# Stringy Singularities

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# References

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# Singularities with Symmetries

## General Setup

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a function with an isolated critical point at 0.

- We also fix $f$ to be quasi–homogenous with $q_i$ the degree of $z_i$. $Q = diag(q_i)$
- $M_f := \mathbb{C}[[\mathbf{z}]]/J_f$ where $J_f$ is the Jacobian ideal.
- Using the Grothendieck residue pairing this is a Frobenius algebra.

# Singularities with Symmetries

## Symmetries

Now also fix $G \subset GL(n, \mathbb{C})$ finite, such that $\forall g \in G, g^*(f) = f$.
Let $Fix(g) \subset \mathbb{C}^n$ be the fixed point set.

$$A_g := M_{f|_{Fix(g)}}, \quad A := \bigoplus A_g$$

## First Goal: Stringy structure

Make $A$ into a $G$–Frobenius algebra (GFA).

The signature structure of a $G$–FA is a $G$–graded multiplication.

$$A_g \otimes A_h \to A_{gh}$$

## Second Goal: Deformations

Give a geometric description of the minversal unfolding.
Noncommutative structure.

# Singularities with Symmetries

## Examples

- $A_n$ singularity. $f = z^{n+1}$, $G = \mathbb{Z}/(n+1)/Z$. Action $z \mapsto \zeta_{n+1}$. $\zeta_n = e^{2\pi i/(n+1)}$
- Quintic. $f = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5$, $G = (\mathbb{Z}/5\mathbb{Z})^{\times 5}$, or $H$ diagonal $\mathbb{Z}/5\mathbb{Z}$
- Symmetric products. $f(\mathbf{x}_1) + f(\mathbf{x}_2) + \dots f(\mathbf{x}_n)$ $G = \Sigma_n$.

# $G$–Frobenius algebras, long definition

## Definition

*A $G$–Frobenius algebra is $< G, A, \circ, 1, \eta, \varphi, \chi >$, where*

| | |
|---|---|
| $G$ | *finite group* |
| $A$ | *finite dim $G$-graded $k$–vector space $A = \bigoplus_{g\in G} A_g$* [1] |
| $\circ$ | *a multiplication on $A$ which respects the grading:* $\circ : A_g \otimes A_h \to A_{gh}$ |
| $1$ | *a fixed element in $A_e$–the unit* |
| $\eta$ | *non-degenerate bilinear form which respects the $G$ grading i.e. $\eta\|_{A_g \otimes A_h} = 0$ unless $gh = e$.* |
| $\varphi$ | *an action of $G$ on $A$ (which will be by algebra automorphisms), $\varphi \in \mathrm{Hom}(G, \mathrm{Aut}(A))$, s.t. $\varphi_g(A_h) \subset A_{ghg^{-1}}$* |
| $\chi$ | *a character $\chi \in \mathrm{Hom}(G, k^*)$* |

[1] $A_e$ is called the untwisted sector and the $A_g$ for $g \neq e$ are called the twisted sectors.

# Axioms for a $G$ Frobenius Algebra

a) *Associativity* $(a_g \circ a_h) \circ a_k = a_g \circ (a_h \circ a_k)$

b) *Twisted commutativity* $a_g \circ a_h = \varphi_g(a_h) \circ a_g$

c) *$G$ Invariant Unit*: $1 \circ a_g = a_g \circ 1 = a_g$ and $\varphi_g(1) = 1$

d) *Invariance of the metric*: $\eta(a_g, a_h \circ a_k) = \eta(a_g \circ a_h, a_k)$

i) *Projective self–invariance of the twisted sectors* $\varphi_g|A_g = \chi_g^{-1} id$

ii) *$G$–Invariance of the multiplication* $\varphi_k(a_g \circ a_h) = \varphi_k(a_g) \circ \varphi_k(a_h)$

iii) *Projective $G$–invariance of the metric* $\varphi_g^*(\eta) = \chi_g^{-2}\eta$

iv) *Projective trace axiom* $\forall c \in A_{[g,h]}$ and $l_c$ left multiplication by $c$: $\chi_h \mathrm{Tr}(l_c \varphi_h|_{A_g}) = \chi_{g^{-1}} \mathrm{Tr}(\varphi_{g^{-1}} l_c|_{A_h})$

# Gauged TFTs and $G$–Frobenius algebras

### Theorem (Turaev, K)

*Topological Field Theories with a finite gauge group $G$ as thought of as projective functors from the rigidified cobordism category of $G$–bundles to vector spaces are in 1–1 correspondence with $G$–Frobenius algebras.*

### Remark

We will need the version [K] which includes a twist by a character $\chi$ of $G$.

# $G$–Frobenius algebras (short definition)

## Theorem

*A $G$–Frobenius algebra $A$ is precisely a unital, associative, commutative algebra object in $D(k[G])$–Mod together with a metric $\eta$ which additionally satisfies*

(S) $\rho(v^{-1}) = \chi^{-1}$ *for a character* $\chi \in \mathrm{Hom}(G, k^*)$

(T) $\forall c \in A_{[g,h]}$ *and* $l_c$ *left multiplication by c:*
$\chi_h \mathrm{Tr}(l_c \varphi_h|_{A_g}) = \chi_{g^{-1}} \mathrm{Tr}(\varphi_{g^{-1}} l_c|_{A_h})$

Here $D(k[G])$ is a quasi-triangular quasi–Hopf algebra and $v$ is the element such that $S^2(u) = vuv^{-1}$.

## Remark

We can also treat just Frobenius algebra objects in the category and use $\tau(\phi, g) := \epsilon(\mu(\phi \otimes id)\Delta(\upsilon(1_k)|_g))$ where $\Delta(\upsilon(1_k)|_g)$ is the bi–degree $(g, g^{-1})$ part of $\Delta(\upsilon(1_k))$ for the trace.

# The twisted Drinfel'd double

## Definition

*For a finite group $G$ and an element $\beta \in Z^3(G, k^*)$, the twisted Drinfel'd double $D^\beta(k[G])$ is the quasi-triangular quasi-Hopf algebra whose*

1. *underlying* vector space has *the basis* $g\underset{x}{\llcorner}$ *with* $x, g \in G$
   $D^\beta(k[G]) = \bigoplus k\; g\underset{x}{\llcorner}$
2. *algebra structure is given by* $g\underset{x}{\llcorner}\, h\underset{y}{\llcorner} = \delta_{g,xhx^{-1}}\theta_g(x,y)\, g\underset{xy}{\llcorner}$
   *where* $\theta_g(x,y) = \frac{\beta(g,x,y)\beta(x,y,(xy)^{-1}g(xy))}{\beta(x,x^{-1}gx,y)}$
3. *the co-algebra structure is given by*
   $\Delta(g\underset{x}{\llcorner}) = \sum_{g_1g_2=g} \gamma_x(g_1,g_2)\, g_1\underset{x}{\llcorner} \otimes g_2\underset{x}{\llcorner}$ *where*
   $\gamma_x(g_1,g_2) = \frac{\beta(g_1,g_2,x)\beta(x,x^{-1}g_1x,x^{-1}g_2x)}{\beta(g_1,x,x^{-1}g_2x)}$

# The twisted Drinfel'd double

4. The Drinfel'd associator $\Phi$ is given by
$\Phi = \sum_{g,h,k \in G} \beta(g,h,k)^{-1} \, g_{\llcorner_e} \otimes h_{\llcorner_e} \otimes k_{\llcorner_e}$
5. The $R$ matrix is given by
$R = \sum_{g \in G} g_{\llcorner_e} \otimes \mathbf{1}_{\llcorner_g}$, where $\mathbf{1}_{\llcorner_g} = \sum_{h \in G} h_{\llcorner_g}$
6. The antipode $S$ is given by
$S(g_{\llcorner_x}) = \frac{1}{\theta_{g^{-1}}(x,x^{-1})\gamma_x(g,g^{-1})} \; x^{-1}gx_{\llcorner_{x^{-1}}}$

## Remark

This is good for gerbe twisting

# Remark: Gerbe twisting

## Theorem

*There are universal gerbe twistings for the TFT with finite gauge group which can be understood as follows.*

| | |
|---|---|
| *0–Gerbes* | *twisting by a character* $\chi$ |
| *1–Gerbes* | *twisting by discrete torsion* |
| *2–Gerbes* | *twisting on the the inertia* |

## Example

For $\beta$ a 2–gerbe on the stack $[pt/G]$
$K^{\beta}_{\mathrm{JKK}full}([pt/G]) \simeq Rep(D^{\beta}(k[G]))$

# Physics background

## $(2,2)$ super-conformal field theory I

- $N = 2$ super-conformal symmetry for both the left and the right movers. This implies that there are four finite rings which are closed under the operator product.
- $(c,c),(a,c),(a,a)$ and $(c,a)$.
- Certain constraints for their eigenvalues with respect to the operators $J_0, \bar{J}_0, L_0, \bar{L}_0$ of the two $N = 2$ super-conformal algebras, which are usually called $q, \bar{q}, h$ and $\bar{h}$, respectively.
- It turns out the rings $(a,a)$ and $(c,a)$ can be recovered from $(c,c)$ and $(a,c)$ by charge conjugation.

# Physics background

## $(2,2)$ super-conformal field theory II

- $|\phi\rangle$ is left chiral if $G^+_{-1/2}|\phi\rangle = 0$ or equivalently $h = \frac{q}{2}$. It is called left anti-chiral if $G^-_{-1/2}|\phi\rangle = 0$ or equivalently $h = -\frac{q}{2}$. Right chiral means that $\bar{G}^+_{-1/2}|\phi\rangle = 0$ or equivalently $\bar{h} = \frac{\bar{q}}{2}$, and finally right anti-chiral means that $\bar{G}^-_{1/2}|\phi\rangle = 0$ or equivalently $\bar{h} = -\frac{\bar{q}}{2}$.
- Thus one confines oneself to study the former two rings. Mirror symmetry as it was originally conceived in physics was an operation which takes one conformal field theory $T$ and produces another conformal field theory $\check{T}$ such that the $(c,c)$ ring of $T$ is isomorphic to the $(a,c)$-ring of $\check{T}$ and vice versa.

# More Physics

## Landau–Ginzburg Theory

$N = 2$ theory which is the conformally invariant fixed point of the Lagrangian

$$\mathcal{L} = \int K(X, \bar{X}) d^2 z d^4 \theta + \int (f(z_i) + \text{ complex conjugate}) \, d^2 z d^2 \theta,$$

where $f$ is a quasi-homogeneous function of fractional degree $q_i$ for $z_i$. This model leads to a trivial $(a, c)$-ring and a $(c, c)$-ring which is given by $\mathbb{C}[\mathbf{z}]/J_f$ where $J_f = (f_{z_i})$ is the Jacobian ideal.
Moreover, the bi-degree $(q, \bar{q})$ for $z_i$ is given by $(q_i, q_i)$.
$(c, c)$-ring is the Milnor ring of the singularity.
$(a, c)$-ring is trivial. (B-Model)

# Mathematical model

## Mathematical model

In [1] we defined a mathematical model for an orbifold Landau–Ginzburg theory for a singularity $f$ with symmetry group $G$. It has an $(a,c)$ and a $(c,c)$ ring which will be denoted $GM_f$ and $(GM_f)^\vee$. Both of these rings have a bi–grading.

## Duality Spectral Flow

The motivation for the dual bi-grading again comes from the physical interpretation of $GM_f$ in orbifold Landau-Ginzburg theory and the dualization being implemented by the spectral flow operator $\mathcal{U}_{(1,0)}$ [IV] which has the natural charge $(d = \hat{c} = \frac{c}{3}, 0)$.

# Mirror Symmetry

## Mirror pairs

We will in certain situations produce mirror pairs who will have interchanged $(a, c)$ and $(c, c)$ rings. This includes the ADE and B,F cases which are mirror–self dual.

## Remark

This means the mirror dual model can be called a "Landau–Ginzburg A–model". Following suggestions of Witten: Fan, Jarvis and Ruan constructed a version of an A-model for special types of Landau–Ginzburg potential. The state space they use is the above $(a, c)$ ring of the corresponding LG orbifold model.

# Special $G$FAs and construction of GFA

## The stringy multiplication problem

Given a $D(k[G])$ module $A = \bigoplus_{g \in G} A_g$ which satisfies (S) for $\chi$ together with

1. A Frobenius algebra structure on each $A_g$
2. Isomorphisms $A_g \simeq A_{g^{-1}}$
3. An (cyclic) $A_e$ module structure on each $A_g$

find the possible $G$–Frobenius algebra structures on $A$

## Definition

*A $GFA$ $A$ is special if each twisted sector is a cyclic module over the untwisted sector.*

# (Re)construction: algebraic solution

## Theorem (K02)

*$A := \bigoplus_{g \in G} A_g$ as above, up to an isomorphism of Frobenius algebras on the $A_g$, then the structures of super $G$-Frobenius algebra on $A$ are in 1-1 correspondence with triples $(\sim, \gamma, \varphi)$.*

- The function $\sim$ is just a super-sign which determines if $1_g$ is even or odd.
- $\gamma$ is a $G$-graded, section-independent cocycle compatible with the metric satisfying the condition of supergrading with respect to the natural $G$ action
- $\varphi$ is a non-Abelian two cocycle with values in $k^*$ which satisfies the condition of discrete torsion with respect to $\sigma$ and the natural $G$ action, such that $(\gamma, \varphi)$ is a compatible pair.
- The cocycle $\gamma$ is a special type of $A_e$ valued group 2-cocycle

# Remarks

1. $\gamma$ defines multiplication on the cyclic generators via
$$1_g \circ 1_h := \gamma(g,h)1_{gh}$$
the extra conditions ensure that the extension of this multiplication using the cyclic $A_e$-module structures is well defined.
2. $\varphi$ gives the $G$-action on the generators
$$\rho(g)(1_h) = \varphi(g,h)1_{ghg^{-1}}$$
3. In the singularity case we will alternatively use
$$\sigma(g) := \tilde{g} + |N_g| \mod 2, \tag{1}$$
where $|N_g| := \mathrm{codim}(Fix(g))$ in $\mathbb{C}^n$. $\sigma \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$
4. In many cases the equations for the cocycles allow one to find a unique multiplication up to the twist by discrete torsion $\epsilon$.

# Singularities with Symmetries

## General Setup

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a function with an isolated critical point at 0.

- We also fix $f$ to be quasi–homogenous with $q_i$ the degree of $z_i$. $Q = diag(q_i)$
- $M_f := \mathbb{C}[[\mathbf{z}]]/J_f$ where $J_f$ is the Jacobian ideal.
- Using the Grothendieck residue pairing this is a Frobenius algebra.

# Singularities with Symmetries

## Symmetries

Now also fix $G \subset GL(n, \mathbb{C})$ finite, such that $\forall g \in G, g^*(f) = f$. Let $Fix(g) \subset \mathbb{C}^n$ be the fixed point set.

- $A_g := M_{f|_{Fix(g)}}$
- $\chi = det(g)$
- Get data for the Stringy Multiplication Problem.

## Euler

$j := \exp(2\pi i Q)$. We call the data $(f, G)$ Euler if $j \in G$. Also $J = \langle j \rangle$.

Notice $J$ is always a symmetry, so we can enlarge $G$ if necessary (quasi–Euler).

# Grading

## Grading Operators

$Q^{(1)}$ inherent grading in the definition of each $A_g$ . For an element $a_g \in A_g$ is

$$\mathcal{Q}(a_g) = Q^{(1)}(a_g) + \frac{1}{2}(s^+(g) + s^-(g)).$$

## Standard grading shift

For a $G$-Frobenius algebra with a choice of linear representation $\rho : G \to GL_n(k)$, $s_g := \frac{1}{2}(s_g^+ + s_g^-)$ with $s_g^+ := d - d_g$ ($d_g$ the degree of co-unit in $A_g$) and

$$s_g^- := \frac{1}{2\pi i}\left(\mathrm{tr}(\log(g)) - \mathrm{tr}(\log(g^{-1}))\right) \quad := \quad \frac{1}{2\pi i}\left(\sum_i l_i(g) - \sum_i l_i(g^{-1})\right)$$

where the $l_i(g)$ are the logarithms of the eigenvalues of $\rho(g)$ using the arguments in $[0, 2\pi)$.

# Bi–grading

## Definition

*Set* $\bar{s}_g := \frac{1}{2}(s_g^+ - s_g^-)$. *Since* $s_g^+ = s_{g^{-1}}^+$ *and* $s_g^- = -s_{g^{-1}}^-$, *it follows that* $\bar{s}_g = s_{g^{-1}}$. *We define the bi-grading* $(\mathcal{Q}, \bar{\mathcal{Q}})$ *by*

$$\mathcal{Q}(a_g) := Q(a_g) + s_g \quad \bar{\mathcal{Q}}(a_g) := Q(a_g) + \bar{s}_g \quad \text{for } a_g \in A_g$$

## Definition

*Fix a graded Frobenius algebra* $A$ *with grading operator* $Q$.
*We define its* $(c,c)$*-realization* $A^{(c,c)}$ *to be given by the Frobenius algebra* $A$ *together with the bi-grading* $(Q,Q)$, *i.e.* $\bar{Q} = Q$.
*We define the* $(a,c)$*-realization of* $A$ *denoted by* $A^{(a,c)}$ *to be given by the Frobenius algebra* $A$ *together with the bi-grading* $(Q,-Q)$, *i.e.,* $\bar{Q} = -Q$.

# (a,c)–ring $(GM_f)^\vee$ via a duality

## The dual $k$-module

Given the $G$-Frobenius algebra $GM_f$, its dual $G$–graded $k$-module $(GM_f)^\vee$ is defined as

$$(GM_f)^\vee = \bigoplus_{g\in G} \check{A}_g : \check{A}_g = A_{gj^{-1}} = M_{f|_{\mathrm{Fix}(gj^{-1})}},$$

where $j$ is the group element defining the exponential of the grading operator $Q$ via $\rho(j) = \exp(2\pi i Q)$.

## The dual $D(k[G])$-module $(GM_f)^\vee$

The $G$-module structure is given by pulling back the action and scaling by $\chi$. In the case of a singularity the character $\chi$ is determined by a choice of sign function $\sigma \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$ given by $\chi(g) = (-1)^{\sigma(g)} \det(g)$.

# Details on the dual $G$–action

## Pull back twisted action

If we denote the $G$-action on $\check{A}$ by $\check{\varphi}$, then using the $k$-module isomorphism $M : A_g \to A_{gj^{-1}}$

$$\check{\varphi}(g)(\check{a}_h) := \chi(g) M \varphi(g) M^{-1}(\check{a}_h) \in \check{A}_{hgh^{-1}}, \quad \text{for } \check{a}_h \in \check{A}_g;$$

or if we denote $M(a) =: \check{a}$ and fix $\sigma \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$, then for $\check{a} \in \check{A}_h$

$$\check{\varphi}(g)(\check{a}) := (-1)^{\sigma(g)} \det(g) (\varphi(g)(a))\check{} \in \check{A}_{ghg^{-1}}$$

## Remark

For $\check{a}_h = M(a1_{hj^{-1}}) \subset \check{A}_h$

$$\check{\varphi}(g)(\check{a}_h) = \check{\varphi}(g) M(a1_{hj^{-1}})$$
$$= \epsilon(g, hj^{-1})(-1)^{\sigma(g)(\sigma(h)+\sigma(j)+1)} \det(g|_{\mathrm{Fix}(hj^{-1})}) M(a1_{ghg^{-1}j^{-1}})$$

# The dual bi–grading

## New bi–grading

If $A$ was initially graded by the operator $Q^{(1)}$ and $Q(a_g) = Q^{(1)}(a_g) + s_g$, then set $\check{s}_g := s_{gj^{-1}} - d$ and $\bar{\check{s}}_g := \bar{s}_{gj^{-1}}$, where we recall that $d$ is the degree of the $G$-Frobenius algebra. We define a bi-grading on $\check{A}$ by

$$\check{\mathcal{Q}}(\check{a}) = Q^{(1)}(a) + \check{s}_g \quad \bar{\check{\mathcal{Q}}} := Q^{(1)}(a) + \bar{\check{s}}_g \quad \text{for } \check{a}_g \in \check{A}_g. \tag{2}$$

## Remark

An Euler $G$-Frobenius algebra $A =< G, A, \circ, 1, \eta, \varphi, \chi, j >$ naturally gives rise to a triple $< A, j, \chi >$ and thus determines a dual $D(k[G])$-module with a non-degenerate pairing and a bi-grading. This transformation is a duality on triples.

# A mirror theorem

## Theorem (K '03)

*Let $f$ be one of the simple singularities $A_n, D_n, E_6, E_7$ and $E_8$ or a Pham singularity with coprime powers, let $J$ be the exponential grading operator and $\Gamma := \langle j \rangle$ with $\rho(j) = J$.*

- *There is a projectively unique, maximally non-degenerate, degenerate $\Gamma$-Frobenius algebra structure of degree $j$ on $(\Gamma M_f)^\vee$.*
- *Moreover the invariants of the $\Gamma$-Frobenius algebra $\Gamma M_f$ are one-dimensional and yield the Frobenius algebra $A_1$, while the invariants of the $(\Gamma M_f)^\vee$ are isomorphic as a bi-graded Frobenius algebra to $M_f^{(a,c)}$.*
- *In short, the $A, D, E$ singularities are mirror self dual in the sense that $((\Gamma M_g)^\vee)^\Gamma$ is the mirror dual.*

| $M_f$ | restriction | $G$ | $\sigma$ | $GM_f^G$ | $((GM_f)^\vee)^G$ |
|---|---|---|---|---|---|
| $A_n$ | | $\mathbb{Z}/(n+1)\mathbb{Z}$ | 0 | $A_1$ | $A_n$ |
| $A_{2n-1}$ | | $\mathbb{Z}/(n+1)\mathbb{Z}$ | 1 | $A_1$ | $B_n$ |
| $A_{2n-1}$ | | $\mathbb{Z}/2\mathbb{Z}$ | 0 | $B_n$ | $I_2(4)$ |
| $A_{2n-1}$ | $n$ odd for dual | $\mathbb{Z}/2\mathbb{Z}$ | 1 | $D_{n+1}$ | $A_1$ |
| $A_{2n-1}$ | | $\mathbb{Z}/n\mathbb{Z}$ | 0 | $I_2(4)$ | $B_n$ |
| $D_{n+1}$ | | $\mathbb{Z}/(2n\mathbb{Z})$ | 0 | $A_1$ | $A_{2n-1}$ |
| $D_{n+1}$ | $n$ even | $\mathbb{Z}/(2n\mathbb{Z})$ | 1 | $A_1$ | $D_{n+1}$ |
| $D_{n+1}$ | $n$ odd | $\mathbb{Z}/n\mathbb{Z}$ | 0 | $A_1$ | $D_{n+1}$ |
| $D_{n+1}$ | $n$ even | $\mathbb{Z}/n\mathbb{Z}$ | 0 | $I_2(4)$ | $B_n$ |
| $D_{n+1}$ | | $\mathbb{Z}/2\mathbb{Z}$ | 0 | $B_n$ | $I_2(4)$ |
| $D_{n+1}$ | $n$ odd for dual | $\mathbb{Z}/2\mathbb{Z}$ | 1 | $A_{2n-1}$ | $I_2(4)$ |
| $A_{k_1-1}\otimes\cdots\otimes A_{k_n-1}$ | $k_i$ coprime | $\mathbb{Z}/k_1\mathbb{Z}\times\ldots\mathbb{Z}/k_n\mathbb{Z}$ | 0 | $A_1$ | $A_{k_1-1}\otimes\cdots\otimes A_{k_n-1}$ |
| $E_6$ | | $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$ | 0 | $A_1$ | $E_6$ |
| $E_7$ | | $\mathbb{Z}/9\mathbb{Z}$ | 0 | $A_1$ | $E_7$ |
| $E_8$ | | $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/5\mathbb{Z}$ | 0 | $A_1$ | $E_8$ |

Table: Since all groups are cyclic, and $\epsilon\equiv 0$, $\mathrm{Hom}(G,\mathbb{Z}/2\mathbb{Z})=e$ or $\mathrm{Hom}(G,\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}$ defining, the entry in the column $\sigma$. The conditions for the duals are the conditions to be quasi-Euler.

# Orbifold mirror philosophy

## Philosophy

Let $T$ be a $N=2$ theory (which for us at the moment means Frobenius algebra) and let $H \subset G$ be symmetry groups with $H$ normal in $G$, then
$(T/H)^H \simeq (((T/H)^H/(G/H))^\vee)^{(G/H)}$.

## Theorem (Mirror self–duality for $D, B, F$)

*The orbifold mirror philosophy produces mirror pairs for the self-dual cases listed in Table 1, with the group $G = \Gamma$ being the group generated by the exponential grading operator and $H = e$.*
*The orbifold mirror philosophy holds for the case of $T = A_{2n-1}, G = \mathbb{Z}/(2n\mathbb{Z}), H = \mathbb{Z}/2\mathbb{Z}$, with $n$ odd, and the choice of $\sigma = 1$ for $\mathbb{Z}/(2n\mathbb{Z})$ which restricts to $\sigma = 1$ for $\mathbb{Z}/2\mathbb{Z}$ and $\sigma = 0$ for $G/H = \mathbb{Z}/n\mathbb{Z}$.*

## Theorem

*An extended orbifold mirror philosophy holds for the entries in Table 2 and produces the additional mirror pairs* $((B_n, I_2(4)), (I_2(4), B_n))$ *and* $((F_4, I_2(4)), (I_2(4), F_4))$.

| $T$ | $G$ | $H$ | $K = G/H$ | $\begin{pmatrix} (T/H)^H, \\ ((T/H)^\vee)^H \end{pmatrix}$ | $\begin{pmatrix} (T/H)/(K))^K, \\ (((T/H)/K)^\vee)^K) \end{pmatrix}$ |
|---|---|---|---|---|---|
| $A_{2n-1}$ $n$ odd | $\mathbb{Z}/(2n\mathbb{Z})$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/n\mathbb{Z}$ | $(D_{n+1}, A_1)$ | $(A_1, D_{n+1})$ |
| $A_{2n-1}$ | $\mathbb{Z}/(2n\mathbb{Z})$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/n\mathbb{Z}$ | $(B_n, I_2(4))$ | $(I_2(4), B_n)$ |
| $D_{n+1}$ $n$ even | $\mathbb{Z}/(2n\mathbb{Z})$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/n\mathbb{Z}$ | $(B_n, I_2(4))$ | $(I_2(4), B_n)$ |
| $E_6$ | $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ | $e \times \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $(F_4, I_2(4))$ | $(I_2(4), F_4)$ |

Table: Mirror pairs from orbifold mirror philosophy

# The case $E_7$

Recall that for $E_7 : x^3 + xy^3$, we have the following degrees $q_1 = q_x = \frac{1}{3}, q_2 = q_y = \frac{2}{9}, d = \frac{8}{9}$.
Fix $\zeta_9 := \exp(2\pi i \frac{1}{9})$, then the $E_7$ singularity has the exponential grading operator $J = \exp(2\pi i Q)$

$$J = \begin{pmatrix} \zeta_9^3 & 0 \\ 0 & \zeta_9^2 \end{pmatrix}.$$

This operator generates a subgroup $\langle J \rangle \subset GL(n, \mathbb{C})$, which is isomorphic to $\mathbb{Z}/9\mathbb{Z}$. We fix a generator $j$ of $\mathbb{Z}/9\mathbb{Z}$ and regard the representation $\rho : \mathbb{Z}/9\mathbb{Z} \to GL(n, \mathbb{C})$ as given by $\rho(j) = J$.
This is also the maximal symmetry group $G_{max} = \langle \Lambda \rangle$

$$\Lambda = \begin{pmatrix} \zeta_9^3 & 0 \\ 0 & \zeta_9^{-1} \end{pmatrix}$$

and $J = \Lambda^7$.

# The $\mathbb{Z}/9\mathbb{Z}$-graded $k$-module $(\mathbb{Z}/9\mathbb{Z})M_f$

The representation is given by $\rho(j^i) = \begin{pmatrix} \zeta_9^{3i} & 0 \\ 0 & \zeta_9^{2i} \end{pmatrix}$.

| $g \in \mathbb{Z}/9\mathbb{Z}$ | $f_g$ | $M_{f_g}$ | $d_g$ | $\nu_1(g)$ | $\nu_2(g)$ | $s_g^+$ | $s_g^-$ | $s_g$ | $\bar{s}_g$ |
|---|---|---|---|---|---|---|---|---|---|
| $e = j^0$ | $x^3 + xy^3$ | $E_7$ | $\frac{8}{9}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $j^1$ | $0$ | $A_1$ | $0$ | $\frac{1}{3}$ | $\frac{2}{9}$ | $\frac{8}{9}$ | $-\frac{8}{9}$ | $0$ | $\frac{8}{9}$ |
| $j^2$ | $0$ | $A_1$ | $0$ | $\frac{2}{3}$ | $\frac{4}{9}$ | $\frac{8}{9}$ | $\frac{2}{9}$ | $\frac{5}{9}$ | $\frac{1}{3}$ |
| $j^3$ | $x^3$ | $A_2$ | $\frac{1}{3}$ | $0$ | $\frac{2}{3}$ | $\frac{5}{9}$ | $\frac{1}{3}$ | $\frac{4}{9}$ | $\frac{1}{9}$ |
| $j^4$ | $0$ | $A_1$ | $0$ | $\frac{1}{3}$ | $\frac{8}{9}$ | $\frac{8}{9}$ | $\frac{4}{9}$ | $\frac{2}{3}$ | $\frac{2}{9}$ |
| $j^5$ | $0$ | $A_1$ | $0$ | $\frac{2}{3}$ | $\frac{1}{9}$ | $\frac{8}{9}$ | $-\frac{4}{9}$ | $\frac{2}{9}$ | $\frac{2}{3}$ |
| $j^6$ | $x^3$ | $A_2$ | $\frac{1}{3}$ | $0$ | $\frac{1}{3}$ | $\frac{5}{9}$ | $-\frac{1}{3}$ | $\frac{1}{9}$ | $\frac{4}{9}$ |
| $j^7$ | $0$ | $A_1$ | $0$ | $\frac{1}{3}$ | $\frac{5}{9}$ | $\frac{8}{9}$ | $-\frac{2}{9}$ | $\frac{1}{3}$ | $\frac{5}{9}$ |
| $j^8$ | $0$ | $A_1$ | $0$ | $\frac{2}{3}$ | $\frac{7}{9}$ | $\frac{8}{9}$ | $\frac{8}{9}$ | $\frac{8}{9}$ | $0$ |

## Lemma

*The elements of bi-degree $(q, q)$ of $(\mathbb{Z}/9\mathbb{Z})E_7$ are exactly the elements in the untwisted sector $A_e$.*

# The $G$-action

For $\mathbb{Z}/9\mathbb{Z}$ we have $\epsilon \equiv 1$ and $\sigma \equiv 0$, so the $G$-action is given by

$$\varphi_{j^i,j^k} = \begin{cases} 1 & \text{if } k = 0 \\ \zeta_9^{-2i} & \text{if } k \in \{3,6\} \\ \zeta_9^{-5i} & \text{else} \end{cases}$$

and the character is

$$\chi(j^i) = \zeta_9^{5i}.$$

**Lemma**

*The $\mathbb{Z}/9\mathbb{Z}$-invariants of the only compatible $D(k[\mathbb{Z}/9\mathbb{Z}])$-module structure are given by the unit $1_e$.*

# The dual bi-grading

The dual grading is given by

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\check{s}_{j^i}$ | 0 | $-\frac{8}{9}$ | $-\frac{8}{9}$ | $-\frac{1}{3}$ | $-\frac{4}{9}$ | $-\frac{2}{9}$ | $-\frac{2}{3}$ | $-\frac{7}{9}$ | $-\frac{5}{9}$ |
| $\bar{\check{s}}_{j^i}$ | 0 | 0 | $\frac{8}{9}$ | $\frac{1}{3}$ | $\frac{1}{9}$ | $\frac{2}{9}$ | $\frac{2}{3}$ | $\frac{4}{9}$ | $\frac{5}{9}$ |

The elements of bi-degree $(-q, q)$ are

$$\langle \check{1}_e, y^2\check{1}_j, \check{1}_{j^2}, \check{1}_{j^3}, \check{1}_{j^5}, \check{1}_{j^6}, \check{1}_{j^8} \rangle.$$

# The dual $\mathbb{Z}/9\mathbb{Z}$-action

The dual $\mathbb{Z}/9\mathbb{Z}$-action is given by

$$\check{\varphi}_{j^i,j^k} = \begin{cases} \zeta_9^{5i} & \text{if } k = 1 \\ \zeta_9^{3i} & \text{if } k \in \{4,7\} \\ 0 & \text{else.} \end{cases}$$

## Lemma

*The $\mathbb{Z}/9\mathbb{Z}$-invariants of the dual $((\mathbb{Z}/9\mathbb{Z})E_7)^\vee$ are given by $\langle \check{1}_e, y^2\check{1}_j, \check{1}_{j^2}, \check{1}_{j^3}, \check{1}_{j^5}, \check{1}_{j^6}, \check{1}_{j^8}\rangle$; they are all of diagonal bi-degree, and their degrees are $(0,0), (-\frac{4}{9}, \frac{4}{9}), (\frac{-8}{9}, \frac{8}{9}), (-\frac{1}{3}, \frac{1}{3}), (-\frac{2}{9}, \frac{2}{9}), (-\frac{2}{3}, \frac{2}{3}), (-\frac{5}{9}, \frac{5}{9})$. The pairing, the bi-grading, and the group grading are the same as those of the anti-chiral realization of $E_7$ under the association $\check{1}_e \mapsto 1, \check{1}_j \mapsto y^2, \check{1}_{j^2} \mapsto x^2y, \check{1}_{j^3} \mapsto x, \check{1}_{j^5} \mapsto y, \check{1}_{j^6} \mapsto x^2, \check{1}_{j^8} \mapsto xy$, so that $E_7$ is self dual.*

# The duality

By inspecting the grading and group grading we have

## Proposition

*There is a unique maximally degenerate G-Frobenius structure of charge j on* $((\mathbb{Z}/9\mathbb{Z})E_7)^\vee$ *whose invariants form the* $(a,c)$*-realization of* $E_7$*. Hence* $(((\mathbb{Z}/9\mathbb{Z})E_7)^\vee)^{\mathbb{Z}/9\mathbb{Z}}$ *is the mirror dual to* $E_7$*.*

# Newer developments I: Results from [K'07]

## Special actually not so special:

We showed that the co-cycle formalism applies in other cases as well. One example is the deRham case. Furthermore we showed we gave a trivialization of the co-cycles using the JKK orbifold K-theory description of the obstruction bundle. [K '07].

## Application to singularities

This works for singularities as well. Get a geometric solution. We have a theory of Chern classes for the obstruction bundles. This defines a geometric multiplication (see [K '07]).

# Newer developments II – Work with A. Libgober.

## A first test in our LG/CY-correspondence

Orbifold Euler characteristic and super trace agree.

## Extending to deformations

We use an orbifold version of the Milnor fibration to get an orbifold version Saito theory. Classical theory by Dubrovin, Manin, Hertling, Givental, Barannikov, Sabbah, ...

# Theorems/Conjectures

## Theorem

*There is a canonical multiplication* $M_{f_{m_1}} \otimes M_{f_{m_2}} \longrightarrow M_{f_{m_1 m_2}}$. *Moreover there is even the structure of a G-Frobenius algebra on* $GM_f$.

## Claim

*Given a pair* $(f, G)$ *as above there is a theory of a miniversal unfolding of this pair which gives rise to a pointed G-Frobenius manifold whose fiber over the special point is the G-Frobenius algebra of the previous conjecture.*

# Summary: older results ('03)

- Good algebraic theory for orbifold LG–models. $(a, c)$ ring $((G_{Ab}M_f)^\vee)^{G_{Ab}}$ is used by [FJR] as state space. $G_{Ab}$ is a maximal Abelian subgroup dictated by the form of the polynomial.
- Mirror symmetry on the level of $D(k[G])$ modules with bi–grading.
- Stringy multiplication reduces to an algebraic co–cycle problem.
- Solved in many cases. Highly non–trivial e.g. $E_7$ case
- A and B models for ADE, BF models including mirror self–duality.

# Summary: newer results ('07) and in progress

- Geometrically defined multiplication for orbifold Landau–Ginzburg models. ('07)
- Work in progress will extend to Frobenius manifolds.
- Can also twist by gerbes and discrete torsion.

# The End

Thank you!