The geometry and algebra of master equations, BV operators and Feynman transforms

Ralph Kaufmann

Purdue University and MPIM Bonn

University of Pittsburgh, May 14, 2014

References

Main papers

with B. Ward and J. Zuniga.

1. *The odd origin of Gerstenhaber, BV and the master equation* arXiv:1208.3266
2. *Feynman categories: arxiv 1312.1269*

Background

Builds on previous work by Harrelson–Zuniga–Voronov, K, K-Schwell, Kimura–Stasheff–Voronov, A. Schwarz, Zwiebach.

Outline

1. Geometry and Physics
   - Gluing surfaces with boundary
   - CFT/CohFT/Riemann surfaces
   - EMOs, or S^1 gluings
2. Chain level Algebra
   - The classic algebra example: Hochschild and the Gerstenhaber bracket
   - An operadic interpretation of the bracket
   - Cyclic generalization
   - Modular and general version
3. Categorical Approach
   - Motivation and the Main Definition
   - Examples

Operad of surfaces with boundary as a model

Basic objects

Consider a surface (topological) Σ with enumerated boundary components $\partial_\Sigma = \amalg_{i=0,\dots,n-1} S^1$.

Standard gluing

Take two such surfaces Σ, Σ' and define $\Sigma \circ_i \Sigma'$ to be the surface obtained from gluing the boundary i of Σ to the boundary 0 of Σ' and enumerate the $n + n' - 1$ remaining boundaries as

$$0, \dots, i-1, 1', \dots, (n'-1)', i+1, \dots, n-1$$

Physics

This is one version of TFT (if one add slightly more structure).

More gluings: cyclic and modular operad

Non–self gluing (cyclic)

Now enumerate the boundaries by a set S. Then define $\Sigma_s \circ_t \Sigma'$ by gluing the boundaries s and t. The new enumeration is by $(S \setminus \{s\}) \amalg (T \setminus \{t\})$

Self–gluing

If $s, s' \in S$ we can define $\circ_{s,s'}\Sigma$ and the surface obtained by gluing the boundary s to the boundary s'. Notice that the genus of the surface increases by one.

Remark

With both these structures, one does get TFT. Surprisingly the cyclic structure is enough. This uses the result of Dijkgraaf and Abrams that a TFT is basically a commutative Frobenius algebra.

CohFT

Gluing at marked points

We can consider Riemann surfaces with possibly double points (or nodes) as singular points and additional marked points. One can then simply glue by attaching at the marked points. This introduces a new double point.

DM spaces

The Deligne–Mumford compactifications $\bar{M}_{g,n}$ form a modular operad. So do their homologies $H_*(\bar{M}_{g,n})$.

Remark

An algebra over $H_*(\bar{M}_{g,n})$ is called a Cohomological Field Theory (Kontsevich-Manin). Gromov–Witten theory yields algebras over this operad.

Gluing with tangent vectors

KSV spaces

Let $\bar{M}^{KSV}_{g,n}$ be the real blowups of the spaces $\bar{M}_{g,n}$ along the compactification divisors.

Elements of these spaces are surfaces with nodes and a tangent vector at each node. More precisely, an element of $(S^1 \times S^1)/S^1$ at each node.

Odd/family gluings [Zwiebach,KSV,HVZ]

Given two elements $\Sigma \in \bar{M}^{KSV}_{g,n}$ and $\Sigma' \in \bar{M}^{KSV}_{g,n}$ and a marked point on each of them, one defines a family by choosing all possible tangent vectors of the surfaces attached to each other at the marked points.

$$\Sigma_i \circ_j \Sigma' : S^1 \to \bar{M}^{KSV}_{g+g',n+n'-2}$$

These give degree–one gluings on the chain level.

Master equation: String field theory

Fundamental property (informal discussion)

Consider the set of fundamental classes $[\bar{M}^{KSV}_{g,n}]$ and their formal sum $S = \sum_{g,n}[\bar{M}^{KSV}_{g,n}]$. This forms a solution to the master equation. (Details to follow).

$$dS + \Delta S + \frac{1}{2}\{S \bullet S\} = 0$$

Parts of the equation

1. d is the geometric boundary, i.e. $d[\bar{M}^{KSV}_{g,n}]$ is the sum over the boundary divisors.
2. Δ is a term coming from self–gluing.
3. $\frac{1}{2}\{S \bullet S\}$ is a term coming from non–self gluing.

Remarks: The devil in the details

Remarks/Caveats

1. One usually adds some parameter, say $\hbar$ to account for genus and make sums "finite".
2. Of course one has to say where the fundamental classes live.
3. The signs are important.
4. One can also use correspondences on the topological level instead of family gluings.
5. One interpretation of the master equation, is that this equation (a) geometrically selects compactifications. and (b) algebraically parameterizes "free" algebra structures over a certain transform (Feynman)

Operads with S^1 action EMOs and the like

Definition/List

We call a modular operad, ...in the topological category an S^1 equivariant operad, if on each $\mathcal{O}(n)$, ...the $\mathbb{S}_n$ action, ..., is augmented to an $\mathbb{S}_n \wr S^1$, ...action and the action is balanced, that is $\rho_i(l)a_i \circ_j b = a_i \circ_j \rho_j(-l)b$

Proposition

Given and S^1 equivariant modular operad, ...has an induced $\mathfrak{K}$–modular, odd ...structure on its S^1 equivariant chains.

Example

The *Arc* operad of KLP and the various operads/PROPs of [K] used in string topology are further examples.

Results

In [KMZ]

We show that for when considering odd or twisted versions of operads, cyclic or modular operads, (wheeled) PROP(erad)s, and the other usual suspects the following holds.

1. Odd non–self–gluings give rise to odd Lie brackets.
2. Odd self–gluings give rise to differential operators.
3. A horizontal multiplication (given by moving to a non–connected (NC) version) turns the odd brackets into odd Poisson or Gerstenhaber brackets and makes the differentials BV operators (on the nose and not just up to homotopy).
4. Algebraically, the master equation classifies dg–algebras over the relevant dual or Feynman transform.
5. Topologically, the master equation drives the compactification.

Results

In [KW]

We define Feynman categories as a generalization of all the examples above.

1. Unified framework which provides all standard theorems and includes twisted versions on equal footing.
2. Universal operations appear naturally (e.g. BV and the brackets.
3. Odd versions if the Feynman category is graded.
4. Co(bar) transforms and dual (Feynman) transforms/master equations.
5. Model category structure.

The classic algebra case: Hochschild cochains

Hochschild cohomology

- **Cochains** A associative, unital algebra
  $CH^n(A, A) = Hom(A^{\otimes n}, A)$
- **Differential** Example: $f \in CH^2(A, A)$

$$df(a_0, a_1, a_2) = a_0 f(a_1, a_2) - f(a_0 a_1, a_2) + f(a_0, a_1 a_2) - f(a_0, a_1) a_2$$

- **Cohomology** $HH^*(A, A) = H^*(CH^*(A, A), d)$

The classic algebra case: Hochschild cochains

Hochschild cohomology

- **Cochains** A associative, unital algebra
  $CH^n(A, A) = Hom(A^{\otimes n}, A)$
- **Differential** Example: $f \in CH^2(A, A)$

$$df(a_0, a_1, a_2) = a_0 f(a_1, a_2) - f(a_0 a_1, a_2) + f(a_0, a_1 a_2) - f(a_0, a_1) a_2$$

- **Cohomology** $HH^*(A, A) = H^*(CH^*(A, A), d)$

Used for deformation theory.
Used in String Topology: If M is a simply connected manifold
$H_*(LM) \simeq HH^*(S^*(M), S_*(M))$

Pre–Lie and bracket

Operad structure

Substituting g in the i–th variable of f, we obtain operations for $i = 1, \ldots, n$:

$$\begin{aligned} \circ_i : CH^n(A, A) \otimes CH^m(A, A) &\to CH^{m+n-1}(A, A) \\ f \otimes g &\mapsto f \circ_i g \end{aligned}$$

Pre-Lie product

For $f \in CH^n(A, A), g \in CH^m(A, A)$ set

$$f \circ g := \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f \circ_i g$$

Pre–Lie and bracket

Operad structure

Substituting g in the i–th variable of f, we obtain operations for $i = 1, \dots, n$:

$$\begin{aligned} \circ_i : CH^n(A,A) \otimes CH^m(A,A) &\to CH^{m+n-1}(A,A) \\ f \otimes g &\mapsto f \circ_i g \end{aligned}$$

Pre-Lie product

For $f \in CH^n(A,A), g \in CH^m(A,A)$ set

$$f \circ g := \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f \circ_i g$$

Mind the signs!

The bracket

The bracket

If $f \in CH^n(A, A)$ set $|f| = n$ and $sf = |f| - 1$ the shifted degree.

$$\{f \bullet g\} = f \circ g - (-1)^{sf\,sg} g \circ f$$

The bracket

The bracket

If $f \in CH^n(A,A)$ set $|f| = n$ and $sf = |f| - 1$ the shifted degree.

$$\{f \bullet g\} = f \circ g - (-1)^{sf\, sg} g \circ f$$

Theorem (Gerstenhaber '60s)

The bracket above is an odd Lie bracket on $CH^*(A,A)$ and a Gerstenhaber bracket on $HH^*(A,A)$.
Gerstenhaber:

- *odd Lie*
  - *odd anti–symmetric $\{f \bullet g\} = -(-1)^{sf\, sg}\{g \bullet f\}$*
  - *odd Jacobi (use shifted signs)*
- *and odd Poisson (derivation in each variable with shifted signs)*

Sign mnemonics

Algebra shift

If L is a Lie algebra then $\Sigma L = L[-1]$ is an odd Lie algebra.

Two ways of viewing the signs

1. shifted signs: f has degree sf
2. f has degree $|f|$ and $\bullet$ has degree 1.

Compatibility

$$-(-1)^{sf\,sg} = -(-1)^{(|f|-1)(|g|-1)} = (-1)^{|f|+|f|\,|g|+|g|}$$

Viewpoint 2 is natural

Sign mnemonics

Algebra shift

If L is a Lie algebra then $\Sigma L = L[-1]$ is an odd Lie algebra.

Two ways of viewing the signs

1. shifted signs: f has degree sf
2. f has degree $|f|$ and $\bullet$ has degree 1.

Compatibility

$$-(-1)^{sf\, sg} = -(-1)^{(|f|-1)(|g|-1)} = (-1)^{|f|+|f|\,|g|+|g|}$$

Viewpoint 2 is natural

- From a geometric point of view:

Sign mnemonics

Algebra shift

If L is a Lie algebra then $\Sigma L = L[-1]$ is an odd Lie algebra.

Two ways of viewing the signs

1. shifted signs: f has degree sf
2. f has degree $|f|$ and $\bullet$ has degree 1.

Compatibility

$$-(-1)^{sf\, sg} = -(-1)^{(|f|-1)(|g|-1)} = (-1)^{|f|+|f|\,|g|+|g|}$$

Viewpoint 2 is natural

- From a geometric point of view: S^1 family

Sign mnemonics

Algebra shift

If L is a Lie algebra then $\Sigma L = L[-1]$ is an odd Lie algebra.

Two ways of viewing the signs

1. shifted signs: f has degree sf
2. f has degree $|f|$ and $\bullet$ has degree 1.

Compatibility

$$-(-1)^{sf\, sg} = -(-1)^{(|f|-1)(|g|-1)} = (-1)^{|f|+|f|\,|g|+|g|}$$

Viewpoint 2 is natural

- From a geometric point of view: S^1 family
- From an algebraic point of view:

Sign mnemonics

Algebra shift

If L is a Lie algebra then $\Sigma L = L[-1]$ is an odd Lie algebra.

Two ways of viewing the signs

1. shifted signs: f has degree sf
2. f has degree $|f|$ and $\bullet$ has degree 1.

Compatibility

$$-(-1)^{sf\, sg} = -(-1)^{(|f|-1)(|g|-1)} = (-1)^{|f|+|f|\,|g|+|g|}$$

Viewpoint 2 is natural

- From a geometric point of view: S^1 family
- From an algebraic point of view: odd operad

Sign mnemonics

Algebra shift

If L is a Lie algebra then $\Sigma L = L[-1]$ is an odd Lie algebra.

Two ways of viewing the signs

1. shifted signs: f has degree sf
2. f has degree $|f|$ and $\bullet$ has degree 1.

Compatibility

$$-(-1)^{sf\,sg} = -(-1)^{(|f|-1)(|g|-1)} = (-1)^{|f|+|f|\,|g|+|g|}$$

Viewpoint 2 is natural

- From a geometric point of view: S^1 family
- From an algebraic point of view: odd operad
- From a categorical point of view:

Sign mnemonics

Algebra shift

If L is a Lie algebra then $\Sigma L = L[-1]$ is an odd Lie algebra.

Two ways of viewing the signs

1. shifted signs: f has degree sf
2. f has degree $|f|$ and $\bullet$ has degree 1.

Compatibility

$$-(-1)^{sf\,sg} = -(-1)^{(|f|-1)(|g|-1)} = (-1)^{|f|+|f|\,|g|+|g|}$$

Viewpoint 2 is natural

- From a geometric point of view: S^1 family
- From an algebraic point of view: odd operad
- From a categorical point of view: odd Feynman category

Essential Ingredients

To define $\{\bullet\}$

we need

- A (graded) collection CH^n.
- $\circ_i : CH^n \otimes CH^m \to CH^{n+m-1}$ operadic, i.e. they satisfing some compatibilities
- Need to be able to use signs and form a sum.

Generalizations we will give

- (odd) operads
- (odd) cyclic operads
- odd modular operads aka. $\mathfrak{K}$–modular operads
- odd functors from Feynman categories

Essential Ingredients

To define $\{\bullet\}$

we need

- A (graded) collection CH^n. (only need $\mathbb{Z}/2\mathbb{Z}$ grading)
- $\circ_i : CH^n \otimes CH^m \to CH^{n+m-1}$ operadic, i.e. they satisfing some compatibilities
- Need to be able to use signs and form a sum.

Generalizations we will give

- (odd) operads
- (odd) cyclic operads
- odd modular operads aka. $\mathfrak{K}$–modular operads
- odd functors from Feynman categories

(Odd) Lie Algebras from operads

Theorem (G,GV,K,KM,..)

Given an operad $\mathcal{O}$ in Vect set $f^n \circ g^m = \sum_{i=1}^{n} f \circ_i g$ *then*

1. $\circ$ *is pre–Lie and* $[f, g] = f \circ g - (-1)^{|f|\,|g|} g \circ f$ *is a Lie bracket on* $\bigoplus_n \mathcal{O}(n)$.
2. *This bracket descends to* $\mathcal{O}_{\mathbb{S}} := \bigoplus_n \mathcal{O}(n)_{\mathbb{S}_n}$
3. *Given an operad $\mathcal{O}$ in g–Vect set* $f^n \circ g^m = \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f \circ_i g$ *then* $\circ$ *is graded pre–Lie and* $\{f \bullet g\} = f \circ g - (-1)^{sf\, sg} g \circ f$ *is an odd Lie bracket on* $\bigoplus_n \mathcal{O}(n)$.

---

a

b

(Odd) Lie Algebras from operads

Theorem (G,GV,K,KM,..)

Given an operad $\mathcal{O}$ in Vect[a] *set* $f^n \circ g^m = \sum_{i=1}^n f \circ_i g$ *then*

1. $\circ$ *is pre–Lie and* $[f, g] = f \circ g - (-1)^{|f|\,|g|} g \circ f$ *is a Lie bracket on* $\bigoplus_n \mathcal{O}(n)$.
2. *This bracket descends to* $\mathcal{O}_{\mathbb{S}} := \bigoplus_n \mathcal{O}(n)_{\mathbb{S}_n}$
3. *Given an operad $\mathcal{O}$ in g–Vect set* $f^n \circ g^m = \sum_{i=1}^n (-1)^{(i-1)(m-1)} f \circ_i g$ *then* $\circ$ *is graded pre–Lie and* $\{f \bullet g\} = f \circ g - (-1)^{sf\, sg} g \circ f$ *is an odd Lie bracket on* $\bigoplus_n \mathcal{O}(n)$.

---

[a] Or in an additive category with direct sums/coproducts $\bigoplus$

[b]

(Odd) Lie Algebras from operads

Theorem (G,GV,K,KM,..)

Given an operad $\mathcal{O}$ in Vect[a] *set* $f^n \circ g^m = \sum_{i=1}^{n} f \circ_i g$ *then*

1. $\circ$ *is pre–Lie and* $[f,g] = f \circ g - (-1)^{|f|\,|g|} g \circ f$ *is a Lie bracket on* $\bigoplus_n \mathcal{O}(n)$.
2. *This bracket descends to* $\mathcal{O}_{\mathbb{S}} := \bigoplus_n \mathcal{O}(n)_{\mathbb{S}_n}$
3. *Given an operad $\mathcal{O}$ in g–Vect set* $f^n \circ g^m = \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f \circ_i g$ *then* $\circ$ *is graded pre–Lie and* $\{f \bullet g\} = f \circ g - (-1)^{sf\, sg} g \circ f$[b] *is an odd Lie bracket on* $\bigoplus_n \mathcal{O}(n)$.

[a] Or in an additive category with direct sums/coproducts $\bigoplus$

[b] Here one should take the total degree

Shifting operads: How to hide the odd origin of the bracket

Operadic shift

For $\mathcal{O}$ in g–Vect: Let $(s\mathcal{O})(n) = \Sigma^{n-1}\mathcal{O} \otimes sign_n$.
Then $s\mathcal{O}(n)$ is an operad.
In the shifted operad $f \tilde{\circ}_i g = (-1)^{(i-1)(|g|-1)} f \circ_i g$ and the degree of f is sf (if $f \in \mathcal{O}(n)$ of degree 0).

Naïve shift

$(\Sigma\mathcal{O})(n) = \Sigma(\mathcal{O}(n))$.
This is *not an operad* as the signs are off. We say that it is an odd operad.

Shifting operads: How to hide the odd origin of the bracket

Operadic shift

For $\mathcal{O}$ in g–Vect: Let $(s\mathcal{O})(n) = \Sigma^{n-1}\mathcal{O} \otimes sign_n$.
Then $s\mathcal{O}(n)$ is an operad.
In the shifted operad $f \tilde{\circ}_i g = (-1)^{(i-1)(|g|-1)} f \circ_i g$ and the degree of f is sf (if $f \in \mathcal{O}(n)$ of degree 0).

Naïve shift

$(\Sigma\mathcal{O})(n) = \Sigma(\mathcal{O}(n))$.
This is *not an operad* as the signs are off. We say that it is an odd operad. A better more highbrow definition will follow

Shifting operads: How to hide the odd origin of the bracket

Operadic shift

For $\mathcal{O}$ in g–Vect: Let $(s\mathcal{O})(n) = \Sigma^{n-1}\mathcal{O} \otimes sign_n$.
Then $s\mathcal{O}(n)$ is an operad.
In the shifted operad $f \tilde{\circ}_i g = (-1)^{(i-1)(|g|-1)} f \circ_i g$ and the degree of f is sf (if $f \in \mathcal{O}(n)$ of degree 0).

Naïve shift

$(\Sigma\mathcal{O})(n) = \Sigma(\mathcal{O}(n))$.
This is *not an operad* as the signs are off. We say that it is an odd operad. A better more highbrow definition will follow

Remark (KWZ)

$CH^*(A, A)$ is most naturally $\Sigma sEnd(A)$, i.e. an odd operad. This explains the degrees & signs, and enables us to do generalizations.

Odd vs. even

Associativity for a graded operad

$$(a \circ_i b) \circ_j c = \begin{cases} (-1)^{(|b|)(|c|)} (a \circ_j c) \circ_{i+l-1} b & \text{if } 1 \le j < i \\ a \circ_i (b \circ_{j-i+1} c) & \text{if } i \le j \le i+m-1 \\ (-1)^{(|b|)(|c|)} (a \circ_{j-m+1} c) \circ_i b & \text{if } i+m \le j \end{cases}$$

The signs come from the commutativity constraint in g–Vect

Associativity for an odd operad

$$(a \bullet_i b) \bullet_j c = \begin{cases} (-1)^{(|b|-1)(|c|-1)} (a \bullet_j c) \bullet_{i+l-1} b & \text{if } 1 \le j < i \\ a \bullet_i (b \bullet_{j-i+1} c) & \text{if } i \le j \le i+m-1 \\ (-1)^{(|b|-1)(|c|-1)} (a \bullet_{j-m+1} c) \bullet_i b & \text{if } i+m \le j \end{cases}$$

1st generalization

(Anti–)Cyclic operads.

A (anti–)cyclic operad is an operad together with an extension of the $\mathbb{S}_n$ action on $\mathcal{O}(n)$ to an $\mathbb{S}_{n+1}$ action such that

1. $T(id) = \pm id$ where $id \in \mathcal{O}(1)$ is the operadic unit.
2. $T(a^n \circ_1 b^m) = \pm(-1)^{|a||b|} T(b) \circ_m T(a)$

where T is the action by the long cycle $(1 \dots n+1)$

Typical examples

- Cyclic operad: End(V) for V a vector space with a non–degenerate symmetric bilinear form.
- Anti–Cyclic operad: End(V) for V a symplectic vector space i.e. with a non–degenerate anti–symmetric bilinear form.

Compositions and bracket

Unbiased definition

Set $\mathcal{O}(S) = [\bigoplus_{S \overset{1-1}{\leftrightarrow} \{1,\dots,|S|\}} \mathcal{O}(|S|)]_{\mathbb{S}_{n+1}}$
Then we get operations

$$ {}_s\circ_t : \mathcal{O}(S) \otimes \mathcal{O}(T) \to \mathcal{O}((S \setminus \{s\}) \amalg (T \setminus \{t\})) $$

Bracket

For $f \in \mathcal{O}(S), g \in \mathcal{O}(T)$ set

$$ [f, g] = \sum_{s \in S, t \in T} f\, {}_s\circ_t g $$

Brackets

Shifts

The operadic shift of an anti–/cyclic operad
$s\mathcal{O}(n) = \Sigma^{n-1}\mathcal{O}(n) \otimes sign_{n+1}$ is cyclic/anti–cyclic.
The odd cyclic/anti–cyclic versions are defined to be the naïve shifts of the anti–cyclic/cyclic ones.

Theorem (KWZ)

Given an anti–cyclic operad [,] induces a Lie bracket on $\bigoplus \mathcal{O}(n)_{\mathbb{S}_{n+1}}$ which lifts to $\bigoplus \mathcal{O}(n)_{C_{n+1}}$
In the odd cyclic case, we obtain an odd Lie bracket.

Brackets

Shifts

The operadic shift of an anti–/cyclic operad $s\mathcal{O}(n) = \Sigma^{n-1}\mathcal{O}(n) \otimes sign_{n+1}$ is cyclic/anti–cyclic.
The odd cyclic/anti–cyclic versions are defined to be the naïve shifts of the anti–cyclic/cyclic ones.

Theorem (KWZ)

Given an anti–cyclic operad $[\ ,\ ]$ *induces a Lie bracket on* $\bigoplus \mathcal{O}(n)_{\mathbb{S}_{n+1}}$ *which lifts to* $\bigoplus \mathcal{O}(n)_{C_{n+1}}$
In the odd cyclic case, we obtain an odd Lie bracket.

Notice: if we take a cyclic operad, use the operadic shift and then the naïve shift, we get an odd Lie bracket.

Compatibility and examples

Compatibility

Let $N = 1 + T + \cdots + T^n$ on $\mathcal{O}(n)$. Then

$$N[f, g]_{\mathrm{cyclic}} = [Nf, Ng]_{\mathrm{non-cyclic}}$$

Examples: Kontsevich/Conant–Vogtman Lie algebras/New

$(\mathcal{O} \otimes \mathcal{V})(n) := \mathcal{O}(n) \otimes \mathcal{V}(n)$ with diagonal $\mathbb{S}_{n+1}$ action.
Then: cyclic $\otimes$ anti–cyclic is anti–cyclic.
Let V^n be a n dimensional symplectic vector space.
For each n get Lie algebras

$(1)\ Comm \otimes End(V) \quad (2)\ Lie \otimes End(V) \quad (3)\ Assoc \otimes End(V)$

Let V^n be a vector space with a symmetric non-degenerate form.
For each n we get a Lie algebra

$$(4) Pre-Lie \otimes End(V)$$

2nd generalization: Modular operads

Modular operads

Modular operads are cyclic operads with extra gluings

$$\circ_{ss'} : \mathcal{O}(S) \to \mathcal{O}(S \setminus \{s, s'\})$$

The operator Δ

For $f \in \mathcal{O}(S)$

$$\Delta(f) := \frac{1}{2} \sum_{(s,s)' \in S} \circ_{ss'}(f)$$

Odd version

We need the odd version. These are $\mathfrak{K}$–modular operads.

Theorems

Theorem (KWZ)

In a $\mathfrak{K}$–modular operad Δ descends to a differential on $\bigoplus \mathcal{O}(n)_{\mathbb{S}_{n+1}}$. There is also the odd Lie bracket form the underlying odd cyclic structure.
In an NC $\mathfrak{K}$–modular operad, the operator Δ becomes a BV operator for the product induced by the horizontal compositions. Moreover this algebra is then GBV.

Remarks

NC modular and NC $\mathfrak{K}$–modular, which are new, means that like in PROPs there is an additional horizontal composition.
There is a more general setup of when such operations exist.

Theorems

Theorem (KWZ)

In a $\mathfrak{K}$–modular operad Δ descends to a differential on $\bigoplus \mathcal{O}(n)_{\mathbb{S}_{n+1}}$. There is also the odd Lie bracket form the underlying odd cyclic structure.
In an NC $\mathfrak{K}$–modular operad, the operator Δ becomes a BV operator for the product induced by the horizontal compositions. Moreover this algebra is then GBV.

Remarks

NC modular and NC $\mathfrak{K}$–modular, which are new, means that like in PROPs there is an additional horizontal composition.
There is a more general setup of when such operations exist.
Wait a couple of minutes.

Comments

1. $\mathfrak{K}$ modular operads were defined by Getzler and Kaparnov
2. The Feynman transform of a modular operad is a $\mathfrak{K}$–modular operad $\rightsquigarrow$ Examples.
3. Restricting the twist to the triples for operads and cyclic operads gives their odd versions. For experts

$$\mathfrak{K} \simeq Det \otimes \mathfrak{D}_s \otimes \mathfrak{D}_\Sigma$$

Here Det is a cocylce with value on a (connected) graph Γ given by $Det(H_1(\Gamma))$.

Comments

1. $\mathfrak{K}$ modular operads were defined by Getzler and Kaparnov
2. The Feynman transform of a modular operad is a $\mathfrak{K}$–modular operad $\rightsquigarrow$ Examples.
3. Restricting the twist to the triples for operads and cyclic operads gives their odd versions. For experts

$$\mathfrak{K} \simeq Det \otimes \mathfrak{D}_s \otimes \mathfrak{D}_\Sigma$$

Here Det is a cocylce with value on a (connected) graph Γ given by $Det(H_1(\Gamma))$.
Somehow not so commenly known.

Master equation

General form

If the “generalized operads” are dg, then one can make sense of the master equation (Details to follow).

$$dS + \Delta S + \frac{1}{2}\{S \bullet S\} = 0$$

This equation parameterizes “free” algebra structures on one hand (e.g. the Feynman transform) and compactifications on the other.

Examples

Feynman transform after Barannikov

The $\mathcal{F}_D P$–algebra structures on V are given by solutions of the master equation

$$dS + \Delta S + \frac{1}{2}\{S \bullet S\} = 0$$

on $(\bigoplus(P(n) \otimes V^{\otimes n+1})^{\mathbb{S}_{n+1}})_0$

Interpertation

The background is that $\mathcal{F}_D P$ is free as an operad, but not a free dg operad. So to specify an algebra over it, there are conditions and the Master equation is the sole equation.

Futher examples

NC version

If we go to the NC version (defined in [KMZ]), then the Δ above becomes a BV operator and the associated bracket Poisson.

Open/closed version

There is an open/closed version due to [HVZ]. This has two BV structures and an extra derivation. The open one has a different mechanism for the signs.

Formal Super–manifolds after Shardrin-Merkl-Merkulov

Algebra structures over a certain wheeled PROP for a formal supermanifold are given by solutions of the master equation.

We need a global view for The Zoo: an inventory of species

Types of operads and graphs

Type	Graphs
Operads	connected rooted trees
Cyclic operads	connected trees
Modular operads	connected graphs
PROPs	directed graphs
NC modular operad	graphs

- This means that for each such diagram there is a unique composition after decorating the vertices of such a graph with "operad" elements.
- These type of graphs will correspond to morphisms in the categorical setup.
- The original setup was algebras over a triple. [Markl, Getzler-Kapranov, . . .] although NC modular is in principle totally new.

Twisted (modular) operads

Twisted/odd operads

In the same setting.

Type	Graphs
odd operad	connected rooted trees
	with orientation on the set of all edges
anti–cyclic	connected trees
	with orientation on each edge
odd cyclic	connected trees
	with orientation on the set of all edges
$\mathfrak{K}$–modular operads	connected graphs
	with an orientation on the set of all edges
$\mathfrak{K}$–modular NC operads	graphs
	with an orientation on the set of all edges

What we want and get

Goal

We need a more general theory of things like operads so encompass all the things we have seen so far.

Generality

We think our approach is just right to fit what one needs.
Our definition fits in between the Borisov–Manin and the Getzler approaches. It is more general than Borisov–Manin as it includes odd and twisted modular versions, EMOs, etc.. It is a bit more strict than Getzler's use of patterns, but there one always has to first prove that the given categories are a pattern.

Feynman categories

Data

1. $\mathcal{V}$ a groupoid
2. $\mathcal{F}$ a symmetric monoidal category
3. $\imath : \mathcal{V} \to \mathcal{F}$ a functor.

Notation

$\mathcal{V}^{\otimes}$ the free symmetric category on $\mathcal{V}$ (words in $\mathcal{V}$).

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\quad \imath \quad} & \mathcal{F} \\ & {\scriptstyle \jmath} \searrow \quad \nearrow {\scriptstyle \imath^{\otimes}} & \\ & \mathcal{V}^{\otimes} & \end{array}$$

Feynman category

Definition

Such a triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ is called a Feynman category if

1. $\imath^{\otimes}$ induces an equivalence of symmetric monoidal categories between $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$.
2. $\imath$ and $\imath^{\otimes}$ induce an equivalence of symmetric monoidal categories $Iso(\mathcal{F} \downarrow \mathcal{V})^{\otimes}$ and $Iso(\mathcal{F} \downarrow \mathcal{F})$.
3. For any $* \in \mathcal{V}$, $(\mathcal{F} \downarrow *)$ is essentially small.

Hereditary condition (ii)

1. In particular, fix $\phi : X \to X'$ and fix $X' \simeq \bigotimes_{v \in I} \imath(*_v)$: there are $X_v \in \mathcal{F}$, and $\phi_v \in Hom(X_v, *_v)$ s.t. the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ \downarrow \simeq & & \downarrow \simeq \\ \bigotimes_{v \in I} X_v & \xrightarrow{\bigotimes_{v \in I} \phi_v} & \bigotimes_{v \in I} \imath(*_v) \end{array} \tag{1}$$

2. For any two such decompositions $\bigotimes_{v \in I} \phi_v$ and $\bigotimes_{v' \in I'} \phi'_{v'}$ there is a bijection $\psi : I \to I'$ and isomorphisms $\sigma_v : X_v \to X'_{\psi(v)}$ s.t. $P_\psi^{-1} \circ \bigotimes_v \sigma_v \circ \phi_v = \bigotimes \phi'_{v'}$ where P_ψ is the permutation corresponding to ψ.
3. These are the only isomorphisms between morphisms.

Hereditary condition (ii)

1. In particular, fix $\phi : X \to X'$ and fix $X' \simeq \bigotimes_{v \in I} \imath(*_v)$: there are $X_v \in \mathcal{F}$, and $\phi_v \in Hom(X_v, *_v)$ s.t. the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\quad\phi\quad} & X' \\ \simeq \downarrow & & \downarrow \simeq \\ \bigotimes_{v\in I} X_v & \xrightarrow{\bigotimes_{v\in I}\phi_v} & \bigotimes_{v\in I} \imath(*_v) \end{array} \qquad (1)$$

2. For any two such decompositions $\bigotimes_{v\in I} \phi_v$ and $\bigotimes_{v'\in I'} \phi'_{v'}$ there is a bijection $\psi : I \to I'$ and isomorphisms $\sigma_v : X_v \to X'_{\psi(v)}$ s.t. $P_\psi^{-1} \circ \bigotimes_v \sigma_v \circ \phi_v = \bigotimes \phi'_{v'}$, where P_ψ is the permutation corresponding to ψ.

3. These are the only isomorphisms between morphisms.

Hereditary condition (ii)

1. In particular, fix $\phi : X \to X'$ and fix $X' \simeq \bigotimes_{v\in I} \imath(*_v)$: there are $X_v \in \mathcal{F}$, and $\phi_v \in Hom(X_v, *_v)$ s.t. the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\quad\phi\quad} & X' \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ \bigotimes_{v\in I} X_v & \xrightarrow{\bigotimes_{v\in I}\phi_v} & \bigotimes_{v\in I} \imath(*_v) \end{array} \tag{1}$$

2. For any two such decompositions $\bigotimes_{v\in I} \phi_v$ and $\bigotimes_{v'\in I'} \phi'_{v'}$ there is a bijection $\psi : I \to I'$ and isomorphisms $\sigma_v : X_v \to X'_{\psi(v)}$ s.t. $P_\psi^{-1} \circ \bigotimes_v \sigma_v \circ \phi_v = \bigotimes \phi'_{v'}$, where P_ψ is the permutation corresponding to ψ.
3. These are the only isomorphisms between morphisms.

Feynman categories

Definition

A Feynman graph category (FGC) is a pair $(\mathcal{F}, \mathcal{V})$ of a monoidal category $\mathcal{F}$ whose objects are sometimes called clusters or aggregates and a comma generating subcategory $\mathcal{V}$ whose objects are sometimes called stars or vertices.

Definition

Let $\mathcal{C}$ be a symmetric monoidal category. Consider the category of strict monoidal functors $\mathcal{F}\text{–}\mathcal{O}ps_{\mathcal{C}} := Fun_{\otimes}(\mathcal{F}, \mathcal{C})$ which we will call $\mathcal{F}$–ops in $\mathcal{C}$ and the category of functors $\mathcal{V}\text{–}\mathcal{M}ods_{\mathcal{C}} := Fun(\mathcal{V}, \mathcal{C})$ which we will call $\mathcal{V}$–modules in $\mathcal{C}$.
If $\mathcal{C}$ and $\mathcal{F}$ respectively $\mathcal{V}$ are fixed, we will only write $\mathcal{O}ps$ and $\mathcal{M}ods$.

Monadicity

Theorem

If $\mathcal{C}$ is cocomplete then the forgetful functor G from $\mathcal{O}ps$ to $\mathcal{M}ods$ has a left adjoint F which is again monoidal.

Corollary

$\mathcal{O}ps$ is equivalent to the algebras over the triple FG.

Morphisms

Given a morphism (functor) of Feynman categories $i : \mathcal{F} \to \mathcal{F}'$ there is pullback i^* of $\mathcal{O}ps$ and if $\mathcal{C}$ is cocomplete there is a left Kan extension which is, as we prove, monoidal. This gives the push–forward i_*.
The free functor is such a Kan extension. So is the PROP generated by an operad and the modular envelope. The passing between biased and unbiased versions is also of this type.

Examples

Basic Example

$(\mathcal{A}gg, \mathcal{C}rl)$: $\mathcal{C}rl$ is the category of S–corollas with the automorphisms $Aut(S)$ and $\mathcal{A}gg$ are disjoint unions of these with morphisms being the graph morphisms between them.

Ordered/Ordered Examples

If a Feynman category has morphisms indexed by graphs, we can define a new Feynman category by using as morphisms pairs of a morphism and an order/orientation of the edges of the graphs.

Ab-version

Enriching over the category of Abelian groups, we get the odd versions.

Examples

Basic Example

$(\mathcal{A}gg, \mathcal{C}rl)$: $\mathcal{C}rl$ is the category of S–corollas with the automorphisms $Aut(S)$ and $\mathcal{A}gg$ are disjoint unions of these with morphisms being the graph morphisms between them.

Ordered/Ordered Examples

If a Feynman category has morphisms indexed by graphs, we can define a new Feynman category by using as morphisms pairs of a morphism and an order/orientation of the edges of the graphs. There is a technical version of this called a Feynman category indexed over $\mathcal{A}gg$.

Ab-version

Enriching over the category of Abelian groups, we get the odd versions.

Enriched versions

Proposition

If $\mathcal{C}$ is Cartesian closed, each coboundary (generalizing the term as used by Getzler and Kapranov) defines an Feynman category enriched over $\mathcal{C}$ (or Ab) such that the $\mathcal{O}ps$ are exactly the twisted modular operads.

Categorical version of EMOs and the like

1. First we can just add a twist parameter to each edge and enrich over *Top*. The $\mathcal{O}ps$ will then have twist gluings.
2. Instead of taking S–corollas, we can take objects with an $Aut(S) \wr S^1$ action that is balanced. Then there is a morphism of Feynman categories, from this category to the first example.

Enriched versions

Proposition

If $\mathcal{C}$ is Cartesian closed, each coboundary (generalizing the term as used by Getzler and Kapranov) defines an Feynman category enriched over $\mathcal{C}$ (or Ab) such that the $\mathcal{O}ps$ are exactly the twisted modular operads.
Note, we do not have to restrict to dgVect here. There is a more general Theorem about these type of enriched categories

Categorical version of EMOs and the like

1. First we can just add a twist parameter to each edge and enrich over Top. The $\mathcal{O}ps$ will then have twist gluings.
2. Instead of taking S–corollas, we can take objects with an $Aut(S) \wr S^1$ action that is balanced. Then there is a morphism of Feynman categories, from this category to the first example.

General version

Theorem (KWZ)

Let $\mathcal{C}$ be cocomplete.

1. *If we have a Feynman category indexed over $\mathcal{A}gg$ with odd self–gluings then for every $\mathcal{O} \in \mathcal{O}ps$ the object $colim_{\mathcal{V}}\mathcal{O}$ carries a differential Δ which is the sum over all self–gluings.*
2. *If we have a Feynman category indexed over $\mathcal{A}gg$ with odd non–self–gluings then for every $\mathcal{O} \in \mathcal{O}ps$ the object $colim_{\mathcal{V}}\mathcal{O}$ carries an odd Lie bracket.*
3. *If we have a Feynman category indexed over $\mathcal{A}gg$ with odd self–gluings and odd non–self–gluings as well as horizontal (NC) compositions, then the operator Δ is a BV operator and induces the bracket.*

More,...

Further results

1. There is also a generalization of Barannikov's result. Using the free functor and the dual notion of Co-$\mathcal{O}ps$, i.e. contravariant functors.
2. There is one more generalization, which replaces $\mathcal{A}gg$ with a category with unary and binary generators with quadratic relations for the morphisms.
   So there not need be any reference at all to graphs in this story.
3. In general to say "quasi–free" (cofibrant), one needs a model category structure, which we give.

More,...

Further results

1. There is also a generalization of Barannikov's result. Using the free functor and the dual notion of Co-$\mathcal{O}ps$, i.e. contravariant functors.
2. There is one more generalization, which replaces $\mathcal{A}gg$ with a category with unary and binary generators with quadratic relations for the morphisms.
   So there not need be any reference at all to graphs in this story.
3. In general to say "quasi–free" (cofibrant), one needs a model category structure, which we give.

More?

More,...

Further results

1. There is also a generalization of Barannikov's result. Using the free functor and the dual notion of Co-$\mathcal{O}ps$, i.e. contravariant functors.
2. There is one more generalization, which replaces $\mathcal{A}gg$ with a category with unary and binary generators with quadratic relations for the morphisms.
   So there not need be any reference at all to graphs in this story.
3. In general to say "quasi–free" (cofibrant), one needs a model category structure, which we give.

More? Sure, but maybe next time.

More,...

Further results

1. There is also a generalization of Barannikov's result. Using the free functor and the dual notion of Co-$\mathcal{O}ps$, i.e. contravariant functors.
2. There is one more generalization, which replaces $\mathcal{A}gg$ with a category with unary and binary generators with quadratic relations for the morphisms.
   So there not need be any reference at all to graphs in this story.
3. In general to say "quasi–free" (cofibrant), one needs a model category structure, which we give.

More? Sure, but maybe next time.

Thank you!