A local estimate from Radon transform and stability of Inverse EIT with partial data

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w/ P. Caro (U. Helsinki) and D. Dos Santos Ferreira (Paris 13)
The purpose of this talk is to address the following points

- To prove a local control of the $L^p$ norm of a function by its local Radon transform.

- To apply the above to prove stability for Calderón Inverse problem in two partial data contexts in which uniqueness holds true.
  (1) Bukhgeim and Uhlmann (Recovering a potential from partial Cauchy data 2002)(semiglobal)
  (2) Kenig, Sjöstrand and Uhlmann (The Calderón Problem for partial data 2007)
Description of Partial data:
Calderón inverse problem.
Let $u$ be the solution of the Dirichlet BVP
\[
\begin{cases}
\nabla \cdot (\gamma \nabla u) = 0 & \text{in } \Omega \\
u|_{\partial \Omega} = f & \in H^\frac{1}{2}(\partial \Omega)
\end{cases}
\] (0.1)

$\gamma$ is a positive function of class $C^2$ on $\bar{\Omega}$.
The Dirichlet-to-Neumann map:
\[
\Lambda_f = \gamma \partial_\nu u|_{\partial \Omega},
\]
where $\partial_\nu$ denotes the exterior normal derivative of $u$.
Partial data: We assume $F, G \subset \partial \Omega$. Data restricted to $f$’s so that $\text{supp } f \subset B$ and measurements $\partial_\nu u$ restricted to $F$.
After standard reduction to Schrödinger equation: $(-\Delta + q)v = 0$. 
Assume $q_1, q_2$ be two bounded potentials on $\Omega$, suppose that 0 is neither a Dirichlet eigenvalue of the Schrödinger operator $-\Delta + q_1$ nor of $-\Delta + q_2$. Given a direction $\xi \in \mathbb{S}^{n-1}$.

Consider the $\xi$-illuminated boundary

$$\partial \Omega_-(\xi) = \{x \in \partial \Omega : \xi \cdot \nu(x) \leq 0\}$$

and the $\xi$-shadowed boundary

$$\partial \Omega_+(\xi) = \{x \in \partial \Omega : \xi \cdot \nu(x) \geq 0\}.$$ 

$\tilde{F}, \tilde{B}$ two open neighborhoods in $\tilde{\Omega}$, respectively of the sets $\partial \Omega_-(\xi)$ and $\partial \Omega_+(\xi)$.

If the two Dirichlet-to-Neumann maps with Dirichlet data $f \in H^{1/2}(\partial \Omega)$ supported in $\tilde{B}$ coincide on $\tilde{F}$

$$\Lambda_{q_1}f(x) = \Lambda_{q_2}f(x), \quad x \in \tilde{F}$$

then the two potentials agree $q_1 = q_2$. 
In this case consider $x_0$ out of Convex hull of $\Omega$. The $x_0$-illuminated boundary

$$\partial \Omega_-(x_0) = \{x \in \partial \Omega : (x - x_0) \cdot \nu(x) \leq 0\}$$

and the $x_0$-shadowed boundary

$$\partial \Omega_+(x_0) = \{x \in \partial \Omega : (x - x_0) \cdot \nu(x) \geq 0\}.$$ 

$\tilde{F}$, $\tilde{B}$ two open neighborhoods in $\partial \Omega$, respectively of the sets $\partial \Omega_-(\xi)$ and $\partial \Omega_+(\xi)$. If the two Dirichlet-to-Neumann maps with Dirichlet data $f \in H^{1/2}(\partial \Omega)$ supported in $\tilde{B}$ coincide on $\tilde{F}$

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Previous results. Stability for partial data

- L. Tzou (2008) (extends to the magnetic case, without the support constrain on Dirichlet data)
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- KSU case
  Nachman and Street (2010) (Reconstruction of Radon-like integrals of the potential)
Our results: Stability in BU partial data

Theorem

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $n \geq 3$ with smooth boundary. Assume given an open set $N$ in $S^{n-1}$ and consider two open neighbourhoods $F, B$ respectively of the front and back sets $\Omega_-(\xi)$ and $\Omega_+(\xi)$ of $\partial \Omega$ for any $\xi \in N$. Let $q_1, q_2$ be two allowable potentials on $\Omega$, suppose that 0 is neither a Dirichlet eigenvalue of the Schrödinger operator $-\Delta + q_1$ nor of $-\Delta + q_2$, then

$$\|q_1 - q_2\|_{L^p} \leq C \left( \log \left| \log \| \Lambda_{q_1} - \Lambda_{q_2} \|_{B \rightarrow F} \right| \right)^{-\lambda/2}$$

Remarks: The norm $\| \Lambda_{q_1} - \Lambda_{q_2} \|_{B \rightarrow F}$ (Nachman-Street)
The allowable potentials: $L^\infty \cap W^{\lambda,p}$
**Theorem**

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $n \geq 3$ with smooth boundary. Assume given an open set $N$ in $\mathbb{R}^3$ and consider two open neighbourhoods $F, B$ respectively of the front and back sets $\Omega - (x_0)$ and $\Omega + (x_0)$ of $\partial \Omega$ for any $x_0 \in N$. Let $q_1, q_2$ be two allowable potentials on $\Omega$, suppose that $0$ is neither a Dirichlet eigenvalue of the Schrödinger operator $-\Delta + q_1$ nor of $-\Delta + q_2$, then

$$\|q_1 - q_2\|_{L^p(G)} \leq C \left( \log \left| \log \|\Lambda_{q_1} - \Lambda_{q_2}\|_{B \rightarrow F} \right| \right)^{-\lambda/2} \quad (0.3)$$

where $G$ is an open neighborhood in $\mathbb{R}^3$ of the penumbra region $F \cap B$

**Remarks:** The norm $\|\Lambda_{q_1} - \Lambda_{q_2}\|_{B \rightarrow F}$

(After F. J. Chung, A Partial Data Result for the Magnetic Schrödinger Inverse Problem. Norm $H^{1/2}(\partial \Omega) \rightarrow H^{-1/2}(\partial \Omega)$)
Steps in the proof

Step 1  Stability from $\text{DtoN}$ map to partial Radon transform.

Step 2  Stability from partial Radon transform to potential
Second step: from limited angle-distance Radon data → local values of the function

Quantitative Helgason-Holmgren Theorem:

Limited data:

- **Distance**
  
  $I = \{ s \in \mathbb{R} : |s| < \alpha \}$ \hspace{1cm} (0.4)

- **Angle** ($\omega_0 \in S^{n-1}$)
  
  $\Gamma = \{ \omega \in S^{n-1} : (\omega \cdot \omega_0)^2 > 1 - \beta^2 \}$ \hspace{1cm} (0.5)
  
  $\Gamma = \{ \omega \in S^{n-1} : d(\omega_0, \omega) < \arcsin \beta \}$. \hspace{1cm} (0.6)

Dependence domain

$$E = \{ x \in \mathbb{R}^n : \omega \cdot (x - y_0) = s, \ s \in I, \ \omega \in \Gamma \}. $$
Allowable potentials

Let $p, 1 \leq p < \infty$, and $\lambda, 0 < \lambda < 1/p$, two functions $q_1, q_2$ satisfy the following conditions:

(a) $1_E q_j \in X \cap L^\infty(\mathbb{R}^n)$ for $j = 1, 2$,

\[ \|q_j\|_{L^\infty(E)} + \|1_E q_j\|_X < M. \]

$X$-norm $q \in L^1(\mathbb{R}^n)$ and

\[ \int_\mathbb{R} (1 + |s|)^n \|R_0 q(s, \cdot)\|_{L^1(S^{n-1})} \, ds < \infty. \]

(b) $(\lambda, p, p)$-Besov regularity on the dependence domain

\[ \int_{\mathbb{R}^n} \frac{\|1_E q_j - (1_E q_j)(\cdot - y)\|_{L^p(\mathbb{R}^n)}^p}{|y|^{n+\lambda p}} \, dy < M^p \]

for $j = 1, 2$. 
\[ y_0 \in \text{supp} (q_1 - q_2) \]

\[ \text{supp} (q_1 - q_2) \subset \{ x \in \mathbb{R}^n : (x - y_0) \cdot \omega_0 \leq 0 \} \]
Theorem 1. Quantitative H-H

Let $M \geq 1$ be constant. Given $y_0 \in \mathbb{R}^n$, $\omega_0 \in S^{n-1}$, $\alpha > 0$ and $\beta \in (0, 1]$ Let $q_i$ be allowable potentials and assume the support condition holds. Then there exists a positive constant $C = C(M, |G|, \alpha, \beta)$, such that

$$
\|q_1 - q_2\|_{L^p(G)} \leq \frac{C}{\left( |\log \int_I (1 + |s|)^n \|\mathcal{R}_{y_0}(q_1 - q_2)(s, \cdot)\|_{L^1(\Gamma)} \, ds \right)^{\frac{1}{2}}},
$$

where

$$
G = \left\{ x \in \mathbb{R}^n : |x - y_0| < \frac{\alpha}{8 \cosh(8\pi/\beta)} \right\}.
$$
Notation:
\( \mathcal{R}_{y_0} q(s, \omega) \) is the \( y_0 \)-centered Radon transform, integral of \( q \) in
\( H = \{ x \in \mathbb{R}^n : \omega \cdot (x - y_0) = s \} \).

Constant

\[
C = C_n M \max \left( 1, |G|^{\frac{1}{p}} \right) \left( 1 + \alpha^{-n} + \beta^{-n} + \alpha^\lambda \right)
\] (0.8)
Sketch of proof 1: uniqueness theorem

Assume $\mathcal{R}_{y_0}(q_1 - q_2)|_{I \times \Gamma} = 0$.
Then such $(y_0, \omega_0)$ does not exits:

- **1 Microlocal Helgason’s theorem:**

$$ (y_0, \omega_0) \notin WF_a(q_1 - q_2) $$

- **2 Microlocal Holmgren’s theorem (Hörmander):**
  From the conditions $y_0 \in \text{supp} (q_1 - q_2)$ and 
  $\text{supp} (q_1 - q_2) \subset \{x \in \mathbb{R}^n : (x - y_0) \cdot \omega_0 \leq 0\}$ one has

$$ (y_0, \omega_0) \in WF_a(q_1 - q_2) $$
1. Quantitative Microlocal Helgason’s theorem. Sketch of proof

The $WF_a(u)$ is characterized by the exponential decay of the wave packet transform:
For any $u \in S'(\mathbb{R}^n)$ define the Wave packet (Segal-Bargmann) transform of $u$ for $\zeta \in \mathbb{C}^n$ as

$$ T u(\zeta) = \langle u, e^{-\frac{1}{2\hbar}}(\zeta - \cdot)^2 \rangle. $$

Where $\langle \cdot, \cdot \rangle$ stands for the duality between $S'(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ – the class of smooth rapidly decreasing functions.
Step 1: From Radon to wave packet

**Proposition**

Consider \( q \in A^\lambda(\mathbb{R}^n) \), \( y_0 \in \mathbb{R}^n \) and \( \omega_0 \in S^{n-1} \). Let \( R_q \) be a positive constant such that \( \text{supp } q \subset \{ x \in \mathbb{R}^n : |x - y_0| \leq R_q \} \). Given \( \alpha > 0 \) and \( \beta \in (0, 1] \) consider the sets

\[
I = \{ s \in \mathbb{R} : |s| < \alpha \}, \quad \Gamma = \{ \omega \in S^{n-1} : |\omega \cdot \omega_0|^2 > 1 - \beta^2 \}.
\]

Then, there exists a positive constant \( C \) such that

\[
|T_q(\zeta)| \leq \frac{C}{h^n} (|\text{Im } \zeta|^n + \alpha^n + R_q^n) e^{\frac{1}{2h}|\text{Im } \zeta|^2}
\]

\[
\times \left[ \| \mathcal{R}_{y_0} q \|_{L^1(I \times \Gamma)} + \| q \|_{L^\infty(\mathbb{R}^n)} \left( e^{-\frac{1}{2h} \frac{\alpha^2}{4}} + e^{-\frac{1}{2h} \frac{\gamma^2 \beta^2}{16}} \right) \right],
\]

for all \( h \in (0, 1] \), \( \zeta \in \mathbb{C}^n \) such that \( |\text{Re } \zeta - y_0| < \alpha/2 \), \( |\text{Im } \zeta| \geq \gamma > 0 \) and \( |\omega_0 \cdot \theta|^2 > 1 - \beta^2/4 \) with \( \theta = |\text{Im } \zeta|^{-1}\text{Im } \zeta \).
Step 2

- **Step 2a (Key step)** We try to remove the condition $|\text{Im } \zeta| \geq \gamma > 0$. Try to go from the wave packets transform of $q$ to the heat evolution of $q$ ($t = h$) by allowing $\text{Im } \zeta = 0$.

- **Step 2b.** Estimate the backward initial value problem for the heat equation. Need a priori Besov regularity on $q$. 
The key Proposition

Proposition

Under the same notation and assumptions of the Theorem. There exists a positive constant $C$ such that

$$e^{-\frac{1}{2h}|\text{Im} \zeta|^2} |T q(\zeta)| < CM_q\left(\left(\frac{2\alpha}{\beta}\right)^n + \alpha^n + R_q^n\right) \|\mathcal{R}_{y_0} q\|_{L^1(I \times \Gamma)}^\kappa,$$

with

$$\kappa := \frac{1}{4 \left(\cosh\left(\frac{8\pi}{\beta}\right)\right)^2}, \quad h := \frac{\alpha^2}{8 |\log \|\mathcal{R}_{y_0} q\|_{L^1(I \times \Gamma)}|},$$

for all $\zeta \in \mathbb{C}^n$ such that

$$|\text{Re} \zeta - y_0| < \frac{\alpha}{8 \cosh\left(\frac{8\pi}{\beta}\right)}, \quad |\text{Im} \zeta| < \frac{2\alpha}{(4 - \beta^2)^{1/2}}.$$
Proof of the Key proposition

Estimates of the wave packet transform:

(K1) \[ |\mathcal{T} q(\zeta)| \leq (2\pi h)^{\frac{n}{2}} \|q\|_{L^\infty(\mathbb{R}^n)} e^{\frac{1}{2h} | \text{Im } \zeta |^2}, \]

(K2) \[ |\mathcal{T} q(\zeta)| \leq (2\pi h)^{\frac{n}{2}} \|q\|_{L^\infty(\mathbb{R}^n)} e^{\frac{1}{2h} | \text{Im } \zeta |^2} e^{-\frac{1}{2h} | \omega_0 \cdot (\text{Re } \zeta - y_0) |^2}, \]

for all \( \zeta \in \mathbb{C}^n \) such that \( \omega_0 \cdot (\text{Re } \zeta - y_0) \geq 0 \) (follows from the hyperplane being supporting)

(K3) \[ |\mathcal{T} q(\zeta)| \leq \frac{C}{h^\frac{n}{2}} (| \text{Im } \zeta |^n + \alpha^n + R_q^n) e^{\frac{1}{2h} | \text{Im } \zeta |^2} \]
\[ \times \left[ \| R_{y_0} q \|_{L^1(I \times \Gamma)} + \|q\|_{L^\infty(\mathbb{R}^n)} e^{-\frac{1}{2h} \frac{\alpha^2}{4}} \right], \]

for all \( h \in (0, 1], \zeta \in \mathbb{C}^n \) such that \( |\text{Re } \zeta - y_0| < \alpha/2, |\text{Im } \zeta| \geq \gamma > 0 \) and \( |\omega_0 \cdot \theta|^2 > 1 - \beta^2/4 \) with \( \theta = |\text{Im } \zeta|^{-1} \text{Im } \zeta \). (Follows from quantitative Helgason-Holmgren with \( \gamma = 2\alpha/\beta \))
Lemma

Let \( a, b \) be positive constants. Consider

\[
R = \{ z \in \mathbb{C} : |\text{Re} z| < a, |\text{Im} z| < b + \varepsilon \},
\]

for some \( \varepsilon > 0 \). Let \( F \) be a sub-harmonic function in \( R \) such that

\[
F(z) < (\min\{0, \text{Re} z\})^2, \text{ for all } z \in \mathbb{C}.
\]

Assume that \( F(z) < -2a^2 \) for \( z \in R \) such that \( |\text{Im} z| \geq b \). Then

\[
F(z) < -\frac{1}{2} \frac{2a^2}{\cosh \left( \pi \frac{b}{a} \right)} \min \left( \frac{1}{\cosh \left( \pi \frac{b}{a} \right)}, \frac{1}{3} \right),
\]

for

\[
|\text{Im} z| < b, \quad |\text{Re} z| < \frac{a}{2} \min \left( \frac{1}{\cosh \left( \pi \frac{b}{a} \right)}, \frac{1}{3} \right).
\]
Proof of lemma

Take the subharmonic function

\[ G(x + iy) = 2a^2 \frac{\cosh \left( \frac{\pi}{a} y \right)}{\cosh \left( \frac{\pi}{a} b \right)} \sin \left( \frac{\pi}{a} (x + \delta) \right) + F(x + iy) - \delta^2, \]

for

\[ \delta = \min \left( \frac{a}{\cosh \left( \frac{\pi}{a} b \right)}, \frac{a}{3} \right) \]

Check that \( G < 0 \) on the boundary of \([-\delta, a - \delta] \times [-b, b]\).

The estimate follows from the maximum principle when we restrict to the values \(|x| < \delta/2\).
Choose \( h := \frac{\alpha^2}{8 \log \| \mathcal{R}_{y_0} q \|_{L^1(I \times \Gamma)}} \), to obtain an appropriate negative exponential in (K3).

Let us denote \( z = \omega_0 \cdot (\zeta - y_0) \in \mathbb{C} \) and write \( \zeta = (z + \omega_0 \cdot y_0) \omega_0 + w \) with \( w \in \mathbb{C}^n \) such that \( \text{Re} w \cdot \omega_0 = \text{Im} w \cdot \omega_0 = 0 \). Use the Lemma for the function

\[
\Phi(z) = |\text{Re} z|^2 - |\text{Im} z|^2 + 2h \log |\mathcal{T} q((z + \omega_0 \cdot y_0) \omega_0 + w)| \\
+ 2h \log \left( \frac{e^{-\frac{1}{2h} |\text{Im} w|^2}}{CM_q (\rho^n + \alpha^n + R^n_q)} \right),
\]

in the rectangle \( a = \alpha/4, b = 2\alpha/\beta \).
Step 2b. The backward estimate

The key point is the fact that the Segal-Bargmann transform restricted to real values is a convolution with the Gaussian. Backward estimate for the heat equation $h = \text{time}$

**Lemma**

*Consider $q \in L^p(\mathbb{R}^n)$ and $G$ an open set in $\mathbb{R}^n$ such that

$$\text{supp } q \cap G \neq \emptyset.$$*

*Assume that there exists $\lambda \in (0, 1)$ such that

$$L_q := \left( \int_{\mathbb{R}^n} \left\| q - q(\cdot - y) \right\|_{L^p(\mathbb{R}^n)}^p \frac{1}{|y|^{n+\lambda p}} \, dy \right)^{1/p} < +\infty.$$*

*Then, there exists a positive constant $C$, only depending on $n$, such that

$$\|q\|_{L^p(G)} \leq C \left( h^{-\frac{n}{2}} \|Tq\|_{L^p(G)} + L_q h^{\frac{\lambda}{2}} \right),$$*
First step: from DtoN map to Radon transform

The uniqueness Results. Locally vanishing Radon Transform.
Stability: Hunk and Wang: Log-log type stability (log stable recovery of the Fourier tranform +log stable continuation of analytic functions)
The Carleman estimates with signed partition of the boundary allow to partially recover the 2-planes X-ray transform in some sets $I \times \Gamma$ which are neighborhood of $(y_0, \omega_0)$ from the partial Dirichlet to Neuman map in a logarithmic stable way (following the approach of DSF, Kenig, Sjoestrand and Uhlmann).
This allows to control the norm of $q_1 - q_2$ in a neighborhood of $y_0$. In the case of BU (semiglobal) one can estimate $\|q_1 - q_2\|_{L^p}$, since we have the control of $\|R_{y_0}(q_1 - q_2)\|_{L^1((0,\infty)\times\Gamma)}$.
In KSU We have the Radon transform if $n = 3$. Giving us the control on $q_1 - q_2$ at points for which we control the Radon transform (the penumbra boundary)
From partial data to Radon transform

We reduce to Schrödinger $-\Delta + q$ equation in the standard way. Use of DSF-K-S-U approach.

Existence of Calderón- Fadeev type solutions. Use of the Carleman estimate with splitting in the illuminated and shadowed regions. Theorem ([BU] and [NS])

**Theorem**

*For $\tau >> 1$ sufficiently large and $g \in \mathcal{C}^\infty(\mathbb{R}^{n-2})$ and $(\xi + i\zeta)^2 = 0$, there exists a unique solution $w_\tau \in H(\Omega; \Delta)$ of the equation $(-\Delta + q)w_\tau = 0$ in $\Omega$, such that $\text{tr}_0 w_\tau \in H(\partial \Omega) \cap \mathcal{E}'(B)$ and which can be written as*

$$w_\tau(x) = e^{\tau(\xi + i\zeta) \cdot x} (g(x'') + R(\tau, x))$$

*where*

$$\|R(\tau, \cdot)\|_{L^2(\Omega)} \leq C\frac{1}{\tau} (\|qg\|_{L^2(\Omega)} + \tau^{1/2} \|\Delta g\|_{L^2(\Omega)}).$$

*The same is true by changing $\tau$ by $-\tau$ and $B$ by $F$.***
The spaces of solutions in $H(\Omega; \Delta)$

Traces (in $H(\partial \Omega) \subset H^{-1/2}(\partial \Omega)$)

Integral formula by using extensions of data

$$(\Lambda_{q_2} - \Lambda_{q_1})(\phi)\psi = \int_{\Omega} P_{q_1}(\phi)(q_1 - q_2) P_{q_2}(\psi). \quad (0.10)$$

Difference ofDtoN maps

**Lemma**

Let $q_j$, $j = 1, 2$ be $L^\infty$ potentials so that $0$ is not a Dirichlet eigenvalue of $-\Delta + q_j$ in $\Omega$. Then $\Lambda_{q_2} - \Lambda_{q_1}$ extends to a continuous map $H(\partial \Omega) \to H(\partial \Omega)^*$. 
2-plane Radon transform estimate

**Proposition**

For any \( g \in C^\infty(\mathbb{R}^{n-2}) \) there exits \( C > 0 \) which only depends on \( \Omega \) and the a priori bound of \( \|q_j\|_{L^\infty} \), such that

\[
\sup_{\xi \in \mathbb{N}, \zeta \in \xi^\perp} \int_{[\xi, \zeta]^\perp} g(x'') \int_{\mathbb{R}^2} q(x'' + t\xi + s\zeta) dt ds dx'' \leq C(\tau^{-1/2} + e^{c\tau} \|\Lambda q_1 - \Lambda q_2\|_{B \rightarrow F}) \|g\|_{H^2(\Omega)}
\]  

(0.11)

By choosing appropriate \( g = \tilde{g}(r\eta) \) for any \( \eta \in [\xi, \zeta]^\perp \)

\[
\sup_{\xi \in \mathbb{N}} \sup_{\eta \in \xi^\perp} \int_{\mathbb{R}} g(r) \mathcal{R}q(r, \eta) dr \leq C(\tau^{-1/2} + e^{c\tau} \|\Lambda q_1 - \Lambda q_2\|_{B \rightarrow F}) \|\tilde{g}\|_{H^2(\mathbb{R})}
\]  

(0.13)
\(L^1\) norm of Radon transform

Interpolation with

\[
\int_{S^{n-1}} \int_{\mathbb{R}} (1 + \tau^2)^{(n-1)/2} |\mathcal{R}q(\cdot, \eta)^{\wedge}(\tau)|^2 d\tau d\sigma(\eta) \leq \|q\|_{L^2}
\]

gives:

**Theorem (Stability of Radon transform)**

\[
\left( \int_M \left( \int_{\mathbb{R}} |\mathcal{R}q(r, \eta)|^2 dr \right)^{\frac{n+3}{4}} d\sigma(\eta) \right)^{\frac{2}{n+3}} \leq C(\tau^{-1/2} + e^{c\tau} \|\Lambda q_1 - \Lambda q_2\|_{B \rightarrow F}^{\frac{n+1}{n+3}}).
\]

(0.15)

Where \(M \subset S^{n-1}\)

\[
M = \bigcup_{\xi \in N} [\xi]^\perp,
\]

(0.16)
Looking for the supporting plane $H$ in quantitative Helgason-Holmgren

Let us take $\eta \in M$ and by translation there exist $r \in \mathbb{R}$ and $y_0 \in \text{supp } q$ such that $H(s, \eta)$ contains $y_0$ and it is a supporting hyperplane for $\text{supp } q$. We are now in the hypothesis of the previous theorem, since $M$ is a neighborhood of $\eta$ in the sphere and we can control the Radon transform por $\omega \in M$ and $s \in \mathbb{R}$. 
Open problem: Cloaking of 0’s

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60 =06
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FELICIDADES GUNTHER