Neumann resonances in linear elasticity for an arbitrary body

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Abstract

We study resonances (scattering poles) associated to the elasticity operator in the exterior of an arbitrary obstacle in $\mathbf{R}^3$ with Neumann boundary conditions. We prove that there exists a sequence of resonances tending rapidly to the real axis.

1 Introduction

Let $\mathcal{O} \subset \mathbf{R}^3$ be a compact set with C^∞-smooth boundary Γ and connected complement $\Omega = \mathbf{R}^3 \setminus \mathcal{O}$. Denote by Δ_e the elasticity operator

$$\Delta_e v = \mu_0 \Delta v + (\lambda_0 + \mu_0)\nabla(\nabla \cdot v),$$

$v = {}^t(v_1, v_2, v_3)$. Here λ_0, μ_0 are the Lamé constants and we assume that

$$\mu_0 > 0, \qquad 3\lambda_0 + 2\mu_0 > 0. \tag{1}$$

Consider Δ_e in Ω with Neumann boundary conditions on Γ

$$\sum_{j=1}^{3} \sigma_{ij}(v)\nu_j|_\Gamma = 0, \qquad i = 1, 2, 3, \tag{2}$$

where $\sigma_{ij}(v) = \lambda_0 \nabla \cdot v \delta_{ij} + \mu_0 \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$ is the stress tensor, ν is the outer normal to Γ. Denote by L the self-adjoint realization of $-\Delta_e$ in Ω with Neumann boundary conditions on Γ. As usual we define *resonances* as the poles of the meromorphic continuation of the cut-off

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resolvent $R_\chi(\lambda) = \chi\,(L-\lambda^2)^{-1}\chi$ from $\mathrm{Im}\,\lambda < 0$ to the whole complex plane $\mathbf{C}$, $\chi \in C_0^\infty$ being a cut-off function equal to 1 near Γ. So we accept the convention that the resonances lie in the upper half-plane.

If one considers the Laplacian with Dirichlet or Neumann boundary conditions, then it is well known that for convex or more generally for non-trapping obstacles the resonances lie above logarithmic curves of the type $\mathrm{Im}\,\lambda = C_1 \ln \mathrm{Re}\,\lambda - C_2$, $C_1 > 0$. There are several special examples of trapping obstacles [I1], [I2], [I3], [G] with resonances tending to the real axis or with a strip of the kind $0 < \mathrm{Im}\,\lambda < C_0$ containing infinitely many resonances. An important open problem in this direction is to prove or reject the Modified Lax and Phillips Conjecture — for any trapping obstacle there is a strip $0 < \mathrm{Im}\,\lambda < C_0$ containing infinitely many resonances.

In [SV2] the authors proved that for the elasticity operator L with Neumann boundary conditions there exists a sequence of resonances tending rapidly to the real axis provided that the obstacle $\mathcal{O}$ is strictly convex. Moreover, below any logarithmic curve $\mathrm{Im}\,\lambda = C_1 \ln|\lambda| - C_2$ there are no other resonances except possibly a finite number. The reason for the existence of almost real resonances are the Rayleigh waves which is a typical phenomenon for the elasticity operator with Neumann boundary conditions. As proven by Taylor [T] (see also [Y]) there are three types of rays for L that carry singularities. The first two types are classical rays reflecting at the boundary according to the laws of geometrical optics and the singularities propagate along them with speeds $c_1 = \sqrt{\mu_0}$, $c_2 = \sqrt{\lambda_0 + 2\mu_0}$. The third type of trajectories lie on the boundary and singularities propagate along them with a slower propagation speed $c_R > 0$ (the Rayleigh speed). Thus any obstacle is trapping for L from the point of view of propagation of singularities and one might expect resonances close to the real axis. The proof in [SV2] is based on a construction of a microlocal parametrix of the corresponding Neumann operator in all of the 5 zones (hyperbolic, mixed, elliptic and two glancing ones) using the calculus of ΨDO-s and FIO-s with large parameter (see e.g. [G]). It turns out that the parametrix is elliptic in the first two zones, can be represented microlocally as a hypoelliptic operator in $L^{0,1}_{2/3,0}$ conjugated with an elliptic FIO in the glancing zone while in the elliptic zone has a characteristic variety of the form $\Sigma = \{\zeta \in T^*\Gamma;\ c_R\|\zeta\| = 1\}$. Therefore, the parametrix is microlocally invertible outside Σ, which is essential for the proof of the pole-free domain, while the proof of the existence of almost real resonances is based on an application of the Phragmén-Lindelöf principle.

In this paper we show the existence of a sequence of resonances of L tending to the real axis for an arbitrary obstacle $\mathcal{O}$. Of course, one can no longer expect a pole-free logarithmic zone as in the case of a strictly convex obstacle [SV2] because there might be resonances near the real axis or more generally in any logarithmic region generated by classical trapped rays. Our main result is the following theorem.

Theorem 1 *There exist two infinite sequences $\{\lambda_j\}$, $\{-\bar{\lambda}_j\}$ of distinct resonances of the elasticity operator L, such that*

$$0 < \mathrm{Im}\,\lambda_j \le C_N |\lambda_j|^{-N} \qquad \textit{for any } N > 0.$$

The proof of Theorem 1 suggests that the reason for the existence of these resonances are the Rayleigh waves. In particular, it provides another proof of Kawashita's result [K] that the elastic wave equation with Neumann boundary conditions does not possess the exponential local energy decay property.

To prove Theorem 1 it suffices to show that for any integer $N \geq 1$ there are infinitely many resonances in $\{\lambda \in \mathbf{C} : \operatorname{Im}\lambda \leq |\lambda|^{-N}, |\operatorname{Re}\lambda| \geq 1\}$. Then, the assumption that there are finitely many resonances in this region would lead to polynomial a priori estimates on $\mathcal{N}^{-1}(\lambda)$, $\mathcal{N}(\lambda)$ being the Neumann operator on Γ, in a smaller region near the real axis. The final step is to show that these a priori estimates cannot hold because the parametrix of the Neumann operator fails to be elliptic at Σ. To do so, we use the calculus of ΨDO-s and FIO-s with large parameter as presented in [G] to construct a parametrix of the Neumann operator in the elliptic zone. Note that this is possible despite the fact that $\mathcal{O}$ is not necessarily convex. In fact, it is sufficient to construct the parametrix in a neighborhood of the characteristic variety Σ. We take a finite number $2m+1$ of terms in the asymptotic expansion of the corresponding amplitude in order to get a parametrix $N_{2m-1}(\lambda)$, such that $\lambda^{2m-1}N_{2m-1}(\lambda)$ is analytic in λ. Then we extend $\lambda^{2m-1}N_{2m-1}(\lambda)$ as an elliptic ΨDO with large parameter $\lambda_1 = \operatorname{Re}\lambda$ globally, thus obtaining a ΨDO $P(\lambda)$ which is an entire function of λ. Applying the Phragmén-Lindelöf principle, we show that $P(\lambda)$ has "zeros", i.e. we have $P(\lambda_j)f_j = 0$ with some λ_j in a logarithmic domain and $\|f_j\| = 1$. Next we show that λ_j are asymptotic zeros of $N_{2m-1}(\lambda)$, i.e. $N_{2m-1}(\lambda_j)f_j = O(|\lambda_j|^{-\infty})$, as well as $\operatorname{Im}\lambda_j = O(|\lambda_j|^{-2m+2})$. Since N_{2m-1} is a parametrix of the Neumann operator $\mathcal{N}$, for m sufficiently large this turns out to be enough to get the desired contradiction.

2 Some a priori estimates on the resolvent

The purpose of this section is to prove the following a priori estimate of the cut-off resolvent which is crucial for our proof of Theorem 1.

Proposition 1 *Assume that $R_\chi(\lambda)$ is analytic in $\{\lambda \in \mathbf{C};\ \operatorname{Im}\lambda \leq |\lambda|^{-N}, |\operatorname{Re}\lambda| \geq C\}$ with some $C > 0$ and integer $N > 0$. Then*

$$\|R_\chi(\lambda)\|_{\mathcal{L}(L^2;H^2)} \leq C_1|\lambda|^{N+7} \quad \text{for} \quad |\operatorname{Im}\lambda| \leq |\lambda|^{-N-6},\ |\operatorname{Re}\lambda| \geq C_2,$$

with some constants $C_1, C_2 > 0$.

Remark 1. It is easy to see from the proof that a similar statement holds in any odd-dimensional space as well and for compactly supported perturbations of the Laplacian. The proof however does not work if the space dimension is even.

Remark 2. As an immediate consequence of Proposition 1 we get that the existence of real quasimodes implies the existence of resonances $\{\lambda_j\}_{j=1}^{\infty}$ with $\operatorname{Im}\lambda_j = O(|\lambda_j|^{-\infty})$. The advantage of this conclusion is that, as shown in [P] (see also [CP], [L]), if there exists an elliptic broken periodic ray (with Poincaré map satisfying some technical conditions), one can construct real quasimodes $k_j \to +\infty$ of the Dirichlet Laplacian in an exterior domain

$\Omega \subset \mathbf{R}^n, n \geq 3$ odd, with C^∞ boundary Γ, i.e. there exist uniformly compactly supported functions $u_j, \|u_j\|_{L^2} = 1$, such that

$$\left\{ \begin{array}{rcll} (\Delta + k_j^2)u_j & = & O(k_j^{-\infty}) & \text{in } \Omega, \\ u_j & = & O(k_j^{-\infty}) & \text{on } \Gamma. \end{array} \right. \tag{3}$$

The proof of Proposition 1 is based on the next two lemmae.

Lemma 1 *Assume that $f(z)$ is analytic in $\{z \in \mathbf{C};\ \operatorname{Im} z < C_1|z|^{-N}, |\operatorname{Re} z| > C_2\}$ and $|f(z)| \leq C_3 e^{C_3|z|^n}$ with some positive constants C_1, C_2, C_3, and integers n, N. Assume moreover that $|f(z)| \leq C_4|z|^m/|\operatorname{Im} z|$ for $-1 < \operatorname{Im} z < 0$, $|z| > 1$ with some C_4, and integer $m \geq 0$. Then*

$$|f(z)| \leq C_5|z|^{m+n+N+2} \qquad \text{for} \quad |\operatorname{Im} z| \leq |z|^{-n-N-2},\ |\operatorname{Re} z| \geq C_5,$$

with some constant $C_5 > 0$.

Proof. Without loss of generality we can assume that $\operatorname{Re} z \geq C_2$. Set

$$u(z) = \exp\{iz^s\} = \exp\{i(\operatorname{Re} z)^s - s(\operatorname{Re} z)^{s-1}\operatorname{Im} z + \ldots\},$$

s being an integer to be chosen latter on. On $\gamma_+ := \{z \in \mathbf{C};\ \operatorname{Im} z = C_1|z|^{-N}, \operatorname{Re} z \geq C_2\}$ we have for $g := fu$

$$\begin{aligned} |g(z)| &\leq C_3 e^{C_3|z|^n} \exp\left\{-s(\operatorname{Re} z)^{s-1}\operatorname{Im} z + C_s(\operatorname{Re} z)^{s-3}(\operatorname{Im} z)^3 - \ldots\right\} \\ &= C_3 e^{C_3|z|^n} \exp\left\{-C_1 s\frac{(\operatorname{Re} z)^{s-1}}{|z|^N} + C_s'\frac{(\operatorname{Re} z)^{s-3}}{|z|^{3N}} - \ldots\right\} \\ &\leq C_3 e^{C_3|z|^n} \exp\left\{-\frac{1}{2}C_1 s\frac{(\operatorname{Re} z)^{s-1}}{|z|^N}\right\} \end{aligned}$$

for $|z|$ sufficiently large. Therefore, if $s > n + N + 1$, we have

$$|g(z)| \leq C \qquad \text{for } z \in \gamma_+. \tag{4}$$

Set $\gamma_- = \{z \in \mathbf{C};\ -\operatorname{Im} z = |z|^{-s}, \operatorname{Re} z \geq C_2\}$. On γ_- we have

$$\begin{aligned} |g(z)| &\leq \frac{C_4|z|^m}{|\operatorname{Im} z|} \exp\left\{s(\operatorname{Re} z)^{s-1}|\operatorname{Im} z| - C_s(\operatorname{Re} z)^{s-3}|\operatorname{Im} z|^3 + \ldots\right\} \\ &\leq C|z|^{m+s}. \end{aligned} \tag{5}$$

We get from (4), (5) that $z^{-m-s}g(z)$ is uniformly bounded on the boundary of the domain between the curves γ_+, γ_- and $\operatorname{Re} z = C_2'$ with $C_2' > C_2$ sufficiently large. Moreover $z^{-m-s}g(z)$ satisfies an a priori exponential estimate in the interior. An application of the Phragmén-Lindelöf principle implies that $z^{-m-s}g(z)$ is uniformly bounded in the interior of that domain as well and therefore

$$\begin{aligned} |f(z)| &\leq C|z|^{m+s} \exp\left\{s(\operatorname{Re} z)^{s-1}\operatorname{Im} z - C_s(\operatorname{Re} z)^{s-3}(\operatorname{Im} z)^3 + \ldots\right\} \\ &\leq C_5|z|^{m+s}, \end{aligned}$$

provided that $|\operatorname{Im} z| \leq 1/(\operatorname{Re} z)^s$. Now it suffices to pick $s = N + n + 2$ in order to complete the proof of the lemma. $\square$

We would like to apply this lemma to the operator-valued function $R_\chi(\lambda)$. To this end we need the following a priori estimate (compare with [SV2, Proposition 5.2]).

Lemma 2 *Assume that $R_\chi(\lambda)$ is analytic in $D_{C_1,C_2} = \{\lambda \in \mathbf{C};\ |\operatorname{Im}\lambda| < C_1|\lambda|^{-N}, |\operatorname{Re}\lambda| > C_2\}$ with some $C_1 > 0$, $C_2 > 0$, $N > 0$. Then for any $C_1' < C_1$, $C_2' > C_2$ we have*

$$\|R_\chi(\lambda)\|_{\mathcal{L}(L^2;H^2)} \leq Ce^{C|\lambda|^4} \tag{6}$$

with some $C > 0$ in $D_{C_1',C_2'} = \{\lambda \in \mathbf{C};\ |\operatorname{Im}\lambda| < C_1'|\lambda|^{-N},\ |\operatorname{Re}\lambda| > C_2'\}$.

Proof. This is a refinement of Proposition 5.2 in [SV2] (see also Lemma 3 in the present paper) and we refer to [SV2] for more details. As in the above cited paper we can find an entire function $h(\lambda)$ of order 3, such that in

$$V = \mathbf{C} \setminus \bigcup_j \left\{\lambda;\ |\lambda - z_j| < |z_j|^{-5-N}\right\}$$

we have

$$\|R_\chi(\lambda)\|_{\mathcal{L}(L^2;H^2)} \leq Ce^{C|\lambda|^4} \qquad \text{for } \lambda \in V, \tag{7}$$

where $\{z_j\}_{j=1}^\infty$ are the zeros of $h(\lambda)$. Let us observe that $\mathbf{C} \setminus V = \bigcup_{k=1}^\infty U_k$, where U_k are disjoint connected sets and each U_k is a union of a finite number of disks, because the series $\sum_{j=1}^\infty |z_j|^{-4}$ converges and hence so does $\sum_{j=1}^\infty |z_j|^{-5-N}$. Clearly, for each k, $\operatorname{diam} U_k < 2M$, where $M := \sum |z_j|^{-4}$, which implies

$$\sum_{|z_j|>A} |z_j|^{-N-5} \leq MA^{-N-1}.$$

Therefore,

$$\operatorname{diam} U_k < 2M[\min\{|\lambda|;\ \lambda \in U_k\}]^{-N-1} \leq 2M(|\lambda| - 2M)^{-N-1} \tag{8}$$

for each $\lambda \in U_k$, and k sufficiently large. Fix $C_1' < C_1$, $C_2' > C_2$ and set $K = \{k \in \mathbf{N};\ D_{C_1',C_2'} \cap U_k \neq \emptyset\}$. For large $k \in K$ we have $U_k \subset D_{C_1,C_2}$ because of (8). Since (7) holds on ∂U_k we can apply the maximum principle to conclude that (7) holds in U_k as well with some other constant $C > 0$ for large k. Thus (7) holds in the entire $D_{C_1',C_2'}$ except perhaps in a bounded set. $\square$

Proof of Proposition 1. We have by the above lemma that the operator-valued function $R_\chi(\lambda) : L^2 \to H^2$ satisfies the first assumption of Lemma 1 with $n = 4$. On the other hand, it is clear that in the lower half-plane R_χ satisfies the estimate

$$\|R_\chi(\lambda)\|_{\mathcal{L}(L^2)} \leq \frac{C}{|\operatorname{Im}\lambda||\lambda|},$$

which easily yields

$$\|R_\chi(\lambda)\|_{\mathcal{L}(L^2;H^2)} \leq \frac{C'|\lambda|}{|\operatorname{Im}\lambda|} \quad \text{for } \operatorname{Im}\lambda < 0.$$

Thus $R_\chi(\lambda) : L^2 \to H^2$ satisfies the second assumption of Lemma 1 as well with $m = 1$. Now Proposition 1 follows from Lemma 1 at once. $\square$

Let us define the Neumann operator $\mathcal{N}(\lambda)$ by the formula

$$\mathcal{N}(\lambda) : H^s(\Gamma) \ni f \mapsto \sum_{j=1}^{3} \boldsymbol{\sigma}_j(v)\nu_j|_\Gamma \in H^{s-1}(\Gamma), \qquad s \geq \frac{3}{2},$$

where $\boldsymbol{\sigma}_j = {}^t(\sigma_{1j}, \sigma_{2j}, \sigma_{3j})$, σ_{ij} is the stress tensor (see (2)) and v solves the following problem

$$\begin{cases} (\Delta_e + \lambda^2)v &= 0 \qquad \text{in } \Omega, \\ v &= f \qquad \text{on } \Gamma, \\ v &- \text{ outgoing.} \end{cases} \tag{9}$$

As $\mathcal{N}^{-1}(\lambda)$ can be easily expressed in terms of $R_\chi(\lambda)$ (see [SV1]), the assumption that $R_\chi(\lambda)$ is holomorphic in $\{\lambda \in \mathbf{C} : \operatorname{Im}\lambda \leq |\lambda|^{-N}, |\operatorname{Re}\lambda| \geq C\}$ implies that so is $\mathcal{N}^{-1}(\lambda)$ and moreover, by Proposition 1,

$$\|\mathcal{N}^{-1}(\lambda)\|_{\mathcal{L}(H^{1/2};H^{3/2})} \leq C|\lambda|^{N+9} \quad \text{for } |\operatorname{Im}\lambda| \leq |\lambda|^{-N-6},\ |\operatorname{Re}\lambda| \geq C'. \tag{10}$$

In the rest of the paper we will find a contradiction to (10).

3 Parametrix for the Neumann operator

We will recall briefly the construction of the parametrix of (9) in the elliptic zone (see [SV2], [CP]). We will use the calculus of ΨDO-s and FIO-s with large parameter as developed in [G]. We choose λ to be the large parameter and we suppose that

$$\lambda \in \Lambda_{C_1,C_2} := \{\lambda \in \mathbf{C};\ |\lambda_2| < C_1 \ln \lambda_1,\ \lambda_1 > C_2\}, \tag{11}$$

where $\lambda_1 = \operatorname{Re}\lambda$, $\lambda_2 = \operatorname{Im}\lambda$, $C_1 > 0$, $C_2 > 0$.

Given an open set X in $\mathbf{R}^n$ denote by $\tilde{C}^\infty(X)$ the space of all functions $u(x, \lambda)$, $\lambda \in \Lambda$ such that $u(\cdot, \lambda) \in C^\infty(X)$ and $p(u(\cdot, \lambda)) = O(|\lambda|^{-\infty})$ for all seminorms p in $C^\infty(X)$. In a similar way we define $\tilde{C}^\infty(K)$, K being a compact, $\tilde{C}_0^\infty(X)$ and $\tilde{\mathcal{D}}'(X)$.

Given two open sets X, Y in $\mathbf{R}^n$, for $m, k \in \mathbf{R}$, $\rho, \delta \in [0, 1)$ we define (see [G, Def. A.I.2]) the class $S^{m,k}_{\rho,\delta}(X \times Y)$ to be the set of all $a(x, y, \eta, \lambda) \in C^\infty(X \times Y \times \mathbf{R}^n)$, such that for any compact $K \subset\subset X \times Y$, all $\alpha, \beta, \gamma \in \mathbf{Z}^n$, $\lambda \in \Lambda$ we have

$$|\partial_x^\alpha \partial_y^\beta \partial_\eta^\gamma a| \leq C_{\alpha,\beta,\gamma,K}|\lambda|^{k+\rho|\gamma|+\delta|\alpha+\beta|}(1 + |\eta|)^{m-|\gamma|}. \tag{12}$$

If $X = Y$, we set $S^{m,k}_{\rho,\delta}(X) = S^{m,k}_{\rho,\delta}(X \times X)$. Given $a \in S^{m,k}_{\rho,\delta}(X \times Y)$, denote by $\operatorname{Op}(a)$ (or $\operatorname{Op}_\lambda(a)$) the operator

$$(\operatorname{Op}(a)u)(x, \lambda) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x-y)\cdot\eta} a(x, y, \eta, \lambda)u(y, \lambda)\, dy\, d\eta. \tag{13}$$

We have well-defined operators in the case where a has bounded support in η and $\lambda \in \Lambda$ or if η is unbounded on $\operatorname{supp} a$, but λ is real. We refer to [G] (see also [SV2]) for more details, as well as for a definition and properties of elliptic ΨDO-s with large parameter, wave front set $\widetilde{\mathrm{WF}}(f)$, etc.

Let us recall [SV2] that operators of the form $\mathrm{Op}_\lambda(a)$ can be represented as ΨDO-s with large parameter $\lambda_1 = \operatorname{Re}\lambda$, provided that $|\eta|$ is bounded on $\operatorname{supp} a$. In other words, $\mathrm{Op}_\lambda(a) = \mathrm{Op}_{\lambda_1}(\tilde{a})$, where

$$\tilde{a}(x, y, \eta, \lambda) = (1 + i\lambda_2/\lambda_1)^n e^{-\lambda_2 (x-y)\cdot\eta} a(x, y, \eta, \lambda).$$

One can regard here $\lambda_2/\ln\lambda_1 \in [-C_1, C_1]$ as an additional parameter. Assuming $\lambda \in \Lambda_{C_1,C_2}$, we get that $a \in S^{0,k}_{\rho,\delta}$ implies $\tilde{a} \in S^{0,k+N}_{\rho+\varepsilon,\delta+\varepsilon}$ for any $\varepsilon > 0$. This follows from the fact that $|e^{-\lambda_2(x-y)\cdot\eta}| \le |\lambda|^N$ with a fixed $N > 0$ and $|\lambda_2| \le C_\varepsilon |\lambda|^\varepsilon$ for any $\varepsilon > 0$. Therefore, $\tilde{a}$ is an amplitude. Using [G, Pr. A.I.4, Pr. A.I.5], we can calculate the symbol of $\tilde{a}$ (depending only on x, η, λ) and we find that actually $\tilde{a} \in S^{0,k}_{\rho,\delta}$ and the principal symbol is $a|_{y=x} - i(\lambda_2/\lambda_1)\eta \cdot \nabla_\eta a|_{y=x}$. Thus if $a \in S^{0,k}_{0,0}$, we have $\mathrm{Op}_\lambda(a) = \mathrm{Op}_{\lambda_1}(\tilde{a})$ with

$$\tilde{a} = a - i\frac{\lambda_2}{\lambda_1}\eta \cdot \nabla_\eta a \qquad \mathrm{mod}\ S^{0,k-1}_{0,0}. \tag{14}$$

Let us now recall the construction of the parametrix in the elliptic zone. Recall (see e.g. [SV2]) that the operator $-\Delta_e$ has two sound speeds c_1, c_2 and the variety $\Sigma = \{\zeta \in T^*\Gamma;\ c_R\|\zeta\| = 1\}$ lies in the elliptic zone $\{\zeta \in T^*\Gamma;\ \|\zeta\| > c_1^{-1}\}$, $\|\cdot\|$ being the norm in $T^*\Gamma$. Let $\zeta^0 \in T^*\Gamma$ with $\|\zeta^0\| > c_1^{-1}$ and from now on we assume that the space dimension is $n = 3$. Let us pick local coordinates such that $\zeta^0 = (0, \eta^0)$, the boundary is given locally by $x_1 = 0$ and the normal derivative at $x = 0$ is given by $\partial/\partial x_1$. Then $x' = (x_2, x_3)$ are local coordinates on Γ. Let $\chi_{\zeta^0}(x', \eta) \in C_0^\infty(T^*\Gamma)$ be a cut-off function equal to 1 near ζ^0. If $\operatorname{supp}\chi_{\zeta^0}$ is sufficiently small, one can construct a local FIO H^{ζ^0} with large parameter $\lambda \in \Lambda_{C_1,C_2}$ such that

$$\begin{cases} (\Delta_e + \lambda^2) H^{\zeta^0} f &= Kf, \\ H^{\zeta^0} f|_\Gamma &= \mathrm{Op}_\lambda(\chi_{\zeta^0}) f, \end{cases} \tag{15}$$

where K has kernel in $\tilde{C}^\infty$. The operator H^{ζ^0} is of the form

$$H^{\zeta^0} f = \left(\frac{\lambda_1}{2\pi}\right)^2 \iint e^{i\lambda(\varphi(x,\eta) - y\cdot\eta)} h(x, \eta, \lambda) f(y, \lambda)\, dy\, d\eta. \tag{16}$$

The phase function φ solves the eikonal equation $(\nabla\varphi)^2 = 1$, $\varphi|_\Gamma = x \cdot \eta$ to infinite order at Γ and $\operatorname{Im}\varphi \ge c x_1$ on $\operatorname{supp}\chi_{\zeta^0}$ with some $c > 0$. This implies that $H^{\zeta^0} f = O(e^{-c\lambda_1 x_1})$. The matrix-valued amplitude h is a solution of the corresponding transport equations and has the form $h = \sum_{j=0}^\infty h_j(x, \eta)\lambda^{-j}$, with h_j formal series in x_1. Set

$$h^{(m)} = \sum_{j=-1}^{m} \lambda^{-j} h_{-j}(x, \eta)$$

and consider the operator $H_m^{\zeta^0}$ associated with $h^{(m)}$. Then $H_m^{\zeta^0}$ solves a problem similar to (14) with K replaced by K plus a FIO of order $-m$. Denote $N_m^{\zeta^0} f = \sum_{j=1}^3 \boldsymbol{\sigma}_j (H_m^{\zeta^0} f)\nu_j|_\Gamma$, where $\boldsymbol{\sigma}_j = {}^t(\sigma_{1j}, \sigma_{2j}, \sigma_{3j})$, σ_{ij} is the stress tensor. Then $N_m^{\zeta^0}$ is a ΨDO with large parameter λ and symbol

$$\sigma(N_m^{\zeta^0}) = \sum_{j=-1}^{m} \lambda^{-j} n_{-j}(x, \eta).$$

The principal symbol is $\lambda n_1(x,\eta) = \lambda\chi_{\zeta^0}(x,\eta)\tilde{n}_1(x,\eta)$, $\tilde{n}_1(x,\eta)$ being a Hermitian matrix with three distinct eigenvalues near Σ. One of the eigenvalues has simple zero at Σ, the other two are elliptic. Moreover, $\tilde{n}_1$ is elliptic everywhere in the elliptic zone outside Σ (see also [CP], [K]).

Thus for any ζ in a neighborhood of Σ in the elliptic zone we constructed an operator H_m^ζ solving (15) provided that $\operatorname{supp}\chi_\zeta$ is contained in a small neighborhood U_ζ of ζ. Let W_1 be a bounded neighborhood of Σ in the elliptic zone and let us pick a partition of unity $\{\chi_{\zeta^j}\}$ associated with $\{U_\zeta\}$ covering W_1 and supported in a slightly larger domain. Using this partition of unity, we construct a solution operator

$$H_m(\lambda) = \sum_j H_m^{\zeta^j}\phi_j, \tag{17}$$

where $\phi_j(x)$ have small supports and $\phi_j(x) = 1$ in a neighborhood of $\pi_x(\operatorname{supp}\chi_{\zeta^j})$. This operator solves

$$\begin{cases} (\Delta_e + \lambda^2)H_m f &= K_m f, \\ H_m f|_\Gamma &= f + Qf, \end{cases} \tag{18}$$

provided that $\widetilde{\mathrm{WF}}(f) \subset W_1$, where $K_m(\lambda)$ is a FIO with amplitude of order $-m$ and

$$\|Qf\|_{H^s} = O(|\lambda|^{-\infty}) \quad \text{for any } s. \tag{19}$$

Set

$$N_m f = \sum_{j=1}^3 \boldsymbol{\sigma}_j (H_m f)\nu_j|_\Gamma.$$

Then $N_m \in L_{0,0}^{0,1}(\Gamma)$ and $\lambda^m N_m$ is holomorphic in λ.

Now we are going to find a relationship between N_m and the Neumann operator $\mathcal{N}$. Following [G], we set

$$\tilde{H}_m(\lambda) = \chi H_m(\lambda) - S_0(\lambda)(\chi K_m(\lambda) + [\Delta_e, \chi]H_m(\lambda)), \tag{20}$$

where $S_0(\lambda)$ is the free outgoing resolvent. Thus $\tilde{H}_m(\lambda)f$ solves the problem

$$\begin{cases} (\Delta_e + \lambda^2)\tilde{H}_m f &= 0, \\ \tilde{H}_m f|_\Gamma &= f + Qf + R_m f, \end{cases} \tag{21}$$

with

$$\|R_m(\lambda)\|_{\mathcal{L}(H^{3/2}(\Gamma))} \le C|\lambda|^{-m+1} \tag{22}$$

provided that $\lambda \in \Lambda_{C_1,C_2}$ with C_1 small enough. If we denote the exact solution to (9) by $\mathcal{H}(\lambda)f$, we see that

$$\mathcal{H}(f + Qf + R_m f) = \tilde{H}_m f.$$

By differentiating this at the boundary, we get

$$\mathcal{N}(f + Qf + R_m f) = N_m f + \tilde{R}_m f, \qquad \widetilde{\mathrm{WF}}(f) \subset W_1, \tag{23}$$

where $\|\tilde{R}_m\|_{\mathcal{L}(H^{3/2},H^{1/2})} = O(|\lambda|^{-m+2})$ and Qf, R_m satisfy (19) and (22), respectively.

4 Proof of the main result

Let $\{\chi_{\zeta^j}\}$ be the partition of unity used to construct H_m (see (17)). Choose an open set $W_2 \subset T^*\Gamma$, such that $\Sigma \subset W_2 \subset\subset W_1$ and pick $\chi_0 \in C_0^\infty(T^*\Gamma)$, such that $\chi_0 = 1$ on W_2 and $\operatorname{supp} \chi_0 \subset W_1$. For each ζ^j let us define a local ΨDO using the special coordinates related to ζ^j by $A^{\zeta^j} = \mathrm{Op}_\lambda(\chi_0 \chi_{\zeta^j})$ and set $A = \sum_j A^{\zeta^j} \phi_j$ (see (17)). Then $A \in L^{0,1}_{0,0}(\Gamma)$, A is an entire function of λ and $\sigma_p(A) = 1$ on W_2, $\sigma_p(A) = 0$ outside W_1. Moreover, the symbol of A in any local coordinates is supported in W_1. Since the symbol of N_m has also compact support, we will extend N_m as an operator elliptic outside W_1 with characteristic variety Σ. To this end fix an integer $m > 0$ and set

$$P(\lambda) = \lambda^{2m-1} N_{2m-1} A + i \left(\lambda^2 - \Delta_\Gamma\right)^m (I - A), \tag{24}$$

Δ_Γ being the Laplacian on Γ. Note first that P is analytic function of λ with values in $\mathcal{L}(H^{s+2m}, H^s)$. Secondly, let us mention that in any logarithmic region Λ_{C_1,C_2}, P can be considered as a ΨDO with large parameter $\lambda_1 = \operatorname{Re} \lambda$ and $P \in L^{2m,2m}_{0,0}(\Gamma)$. We claim that P is elliptic outside Σ. Indeed, for the principal symbol of P we have

$$\sigma_p(P) = \lambda^{2m-1} \sigma_p(A) \sigma_p(N_{2m-1}) + i \lambda_1^{2m} \left(1 - \sigma_p(A)\right) \left((1 + i\lambda_2/\lambda_1)^2 + |\eta|_x^2\right)^m,$$

where $\sigma_p(A)$ is a function supported in W_1 and $|\eta|_x$ denotes the norm of the covector (x, η). Note that here we consider N_{2m-1}, A as ΨDO-s with large parameter λ_1, not λ, and respectively $\sigma_p(N_{2m-1})$, $\sigma_p(A)$ are the principal symbols of these operators obtained by using (14). In $W_2 \setminus \Sigma$ the principal symbol $\sigma_p(P)$ is elliptic, because $\sigma_p(A)|_{W_2} = 1$ is elliptic. Outside W_1, $\sigma_p(P) = \lambda_1^{2m} \left((1 + i\lambda_2/\lambda_1)^2 + |\eta|_x^2\right)^m$ is elliptic as well, including at the infinite points of $\hat{T}^*\Gamma$. Finally, on $W_1 \setminus W_2$ our claim follows from the fact that for any Hermitian elliptic matrix B we have $|\alpha B x + i\beta x|^2 = \alpha^2 |Bx|^2 + \beta^2 |x|^2 \geq c|x|^2$ provided that $\alpha + \beta = 1$.

Next proposition establishes existence of “resonances” of P.

Proposition 2 *There exist λ_j and $f_j \in C^\infty(\Gamma)$, $j = 1, 2, \ldots$, such that*
a) $P(\lambda_j) f_j = 0$,
b) $|\operatorname{Im} \lambda_j| \leq C \ln |\lambda_j|$ *and* $|\lambda_j| \to \infty$, *as* $j \to \infty$,
c) $\|f_j\|_{H^{3/2}(\Gamma)} = 1$, $\widetilde{\mathrm{WF}}(f) \subset \Sigma$, *where* $f(x, \lambda) := f_j(x)$, $\lambda = \lambda_j$.

To prove Proposition 2 we are going to apply the Phragmén-Lindelöf principle to $P^{-1}(\lambda)$. To this end we need the following two lemmae. First we prove an a priori exponential estimate of $P^{-1}(\lambda)$ similar to that in Lemma 2 (see also [SV2], Proposition 5.2).

Lemma 3 *Assume that $P^{-1}(\lambda)$ has no poles in Λ_{C_1,C_2} with some $C_1 > 0$, $C_2 > 0$. Then*

$$\|P^{-1}(\lambda)\|_{\mathcal{L}(L^2(\Gamma))} \leq Ce^{C|\lambda|^4}, \quad \lambda \in \Lambda_{C_1/2,2C_2}.$$

Proof. Let us rewrite $P(\lambda)$ in the form

$$P(\lambda) = i\,(I - \Delta_\Gamma)^m\,(I + K(\lambda)), \tag{25}$$

where $K(\lambda) = K_1(\lambda) + K_2(\lambda)$ with

$$\begin{aligned} K_1(\lambda) &= -\left[I + (\lambda^2 - 1)\,(I - \Delta_\Gamma)^{-1}\right]^m A - i\,(I - \Delta_\Gamma)^{-m}\,\lambda^{2m-1} N_{2m-1} A, \\ K_2(\lambda) &= \left[I + (\lambda^2 - 1)\,(I - \Delta_\Gamma)^{-1}\right]^m - I. \end{aligned}$$

Clearly, $K(\lambda)$ is an entire family of compact operators on $L^2(\Gamma)$. Moreover, the operator $K^2(\lambda)$ is of trace class and we can consider the entire function $h(\lambda) = \det(I - K^2(\lambda))$. As in the proof of Proposition 5.2 in [SV2] first we will prove the following a priori estimate

$$\|P^{-1}(\lambda)\|_{\mathcal{L}(L^2(\Gamma))} \leq Ce^{C|\lambda|^4}, \quad \lambda \in V, \tag{26}$$

where $V = \mathbf{C} \setminus \cup\{\lambda \in \mathbf{C};\ |\lambda - z_j| \leq |z_j|^{-4}\}$, z_j being the zeros of h. To this end we will prove first that $h(\lambda)$ is of order 3. We have

$$\begin{aligned} |h(\lambda)| &= \left|\det\left(I - K^2(\lambda)\right)\right| \leq \prod_{j=1}^{\infty}\left(1 + \mu_j(K^2(\lambda))\right) \\ &\leq \prod_{j=1}^{\infty}\left(1 + \mu_j(\tilde{K}_1(\lambda))\right)^2 \prod_{j=1}^{\infty}\left(1 + \mu_j(K_2^2(\lambda))\right)^2, \end{aligned} \tag{27}$$

where $\mu_j(K)$ denote the characteristic values of K and $\tilde{K}_1 = K_1^2 + K_1K_2 + K_2K_1$. Let us first estimate $\mu_j(\tilde{K}_1(\lambda))$. Clearly,

$$\mu_j(\tilde{K}_1(\lambda)) \leq \|\tilde{K}_1(\lambda)\| \leq Ce^{C|\lambda|} \quad \forall \lambda \in \mathbf{C}, \forall j. \tag{28}$$

On the other hand,

$$\begin{aligned} \mu_j(\tilde{K}_1(\lambda)) &\leq \mu_{[\frac{j}{2}]}(K_1^2 + K_1K_2) + \mu_{[\frac{j}{2}]}(K_1K_2) \\ &\leq \|K_1 + K_2\|\mu_{[\frac{j}{2}]}(K_1) + \|K_2\|\mu_{[\frac{j}{2}]}(K_1) \\ &\leq Ce^{C|\lambda|}\mu_{[\frac{j}{2}]}(A). \end{aligned} \tag{29}$$

Let us recall that A is a finite sum of operators $A^\zeta\phi$ with kernels of the kind

$$\left(\frac{\lambda}{2\pi}\right)^2 \chi(x,\eta)e^{i\lambda(x-y)\cdot\eta}\phi(y),$$

where χ is a cut-off function supported near ζ in the elliptic zone, $\phi \in C_0^\infty$ and $\phi(y) = 1$ in a neighborhood of $\pi_x(\operatorname{supp}\chi)$. Set $M := \pi_\eta(\operatorname{supp}\chi)$. We have $A^\zeta = A_1A_2$, where

$A_1 : L^2(M) \to L^2(\Gamma)$, $A_2 : L^2(\Gamma) \to L^2(M)$ have kernels $(\lambda/2\pi)^2\chi(x,\eta)e^{i\lambda x\cdot\eta}$ and $e^{-i\lambda y\cdot\eta}\phi(y)$, respectively. For the kernel of A_2 we have

$$\sup_{y\in\mathrm{supp}\phi,\,\eta\in M}\left|(I-\Delta_\eta)^k e^{-i\lambda y\cdot\eta}\phi(y)\right| \le Ce^{C|\lambda|}\left((C|\lambda|)^{2k}+(2k)^{2k}\right).$$

Therefore, we get for any $k \ge 0$ and for any $j > 0$,

$$\mu_j(A_2) \le \mu_j\left((I-\Delta_\eta)^{-k}\right)\|(I-\Delta_\eta)^k A_2\| \le Cj^{-k}e^{C|\lambda|}\left((C|\lambda|)^{2k}+(2k)^{2k}\right).$$

Taking $k = [|\lambda|/2]$, $j \ge C(q)|\lambda|^2$ gives

$$\mu_j(A_2) \le j^{-2}e^{-q|\lambda|} \quad \text{for any } q > 0 \text{ and } j \ge C(q)|\lambda|^2. \tag{30}$$

By (30) and the estimate $\|A_1\| \le Ce^{C|\lambda|}$ we get the same type of estimate for A^ζ, and hence for A. Thus, choosing q properly, in view of (26) we obtain

$$\mu_j(\tilde{K}_1(\lambda)) \le Cj^{-2} \quad \text{for } j \ge C'|\lambda|^2. \tag{31}$$

Combining (28) and (31) yields

$$\prod_{j=1}^{\infty}\left(1+\mu_j(\tilde{K}_1(\lambda))\right)^2 \le \prod_{j\le C'|\lambda|^2}\left(Ce^{C|\lambda|}\right)^2 \prod_{j>C'|\lambda|^2}\left(1+C'j^{-2}\right)^2 \le e^{C|\lambda|^3}. \tag{32}$$

It remains to estimate $\mu_j(K_2^2)$. We have

$$\mu_j(K_2^2) \le \mu^2_{[\frac{j}{2}]}(K_2). \tag{33}$$

On the other hand,

$$K_2 = \sum_{p=1}^{m}\binom{m}{p}(\lambda^2-1)^p(I-\Delta_\Gamma)^{-p},$$

thus, setting $\langle\lambda\rangle := (1+|\lambda|^2)^{1/2}$, we get

$$\begin{aligned}\mu_j(K_2) &\le \sum_{p=1}^{m}\binom{m}{p}\langle\lambda\rangle^{2p}\mu_{[\frac{j}{m}]}\left((I-\Delta_\Gamma)^{-p}\right)\\ &\le \sum_{p=1}^{m}\binom{m}{p}\langle\lambda\rangle^{2p}\left(\frac{cj}{m}\right)^{-p}\\ &\le C\langle\lambda\rangle^2 j^{-1} \quad \text{for } j \ge \langle\lambda\rangle^2/2.\end{aligned}$$

Thus we get from (33) and the estimate above

$$\mu_j(K_2^2) \le C\langle\lambda\rangle^4 j^{-2} \quad \text{for } j \ge \langle\lambda\rangle^2. \tag{34}$$

Using (34) we deduce

$$\begin{aligned}\prod_{j=1}^{\infty}\Big(1+\mu_j(K_2^2(\lambda))\Big) &\leq \prod_{j\leq\langle\lambda\rangle^2}\Big[C\langle\lambda\rangle^{2m}\Big]\prod_{j>\langle\lambda\rangle^2}\Big(1+C\langle\lambda\rangle^4 j^{-2}\Big)\\ &\leq \exp\Big[C\langle\lambda\rangle^2\ln\langle\lambda\rangle\Big]\exp\Big[C\langle\lambda\rangle^4\sum_{j>\langle\lambda\rangle^2}j^{-2}\Big]\\ &\leq \exp\Big[C\langle\lambda\rangle^2\ln\langle\lambda\rangle\Big]\exp\Big[C\langle\lambda\rangle^2\Big]\\ &\leq Ce^{C|\lambda|^3}. \end{aligned} \tag{35}$$

Now (32) and (35) together imply

$$|h(\lambda)| \leq \prod_{j=1}^{\infty}\Big(1+\mu_j(K^2(\lambda))\Big) \leq Ce^{C|\lambda|^3}, \quad \lambda\in\mathbf{C}. \tag{36}$$

We will complete the proof of the lemma as in [SV2]. By [Ti, Ch. VIII] we conclude from (36) that

$$|h^{-1}(\lambda)| \leq Ce^{C|\lambda|^4}, \qquad \lambda\in V. \tag{37}$$

On the other hand, we have (see e.g. [GK, Thm. 5.1])

$$\Big|\det\Big(I-K^2(\lambda)\Big)\Big| . \Big\|\Big(I-K^2(\lambda)\Big)^{-1}\Big\| \leq \prod_{j=1}^{\infty}\Big(1+\mu_j(K^2(\lambda))\Big) \leq Ce^{C|\lambda|^3}. \tag{38}$$

By (36) and (37) we obtain

$$\Big\|\Big(I-K^2(\lambda)\Big)^{-1}\Big\| \leq Ce^{C|\lambda|^4}, \qquad \lambda\in V,$$

which implies immediately (26). As in the proof of Lemma 2 if we assume that P^{-1} is free of poles in some logarithmic domain, we will get that (26) holds in a slightly shrunken domain. □

Denote $l_\pm = \{\lambda\in\mathbf{C};\ \mathrm{Re}\,\lambda \geq C_2,\ \mathrm{Im}\,\lambda = \pm C_1\ln(\mathrm{Re}\,\lambda)\}$. Let us assume that $C_2>1$, so that $\ln(\mathrm{Re}\,\lambda)>0$.

Lemma 4 *For any $C_1>0$ there exists $C_2>1$, such that the operator $P(\lambda)$ is invertible on $l_\pm$ and*

$$\|P^{-1}(\lambda)\|_{\mathcal{L}(L^2(\Gamma))} \leq \frac{C}{\ln|\lambda|}|\lambda|^{-2m+1}, \qquad \lambda\in l_\pm.$$

Proof. Let $\zeta\notin W_2$ and $\chi\in C_0^\infty(T^*\Gamma)$ be a cut-off function with sufficiently small support in $T^*\Gamma\setminus W_2$. Since P is a ΨDO with large parameter λ_1 elliptic outside W_2, we get

$$\|\mathrm{Op}_{\lambda_1}(\chi)f\| \leq \frac{C}{|\lambda|^{2m}}\|Pf\| + C_N|\lambda|^{-N}\|f\| \tag{39}$$

for any $N > 0$ and $\lambda \in l_\pm$, where $\mathrm{Op}_{\lambda_1}(\chi)$ is the ΨDO with symbol χ written in the special coordinates related to ζ. The same estimate holds if χ is supported near the infinite points in $\hat{T}^*\Gamma$, i.e. for $\chi = \chi'(x)\chi''(\eta)$, where $\mathrm{supp}\,\chi'$ is close to a point $x^0 \in \Gamma$, while $\mathrm{supp}\,\chi_2 \subset \{\eta;\ c_R|\eta| > 2\}$.

Let us now choose $\zeta \in W_1$ and pick $\chi \in C_0^\infty(T^*\Gamma)$ supported in W_1, such that $\chi = 1$ in a neighborhood of ζ. The principal symbol of P considered as a ΨDO with large parameter $\lambda \in l_\pm$ is $\lambda^{2m}\tilde{n}_1(x,\eta)$ (see section 3). In a neighborhood U_ζ of $\mathrm{supp}\,\chi$ we have

$$T^*\tilde{n}_1 T = \mathrm{diag}(c_R^2|\eta|_x^2 - 1, 1, 1)S, \tag{40}$$

where $S = \mathrm{diag}(a_1', a_2, a_3)$ is elliptic and $a_1 = (c_R^2|\eta|_x^2 - 1)a_1'$, a_2, a_3 are the eigenvalues of $\tilde{n}_1$. Here T is a unitary matrix. Let us now consider $\mathrm{Op}_\lambda\,(c_R^2|\eta|_x^2 - 1)$ as a ΨDO with large parameter λ_1. From (14) we deduce that the principal symbol of the latter reads

$$\frac{1}{\lambda^2}\left(\lambda_1^2 c_R^2|\eta|_x^2 - \lambda_1^2 - 2i\lambda_1\lambda_2\right).$$

Therefore, modulo $S_{0,0}^{0,-1}$ this operator coincides with

$$\frac{1}{\lambda^2}\left(-\Delta_\Gamma - \lambda_1^2 - 2i\lambda_1\lambda_2\right).$$

Observe that for any g and C_2 sufficiently large

$$\|\frac{1}{\lambda^2}\left(-\Delta_\Gamma - \lambda_1^2 - 2i\lambda_1\lambda_2\right)g\| \geq \frac{|\lambda_2|}{|\lambda|}\|g\| = C_1\frac{\ln\lambda_1}{|\lambda|}\|g\|.$$

This inequality together with (40) shows that for any $N > 0$ we have

$$\|\mathrm{Op}_\lambda(\chi)f\| \leq \frac{C}{|\lambda|^{2m-1}\ln|\lambda|}\|Pf\| + C_N|\lambda|^{-N}\|f\|.$$

Choose $\chi_1 \in C_0^\infty(T^*\Gamma)$, such that $\mathrm{supp}\,\chi_1 \subset \{\zeta;\ \chi(\zeta) = 1\}$. Then one easily gets

$$\|\mathrm{Op}_{\lambda_1}(\chi_1)f\| \leq \frac{C}{|\lambda|^{2m-1}\ln|\lambda|}\|Pf\| + C_N|\lambda|^{-N}\|f\|. \tag{41}$$

By (39), (41),

$$\|f\| \leq \frac{C}{|\lambda|^{2m-1}\ln|\lambda|}\|Pf\|$$

for any f and $|\lambda|$ sufficiently large. Since we can prove the same type of estimate for P^*, we get that $P^{-1}(\lambda)$ exists for $\lambda \in l_\pm$ sufficiently large and satisfies the desired estimate. □

Proof of Proposition 2. Assume now that there is a finite number of poles of P^{-1} in some logarithmic domain Λ_{C_1,C_2}. Taking C_2 sufficiently large we can assume that Λ_{C_1,C_2} is free of poles and Lemma 4 holds. Let us apply the Phragmén-Lindelöf principle to the function $\lambda^{2m-1}(\log\lambda)P^{-1}(\lambda)$ in $\Lambda_{C_1/2,2C_2}$. By Lemma 3 it satisfies the a priori exponential estimate in $\Lambda_{C_1/2,2C_2}$, while by Lemma 4 it is uniformly bounded on the boundary. Therefore, it is uniformly bounded in $\Lambda_{C_1/2,2C_2}$ as well, i.e.

$$\|P^{-1}(\lambda)\|_{\mathcal{L}(L^2(\Gamma))} \leq \frac{C}{|\lambda|^{2m-1}\ln|\lambda|}, \quad \lambda \in \Lambda_{C_1/2,2C_2}. \tag{42}$$

The final step of the proof of Proposition 2 is to show that (42) leads to contradiction for real λ. We will do this in exactly the same way as in [SV2]. Let $\{\mu_j^2\}$ be the eigenvalues of $-c_R^2\Delta_\Gamma$ and denote by φ_j, $\|\varphi_j\| = 1$ the corresponding eigenfunctions. Fix $\zeta^0 \in \Sigma$ and let χ be supported in a small neighborhood U of ζ^0 in the elliptic region. Let $\Pi(x,\eta)$, $(x,\eta) \in U$ be the projection onto the eigenspace corresponding to the first eigenvalue $a_1 = (c_R^2|\eta|_x^2 - 1)a_1'$. Set

$$f_k(\,\cdot\,,\mu_j) = \mathrm{Op}_{\mu_j}(\chi\Pi)e_k\varphi_j, \tag{43}$$

$\{e_k\}_{k=1}^3$ being the standard base in $\mathbf{R}^3$. Denote $\Theta = \{\mu_j\}_{j=1}^\infty$ and $f_k(x,\lambda) = f_k(x,\mu_j)$, $\varphi(x,\lambda) = \varphi_j(x)$ for $\lambda \in \Theta$. Consider all ΨDO-s bellow as ΨDO-s with large parameter $\lambda \in \Theta$. Then

$$Pf_k = Ge_k\varphi, \tag{44}$$

where $G \in L_{0,0}^{0,2m}(\Gamma)$, $\sigma_p(G) = \lambda^{2m}(c_R^2|\eta|_x^2 - 1)a_1'\chi\Pi$. Since the principal symbol of $-c_R^2\Delta_\Gamma - \lambda^2$ is $\lambda^2(c_R^2|\eta|_x^2 - 1)$, we have

$$Pf_k = \lambda^{2m-2}\mathrm{Op}(\chi a_1'\Pi)\left(-c_R^2\Delta_\Gamma - \lambda^2\right)e_k\varphi + Be_k\varphi = Be_k\varphi, \tag{45}$$

where $B \in L_{0,0}^{0,2m-1}(\Gamma)$. Thus

$$\|Pf_k\| \leq C\lambda^{2m-1} \qquad \text{for } k = 1,2,3;\ \ \lambda \in \Theta. \tag{46}$$

According to (42), (43),

$$\|\mathrm{Op}(\chi\Pi)e_k\varphi\| \leq \frac{C}{\ln\lambda} \qquad \text{for } k = 1,2,3;\ \lambda \in \Theta. \tag{47}$$

Since the projection $\Pi(\zeta)$ is well defined and does not vanish near Σ, we have that $\sum|\Pi_{ij}|^2$ is elliptic in U provided that U is sufficiently close to Σ. Thus from (47) we deduce that

$$\|\mathrm{Op}(\chi'\chi'')\varphi\| \leq \frac{C}{\ln\lambda}, \tag{48}$$

where $\chi' = \chi'(x)$, $\chi'' = \chi''(\eta)$ and $\chi'(x) = 1$, $\chi''(\eta) = 1$ for (x,η) close to ζ^0, $\operatorname{supp}\chi'\chi'' \subset \{\chi = 1\}$. On the other hand, $(-c_R^2\Delta_\Gamma - \lambda^2)\varphi = 0$ and $-c_R^2\Delta_\Gamma - \lambda^2$ is a ΨDO on Γ in $L_{0,0}^{2,2}(\Gamma)$ with principal symbol $\lambda^2(c_R^2|\eta|_x^2 - 1)$ elliptic outside Σ. Therefore, $\widetilde{\mathrm{WF}}(\varphi) \subset \Sigma$. Hence,

$$\|\mathrm{Op}(\chi'(1-\chi''))\varphi\| \leq C_N\lambda^{-N}, \qquad \forall N > 0. \tag{49}$$

Combining (48) and (49) we get

$$\|\chi'\varphi\| \leq \frac{C}{\ln\lambda}, \qquad \lambda \in \Theta,$$

for any cut-off function χ', such that $\chi' = 1$ near $x^0 = \pi_x(\zeta^0)$ and $\operatorname{supp} \chi'$ is sufficiently small. Since $\zeta^0 \in \Sigma$ was arbitrary, we get $\|\varphi\| \leq C/\ln \lambda$ which contradicts the fact that $\|\varphi\| = 1$.

Thus, there exists a sequence $\{\lambda_j\}$ of poles of $P^{-1}(\lambda)$ satisfying (b). Going back to the representation (25) we conclude by the Fredholm alternative that for any pole λ_j there is a function $f_j \neq 0$ such that $P(\lambda_j)f_j = 0$. Since P, considered as a ΨDO with large parameter λ_1 (with λ satisfying (b)) is elliptic outside Σ, we get (c). □

Next we show that Proposition 2 implies existence of asymptotic zeros of N_{2m-1}.

Proposition 3 *Let λ_j, f_j, $j = 1, 2, \ldots$ be as in Proposition 2. Then*
(a) $N_{2m-1}(\lambda_j)f_j = O(|\lambda_j|^{-\infty})$,
(b) $|\operatorname{Im} \lambda_j| \leq C|\lambda_j|^{-2m+2}$ with some $C > 0$.

Proof. It follows from Proposition 2 (c) that $(I - A)f_j = O(|\lambda_j|^{-\infty})$, which in view of (24) yields (a). To prove (b), set $f(x, \lambda) = f_j(x)$, $\lambda \in \Theta := \{\lambda_j\}_{j=1}^{\infty}$ and recall that H_{2m-1} solves (18) with $2m - 1$ instead of m. Arguing as in [SV2], we get that

$$(\Delta_e + \lambda^2)\phi H_{2m-1}f = [\Delta_e, \phi]H_{2m-1}f + \phi K_{2m-1}f, \tag{50}$$

where $\phi \in C_0^\infty$, $\phi = 1$ near Γ. Multiply (50) by $\phi H_{2m-1}f$ and integrate by parts. Using (18), we get that

$$\operatorname{Im} \lambda^2 \leq \frac{\|[\Delta_e, \phi]H_{2m-1}f\| + C|\lambda|^{-2m+1}}{\|\phi H_{2m-1}f\|}.$$

Now, we use the facts that $[\Delta_e, \phi]$ is a first order differential operator with compactly supported coefficients vanishing near Γ and that the parametrix in the elliptic zone decays exponentially in $|\lambda|x_1$. This implies

$$\|[\Delta_e, \phi]H_{2m-1}f\| \leq Ce^{-\gamma|\lambda|}\|f\| \leq Ce^{-\gamma|\lambda|}$$

with some $\gamma > 0$ (recall that $\|f\|_{H^{3/2}} = 1$). On the other hand, by trace theorem, in view of (17) and using the fact that the operator Δ_e with Dirichlet boundary conditions is coercive, we have

$$\begin{aligned} \|f + Qf\|_{H^{3/2}} &\leq C\|\phi H_{2m-1}f\|_{H^2} \leq C'(\|\Delta_e \phi H_{2m-1}f\| + \|\phi H_{2m-1}f\|) \\ &\leq C''|\lambda|^2\|\phi H_{2m-1}f\| + C''|\lambda|^{-2m+1}, \end{aligned}$$

which gives for large $\lambda \in \Theta$

$$1 \leq 2C''|\lambda|^2\|\phi H_{2m-1}f\|.$$

Combining the above estimates implies (b) at once. □

We are ready now to conclude the proof of Theorem 1. Let us see first that for any integer $N \geq 1$ there are infinitely many resonances in $\{\lambda \in \mathbf{C} : \operatorname{Im} \lambda \leq |\lambda|^{-N}, |\operatorname{Re} \lambda| \geq 1\}$. Assume the contrary, i.e. $R_\chi(\lambda)$ is holomorphic in $\{\lambda \in \mathbf{C} : \operatorname{Im} \lambda \leq |\lambda|^{-N}, |\operatorname{Re} \lambda| \geq C_0\}$ for some constant $C_0 > 0$. Choose m so that $2m - 3 > N + 9$. Let f_j, λ_j be as in Propositions

2 and 3, and set $f(x,\lambda) = f_j(x)$, $\lambda \in \Theta := \{\lambda_j\}_{j=1}^{\infty}$. In view of Proposition 2(c) we can use (23) to obtain

$$N_{2m-1}f + \tilde{R}_{2m-1}f = \mathcal{N}(f + Qf + R_{2m-1}f), \quad \lambda \in \Theta.$$

Since $\Theta \cap \{\lambda;\ |\mathrm{Re}\,\lambda| \geq C_1\} \subset \{\lambda \in \mathbf{C} :\ |\mathrm{Im}\,\lambda| \leq |\lambda|^{-N-6},\ |\mathrm{Re}\,\lambda| \geq C_1\}$ for C_1 large enough, this estimate combined with (10) yields

$$\|f + Qf + R_{2m-1}f\|_{H^{3/2}} \leq C|\lambda|^{N+9}\|N_{2m-1}f + \tilde{R}_{2m-1}f\|_{H^{1/2}} \leq C'|\lambda|^{N+9-(2m-3)}.$$

On the other hand, in view of (19), (22),

$$1 = \|f\|_{H^{3/2}} \leq \|f + Qf + R_{2m-1}f\|_{H^{3/2}} + C|\lambda|^{-1},$$

therefore we get a contradiction for large $\lambda = \lambda_j$.

We will now choose our sequence of resonances by induction. Assume that we have already chosen $\lambda_1, ..., \lambda_{k-1}$. It follows from the above analysis that there exists a resonance λ_k satisfying $|\lambda_k| > |\lambda_{k-1}| + 1$ and

$$0 < \mathrm{Im}\,\lambda_k \leq |\lambda_k|^{-k}. \tag{51}$$

Thus we have an infinite sequence of different resonances $\{\lambda_k\}$ satisfying (51) for each $k \geq 1$. It is easy to see now that

$$0 < \mathrm{Im}\,\lambda_k \leq C_N|\lambda_k|^{-N}, \quad \forall k, \tag{52}$$

for any integer $N \geq 1$ with $C_N = |\lambda_N|^N$. Indeed, for $k \geq N$ (52) follows from (51) at once, while for $k \leq N$ we have

$$\mathrm{Im}\,\lambda_k \leq |\lambda_k|^{-k} \leq |\lambda_N|^N|\lambda_k|^{-N},$$

which completes the proof of (52), and hence of Theorem 1.

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