Stringy orbifold K-Theory

Ralph Kaufmann

Purdue University

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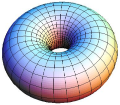
Orbifolds

Varieties/manifolds

Locally like *Spec*(A) e.g. $\mathbb{A}^n$, $\mathbb{R}^n$. Usually want a "nice" variety, i.e. smooth projective variety. Or smooth structure



Deligne Mumford stacks/Orbifolds

Definition of stacks technically complicated. "Nice" stacks are Deligne–Mumford stacks with projective coarse moduli space. Locally think of *Spec*$(A)/G$ — G a group of local automorphisms.

In the differential category, being locally modeled by $\mathbb{R}^n/G$ is one definition of an orbifold.

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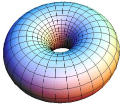

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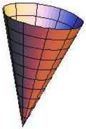

Nice Stacks/Orbifolds

Global Quotients

A nice class of examples are *global quotients* $[V/G]$ of a variety by a finite group action. We sometimes write (V, G).

Lie quotients

Interesting stacks in the differential category are given by $M/\mathcal{G}$ where $\mathcal{G}$ is a Lie group that acts with finite stabilizers. In the algebraic category the corresponding spaces are nice, that is they are examples of DM-stacks with a projective coarse moduli space.

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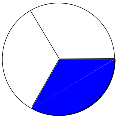
Orbifolds/Group actions

Example 1: Cone

Let C_n be the cyclic group of n–th roots of unity and let ζ_n be a generator.
An example over $\mathbb{C}$:

$$C_3 = \{1, e^{2\pi i/3}, e^{4\pi i/3}\},\ C_4 = \{1, i, -1, -i\}$$

C_n acts by rotation.



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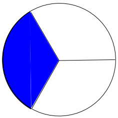

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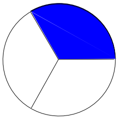

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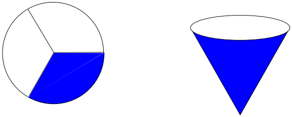

The example of C_3: Action and Invariants

Orbifolds

Example 2: Symmetric Products

$X^{\times n} = X \times \cdots \times X$, $\mathbb{S}_n$ group of permutations.
$(X^{\times n}, \mathbb{S}_n)$, where acts via
$\sigma \in \mathbb{S}_n : \sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

A picture for $n = 2$

- $Y = X \times X$
- $\Delta = \{(x,x) | x \in X\} \subset Y$
- $\tau = (12)$
- $\tau(x,y) = (y,x)$

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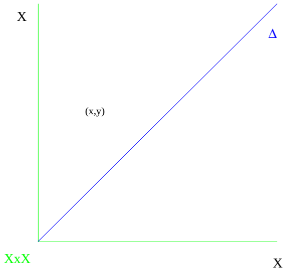

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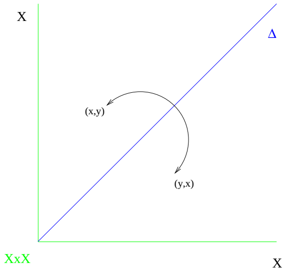


Global quotients: a stringy perspective

Classical perspective: *Invariants*

That is for (X, G) look at X/G, where X/G is the space of orbits.

Stringy Perspective: *Inertia variety*

For (X, G) study

$$I(X, G) = \amalg_{g \in G} X^g$$

where $X^g = \{x \in X : g(x) = x\}$ is the set of g–fixed points, and $\amalg$ is the disjoint union.

Global quotients a stringy perspective

The space $I(X,G)$ has a natural G action

We have $h(X^g) \subset X^{hgh^{-1}}$. Let $x \in X^g$ then

$$(hgh^{-1})(h(x)) = h(g((hh^{-1})(x))) = h(g(x)) = h(x)$$

Inertia stack I_G

$$I_G(X,G) := [I(X,G)/G]$$

Let $C(G)$ be the set of conjugacy classes of G, then

$$I_G(X,G) = \amalg_{[g]\in C(G)}[X^g/Z(g)]$$

where $[g]$ runs through a system of representatives, and $Z(g)$ is the centralizer of g.

Inertia stack

Structure of inertia orbifold/stack

The inertia variety exists only for a global quotient, while the inertia stack can be defined for a DM stack

$$\widetilde{\mathscr{X}} := \coprod_{(g)} \mathscr{X}_{(g)},$$

where the indices run over conjugacy classes of local automorphisms, and $\mathscr{X}_{(g)} = \{(x,(g)) | g \in G_x\}/Z_{G_x}(g)$.

Examples of the stringy spaces

Example 1: Cone $X = \mathbb{C}$, $G = C_3$

1. $X/G =$ Cone
2. $I(X,G) = \mathbb{C} \amalg \text{Vertex} \amalg \text{Vertex}$
3. $I_G(Y,G) = \text{Cone} \amalg \text{Vertex} \amalg \text{Vertex}$.

Example 2: Symmetric Products

$Y = X \times X$, $G = \mathbb{S}_2 = \langle 1, \tau : \tau^2 = 1 \rangle$.
$\Delta : X \to X \times X; x \mapsto (x,x)$ the diagonal.

1. $X/G = \{\{x,y\} : x,y \in X\}$ the set of unordered pairs
2. $I(Y,G) = I(X \times X, \mathbb{S}_2) = (X \times X) \amalg \Delta(X)$
3. $I_G(Y,G) = \{\{x,y\} : x,y \in X\} \amalg \Delta(X)$

Motivation for the stringy perspective

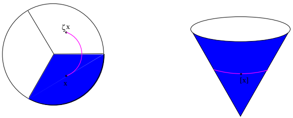
Physics

- "Twist fields" from orbifold conformal/topological field theory.
- Orbifold Landau–Ginzburg theories.
- Orbifold string theory.

Mathematics

- Cobordisms of bordered surfaces with principal G–bundles.
- Orbifolded singularities with symmetries.
- Orbifold Gromov–Witten Invariants.

An open interval in X becomes a closed S^1 under the action of G.



Mathematical Motivation

1. New methods/invariants for singular spaces.
2. The slogan: "string smoothes out singularities".
   Orbifold K–Theory Conjecture: The orbifold K–Theory of (X, G) is isomorphic to the K–Theory of a certain Y, such that $Y \to X$ is a crepant resolution of singularities.
3. Mirror–Symmetry. One can expect a general construction using orbifolds as the mathematical equivalent for orbifold Landau–Ginzburg models.
4. New methods/applications for representation theory.

Stringy Multiplication

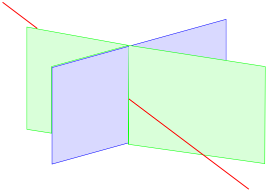
1. Given: (X, G) and a functor, which has a multiplication (E.g. K, A^*, K^*_{top}, H^*).
   Let's fix coefficients to lie in $\mathbb{Q}$.
2. Fact: $\mathscr{K}(X, G) := K(I(X, G)) = \bigoplus_{g \in G} K(X^g)$ is a vector space. (The same holds for H^*, A^*, K^*_{top})
3. Problem: From the stringy (cobordism/"twist field") point of view one expects a *stringy G–graded* multiplication.

$$K(X^g) \otimes K(X^h) \to K(X^{gh})$$

Pushing, Pulling and Obstructions

Again, let's fix $\mathbb{Q}$ as coefficients.
We will first deal with the functor K.
Set $X^{\langle g,h \rangle} = X^g \cap X^h$

$$\begin{array}{ccccc} X^g & & X^h & & X^{gh} \\ e_1 \nwarrow & & e_2 \uparrow & & \nearrow e_3 \\ & & X^{\langle g,h\rangle} & & \end{array}$$



1. One naturally has pull–back operations e_i^*, since the functors are contra-variant. This is basically just the restriction.
2. On also has push–forward operations e_{i*}.

The Stringy Multiplication

Ansatz (basic idea)

For $\mathscr{F}_g \in K(X^g), \mathscr{F}_h \in K(X^h)$

$$\begin{array}{ccccc} K(X^g) & & K(X^h) & & K(X^{gh}) \\ & e_1^* \searrow & e_2^* \downarrow & \nearrow e_{3*} & \\ & & K(X^{\langle g,h \rangle}) & & \end{array}$$

$$\mathscr{F}_g \cdot \mathscr{F}_h := e_{3*}(e_1^*(\mathscr{F}_g) \otimes e_2^*(\mathscr{F}_h) \otimes Obs_K(g,h))$$

The Obstruction

The above formula without $Obs_K(g,h)$ will not give something associative in general. There are also natural gradings which would not be respected without $Obs_K(g,h)$ either.

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The solution

$$\mathrm{Obs}_{\mathrm{K}}(g,h) = \lambda_{-1}(\mathscr{R}(g,h)^*)$$

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Two sources for the obstruction bundle $\mathscr{R}(g,h)$

1. $\mathscr{R}(g,h)$ from GW theory/mapping of curves [CR02/04,FG03,JKK05] (Initially only for H^*).
2. $\mathscr{R}(g,h)$ from K-theory and representation theory. [JKK07 (Inv. Math)].

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When both definitions apply, they agree.

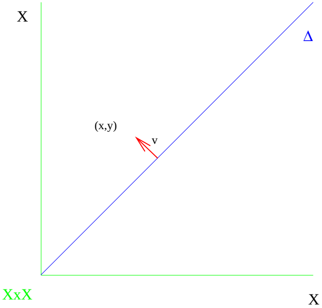
The obstruction bundle

Eigenspace decomposition

Let $g \in G$ have the order r. $\langle g \rangle \subset G$ acts on X and leaves X^g invariant. So $\langle g \rangle$ acts on the restriction of the tangent bundle $TX|_{X^g}$ and the latter decomposes into Eigenbundles $W_{g,k}$ whose Eigenvalue is $\exp(-2\pi ki/r)$ for the action of g.

$$TX|_{X^g} = \bigoplus_{k=1}^{r-1} W_{g,k}$$

Example: The symmetric product



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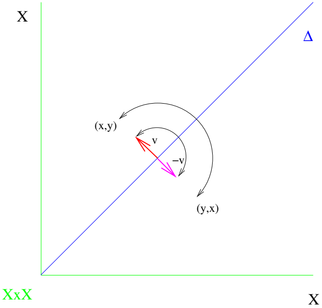

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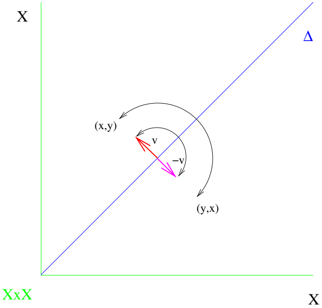

The Eigenvalues are

$$1 = exp(2\pi i 0) \text{ und } -1 = exp(2\pi i \frac{1}{2})$$

Stringy K-Theory

The bundles $\mathscr{S}$

$$\mathscr{S}_g := \bigoplus_{k=1}^{r-1} \frac{k}{r} W_{g,k} \in K(X^g)$$

Theorem (JKK)

Let X be a smooth projective variety with an action of a finite group G, then

$$\mathscr{R}(g,h) = TX^{\langle g,h\rangle} \ominus TX|_{X^{\langle g,h\rangle}} \oplus \mathscr{S}_g|_{X^{\langle g,h\rangle}} \oplus \mathscr{S}_h|_{X^{\langle g,h\rangle}} \oplus \mathscr{S}_{(gh)^{-1}}|_{X^{\langle g,h\rangle}}$$

defines a stringy multiplication on $\mathscr{K}(X,G)$ (actually a categorical–G–Frobenius algebra).

The "stringy" Chern–Character

Chow-Ring

Let A be the Chow-Ring $\mathscr{A}(X,G) := A(I(X,G))$. For $v_g \in A(X^g), v_h \in A(X^h)$, $Obs_A(g,h) = c_{\mathrm{top}}(\mathscr{R}(g,h))$ set

$$v_g \cdot v_h := e_{3*}(e_1^*(v_g) \otimes e_2^*(v_h) \otimes Obs_A(g,h))$$

then this defines an associative multiplication (categorical-G-Frobenius algebra).

The "stringy" Chern–Character

Chern–Character

Let X be a smooth, projective variety with an action of G.
Define $\mathscr{C}\mathbf{h} : \mathscr{K}(X, G) \to \mathscr{A}(X, G)$ via

$$\mathscr{C}\mathbf{h}(\mathscr{F}_g) := \mathbf{ch}(\mathscr{F}_g) \cup \mathbf{td}^{-1}(\mathscr{S}_g)$$

Here $\mathscr{F}_g \in K(X^g)$, **td** is the Todd class and **ch** is the usual Chern–Character.

Theorem (JKK)

$\mathscr{C}\mathbf{h} : \mathscr{K}(X, G) \to \mathscr{A}(X, G)$ is an isomorphism. An isomorphism of categorical–G–Frobenius algebras.

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Variations

Theorems

1. If X is an equivariantly stable almost complex manifold, then analogous results hold for the topological K–theory K^*_{top} and cohomology H^* yielding an isomorphism of G–Frobenius algebras.
2. For a nice[a] stack $\mathscr{X}$ there are respective versions for the K–theory, the Chow Rings und and the Chern–Characters using the inertia *stacks*.
3. Notice the stringy K is usually "bigger" than stringy Chow and $\mathscr{C}\mathbf{h}$ is only a ring *homomorphism*. For a global quotient (X/G) we have that $\mathscr{K}(X,G)^G$ embeds into the full stringy K.

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Compatibility

Compatibility with the curve constructions in the case of H^*

1. If X is a complex manifold, then the multiplication on $\mathcal{H}^*$ coincides with that defined by Fantechi–Goettsche on $H^*(I(X,G))$.
2. In the orbifold case, i.e. $[X/\mathscr{G}]$ as above, the multiplication on $\mathcal{H}^*$ coincides with that defined by Chen–Ruan.

Orbifold K-Theory Conjecture

For a K3–surface X, $Y = (X^{\times n}, \mathbb{S}_n)$ and $Z = Hilb^{[n]}(X)$ as its resolution $Z \to Y$ the Orbifold K-theory conjecture holds, with a concrete choice of discrete torsion.

Simple examples

pt/G

Let pt be a point with a trivial operation of G and let's take coefficients in $\mathbb{C}$: $\forall g \in G : g(pt) = pt$. Now, $I(pt, G) = \amalg_{g \in G} pt$ and

$$\mathcal{H}^*(pt, G) = \bigoplus_{g \in G} \mathbb{C} \simeq \mathbb{C}[G].$$

The G–invariants are the class functions.

Symmetric Products (Second quantization [K04])

Let X be a smooth projective variety. The diagonal $\Delta : X \to X \times X; x \mapsto (x, x)$ defines a push–forward $\Delta_* : H^*(X) \to H^*(X \times X) \simeq H^*(X) \otimes H^*(X)$

$$\mathcal{H}^*(X \times X, \mathbb{S}_2) = (H^*(X) \otimes H^*(X)) \oplus H^*(X)$$

with the multiplication $x_\tau \cdot y_\tau = x \otimes y \cdot \Delta_*(1) \in H^*(X) \otimes H^*(X)$.

Frobenius Algebras

Definition

A Frobenius algebra is a finite dimensional, commutative, associative unital algebra R with a non–degenerate symmetric paring $\langle\ ,\ \rangle$ which satisfies

$$\langle ab, c\rangle = \langle a, bc\rangle$$

Example

$H^*(X, k)$ for X a compact oriented manifold or a smooth projective smooth variety. The pairing is the Poincaré paring which is essentially given by the integral over X. Notice over $\mathbb{Q}$, K^{top} and H^* are isomorphic, but not isometric (Hirzebruch-Riemann-Roch).

Notice that if L_v is the left multiplication by v, there is a trace $\tau : R \to k$: $\tau(v) := Tr(L_v)$

Pre-Frobenius algebras

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Example

$A^*(X)$ and $K^*(X)$ for a smooth projective variety. The trace element is given by the integral and Euler characteristic respectively.

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Remark

For these examples we could also use an alternative definition of a Frobenius object, which is given as an algebra μ and a co-algebra Δ with unit η and co-unit ϵ which satisfy

$$\Delta \circ \mu = (\mu \otimes id) \circ (id \otimes \Delta)$$

Then $\tau(v) = \epsilon(\mu(v, \Delta \circ \eta(1))$.

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For these examples we could also use an alternative definition of a Frobenius object, which is given as an algebra μ and a co-algebra Δ with unit η and co-unit ϵ which satisfy

$$\Delta \circ \mu = (\mu \otimes id) \circ (id \otimes \Delta)$$

Then $\tau(v) = \epsilon(\mu(v, \Delta \circ \eta(1))$. This fits well with TFT.

G–Frobenius algebras

Definition (concise form) [K-Pham]

A strict categorical G–Frobenius algebra is a Frobenius objects in the braided monoidal category of modules over the quasi–triangular quasi–Hopf algebra defined by the Drinfeld double of the group ring $D(k[G])$ which satisfies two additional axioms:

T Invariance of the twisted sectors.

S The trace axiom.

Pre– and Non–Strict

- There is a notion of pre-G–FA, which uses trace elements which satisfy T as extra data.
- Non–strict, means that there are certain characters which appear. (This is needed in Singularity Theory. Today only strict).

More details/examples

Some details/consequences

If R is a GFA then

1. $R = \bigoplus_{g\in G} R_g$ is a G–graded algebra
2. R has a G action ρ such that $\rho(g)(R_h) \subset R_{ghg^{-1}}$ (Drinfel'd–Yetter)
3. $a_g b_h = \rho(g)(b_h)a_g$ (twisted commutativity)
4. (T) $g|_{R_g} = id$
5. (S) If $v \in R_{[g,h]}$ $Tr_{R_h}(\rho(g^{-1}) \circ L_v) = Tr_{R_g}(L_v \circ \rho(h))$

Examples

$k[G] = \mathcal{H}^*(pt, G)$ and $k^\alpha[G]$ for $\alpha \in Z^2(G, k^*)$.

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Examples

$k[G] = \mathcal{H}^*(pt, G)$ and $k^\alpha[G]$ for $\alpha \in Z^2(G, k^*)$.
$\rightsquigarrow$ discrete torsion [K04]

Quantum and/or stringy

Gromov-Witten invariants of a variety V

[Ruan-Tian, Kontsevich-Manin, Behrend-Fantechi, Li-Tian, Siebert, ...]

$$\begin{array}{ccc} \overline{M}_{g,n}(V,\beta) & \xrightarrow{ev_i} & V \\ \downarrow & & \\ \overline{M}_{g,n} & & \end{array}$$

lead to operations on cohomology, for $\Delta_i \in H^*(V)$

$$< \Delta_1, \dots, \Delta_n > := \int_{[\overline{M}_{g,n}]^{virt}} \prod ev_i^*(\Delta_i)$$

Quantum and/or stringy

Quantum K-theory [Givental-Lee]

Realization of Givental:

$$``[\overline{M}_{g,n}]^{virt} = c_{top}(\mathcal{O}bs)"$$

in good cases $\rightsquigarrow$ quantum K-theory: Operations on $K(V)$

$$< \mathcal{F}_1, \dots, \mathcal{F}_n > := \chi(\prod ev_i^*(\mathcal{F}_i)\lambda_{-1}(\mathcal{O}bs^*))$$

in both cases get classical Frobenius algebra for $n = 3, \beta = 0$. The other operations give a deformation, viz. Frobenius manifold or *CohFT*.

Orbifold GW theory, stringy mutliplication

Orbifold GW theory [Chen-Ruan]

X orbifold (locally $\mathbb{R}^n/G$ where G finite group). They constructed moduli spaces of orbifold stable maps to get operations on

$$H^*_{CR}(X) = H^*(I)_G(X,G))$$

where $I_G(X,G)$ is the inertia orbifold or stack.
New classical limit $(n=3, \beta=0)$ plus new "stringy" ring structure.

Algebraic setting [Abramovich-Graber-Vistoli]

$\mathscr{X}$ a DM-stack over a field of characteristic zero.

Global orbifolds

Global orbifold cohomology I [Fantechi-Göttsche]

For $Y = X/G$ with G finite defined a ring structure and G-action on $H^*_{FG}(X) = \bigoplus_{g \in G} H^*(X^g)$ such that $H^*_{FG}(X)^G \simeq H^*_{CR}(X)$.

Global orbifold cohomology II [Jarvis, K, Kimura]

Gave a construction for the spaces $\overline{M}^G_{g,n}$ and $\overline{M}^G_{g,n}(V, \beta)$, constructed the virtual fundamental class for $\beta = 0$ (and all β for a trivial action) and showed that $H^*_{JKK}(X)$ is a G-Frobenius algebra and that $H^*_{JKK}(X) \simeq H^*_{FG}(X)$. Also gave analog of Frobenius manifold deformation viz. $G - CohFT$ and showed that G-invariants of a $G - CohFT$ are a $CohFT$, i.e. a Frobenius manifold.

The obstruction bundle: curve case

The Obstruction bundle II

The principal $\langle \mathbf{m} \rangle$-bundle over $\mathbb{P}^1 - \{0, 1, \infty\}$ with monodromies m_i extends to a smooth connected curve E. $E/\langle m_1, m_2 \rangle$ has genus zero, and the natural map $E \to E/\langle \mathbf{m} \rangle$ is branched at the three points p_1, p_2, p_3 with monodromy m_1, m_2, m_3, respectively. Let $\pi : E \times X^{\langle m_1, m_2 \rangle} \to X^{\langle m_1, m_2 \rangle}$ be the second projection. We define the obstruction bundle $\mathscr{R}(m_1, m_2)$ on $X^{\langle m_1, m_2 \rangle}$ to be

$$\mathscr{R}(m_1, m_2) := R^1 \pi_*^{\langle m_1, m_2 \rangle} (\mathscr{O}_E \boxtimes TX|_{X^{\langle m_1, m_2 \rangle}}).$$

Theorem (JKK)

When both definitions of $\mathscr{R}$ make sense, they agree.

Summary

Constructions

We have stringy versions for the functors A^*, K^*, H^*, K^*_{top} and Chern characters, in two settings

1. Global Orbifolds. $\mathscr{C}\mathbf{h}$ is a ring isomorphism of G-FAs.
2. Nice stacks. $\mathscr{C}\mathbf{h}$ is a ring homomorphism. ($\mathcal{K}$ carries more data).

These results hold in

1. In the algebraic category
2. In the stable almost complex category

Features

We can define these without reference to moduli spaces of maps.
Get examples of the crepant resolution conjecture.

Newer Developments, based on our formalism

A slew of activities

1. Orbifold deRham Theory (non–Abelian case) [K07].
2. Hochschild, S^1 equivariant version [Pflaum et al. 07].
3. Stringy Singularity Theory [K,Libgober in prep].
4. Symplectic theory [Goldin, Holm, Knudsen, Harada, Kimura, Matusumura 07/09].
5. Equivariant versions [Jarvis Kimura Edidin 09] [K Edidin].
6. Stringy orbifold string topology [Gonzáles et al.].
7. Gerbe Twists [Adem, Ruan, Zhang 06]
8. Global Gerbe twists using twisted Drinfel'd double [K, Pham 08]
9. Higher twists [Pham 09]
10. Wreath products [Matsumura 06]

Still more to come, so check back ...

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Thanks!!!