Feynman categories

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References

References

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Goals

Main Objective

Provide a *lingua universalis* for operations and relations in order to understand their structure.

Internal Applications

1. Realize universal constructions (e.g. free, push–forward, pull–back, plus construction, decorated).
2. Construct universal transforms. (e.g. bar,co–bar) and model category structure.
3. Distill universal operations in order to understand their origin (e.g. Lie brackets, BV operatos, Master equations).
4. Construct secondary objects, (e.g. Lie algebras, Hopf algebras).

Applications

Applications

- Find out information of objects with operations. E.g. Gromov-Witten invariants, String Topology, etc.
- Find out where certain algebra structures come from naturally: pre-Lie, BV, ...
- Find out origin and meaning of (quantum) master equations
- Find background for certain types of Hopf algebras.
- Find formulation for TFTs.
- Transfer to other areas such as algebraic geometry, algebraic topology, mathematical physics, number theory.

Plan

Warm up I

Operations and relations for Associative Algebras

- Data: An object A and a multiplication $\mu : A \otimes A \to A$
- An associativity equation $(ab)c = a(bc)$.
- Think of μ as a 2-linear map. Let $\circ_1$ and $\circ_2$ be substitution in the 1st resp. 2nd variable: The associativity becomes

$$\boxed{\mu \circ_1 \mu = \mu \circ_2 \mu : A \otimes A \otimes A \to A}.$$

$$\mu \circ_1 \mu(a, b, c) = \mu(\mu(a, b), c) = (ab)c$$
$$\mu \circ_2 \mu(a, b, c) = \mu(a, \mu(b, c)) = a(bc)$$

- We get n–linear functions by iterating μ: $a_1 \otimes \cdots \otimes a_n \to a_1 \ldots a_n$.
- There is a permutation action $\tau\mu(a, b) = \mu \circ \tau(a, b) = ba$
- This give a permutation action on the iterates of μ. It is a free action there and there are $n!$ n–linear morphisms generated by μ and the transposition.

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Warm up II

Categorical formulation for representations of a group G.

- $\underline{G}$ the category with one object $*$ and morphism set G.
- $f \circ g := fg$.
- This is associative ✓
- Inverses are an extra structure $\Rightarrow$ $\underline{G}$ is a groupoid.
- A representation is a functor ρ from $\underline{G}$ to $\mathcal{V}ect$.
- $\rho(*) = V$, $\rho(g) \in Aut(V)$
- Induction and restriction now are pull–back and push–forward (Lan) along functors $\underline{H} \to \underline{G}$.

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Feynman categories

Data

1. $\mathcal{V}$ a groupoid
2. $\mathcal{F}$ a symmetric monoidal category
3. $\imath : \mathcal{V} \to \mathcal{F}$ a functor.

Notation

$\mathcal{V}^{\otimes}$ the free symmetric category on $\mathcal{V}$ (words in $\mathcal{V}$).

$$\begin{tikzcd} \mathcal{V} \arrow[rr, "\imath"] \arrow[dr, "\jmath"'] & & \mathcal{F} \\ & \mathcal{V}^{\otimes} \arrow[ur, "\imath^{\otimes}"'] & \end{tikzcd}$$

Feynman category

Definition

Such a triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ is called a Feynman category if

1. $\imath^{\otimes}$ induces an equivalence of symmetric monoidal categories between $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$.
2. $\imath$ and $\imath^{\otimes}$ induce an equivalence of symmetric monoidal categories between $Iso(\mathcal{F} \downarrow \mathcal{V})^{\otimes}$ and $Iso(\mathcal{F} \downarrow \mathcal{F})$.
3. For any $* \in \mathcal{V}$, $(\mathcal{F} \downarrow *)$ is essentially small.

"Algebras" over Feynman categories: $\mathcal{O}ps$ and $\mathcal{M}ods$

Definition

Fix a symmetric monoidal category $\mathcal{C}$ and $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ a Feynman category.

- Consider the category of strong symmetric monoidal functors $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} := Fun_{\otimes}(\mathcal{F}, \mathcal{C})$ which we will call $\mathcal{F}$–ops in $\mathcal{C}$
- $\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}} := Fun(\mathcal{V}, \mathcal{C})$ will be called $\mathcal{V}$-modules in $\mathcal{C}$ with elements being called a $\mathcal{V}$–mod in $\mathcal{C}$.

Theorem

The forgetful functor $G : \mathcal{O}ps \to \mathcal{M}ods$ has a left adjoint F (free functor) and this adjunction is monadic.

Theorem

Feynman categories form a 2–category and it has push–forwards $f_* = f_!$ and pull–backs f^* for $\mathcal{O}ps$ and $\mathcal{M}ods$.

Examples based on $\mathfrak{G}$: morphisms have underlying graphs

$\mathfrak{F}$	Feynman category for	condition on graphs additional decoration
$\mathfrak{O}$	operads	rooted trees
$\mathfrak{O}_{mult}$	operads with mult.	b/w rooted trees.
$\mathfrak{C}$	cyclic operads	trees
$\mathfrak{G}$	unmarked nc modular operads	graphs
$\mathfrak{G}^{ctd}$	unmarked modular operads	connected graphs
$\mathfrak{M}$	modular operads	connected + genus marking
$\mathfrak{M}^{nc,}$	nc modular operads	genus marking
$\mathfrak{D}$	dioperads	connected directed graphs w/o directed loops or parallel edges
$\mathfrak{P}$	PROPs	directed graphs w/o directed loops
$\mathfrak{P}^{ctd}$	properads	connected directed graphs w/o directed loops
$\mathfrak{D}^{\circlearrowleft}$	wheeled dioperads	directed graphs w/o parallel edges
$\mathfrak{P}^{\circlearrowleft,ctd}$	wheeled properads	connected directed graphs
$\mathfrak{P}^{\circlearrowleft}$	wheeled props	directed graphs

Table: List of Feynman categories with conditions and decorations on the graphs, yielding the zoo of examples

Examples on $\mathfrak{G}$ with extra decorations

Decoration and restriction allows to generate the whole zoo and even new species

$\mathfrak{F}_{dec\mathcal{O}}$	Feynman category for	decorating $\mathcal{O}$	restriction
$\mathfrak{F}^{dir}$	directed version	$\mathbb{Z}/2\mathbb{Z}$ set	edges contain one input and one output flag
$\mathfrak{F}^{rooted}$	root	$\mathbb{Z}/2\mathbb{Z}$ set	vertices have one output flag.
$\mathfrak{F}^{genus}$	genus marked	$\mathbb{N}$	
$\mathfrak{F}^{c-col}$	colored version	c set	edges contain flags of same color
$\mathfrak{O}^{\neg\Sigma}$	non-Sigma-operads	*Ass*	
$\mathfrak{C}^{\neg\Sigma}$	non-Sigma-cyclic operads	*CycAss*	
$\mathfrak{M}^{\neg\Sigma}$	non–Signa-modular	*ModAss*	
$\mathfrak{C}^{dihed}$	dihedral	*Dihed*	
$\mathfrak{M}^{dihed}$	dihedral modular	*ModDihed*	

Table: List of decorates Feynman categories with decorating $\mathcal{O}$ and possible restriction. $\mathfrak{F}$ stands for an example based on $\mathfrak{G}$ in the list.

Hereditary condition (ii)

1. In particular, fix $\phi : X \to X'$ and fix $X' \simeq \bigotimes_{v \in I} \imath(*_v)$: there are $X_v \in \mathcal{F}$, and $\phi_v \in Hom(X_v, *_v)$ s.t. the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\simeq"'] & X' \arrow[d, "\simeq"] \\ \bigotimes_{v \in I} X_v \arrow[r, "\bigotimes_{v \in I} \phi_v"] & \bigotimes_{v \in I} \imath(*_v) \end{tikzcd} \quad (1)$$

2. For any two such decompositions $\bigotimes_{v \in I} \phi_v$ and $\bigotimes_{v' \in I'} \phi'_{v'}$ there is a bijection $\psi : I \to I'$ and isomorphisms $\sigma_v : X_v \to X'_{\psi(v)}$ s.t. $P_\psi^{-1} \circ \bigotimes_v \sigma_v \circ \phi_v = \bigotimes \phi'_{v'}$ where P_ψ is the permutation corresponding to ψ.
3. These are the only isomorphisms between morphisms.

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Simplification: weak hereditary condition

Proposition

If $(\mathcal{F}, \otimes)$ *has a fully faithful functor to* $(\mathcal{S}et, \amalg)$ *then it is enough to check that* (1) *exists and that is unique up to isomorphism.*
Moreover the existence of (1) *is equivalent to*

Remark

This is not the case for k–linear $\mathcal{F}$.
It is the case for the usual versions of operad–like objects, which all have combinatorial Feynman categories.

Example 1

$\mathcal{F} = Sur$, $\mathcal{V} = \mathbb{I}$

- Sur the category of finite sets and surjection with $\amalg$ as monoidal structure
- $\mathbb{I}$ the trivial category with one object $*$ and one morphism id_*.
- $\mathbb{I}^{\otimes}$ is equivalent to the category with objects $\bar{n} \in \mathbb{N}_0$ and $Hom(\bar{n}, \bar{n}) \simeq \mathbb{S}_n$, where we think $\bar{n} = \{1, \dots, n\} = \{1\} \amalg \dots \amalg \{1\}$, $1 = \imath(*)$.
- $\mathbb{I}^{\otimes} \simeq Iso(Sur)$.

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- $\mathbb{I}^{\otimes} \simeq Iso(Sur)$. ✓
- $T \simeq \{1, \dots, n\}$.

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "\simeq"] & T \arrow[d, "\simeq"] \\
\amalg_{i=1}^{|T|} f^{-1}(i) \arrow[r, "\amalg f|_{f^{-1}(i)}"] & \amalg_{i=1}^{|T|} \imath(*)
\end{tikzcd}$$

Further examples

More examples of this type

1. Finite sets and injections.
2. $\Delta_+ S$ crossed simplicial group.

There is a non–symmetric monoidal version

Example: Δ_+, Order preserving surjections/injections. Joyal duality.

Examples

$\mathcal{M}ods$ and $\mathcal{O}ps$ for Example 1

$\mathcal{M}ods_{\mathcal{C}}$ is just $Obj(\mathcal{C})$ and $\mathcal{O}ps$ are associative algebra objects or monoids in $\mathcal{C}$.

Tautological example

$(\mathcal{V}, \mathcal{V}^{\otimes}, \jmath)$. $\mathcal{M}ods_{\mathcal{C}} \simeq \mathcal{O}ps_{\mathcal{C}}$.
If $\mathcal{V} = \underline{G}$, we recover the motivating example of group theory.
Not so trivial: there is always a morphism of Feynman categories $(\mathcal{V}, \mathcal{V}^{\otimes}, \jmath) \to (\mathcal{V}, \mathcal{F}, \imath)$ and the push–forward along it is the free construction.

Trival $\mathcal{O}$

Let $\mathcal{O} : \mathcal{F} \to \mathcal{C}$ be the functor that assigns $\mathbb{I} \in Obj(\mathcal{C})$ to any object in $\mathcal{V}$, and which sends morphisms to the identity or the unit constraints.

Example 2

The Borisov-Manin category of graphs.

1. A graph Γ is a tuple $(F, V, \partial, \imath)$ of flags F, vertices V, incidence $\partial : F \to V$ and flag gluing $\imath : F^{\circlearrowleft}$. $\imath^2 = id$. Either glue two half-edges to an edge or keep a tail.
2. A graph morphism $\phi : \Gamma \to \Gamma'$ is a triple $(\phi_V, \phi^F, \imath_\phi)$, where $\phi_V : V \to V'$ is a surjection on vertices, $\phi^F : F' \to F$ is an injection and $\imath_\phi : F \setminus \phi^F(F')^{\circlearrowleft}$ a pairing (ghost edges).
3. A graph morphism from a collection of corollas Γ to a corolla $*$ has a ghost graph $\mathbb{\Gamma} = (V_\Gamma, F_\Gamma, \imath_\phi)$.

$\mathfrak{G} = (\mathcal{C}rl, \mathcal{A}gg, \imath)$

$\mathcal{C}rl$ the category of corollas with isomorphisms. $\mathcal{A}gg$ the full subcategory whose objects are aggregates of corollas.

Examples

Roughly (in the connected case and up to isomorphism)

The source of a morphism are the vertices of the ghost graph $\mathbb{\Gamma}$ and the target is the vertex obtained from $\mathbb{\Gamma}$ obtained by contracting all edges. If $\mathbb{\Gamma}$ is not connected, one also needs to merge vertices according to ϕ_V.

Composition corresponds to insertion of ghost graphs into vertices.

$$\begin{tikzcd} X \arrow[r, "\phi_2"] \arrow[rr, bend right, "\phi_0"'] & Y \arrow[r, "\phi_1"] & * \end{tikzcd}$$

up to isomorphisms (if $\mathbb{\Gamma}_0, \mathbb{\Gamma}_1$ are connected) corresponds to inserting $\mathbb{\Gamma}_v$ into $*_v$ of $\mathbb{\Gamma}_1$ to obtain $\mathbb{\Gamma}_0$.

$$\begin{tikzcd} \amalg_v \amalg_{w \in V_v} *_w \arrow[r, "\amalg_v \mathbb{\Gamma}_v"] \arrow[rr, bend right, "\mathbb{\Gamma}_0"'] & \amalg_v *_v \arrow[r, "\mathbb{\Gamma}_1"] & * \end{tikzcd}$$

Graph Examples

$\mathcal{O}ps$

We can restrict the underlying ghost graphs of maps to corollas to obtain several Feynman categories. The $\mathcal{O}ps$ will then yield types of operads or operad like objects.

Types of operads and graphs

$\mathcal{O}ps$	Graphs
Operads	rooted trees
Cyclic operads	trees
Modular operads	connected graphs (add genus marking)
PROPs	directed graphs (and input output marking)
NC modular operad	graphs (and genus marking)
Broadhurst-Connes -Kreimer	1-PI graphs
...	...

Other versions

Enriched version

We can consider Feynman categories and target categories enriched over another monoidal category, such as $\mathcal{T}op$, $\mathcal{A}b$ or $dg\mathcal{V}ect$. Note there are two cases. Either the enrichment is Cartesian, then we simply have to replace all limits by indexed limits. Or, the enrichment is not Cartesian, then there is an extra condition replacing the groupoid condition.

Cartesian case

We proved that in the non–enriched case we can equivalently replace (ii) by (ii').

(ii') The pull-back of presheaves $\imath^{\otimes\wedge}: [\mathcal{F}^{op}, Set] \to [\mathcal{V}^{\otimes op}, Set]$ *restricted to representable presheaves* is monoidal.

This is then yields the definition in the Cartesian case.

Examples on the simple structure

Theorem

The category of Feynman categories with trivial $\mathcal{V}$ enriched over $\mathcal{E}$ is equivalent to the category of operads (with the only iso in $\mathcal{O}(1)$ being the identity) in $\mathcal{E}$ with the correspondence given by $O(n) :=: Hom(\bar{n}, \bar{1})$. The $\mathcal{O}ps$ are now algebras over the underlying operad.

Examples

1. Operad of surjections (corollas), non–symmetric version ordered surjections (planar corollas), simplices (Joyal dual). Operad of leaf labelled rooted trees (gluing at leaves), non–symmetric version planar rooted trees.
2. linear operads. e.g. Ass, Com, Lie, A_∞.
3. $E(k)$, topological, semi-simple operads etc.

Non–trivial examples

Definition

Let $\mathfrak{F}$ be a Feynman category. An enrichment functor is a lax 2–functor $\mathcal{D} : \mathcal{F} \to \underline{\mathcal{E}}$ with the following properties

1. *$\mathcal{D}$ is strict on compositions with isomorphisms.*
2. *$\mathcal{D}(\sigma) = \mathbb{I}_{\mathcal{E}}$ for any isomorphism.*
3. *$\mathcal{D}$ is monoidal, that is $\mathcal{D}(\phi \otimes_{\mathcal{F}} \psi) = \mathcal{D}(\phi) \otimes_{\mathcal{E}} \mathcal{D}(\psi)$*

Theorem

The indexed enriched (over $\mathcal{E}$) Feynman category structures on a given FC $\mathfrak{F}$ are in 1–1 correspondence with $\mathfrak{F}^{hyp}$-$\mathcal{O}ps$ and these are in 1–1 correspondence with enrichment functors.

Twisted (modular) operads.

Looking at $\mathfrak{F} = \mathfrak{M}$, we recover the notion of twisted modular operad. In the cyclic case, an example are anti–cyclic operads.

Odd versions

Odd versions

Given a well-behaved presentation of a Feynman category (generators+relations for the morphisms) we can define an odd version which is enriched over $\mathcal{A}b$.

Odd Feynman categories over graphs

In the case of underlying graphs for morphisms, odd usually means that edges get degree 1, that is we use a Kozsul sign with that degree. More later.

Suspension vs. odd

Suspensions

There is also a twist which realizes suspensions. These are equivalent to the odd version *if* we are in the directed case, see [KWZ12] .

Examples

1. Operads are very special they are equivalent to their odd version.
2. The odd cyclic operads are equivalent to anti–cyclic operads.
3. For modular operads the suspended version is not equivalent to the odd versions a.k.a $\mathfrak{K}$–modular operads. The difference is given by the twist $H_1(\Gamma(\phi))$ (Barannikov, Getzler–Kapranov).

Examples

$\mathfrak{F}$	Feynman category for	condition on graphs additional decoration
$\mathfrak{C}^{odd}$	odd cyclic operads	trees + orientation of set of edges
$\mathfrak{M}^{odd}$	$\mathfrak{K}$–modular	connected + orientation on set of edges + genus marking
$\mathfrak{M}^{nc,odd}$	nc $\mathfrak{K}$-modular	orientation on set of edges + genus marking
$\mathfrak{D}^{\circlearrowleft odd}$	odd wheeled dioperads	directed graphs w/o parallel edges + orientations of edges
$\mathfrak{P}^{\circlearrowleft,ctd,odd}$	odd wheeled properads	connected directed graphs w/o parallel edges + orientation of set of edges
$\mathfrak{P}^{\circlearrowleft,odd}$	odd wheeled props	directed graphs w/o parallel edges + orientation of set of edges

Table: List of Feynman categories with conditions and decorations on the graphs

Physics connection

Feynman graphs

are the morphisms in the Feynman category. The possible vertices are the objects.

S–matrix

The external lines are given by the target of the morphism. The comma/slice category over a given target is then a graphical version of the S–matrix.

Correlation functions

These are given by the functors $\mathcal{O}$.

Open Questions

What corresponds to algebras and plus construction, functors. Possible answers via Rota–Baxter (in progress).

Constructions yielding Feynman categories

A partial list

1. $+$ construction: Twisted modular operads, twisted versions of any of the previous structures. Quotient gives $\mathfrak{F}^{hyp}$.
2. $\mathcal{F}_{dec\mathcal{O}}$: non–Sigma and dihedral versions.It also yields all graph decorations.
3. free constructions $\mathfrak{F}^{\boxtimes}$, s.t. $\mathfrak{F}^{\boxtimes}$-$\mathcal{O}ps_{\mathcal{C}} = Fun(\mathcal{F}, \mathcal{C})$. Used for the simplicial category, crossed simplicial groups and FI–algebras.
4. Non–connected construction $\mathfrak{F}^{nc}$, whose $\mathcal{F}^{nc}$-$\mathcal{O}ps$ are equivalent to lax monoidal functors of $\mathcal{F}$.
5. The Feynman category of universal operations on $\mathfrak{F}$–$\mathcal{O}ps$.
6. Cobar/bar, Feynman transforms in analogy to algebras and (modular) operads.
7. W–construction.

$+$–construction

In general

there is a "$+$" construction, like for polynomial monads, that produces a new Feynman category out of an old one. Inverting isomorphisms one obtains $\mathfrak{F}^{hyp}$.
The main theorem is that enrichments of $\mathfrak{F}$ are in 1–1 correspondence with $\mathfrak{F}^{hyp}$–$\mathcal{O}ps$.

Examples

$\mathfrak{F}^{hyp}_{modular} = \mathfrak{F}_{hyper}$ and twisted modular operads as algebras over the twisted triple. $\mathfrak{F}^{+}_{surj} = \mathfrak{F}_{Mayoperads}$, $\mathfrak{F}^{hyp}_{surj} = \mathfrak{D}$, $\mathfrak{F}^{+}_{triv} = \mathfrak{F}_{surj}$.
(Slightly more complicated)

Algebras

The $\mathcal{F}^{hyp}$–$\mathcal{O}ps$ then give enrichments for $\mathcal{F}$. Given such an $\mathcal{O} \in \mathcal{F}^{hyp}$-$\mathcal{O}ps$ the $\mathcal{F}_{\mathcal{O}}$-$\mathcal{O}ps$ are (by definition) algebras over $\mathcal{O}$.

$\mathfrak{F}_{dec\mathcal{O}}$ joint w/ Jason Lucas

Theorem

Given an $\mathcal{O} \in \mathcal{F}\text{–}\mathcal{O}ps$, then there is a Feynman category $\mathcal{F}_{dec\mathcal{O}}$ which is indexed over $\mathcal{F}$. It objects are pairs $(X, dec \in \mathcal{O}(X))$ and $Hom_{\mathcal{F}_{dec\mathcal{O}}}((X, dec), (X', dec'))$ is the set of $\phi : X \to X'$, s.t. $\mathcal{O}(\phi) : dec \to dec'$. This construction works a priori for Cartesian $\mathcal{C}$, but with modifications it also works for the non–Cartesian case.

Examples

Non–sigma operads, cyclic non–Sigma operads, non–Sigma modular operads.
Here $\mathcal{O}$ is $\mathcal{A}ssoc$, $\mathcal{C}yc\mathcal{A}ssoc$, $\mathcal{M}od\mathcal{C}yc\mathcal{A}ssoc$.
There is a general theorem saying that the decoration by the push–forward exists and how such push–forwards factor. This recovers e.g. that the modular envelope of $\mathcal{C}yc\mathcal{A}ssoc$ factors through non–Sigma modular operads (Result of Markl).

Results

Theorem

Theorem there commutative squares which are natural in $\mathcal{O}$

$$\begin{tikzcd}
\mathfrak{F}_{dec\mathcal{O}} \arrow[r, "f^{\mathcal{O}}"] \arrow[d, "forget"'] & \mathfrak{F}'_{dec\, f_*(\mathcal{O})} \arrow[d, "forget'"] \\
\mathfrak{F} \arrow[r, "f"] & \mathfrak{F}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{F}_{dec\mathcal{O}} \arrow[r, "\sigma_{dec}"] \arrow[d, "f^{\mathcal{O}}"'] & \mathfrak{F}_{dec\mathcal{P}} \arrow[d, "f^{\mathcal{P}}"] \\
\mathfrak{F}'_{decf_*(\mathcal{O})} \arrow[r, "\sigma'_{dec}"] & \mathfrak{F}'_{decf_*(\mathcal{P})}
\end{tikzcd}
\tag{2}$$

On the categories of monoidal functors to $\mathcal{C}$, we get the induced diagram of adjoint functors.

$$\begin{tikzcd}
\mathcal{F}_{dec\mathcal{O}}\text{-}\mathcal{O}ps \arrow[r, shift left, "f^{\mathcal{O}}_*"] \arrow[d, shift right, "forget_*"'] & \mathcal{F}'_{dec\, f_*(\mathcal{O})}\text{-}\mathcal{O}ps \arrow[l, shift left, "f^{\mathcal{O}*}"] \arrow[d, shift left, "forget'_*"] \\
\mathcal{F}\text{-}\mathcal{O}ps \arrow[r, shift left, "f_*"] \arrow[u, shift right, "forget^*"'] & \mathcal{F}'\text{-}\mathcal{O}ps \arrow[l, shift left, "f^*"] \arrow[u, shift left, "forget'^*"]
\end{tikzcd}
\tag{3}$$

More $\mathcal{F}_{dec\mathcal{O}}$

Theorem

If $\mathcal{T}$ is a terminal object for $\mathcal{F}$-$\mathcal{O}ps$ and forget : $\mathcal{F}_{dec\mathcal{O}} \to \mathcal{F}$ is the forgetful functor, then forget$^*(\mathcal{T})$ is a terminal object for $\mathcal{F}_{dec\mathcal{O}}$-$\mathcal{O}ps$. We have that forget$_*$forget$^*(\mathcal{T}) = \mathcal{O}$.

Definition

We call a morphism of Feynman categories $i : \mathfrak{F} \to \mathfrak{F}'$ a minimal extension over $\mathcal{C}$ if $\mathfrak{F}$-$\mathcal{O}ps_{\mathcal{C}}$ has a a terminal/trivial functor $\mathcal{T}$ and $i_*\mathcal{T}$ is a terminal/trivial functor in $\mathfrak{F}'$-$\mathcal{O}ps_{\mathcal{C}}$.

Proposition

If $f : \mathfrak{F} \to \mathfrak{F}'$ is a minimal extension over $\mathcal{C}$, then $f^{\mathcal{O}} : \mathfrak{F}_{dec\mathcal{O}} \to \mathfrak{F}'_{decf_*(\mathcal{O})}$ is as well.

Example

Markl's Non-Σ modular (see also [KP06])

$$
\begin{tikzcd}
\mathcal{F}_{dec\ CycAss} = \mathfrak{C}^{\neg\Sigma} \arrow[rr, "i^{CycAss}"] \arrow[d, "forget"'] & & \mathfrak{M}_{dec\ i_*(CycAss)} = \mathfrak{M}^{\neg\Sigma} \arrow[d, "forget"] \\
\mathfrak{C} \arrow[rr, "i"] & & \mathfrak{M}
\end{tikzcd}
\tag{4}
$$

1. On the left side, if $*_C$ is final for $\mathfrak{C}$ and hence $forget^*(*_C) = \underline{*}_C$ is final for $\mathfrak{C}^{\neg\Sigma}$. The pushforward $forget_*(\underline{*}_C) = CycAss$.
2. On the right side, if $*_M$ is final for $\mathfrak{M}$ and hence $forget^*(*_M) = \underline{*}_M$ is final for $\mathfrak{M}^{\neg\Sigma}$. The pushforward $forget_*(\underline{*}_M) = ModAss$.
3. The inclusion i is a minimal extension.
4. Hence i^{CycAss} is also a minimal extension.

$\mathcal{F}_{dec\mathcal{O}}$ 2.0

Further applications

Further applications will be

1. New decorated interpretation moduli space operations generalizing those of R.K. *Moduli space actions on Hochschild Cochains*
2. The Stolz–Teichner setup for twisted field theories.
3. Kontsevich's graph comlexes.

Universal operations

Cocompletion

Let $\hat{\mathcal{F}}$ be the cocompletion of $\mathcal{F}$. This is monoidal with Day convolution $\circledast$. If $\mathcal{C}$ is cocomplete, and $\mathcal{O} \in \mathcal{O}ps$ factors.

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\mathcal{O}"] \arrow[dr, "\jmath"'] & & \mathcal{C} \\ & \hat{\mathcal{F}} \arrow[ur, "\hat{\mathcal{O}}"'] & \end{tikzcd}$$

Theorem

Let $\mathbb{I} := \mathrm{colim}_{\mathcal{V}} \jmath \circ \imath \in \hat{\mathcal{F}}$ *and let* $\mathcal{F}_{\mathcal{V}}$ *the symmetric monoidal subcategory generated by* $\mathbb{I}$. *Then* $\mathfrak{F}_{\mathcal{V}} := (\mathcal{F}_{\mathcal{V}}, \mathbb{I}, \imath_{\mathcal{V}})$ *is a Feynman category. (This gives an underlying operad of universal operations).*

Examples

Operads

$\mathfrak{O}$ the Feynman category for operads, $\mathcal{C} = dg\mathcal{V}ect$.

- Then $\hat{\mathcal{O}}(\mathbb{I}) = \bigoplus_n \mathcal{O}(n)_{\mathbb{S}_n}$ and the Feynman category is (weakly) generated by $\circ := [\sum \circ_i]$. (This is a two line calculation).
- This gives rise to the Lie bracket by using the anti–commutator. The operations go back to Gerstenhaber and Kapranov-Manin.
- It lifts to the non-Sigma case i.e. a pre–Lie structure on $\bigoplus_n \mathcal{O}(n)_{\mathbb{S}_n}$.

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Examples

Odd/anti–cyclic Operad

The universal operations are (weakly) generated by a Lie bracket. $[\ ,\ ] := [\sum_{st} \circ_{st}]$, (see [KWZ]). This actually lifts to cyclic coinvariants (non–sigma cyclic operads).
Specific examples:

- $End(V)$ for a symplectic vector space is anti–cyclic.
- Any tensor product: $(\mathcal{O} \otimes \mathcal{P})(n) := \mathcal{O}(n) \otimes \mathcal{P}(n)$ with $\mathcal{O}$ cyclic and $\mathcal{P}$ anti–cyclic is anti–cyclic.

Three geometries (Kotsevich, Conant-Vogtmann)

Fix V^n n–dim symplectic $V^n \to V^{n+1}$. For each n get Lie algebras
(1) $Comm \otimes End(V^n)$ (2) $Lie \otimes End(V^n)$ (3) $Assoc \otimes End(V^n)$
Take the limit as $n \to \infty$.

Universal Operations

$\mathfrak{F}$	Feynman cat for	$\mathfrak{F}, \mathfrak{F}_{\mathcal{V}}, \mathfrak{F}_{\mathcal{V}}^{nt}$	weak gen. subcat.
$\mathfrak{O}$	Operads	rooted trees	$\mathfrak{F}_{pre-Lie}$
$\mathfrak{O}^{odd}$	odd operads	rooted trees + orientation of set of edges	odd pre-Lie
$\mathfrak{O}^{pl}$	non-Sigma operads	planar rooted trees	all $\circ_i$ operations
$\mathfrak{O}_{mult}$	Operads with mult.	b/w rooted trees	pre-Lie + mult.
$\mathfrak{C}$	cyclic operads	trees	com. mult.
$\mathfrak{C}^{odd}$	odd cyclic operads	trees + orientation of set of edges	odd Lie
$\mathfrak{M}^{odd}$	$\mathfrak{K}$–modular	connected + orientation on set of edges	odd dg Lie
$\mathfrak{M}^{nc,odd}$	nc $\mathfrak{K}$-modular	orientation on set of edges	BV
$\mathfrak{D}$	Dioperads	connected directed graphs w/o directed loops or parallel edges	Lie–admissible

Table: Here $\mathfrak{F}_{\mathcal{V}}$ and $\mathfrak{F}_{\mathcal{V}}^{nt}$ are given as $\mathcal{F}_{\mathcal{O}}$ for the insertion operad. The former for the type of graph with unlabelled tails and the latter for the version with no tails.

Hopf algebras

Basic structures

Assume $\mathcal{F}$ is decomposition finite. Consider $\mathcal{B} = Hom(Mor(\mathcal{F}), \mathbb{Z})$. Let μ be the tensor product with unit $id_{\mathbb{I}}$.

$$\Delta(\phi) = \sum_{(\phi_0,\phi_1):\phi=\phi_1\circ\phi_0} \phi_0 \otimes \phi_1$$

and $\epsilon(\phi) = 1$ if $\phi = id_X$ and 0 else.

Theorem (Galvez-Carrillo, K , Tonks)

$\mathcal{B}$ together with the structures above is a bi–algebra. Under certain mild assumptions, a canonical quotient is a Hopf algebra

Examples

In this fashion, we can reproduce Connes–Kreimer's Hopf algebra, the Hopf algebras of Goncharov and a Hopf algebra of Baues that he defined for double loop spaces. This is a non–commutative graded version. There is a three-fold hierarchy. A non-commutative version, a commutative version and an "amputated" version.

Details I

Non–commutative version

Use Feynman categories whose underlying tensor structure is only monoidal (not symmetric). $\mathcal{V}^{\otimes}$ is the the free monoidal category.

Key Lemma

The bi–algebra equation holds due to the hereditary condition.

Unit

The unit of the co–algebra is given by $1 = id_{\emptyset}$, i.e. the identity morphism of the empty word.

Quotient by Isomorphisms

If there are any isomorphism in $\mathcal{V}$ then $\mathcal{F}$ one can quotient out the co–ideal defined by equiv. rel. generated by isomorphism diagrams of type (1). The result is called almost connected. (This is automatic if there are no isomorphism except for identities in $\mathcal{V}$).

Details II

Theorem

For the almost connected version let $\mathscr{I}$ be the ideal generated by $1 - id_X$. Then this is a co–ideal and the quotient $\mathcal{B}/\mathscr{I}$ is a connected Hopf algebra and hence a bi–algebra. Goncharov and Baues (shifted co–bar version), planar Connes-Kreimer with external lines (both tree and 1-PI).

Commutative version

For the commutative version, one looks at the co–invariants in the symmetric case. Non–planar Connes–Kreimer with external lines.

Amputated version

For this one needs a semi–cosimplicial structure, i.e. one must be able to forget external legs coherently. Then there is a colimit, in which all the external legs can be forgotten. Connes–Kreimer without external legs (e.g. the original tree version).

Details III

Generalization of special case: co–operad with multiplication

In a sense the above examples were free. One can look at a more general setting where this is not the case. The length of an object is the replaced by a depth filtration. The algebras are then deformations of their associated graded. Main example (cooperad with multiplication) generalizes enrichment of F_{surj}.

Grading/Filtration

Co-operad with multiplication	operad degree − depth
Amputated version	co-radical degree + depth

q deformation - infinitesimal version

Taking a slightly different quotient, one can get a non–unital, co–unital bi–algebra and a q–filtration. Sending $q \to 1$ recovers $\mathcal{H}$.

Coproduct for cooperad with multiplication

Theorem

Let $\check{\mathcal{O}}$ be a co-operad with compatible associative multiplication $\mu : \check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(m) \to \check{\mathcal{O}}(n+m)$ in an Abelian symmetric monoidal category with unit $\mathbb{I}$. Then $\mathcal{B} := \bigoplus_n \check{\mathcal{O}}(n)$ is a (non-unital, non-co-unital) bialgebra, with multiplication μ and comultiplication Δ given by $(\mathbb{I} \otimes \mu)\check{\gamma}$:

$$\begin{tikzcd} \check{\mathcal{O}}(n) \arrow[r, "\check{\gamma}"] \arrow[dr, bend right, "\Delta := (\mathbb{I}\otimes\mu)\check{\gamma}"'] & \bigoplus_{\substack{k\geq 1,\\ n=m_1+\cdots+m_k}} \left(\check{\mathcal{O}}(k) \otimes \bigotimes_{r=1}^{k} \check{\mathcal{O}}(m_r) \right) \arrow[d, "\mathbb{I}\otimes\mu^{k-1}"] \\ & \bigoplus_{k\geq 1} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n). \end{tikzcd} \tag{5}$$

Example

Free cooperad with multiplication on a cooperad

$\check{\mathcal{O}}^{nc}(n) = \bigoplus_k \bigoplus_{(n_1,\dots,n_k):\sum n_i=n} \check{\mathcal{O}}(n_1) \otimes \dots \otimes \check{\mathcal{O}}(n_k)$
Multiplication given by $\mu = \otimes$.

Hopf algebras/(co)operads/Feynman category

H_{Gont}	$Inj_{*,*} = Surj^*$	$\mathfrak{F}_{Surj}$
H_{CK}	leaf labelled trees	$\mathfrak{F}_{Surj,\mathcal{O}}$
$H_{CK,graphs}$	graphs	$\mathfrak{F}_{graphs}$
H_{Baues}	$Inj^{gr}_{*,*}$	$\mathfrak{F}_{Surj,odd}$

(Co)Bar Feynman transform

Algebra case

- C associative co–algebra. $\Omega C := Free_{alg}(\Sigma^{-1}\bar{C})+$ differential coming from co–algebra structure
- A associative algebra. $BA = T\Sigma^{-1}\bar{A}$ + co–differential from algebra structure
- ΩBA is a free resolution.
- A say finite dim or graded with finite dim pieces $\check{A}$ its dual. $FA := \Omega\check{A}$ + differential from multiplication. FFA a resolution.

We can define the same transformation for elements of $\mathcal{O}ps$ for well–presented Feynman categories

- The result of a Feynman transform is an *op* over the odd version of the Feynman category
- For the freeness we need model structures, which we give.

Bar/Cobar/Feynman transform

Presentations

In order to define the transforms, one has to fix a version $\mathfrak{F}^{odd}$ of $\mathfrak{F}$. This is analogous to the suspension in the usual bar transforms. In fact, the following is more natural, see [KW15, KWZ12]. The degree is 1 for each bar.

Degrees of morphisms

For the operads or modular operads, the degree is 1 for each *edge*. This puts a degree on morphisms. A morphism of degree n has a ghost graph with n edges.

Basic example

In $\mathfrak{G}$

1. There are 4 types of basic morphisms: Isomorphisms, simple edge contractions, simple loop contractions and mergers. Call this set Φ.
2. These one–comma generate all morphisms. Furthermore, isomorphisms act transitively on the other classes. The relations on the generators are given by commutative diagrams.
3. The relations are quadratic for edge contractions as are the relations involving isomorphisms. Finally there is a non–homogenous relation coming from a simple merger and a loop contraction being equal to a edge contraction.
4. We can therefore assign degrees as 0 for isomorphisms and mergers, 1 for edge or loop contractions and split Φ as $\Phi^0 \amalg \Phi^1$. This gives a degree to any morphism.

Setup

Summary

Up to isomorphism any morphism of degree n can be written in $n!$ ways up to morphisms of degree 0. These are the enumerations of the edges of the ghost graph.

Setup

$\mathfrak{F}$ be a Feynman category enriched over $\mathcal{A}b$ and with an ordered presentation and let $\mathfrak{F}^{odd}$ be its corresponding odd version. Furthermore let Φ^1 be a resolving subset of one-comma generators and let $\mathcal{C}$ be an additive category, i.e. satisfying the analogous conditions above.

Differential

$d_{\Phi^1} = \sum_{[\phi_1]\in\Phi^1/\sim} \phi_1\circ$ defines a differential on the Abelian group generated by the isomorphism classes morhpisms. The non–defined terms are set to zero.

Bar/Cobar/Feynman transform

The bar construction

is the functor

$$\mathrm{B}\colon \mathcal{F}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})} \to \mathcal{F}^{odd}\text{-}\mathcal{O}ps_{Kom(\mathcal{C}^{op})}$$

$$\mathrm{B}(\mathcal{O}) := \imath_{\mathfrak{F}^{odd}\,*}(\imath^*_{\mathfrak{F}}(\mathcal{O}))^{op}$$

together with the differential $d_{\mathcal{O}^{op}} + d_{\Phi^1}$.

The cobar construction

is the functor

$$\Omega\colon \mathcal{F}^{odd}\text{-}\mathcal{O}ps_{Kom(\mathcal{C}^{op})} \to \mathcal{F}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})}$$

$$\Omega(\mathcal{O}) := \imath_{\mathfrak{F}\,*}(\imath^*_{\mathfrak{F}^{odd}}(\mathcal{O}))^{op}$$

together with the co-differential $d_{\mathcal{O}^{op}} + d_{\Phi^1}$.

Bar/Cobar/Feynman transform

Feynman transform

Assume there is a duality equivalence $\vee\colon \mathcal{C} \to \mathcal{C}^{op}$. The Feynman transform is a pair of functors, both denoted FT,

$$\mathrm{FT}\colon \mathcal{F}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})} \leftrightarrows \mathcal{F}^{odd}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})} : \mathrm{FT}$$

defined by

$$\mathrm{FT}(\mathcal{O}) := \begin{cases} \vee \circ \mathrm{B}(\mathcal{O}) & \text{if } \mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})} \\ \vee \circ \Omega(\mathcal{O}) & \text{if } \mathcal{O} \in \mathcal{F}^{odd}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})} \end{cases}$$

Master equations

Theorem

([Bar07],[MV09],[MMS09],[KWZ12]) Let $\mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ *and* $\mathcal{P} \in \mathcal{F}^{odd}\text{-}\mathcal{O}ps_{\mathcal{C}}$ *for an* $\mathcal{F}$ *represented in Table 2. Then there is a bijective correspondence:*

$$Hom(\mathrm{FT}(\mathcal{P}), \mathcal{O}) \cong ME(lim(\mathcal{P} \otimes \mathcal{O}))$$

This holds in general for the master equation given by

$$d_{\mathcal{Q}} + \sum_n \Psi_{\mathcal{Q},n} = 0$$

Master equations

The Feynman transform is quasi–free. An algebra over $F\mathcal{O}$ is dg–if and only if it satisfies the following master equation.

Name of $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$	Algebraic Structure of $F\mathcal{O}$	Master Equation (ME)
operad ,[GJ94]	odd pre-Lie	$d(-) + - \circ - = 0$
cyclic operad [GK95]	odd Lie	$d(-) + \frac{1}{2}[-,-] = 0$
modular operad [GK98]	odd Lie + Δ	$d(-) + \frac{1}{2}[-,-] + \Delta(-) = 0$
properad [Val07]	odd pre-Lie	$d(-) + - \circ - = 0$
wheeled properad [MMS09]	odd pre-Lie + Δ	$d(-) + - \circ - + \Delta(-) = 0$
wheeled prop [KWZ12]	dgBV	$d(-) + \frac{1}{2}[-,-] + \Delta(-) = 0$

Bar/Cobar

Lemma

The bar and cobar construction form an adjunction:

$$\Omega\colon \mathcal{F}^{odd}\text{-}\mathcal{O}ps_{Kom(\mathcal{C}^{op})} \rightleftarrows \mathcal{F}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})} : \mathrm{B}$$

Theorem

Let $\mathfrak{F}$ be a quadratic Feynman category and $\mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{Kom(\mathcal{C})}$. Then the counit $\Omega\mathrm{B}(\mathcal{O}) \to \mathcal{O}$ of the above adjunction is a levelwise quasi-isomorphism.

Model structure

Theorem

Let $\mathfrak{F}$ be a Feynman category and let $\mathcal{C}$ be a cofibrantly generated model category and a closed symmetric monoidal category having the following additional properties:

1. *All objects of $\mathcal{C}$ are small.*
2. *$\mathcal{C}$ has a symmetric monoidal fibrant replacement functor.*
3. *$\mathcal{C}$ has $\otimes$-coherent path objects for fibrant objects.*

Then $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ is a model category where a morphism $\phi\colon \mathcal{O} \to \mathcal{Q}$ of $\mathcal{F}$-ops is a weak equivalence (resp. fibration) if and only if $\phi\colon \mathcal{O}(v) \to \mathcal{Q}(v)$ is a weak equivalence (resp. fibration) in $\mathcal{C}$ for every $v \in \mathcal{V}$.

Examples

Examples

1. Simplicial sets. (Straight from Theorem)
2. $dgVect_k$ for $char(k) = 0$ (Straight from Theorem)
3. Top (More work)

Remark

Condition (i) is not satisfied and so we can not directly apply the theorem. Instead, we follow [Fre10] and use the fact that all objects in Top are small with respect to topological inclusions.

Theorem

Let $\mathcal{C}$ be the category of topological spaces with the Quillen model structure. The category $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ has the structure of a cofibrantly generated model category in which the forgetful functor to $\mathcal{V}\text{-}Seq_{\mathcal{C}}$ creates fibrations and weak equivalences.

Quillen adjunctions from morphisms of Feynman categories

Adjunction from morphisms

We assume $\mathcal{C}$ is a closed symmetric monoidal and model category satisfying the assumptions of Theorem above. Let $\mathfrak{E}$ and $\mathfrak{F}$ be Feynman categories and let $\alpha\colon \mathfrak{E} \to \mathfrak{F}$ be a morphism between them. This morphism induces an adjunction

$$\alpha_L : \mathcal{E}\text{-}\mathcal{O}ps_{\mathcal{C}} \leftrightarrows \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} : \alpha_R$$

where $\alpha_R(\mathcal{A}) := \mathcal{A} \circ \alpha$ is the right adjoint and $\alpha_L(\mathcal{B}) := Lan_\alpha(\mathcal{B})$ is the left adjoint.

Lemma

Suppose α_R restricted to $\mathcal{V}_{\mathfrak{F}}\text{-}\mathcal{M}ods_{\mathcal{C}} \to \mathcal{V}_{\mathfrak{E}}\text{-}\mathcal{M}ods_{\mathcal{C}}$ preserves fibrations and acyclic fibrations, then the adjunction (α_L, α_R) is a Quillen adjunction.

Example

1. Recall that $\mathfrak{C}$ and $\mathfrak{M}$ denote the Feynman categories whose *ops* are cyclic and modular operads respectively and that there is a morphism $i\colon \mathfrak{C} \to \mathfrak{M}$ by including as genus zero.
2. This morphism induces an adjunction between cyclic and modular operads
$$i_L\colon \mathfrak{C}\text{-}\mathcal{O}ps_{\mathcal{C}} \leftrightarrows \mathfrak{M}\text{-}\mathcal{O}ps_{\mathcal{C}} : i_R$$
and the left adjoint is called the modular envelope of the cyclic operad.
3. The fact that the morphism of Feynman categories is inclusion means that i_R restricted to the underlying $\mathcal{V}$-modules is given by forgetting, and since fibrations and weak equivalences are levelwise, i_R restricted to the underlying $\mathcal{V}$-modules will preserve fibrations and weak equivalences.
4. Thus by the Lemma above this adjunction is a Quillen adjunction.

Cofibrant replacement

Theorem

The Feynman transform of a non-negatively graded dg $\mathcal{F}$-op is cofibrant.
The double Feynman transform of a non-negatively graded dg $\mathcal{F}$-op in a quadratic Feynman category is a cofibrant replacement.

Setup: quadratic Feynman category $\mathfrak{F}$

The category $w(\mathfrak{F}, Y)$, for $Y \in \mathcal{F}$ Objects:

Objects are the set $\coprod_n C_n(X,Y) \times [0,1]^n$, where $C_n(X,Y)$ are chains of morphisms from X to Y with n degree ≥ 1 maps modulo contraction of isomorphisms.
An object in $w(\mathfrak{F}, Y)$ will be represented (uniquely up to contraction of isomorphisms) by a diagram

$$X \xrightarrow[f_1]{t_1} X_1 \xrightarrow[f_2]{t_2} X_2 \to \cdots \to X_{n-1} \xrightarrow[f_n]{t_n} Y$$

where each morphism is of positive degree and where $t_1, \ldots, t_n$ represents a point in $[0,1]^n$. These numbers will be called weights. Note that in this labeling scheme isomorphisms are always unweighted.

Setup: quadratic Feynman category $\mathfrak{F}$

The category $w(\mathfrak{F}, Y)$, for $Y \in \mathcal{F}$ Morphisms:

1. Levelwise commuting isomorphisms which fix Y, i.e.:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\cong"] & X_1 \arrow[r] \arrow[d, "\cong"] & X_2 \arrow[r] \arrow[d, "\cong"] & \dots \arrow[r] & X_n \arrow[r] \arrow[d, "\cong"] & Y \\
X' \arrow[r] & X'_1 \arrow[r] & X'_2 \arrow[r] & \dots \arrow[r] & X'_n \arrow[ur] &
\end{tikzcd}$$

2. Simultaneous $\mathbb{S}_n$ action.
3. Truncation of 0 weights: morphisms of the form $(X_1 \xrightarrow{0} X_2 \to \cdots \to Y) \mapsto (X_2 \to \cdots \to Y)$.
4. Decomposition of identical weights: morphisms of the form $(\cdots \to X_i \xrightarrow{t} X_{i+2} \to \dots) \mapsto (\cdots \to X_i \xrightarrow{t} X_{i+1} \xrightarrow{t} X_{i+2} \to \dots)$ for each (composition preserving) decomposition of a morphism of degree ≥ 2 into two morphisms each of degree ≥ 1.

W-construction

Definition

Let $\mathcal{P} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{T}op}$. For $Y \in ob(\mathcal{F})$ we define

$$W(\mathcal{P})(Y) := colim_{w(\mathfrak{F},Y)} \mathcal{P} \circ s(-)$$

Theorem

Let $\mathfrak{F}$ be a simple Feynman category and let $\mathcal{P} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{T}op}$ be ρ-cofibrant. Then $W(\mathcal{P})$ is a cofibrant replacement for $\mathcal{P}$ with respect to the above model structure on $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{T}op}$.

Here "simple" is a technical condition satisfied by all graph examples.

Geometry and moduli spaces

Modular Operads

The typical topological example are $\bar{M}_{gn}$. These give rise to chain and homology operads.

- Gromov–Witten invariants make $H^*(V)$ and algebra over $H_*(\bar{M}_{g,n})$

Odd Modular

The canonical geometry is given by $\bar{M}^{KSV}$ which are real blowups of $\bar{M}_{gn}$ along the boundary divisors.

- We get 1-parameter gluings parameterized by S^1. Taking the full S^1 family on chains or homology gives us the structure of an odd modular operad.
- Going back to Sen and Zwiebach, a viable string field theory action S is a solution of the quantum master equation.

Next steps

- Formalize the dual pictures of primitive elements and $+$ construction as well as universal operations and PBW.
- Connect to Tannakian categories. E.g. find out the role of fibre functors or special large/small object. (Idea: special properties of $\mathcal{H}_{CK}$).
- Connect to Rota–Baxer, Dynkin-operators, B^+-operators (we can do this part) etc.
- Construct Feynman category for the open/closed version of Homological Mirror symmetry.
- Find action of Grothendieck-Teichmüller group (GT).
- ...

The end

Thank you!

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