$$H_{CK}=U^*(L)$$

where L is the Lie algebra associated to the free pre-Lie algebra in one generator.

Theorem 7 [K]: The shifted symmetric top-dimensional chains of $CC_*(Cact)$ form a Lie algebra whose S_n coinvariants are isomorphic to the Lie algebra L above.

By [K] there is a canonically associated Hopf algebra to an operad which affords a direct sum

$$\mathcal{O} \rightarrow \mathcal{L} \rightarrow U^*(\mathcal{L})$$

generalizing the formula

$$L \rightarrow U^*(L)$$

for Lie Algebras.

Theorem 8 [K]: The symmetric top-dimensional chains of $CC_*(Cact)$ form an operad which is isomorphic to the operad whose algebras are graded pre-Lie algebras.

The shifted symmetric chains are isomorphic to the operad for pre-Lie algebra.

The S_n-coinvariants.of the Hopf algebra of this operad are isomorphic to H_{CK}.

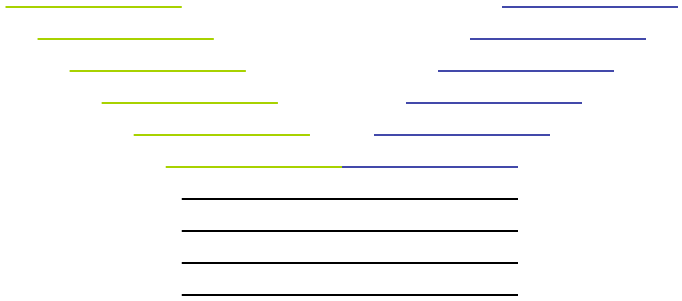
Selected References

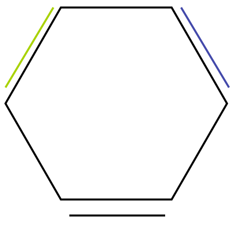
1. Kaufmann, Ralph M.; Livernet, Muriel and Penner, Robert C. *"Arc Operads and Arc Algebras".* Geometry and Topology 7 (2003), 511-568.
2. Kaufmann, Ralph M. *"On several varieties of cacti and their relations''.* AGT 5 (2005), 237–300.
3. Kaufmann, Ralph M. *"On Spineless Cacti, Deligne's Conjecture and Connes-Kreimer's Hopf Algebra." Preprint, math.QA/0308005. Topology to appear.*
4. Kaufmann, Ralph M *. "A proof of a cyclic version of Deligne's conjecture''. Preprint math.QA/0403340*
5. Kaufmann, Ralph M. "*Moduli Actions on the Hochschild co-chains of a Frobenius algebra I:Cell models*". math.AT/0606064
6. Kaufmann, Ralph M. "*Moduli Actions on the Hochschild co-chains of a Frobenius algebra II:correlators*". Math.AT/0606064

An overview of operads and the results above as well as an introduction to operads will appear in

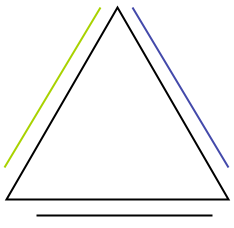
7. Kaufmann, Ralph M. *Operads, Strings and Deligne'sconjecture. Advanced Series on Mathematical Physics. WorldScientific.*

Open strings
Moving & Joining





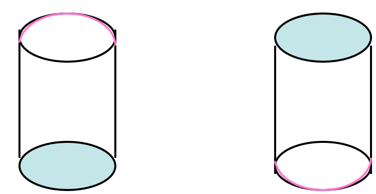
or



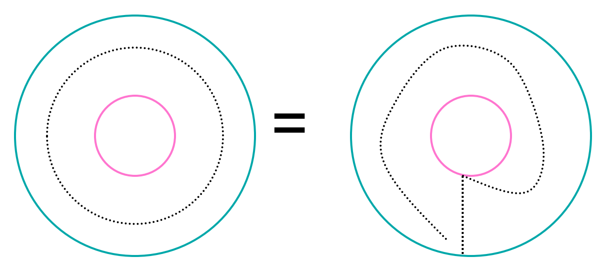
What about open strings

- Closed -> commuative Frobenius algebra +deformations or operations on its Hochschild complex.
- Open -> symmetric algebras +? or ?
- Open/closed a tuple(A,C) of a symmetric and a commutative Frobenius algebra with operations and relations +? or ?

- Open string closes
- Closed string opens



- Cardy eqation



Note: different colors<->D-brane labels

Closed/open String diagrammatics (with Penner)

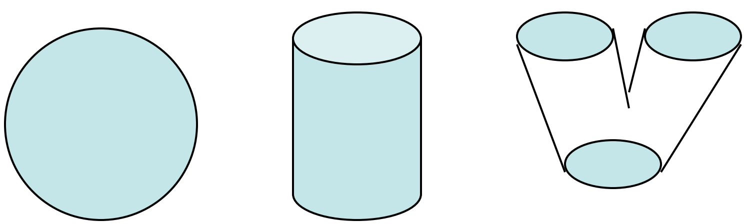
- Solved one of the ?’s.
- A model for open/closed TCTF
- **Theorem:** There is an (new type of) operad/ PROP structure on an open/closed string version using arcs. This recovers Cardy and all known equations *and* gives new operations and ***new equations***
- **BRST:** Get the right closed BRST operator. It pulls back to an open BRST, which seems to exhibit the right properties, i.e. reproduces the Warner term (further study needed)

Cobordisms & TFT

I data

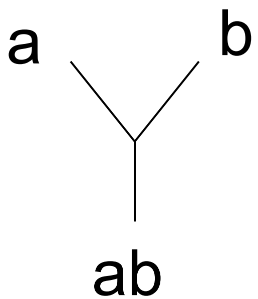
Geometry

- Surfaces with boundary.
- Decompostions gives the elementary pieces



Algebra

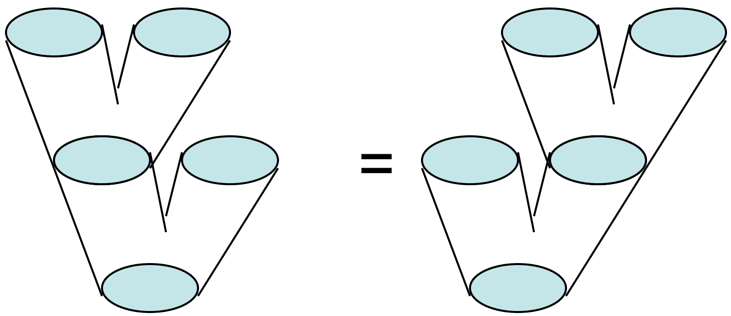
- Boundaries=inputs/ outputs of multilinear operations on a vector space -> flow chart
- Three operations -> Algebra with unit
- Unit: K->V, id: V-> V and m: $V\otimes V$ ->V



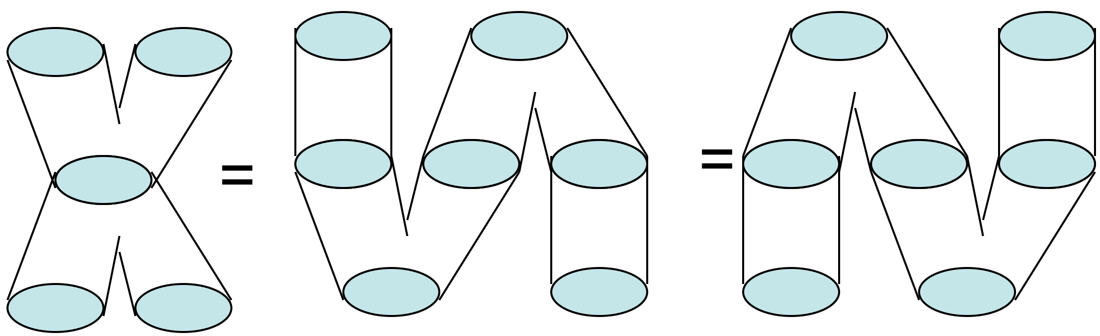
Note: To go from left to right one uses functors and equivalences of categories

Cobordisms & TFT

II axioms



- V is an associative algebra (ab)c=a(bc)

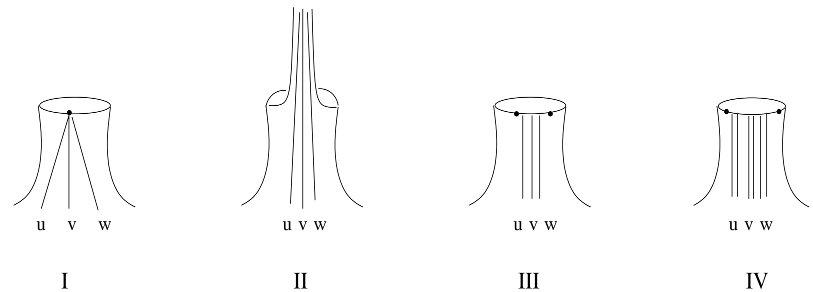


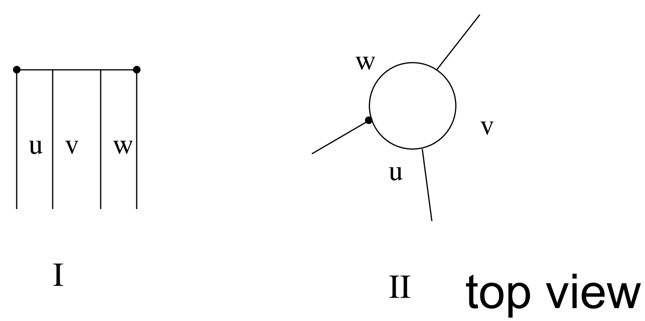
- V is a Frobenius algebra.

Note: If one adds orientation one gets an algebra with a perfect-paring < , > and <ab,c>=<a,bc>

Different pictures of Arcs

- Using projectivized barycentric coordinates: A point in $A_{g,s,r}$ is described by a surface with projectively weighted arcs.
- There are different pictures for arcs. Choosing a measure on *F* we can think of arcs as bands given by a partially measured foliation.
- We will also contract the outside of the window and the space between the bands
- The weight of a boundary is the sum of the weights of the bands incident to the boundary.
- Notice in picture I we get a surface with a graph whose vertices are the marked points on the boundary.



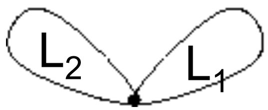


String Topology

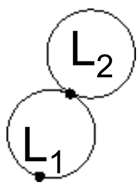
The basic idea:

Three compositions for loops

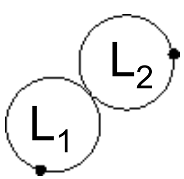
1. Compose loops in the based loop space



2. Compose loops L_1 and L_2 if the basepoint of L_2 lies on L_1.



3. Compose loops L1 and L2 if L1 and L2 intersect in a one point.



Chas-Sullivan's String topology:

Claim [CS]: 2 and 3 lead to operations on the chain level of the free loop space of a compact manifold and thus to a GBV structure on the homology of the loop space space.

With the formulation of [CJ,V,K,KLP]

1. is described by corollas
2. is described by spineless cacti
3. is described by cacti

Theorems:

[Merk] gives the multiplication in the case of a simply connected manifold via Hochschild.

[CJ] give the multiplicative structure.

[CV] explain BV structure.

Idea: [K] BV structure via the cyclic Deligne conjecture.