Stability of the Gauge Equivalent Classes in Inverse Stationary Transport in Refractive Media

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Abstract. In the inverse stationary transport problem through anisotropic attenuating, scattering, and refractive media, the albedo operator stably determines the gauge equivalent class of the attenuation and scattering coefficients.

1. Introduction

This paper concerns the problem of recovering the absorption and scattering properties of a refractive medium from boundary knowledge of the albedo operator. The medium $M \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with smooth boundary, endowed with a known Riemannian metric g. The free moving particles travel through M along the geodesics. In the stationary case the propagation of particles is modeled by the linear transport equation

$$-\mathcal{D}u(x,v) - a(x,v)u(x,v) + \int_{S_xM} k(x,v',v)u(x,v')\, d\omega_x(v') = 0. \tag{1.1}$$

In the equation above $u(x,v)$ denotes the density of particles at position x with velocity v in S_xM, the unit tangent sphere at x. The operator $\mathcal{D}$ is the derivative along the geodesic flow: For a given point $(x,v) \in S_xM$, if $\gamma_{(x,v)}(\cdot)$ denotes the geodesic starting at $\gamma_{(x,v)}(0) = x$ with initial velocity $\dot\gamma_{(x,v)}(0) = v$, then

$$\mathcal{D}u(x,v) := \frac{\partial}{\partial t}\Big|_{t=0} u(\vec\gamma_{(x,v)}(t)), \tag{1.2}$$

where, for brevity, we use the notation $\vec\gamma_{(x,v)}(t) = (\gamma_{(x,v)}(t), \dot\gamma_{(x,v)}(t))$. If g is Euclidean then $\mathcal{D}$ is the directional derivative: $\mathcal{D}u(x,v) = v \cdot \nabla_x u(x,v)$. The measure $d\omega_x(v')$ in (1.1) is the volume form on S_xM induced from the volume form on T_xM (the tangent space to M at x) determined by g at x. The resulting (Liouville) form on SM is preserved under the geodesic flow of g, see [**22**]. The attenuation coefficient $a(x,v)$ in (1.1) quantifies the rate at which particles are lost from the point (x,v) in phase space due to absorption and scattering into new directions. The scattering coefficient $k(x,v',v)$ is proportional to the probability that a particle at position x with velocity $v' \in S_xM$ will scatter to have new velocity $v \in S_xM$.

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The boundary measurements are described by the albedo operator $\mathcal{A}$: Let $\Gamma_\pm = \partial_\pm SM = \{(x,v) \in \partial SM \ : \ \pm\langle v, \nu_x\rangle > 0\}$ denote the "incoming" and "outgoing" bundles, where ν_x is the unit outer normal vector to the boundary ∂M at x and $\langle\cdot,\cdot\rangle$ is the inner product, each with respect to g at x. The medium is probed with the given radiation

$$u|_{\Gamma_-} = u_-, \tag{1.3}$$

and the exiting radiation is detected on Γ_+. The albedo operator takes the incoming flux to the outgoing flux at the boundary: $\mathcal{A}u_- = u|_{\Gamma_+}$.

The inverse boundary value problem is to determine the coefficients a and k from the knowledge of $\mathcal{A}$.

When the attenuation a is isotropic (v-independent), there is a large collection of uniqueness results under varying assumptions on the parameters; see [**2**] for a comprehensive account, and [**20**] for a recent survey of numerical methods. The works below are based on the singular decomposition of the Schwartz kernel of $\mathcal{A}$, an idea first introduced in [**6**] and [**7**]; see also [**5**]. In the Euclidean setting, uniqueness of $a(x)$ (dimensions two and above) and $k(x,v',v)$ (dimensions three and above) was proven in [**7**] under some minimal restrictions guaranteeing only that the forward problem is well-posed. For sufficiently small k this result was extended to dimension two in [**24**]; see also [**26, 27**]. Stability results are proven in [**24, 10, 21, 28**] with the most general result (in Euclidean geometry) in [**3**]. When no angular resolution is measured in the outgoing flux, the singular decomposition of the new boundary operator has been used to recover an isotropic coefficient a and the spatial part of k in [**10, 4**].

The case of a Euclidean metric corresponds to transport in materials with a constant index of refraction. If the index of refraction is isotropic, but varying, then (1.1) can be derived as a limiting case of Maxwell's equations with non-constant (but isotropic) permeability, resulting in a metric which is conformal to the Euclidean metric ([**1**]). For a general metric, we consider (1.1) as a model for transport in a medium with varying, anisotropic index of refraction. When the attenuation is assumed isotropic, uniqueness results in Euclidean geometry are extended to the Riemannian metric setting in [**11, 13, 14, 15**]. There and here, the manifold is assumed to be *simple* as follows.

Definition 1.1. (M,g) is called *simple* if it is strictly convex, and for any $x \in \overline{M}$ the exponential map $\exp_x : \exp_x^{-1}(\overline{M}) \to \overline{M}$ is a diffeomorphism. If M is two dimensional we have the following additional assumption: Let κ be the maximum sectional curvature of M. If $\kappa > 0$, then we also assume $\operatorname{diam}(M) < \pi/\sqrt{\kappa}$.

The works mentioned above concern the media with an isotropic attenuation character. However, since the attenuation is a combination of absorption and loss of particles due to scattering: $a(x,v) = \sigma(x,v) + \int_{S_xM} k(x,v,v')d\omega_x(v')$, even when the absorption part is isotropic ($\sigma = \sigma(x)$), if k depends on two independent directions the resulting attenuation is anisotropic. Evidence of anisotropy in biological tissue has been observed experimentally, see [**12**].

When the attenuation coefficient is anisotropic, it is possible to have media of differing attenuation and scattering properties which yield the same albedo operator. Moreover the non-uniqueness is characterized by the action of a gauge transformation [**23**]: see (1.4) below. The same algebraic structure of non-uniqueness is valid in refractive media [**16**].

In Theorems 4.1 and 4.2 we show the stability of the gauge equivalent classes occurring in refractive media, thus extending the results from the Euclidean case in [**17**]. We also generalize a stability result in [**3**] from Euclidean to Riemannian geometry.

The algebraic structure of non-uniqueness in [**23, 16**] can be readily observed. Indeed, if $\phi \in L^\infty(SM)$ is positive with $1/\phi \in L^\infty(SM)$, $\mathcal{D}\phi \in L^\infty(SM)$ and such that $\phi = 1$ on ∂SM. Set

$$\tilde{a}(x,v) = a(x,v) - \mathcal{D}\log\phi(x,v), \quad \tilde{k}(x,v',v) = \frac{k(x,v',v)\phi(x,v)}{\phi(x,v')}. \tag{1.4}$$

Then u satisfies (1.1) if and only if $\tilde{u} = \phi u$ solves

$$-\mathcal{D}\tilde{u}(x,v) - \tilde{a}(x,v)\tilde{u}(x,v) + \int_{S_xM} \tilde{k}(x,v',v)\tilde{u}(x,v')\, d\omega_x(v') = 0.$$

Since $\phi = 1$ on Γ, $u = \tilde{u}$ there, so the albedo operator $\mathcal{A}$ for the parameters (a,k) is indistinguishable from the albedo operator $\tilde{\mathcal{A}}$ for the pair $(\tilde{a},\tilde{k})$, i.e. $\mathcal{A} = \tilde{\mathcal{A}}$. This motivates the following definition in [**23, 16**].

Definition 1.2. Two pairs of coefficients (a,k) and $(\tilde{a},\tilde{k})$ are called *gauge equivalent* if there exists a positive map $\phi \in L^\infty(SM)$ with $1/\phi \in L^\infty(SM)$, $\mathcal{D}\phi \in L^\infty(SM)$, and $\phi = 1$ on Γ, such that (1.4) holds. We denote this equivalence by $(a,k) \sim (\tilde{a},\tilde{k})$.

The relation defined above is reflexive since $(a,k) \sim (a,k)$ via $\phi \equiv 1$; it is symmetric since $(a,k) \sim (\tilde{a},\tilde{k})$ via ϕ yields $(\tilde{a},\tilde{k}) \sim (a,k)$ via $1/\phi$; and it is transitive since if $(a,k) \sim (\tilde{a},\tilde{k})$ via ϕ and $(\tilde{a},\tilde{k}) \sim (a',k')$ via $\tilde{\phi}$ then $(a,k) \sim (a',k')$ via $\phi\tilde{\phi}$. Therefore one has the multiplicative group of gauges acting transitively (since any equivalent pair are related by some gauge ϕ [**23, 16**]) on the equivalent class of a pair of coefficients. We denote the equivalence class of (a,k) by $\langle a,k\rangle$.

2. Transport of the data to a larger domain

Due to the method of proof, the total travel time of each particle in M has to be uniformly bounded away from zero. This can be done without loss of generality by doing the measurements away from the boundary ∂M. More precisely, let M_0 be a slightly larger domain strictly containing M. The metric g can be extended to g_0 on M_0 in such a way that (M_0, g_0) still remains simple [**25**].

As in [**17**], we reduce the problem in M to one in M_0: Let (a,k) and $(\tilde{a},\tilde{k})$ be coefficients for which the forward problems in (M,g) are well-posed, and $\mathcal{A}$ and $\tilde{\mathcal{A}}$ be their corresponding albedo operators. Defining $a = \tilde{a} = k = \tilde{k} = 0$ in $M_0 \setminus M$, the forward problems in (M_0, g_0) are also well-posed and the albedo operators $\mathcal{A}_0$ and $\tilde{\mathcal{A}}_0$ are well defined maps between functions on

$$\Gamma^0_\pm := \partial_\pm SM_0 = \{(x,v) \in \partial SM_0 \;:\; \pm\langle v, \nu_x\rangle > 0\}$$

(now ν_x is the outer unit normal vector to ∂M_0 at x, with respect to the extended metric g_0). As in [**17**], when the two forward problems for M are well-posed in L^p, $1 \le p \le \infty$, the following isometric property holds:

$$\|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^p(\Gamma_-,d\mu);L^p(\Gamma_+,d\mu))} = \|\mathcal{A}_0 - \tilde{\mathcal{A}}_0\|_{\mathcal{L}(L^p(\Gamma^0_-,d\mu_0);L^p(\Gamma^0_+,d\mu_0))}. \tag{2.1}$$

The measure $d\mu$ (and, analogously, $d\mu_0$) in (2.1) is defined as follows: Let $d\Sigma^{2n-2}$ be the volume form on $\Gamma_\pm$ obtained by the natural restriction of the volume form on SM to $\Gamma_\pm$. Then, by extending $d\Sigma^{2n-2}$ as a homogeneous form of order $n-1$ in $|v'|$, we have that $d|v'|\,d\Sigma^{2n-2}(x',v')$ coincides with the volume form on SM; see [**22**] for details. We define

$$d\mu(x',v') = |\langle v',\nu_{x'}\rangle|\, d\Sigma^{n-2}(x',v'). \tag{2.2}$$

Remark: The proof of (2.1) is essentially identical to that in the Euclidean case as presented in [**17**] due to the following invariant property of the form $d\mu$: Fix $(x_0',v_0') \in \Gamma_\pm$. Let $\partial\tilde{\Omega}$ be any surface so that the geodesic issued from (x_0',v_0') hits it transversally. Then the geodesic flow defines a natural local "projection" near (x_0',v_0'), of $\Gamma_\pm$ onto $\tilde{\Gamma}_\pm$. Let $d\tilde{\mu}$ be the measure on $\tilde{\Gamma}_\pm$ defined analogously to (2.2). Then the pull back of $d\tilde{\mu}$ is $d\mu$ (see the proof of Lemma 4.2.2 in [**22**]).
Given $(x,v)\in SM$, we define the "forward/backward travel time to the boundary" functions

$$\tau_\pm(x,v) = \min\{t\geq 0 \,:\, \gamma_{(x,v)}(\pm t)\in \partial M_0\},$$

and let $\tau(x,v) = \tau_+(x,v)+\tau_-(x,v)$ be the total travel time of the free particle (of velocity v at x) through M_0. Since $\mathrm{dist}_{g_0}(\overline{M},\partial M_0)>0$, we have

$$c_0 := \inf\{\tau(x,v) \,:\, (x,v)\in \overline{SM}\} > 0. \tag{2.3}$$

Using the isometry property (2.1), we can considered the inverse problem in the larger domain (M_0,g_0) with the albedo operators now acting between $\Gamma_\pm^0$. Equivalently, for the original problem in (M,g) we may work without loss of generality with coefficients a,k of (a priori fixed) compact support in M.

To simplify notation, while still working in the larger domain, we drop the 0 index throughout the remaining of the paper: thus M_0 becomes M, $\Gamma_\pm^0$ becomes $\Gamma_\pm$, etc.

3. The singular structure of the albedo operator's kernel

In this section we recall the singular decomposition of the Schwartz kernel of the albedo operator for the two cases separated by dimension.

We work within the class of *admissible* coefficients: For $n\geq 3$

$$(a,k)\in L^\infty(SM)\times L^\infty(SM, L^1(S_xM)), \tag{3.1}$$

and for $n=2$

$$(a,k)\in L^\infty(SM)\times L^\infty(S^2M), \tag{3.2}$$

where $S^2M := \{(x,v',v) \,:\, x\in M,\ v',v\in S_xM\}$. Note that the gauge transformations (1.4) preserve the admissible classes in (3.1) and (3.2).

Moreover, either one of the following *subcritical* conditions that yield well-posedness for the boundary value problem (1.1) and (1.3) is assumed to hold:

$$\operatorname{ess\,sup}_{(x,v)\in SM}\left|\tau(x,v)\int_{S_xM} k(x,v,v')\,d\omega_x(v')\right| < 1, \tag{3.3}$$

or

$$a(x,v) - \int_{S_xM} k(x,v,v')\,d\omega_x(v') \geq 0, \quad a.e.\ (x,v)\in SM; \tag{3.4}$$

see, e.g., [**3, 7, 8, 18, 19**].

The right hand side of (1.1) defines a closed, unbounded operator on $L^1(SM_R)$ with the domain $\{u\in L^1(SM_R) \,:\, \mathcal{D}u\in L^1(SM_R),\ u|_{\Gamma_-}=0\}$; see [**7, 16**].

Proposition 3.1 below, which describes the terms in the expansion of the kernel of $\mathcal{A}$, is proven in [**13**], the Euclidean equivalent appearing in [**7**]. We denote by $\delta_{\{x',v'\}}(x,v)$ the delta-distribution on Γ_+ with respect to the measure $d\mu$ defined by

$$\int_{\Gamma_+} \varphi(x,v)\delta_{\{x',v'\}}(x,v)\, d\mu(x,v) = \varphi(x',v'), \qquad \varphi \in C_c^\infty(\Gamma_+).$$

Similarly, $\delta_{\{x\}}(y)$ is the delta distribution on M supported at x.

If $x, y \in M$ let $v(x,y) \in S_xM$ denote the tangent vector at x of the unit speed geodesic joining x to y (uniquely defined since M is simple), and $d(x,y)$ be the Riemannian distance between x and y. Denote the total attenuation along the geodesic from x to y by

$$E(x,y) := \exp\Big\{-\int_0^{d(x,y)} a\big(\vec{\gamma}_{(x,v(x,y))}(t)\big)\, dt\Big\}. \tag{3.5}$$

Note that $\dot\gamma_{(y,v(y,x))}(d(y,x)-s) = -\dot\gamma_{(x,v(x,y))}(s)$, so when a depends on direction, $E(x,y) \neq E(y,x)$.

Proposition 3.1. [**13**] Let (M,g) be a smooth simple Riemannian manifold of dimension $n \geq 3$. Assume that (a,k) are admissible and subcritical so that the forward problem is well-posed. Then the albedo operator $\mathcal{A} : L^1(\Gamma_-, d\mu) \to L^1(\Gamma_+, d\mu)$ is bounded and its Schwartz kernel $\alpha(x,v,x',v')$, considered as a distribution on Γ_+ parameterized by $(x',v') \in \Gamma_-$, has the expansion $\alpha = \alpha_0 + \alpha_1 + \alpha_2$, where

$$\alpha_0 = E\big(\gamma_{(x,v)}(-\tau_-(x,v)), x\big)\delta_{\{\vec{\gamma}_{(x',v')}(\tau(x',v'))\}}(x,v), \tag{3.6}$$

$$\alpha_1 = \int_0^{\tau_+(x',v')}\int_0^{\tau_-(x,v)} E(y,x)E(x',z)k(z,\dot z,\dot y)\delta_{\{y\}}(z)\, ds\, dt \tag{3.7}$$

$$y = y(s) = \gamma_{(x,v)}(s-\tau_-(x,v)),\ z = z(t) = \gamma_{(x',v')}(t),$$

$$\alpha_2 \in L^\infty(\Gamma_-; L^1(\Gamma_+, d\mu)). \tag{3.8}$$

We note that $k(z(t),\dot z(t),\dot y(s))$ is only defined on the support of the integrand, namely when $y(s) = z(r)$.

When $n = 2$ the left hand side of (1.1) defines an unbounded operator of domain $\{u \in L^\infty(SM) : \mathcal{D}u \in L^\infty(SM),\ u|_{\Gamma_-} = 0\}$. Provided that (3.2) holds, and we have subcriticality (3.3) or (3.4), this operator has a bounded inverse in $L^\infty(SM)$, see [**14**], and the singular decomposition of the albedo kernel is more explicit as follows.

Given $(x,v,x',v') \in \Gamma_+ \times \Gamma_-$, define $\chi : \Gamma_+ \times \Gamma_- \to \{0,1\}$ by $\chi(x,v,x',v') = 1$ if there exist $0 \leq s = s(x,v,x',v') \leq \tau_-(x,v)$ and $0 \leq t = t(x,v,x',v') \leq \tau_+(x',v')$ such that $\gamma_{(x,v)}(s-\tau_-(x,v)) = \gamma_{(x',v')}(t)$ (i.e., the geodesics intersect in M), and $\chi(x,v,x',v') = 0$ otherwise. When $\chi(x,v,x',v') = 1$, let $\psi(x,v,x',v')$ be the angle $\arccos\langle\dot\gamma_{(x,v)}(s-\tau_-(x,v)), \dot\gamma_{(x',v')}(t)\rangle$ between the tangent vectors of these geodesics at the point of intersection.

Proposition 3.2. [**14**] Let (M,g) be a two dimensional simple Riemannian manifold. Assume that (a,k) are admissible and that (3.2) holds. Then the albedo operator $\mathcal{A} : L^\infty(\Gamma_-, d\mu) \to L^\infty(\Gamma_+, d\mu)$ is bounded and its Schwartz kernel $\alpha(x,v,x',v')$, considered as a distribution on Γ_+ parameterized by $(x',v') \in \Gamma_-$, has the expansion $\alpha = \alpha_0 + \alpha_1 + \alpha_2$, where

$$\alpha_0 = E\big(\gamma_{(x,v)}(-\tau_-(x,v)), x\big)\delta_{\{\vec{\gamma}_{(x',v')}(\tau(x',v'))\}}(x,v),$$

$$\alpha_1 = \chi(x,v,x',v')E(x',\gamma_{(x',v')}(t))E(\gamma_{(x',v')}(t),x)\mathcal{J}\frac{k\big(\vec{\gamma}_{(x',v')}(t), \dot{\gamma}_{(x,v)}(s-\tau_-(x,v))\big)}{|\sin(\psi(x,v,x',v'))|},$$

$$0 \le \alpha_2\chi \le C\|k\|^2_{L^\infty(S^2M)}\Big(1+\log\frac{1}{|\sin(\psi(x,v,x',v'))|}\Big).$$

Here, $\mathcal{J} = \mathcal{J}(x,v,x',v')$ is a function uniformly bounded $0 < m_1 \le \mathcal{J} \le m_2 < \infty$ on $\Gamma_+ \times \Gamma_-$ (see [**14**, Proposition 4]).

4. Statement of the main results

Let $(B_a, \|\cdot\|_{B_a})$ and $(B_k, \|\cdot\|_{B_k})$ be Banach spaces in which the attenuation and, respectively, the scattering kernel are considered, $(a,k), (\tilde{a},\tilde{k}) \in B_a \times B_k$. The *distance* Δ between equivalence classes *with respect to* $B_a \times B_k$ is defined by the infimum of the distances between all possible pairs of representatives:

$$\Delta(\langle a,k\rangle, \langle\tilde{a},\tilde{k}\rangle) := \inf_{(a',k')\in\langle a,k\rangle,\ (\tilde{a}',\tilde{k}')\in\langle\tilde{a},\tilde{k}\rangle} \max\{\|a'-\tilde{a}'\|_{B_a}, \|k'-\tilde{k}'\|_{B_k}\}.$$

The following norms are used throughout

$$\begin{aligned}
\|a\|_\infty &= \operatorname{ess\,sup}_{(x,v)\in SM}|a(x,v)|,\\
\|k\|_{\infty,1} &= \operatorname{ess\,sup}_{(x,v')\in SM}\int_{S_xM}|k(x,v',v)|\,d\omega_x(v),\\
\|k\|_\infty &= \operatorname{ess\,sup}_{(x,v',v)\in S^2M}|k(x,v',v)|\,,\\
\|k\|_1 &= \int_M\int_{S_xM}\int_{S_xM}|k(x,v',v)|\,dx\,d\omega_x(v')\,d\omega_x(v).
\end{aligned}$$

Case $n \ge 3$**:** Define the class

$$U_{\Sigma,\rho} := \{(a,k) \text{ as in (3.1)} \ :\ \|a\|_\infty \le \Sigma,\ \|k\|_{\infty,1} \le \rho\}.$$

Theorem 4.1. *Let (M,g) be simple Riemannian manifold of dimension $n \ge 3$. Let $(a,k), (\tilde{a},\tilde{k}) \in U_{\Sigma,\rho}$ be such that the corresponding forward problems are well posed. Then*

$$\Delta(\langle a,k\rangle, \langle\tilde{a},\tilde{k}\rangle) \le C\|\mathcal{A}-\tilde{\mathcal{A}}\|_{\mathcal{L}(L^1(\Gamma_-,d\mu);L^1(\Gamma_+,d\mu))},$$

where Δ is with respect to $L^\infty(SM)\times L^1(S^2M)$, and C is a constant depending only on Σ, ρ, n, and c_0 in (2.3). More precisely, there exists a representative $(a',k') \in \langle a,k\rangle$ such that

$$\|a'-\tilde{a}\|_\infty \le C\|\mathcal{A}-\tilde{\mathcal{A}}\|_{\mathcal{L}(L^1(\Gamma_-,d\mu);L^1(\Gamma_+,d\mu))}, \tag{4.1}$$

$$\|k'-\tilde{k}\|_1 \le C\|\mathcal{A}-\tilde{\mathcal{A}}\|_{\mathcal{L}(L^1(\Gamma_-,d\mu);L^1(\Gamma_+,d\mu))}. \tag{4.2}$$

Case $n = 2$**:** From Proposition 3.2 above recall the Schwartz kernel of the albedo operator in the form

$$\alpha = A_0(x,v)\delta_{\{\vec{\gamma}_{(x',v')}(\tau(x',v'))\}}(x,v) + \beta$$

where

$$\begin{aligned}
&A_0(x,v) = E\big(\gamma_{(x,v)}(-\tau_-(x,v)), x\big) \in L^\infty(\Gamma_+)\\
&\beta(x,v,x',v')\chi|\sin\psi(x,v,x',v')| \in L^\infty(\Gamma_+^R \times \Gamma_-^R).
\end{aligned}$$

We define

$$\|\mathcal{A}\|_* = \max\{\|A_0\|_\infty, \|\beta\chi|\sin\psi|\|_\infty\}. \tag{4.3}$$

By using the Remark in Section 2, the proof in [**17**] carries through verbatim to show that $\|\mathcal{A}-\tilde{\mathcal{A}}\|_*$ is preserved when transported from the boundary of the inner domain M to the boundary of the larger domain.

Define the class

$$V_{\Sigma,\rho} := \big\{(a,k) \text{ as in (3.2) } : \ \|a\|_\infty \le \Sigma, \ \|k\|_\infty \le \rho\big\}.$$

THEOREM 4.2. *Let (M,g) be a two dimensional simple Riemannian manifold. For any $\Sigma > 0$ there exists $\rho > 0$ depending only on Σ and (M,g) such that the following holds: if (a,k), $(\tilde{a},\tilde{k}) \in V_{\Sigma,\rho}$ then*

$$\Delta(\langle a,k\rangle, \langle \tilde{a},\tilde{k}\rangle) \le C\|\mathcal{A}-\tilde{\mathcal{A}}\|_*$$

where Δ is with respect to $L^\infty(SM)\times L^\infty(S^2M)$ and C is a constant depending only on Σ and (M,g).

Note that ρ sufficiently small already yields a subcritical regime as in (3.3).

REMARK 4.3. If a is isotropic (dependent on x only), Theorem 4.1 remains true, of course. On the other hand, within the class of isotropic a, there is uniqueness under the simplicity assumption (definition 1.1), based on the inversion of the geodesic X-ray transform. It is interesting to see whether one can get stability for isotropic a without the gauge invariance. This can indeed be done, along the following lines. Estimate (5.3) implies an estimate of the form $\|I_g(a-\tilde{a})\|_{L^\infty(\Gamma_-)} \le C\|\mathcal{A}-\tilde{\mathcal{A}}\|$, where I_g denotes the geodesic X-ray transform. The geodesic X-ray transform on simple manifolds is invertible [**22**], and stability estimates are known in various norms, see [**22, 25**]. Using interpolation as in [**3, 28**], one can estimate $\|a-\tilde{a}\|_{L^2}$ in terms of a fractional power of $\|\mathcal{A}-\tilde{\mathcal{A}}\|$, under the *a-priori* assumptions as in Theorem 4.1. The proof for stability of the recovery of k, without gauge invariance, is similar to the proof in the theorem.

5. Preliminary estimates

In this section we extend a result from the Euclidean to Riemannian setting; see [**3**, Theorem 3.2] for contrast. In addition, the proof below allows for discontinuous coefficients, which is needed when transporting the albedo operator to the larger domain (and no boundary knowledge of the coefficients is available).

LEMMA 5.1. *There is a family of maps $\phi_{\varepsilon,x_0',v_0'} \in L^1(\Gamma_-, d\mu)$, for $(x_0',v_0') \in \Gamma_-$ and $\varepsilon > 0$, such that $\|\phi_{\varepsilon,x_0',v_0'}\|_{L^1(\Gamma_-,d\mu)} = 1$ and, for any $f \in L^\infty(\Gamma_-, d\mu)$ given,*

$$\lim_{\varepsilon\to 0}\int_{\Gamma_-} \phi_{\varepsilon,x_0',v_0'}(x',v')f(x',v')\,d\mu(x',v') = f(x_0',v_0'), \tag{5.1}$$

whenever (x_0',v_0') is in the Lebesgue set of f. In particular, (5.1) *holds for almost every $(x_0',v_0') \in \Gamma_-$.*

For a measurable function f on $\mathbb{R}^n$, The Lebesgue set of f is

$$L_f := \Big\{x \ : \ \lim_{r\to 0}\frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)-f(x)|\,dy = 0\Big\}$$

where $B_r(x)$ is the ball of radius r centered at x, and where $|\cdot|$ denotes Lebesgue measure. We point out that if $\in L^1_{loc}(\mathbb{R}^n)$, then $|\mathbb{R}^n \setminus L_f| = 0$ ([**9**, Theorem 3.20]).

PROOF. For $(x_0', v_0') \in \Gamma_-$ and $\varepsilon > 0$ sufficiently small, let $(x', v') : U \times W \subset \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to \partial SM$ with $x'(U) \subset \partial M$ be a coordinate chart near $(x_0', v_0') = (x'(0), v'(0))$. Let $d\Sigma^{2n-2}(x', v') = \sqrt{g_-}\, du\, dw$ be the local coordinate expression for the volume element (see (2.2)). For $(x', v') \in \Gamma_-$, define

$$\phi_{\varepsilon, x_0', v_0'}(x', v') = \frac{1}{|\langle \nu_{x'}, v'\rangle|} \frac{1}{\sqrt{g_-(x', v')}} \varphi_\varepsilon(u(x')) \varphi_\varepsilon(w(x', v')),$$

where $\varphi(u) \equiv 1/(\omega_{n-1})$ for $|u| < 1$, $\varphi(u) \equiv 0$ for $|u| \geq 1$, and $\varphi_\varepsilon(u) = \varepsilon^{-n+1}\varphi(u/\varepsilon)$. By ω_{n-1} we denoted the volume of the unit ball in R^{n-1}. Then, for any $\varepsilon > 0$, $\int \varphi_\varepsilon(u) du = 1$ and, by using (2.2), we obtain

$$\begin{aligned}
&\int_{\Gamma_-} \phi_{\varepsilon, x_0', v_0'}(x', v') f(x', v')\, d\mu(x', v') \\
&\qquad = \int_{\Gamma_-} \varphi_\varepsilon(u(x')) \varphi_\varepsilon(w(x', v')) f(x', v') \frac{1}{\sqrt{g_-(x', v')}}\, d\Sigma^{n-2}(x', v') \\
&\qquad = \int_{\mathbb{R}^{2n-2}} \varphi_\varepsilon(u) \varphi_\varepsilon(w) f(x'(u), v'(u, w))\, du\, dw.
\end{aligned}$$

Apply the equality above to $f \equiv 1$ to get $\|\phi_{\varepsilon, x_0', v_0'}\|_{L^1(\Gamma_+, d\mu)} = 1$. The conclusion follows from the approximation of identity for L^p maps, see, e.g., [**9**, Theorem 8.15]. $\square$

Consider $F : \partial M \times SM \to \mathbb{R}$ defined by

$$F(x', y, w) = E(x', y) E\big(y, \gamma_{(y,w)}(\tau_+(y, w))\big), \tag{5.2}$$

with E as in (3.5). $F(x', y, w)$ represents the total attenuation along the broken geodesics from $x' \to y \to \gamma_{(y,w)}(\tau_+(y, w))$.

Let $(a, k), (\tilde{a}, \tilde{k})$ be admissible pairs as in (3.1). Recall that, after the extension of the domain, the coefficients have fixed compact support away from the boundary ∂M. All the operators bearing the tilde refer to $(\tilde{a}, \tilde{k})$ and are defined in a similar way to the ones for (a, k), i.e., $\tilde{\mathcal{A}}$ is the albedo operator corresponding to $(\tilde{a}, \tilde{k})$. Recall that n is the dimension of the space. To simplify notation let

$$\|\mathcal{A} - \tilde{\mathcal{A}}\| := \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1(\Gamma_-, d\mu); L^1(\Gamma_+, d\mu))}.$$

THEOREM 5.2. *Let $(a, k), (\tilde{a}, \tilde{k})$ be as in (3.1). For almost every $(x_0', v_0') \in \Gamma_-$ the following estimates hold: For $n \geq 2$,*

$$\left| e^{-\int_0^{\tau_+(x_0', v_0')} a(\vec{\gamma}_{(x_0', v_0')}(s))\, ds} - e^{-\int_0^{\tau_+(x_0', v_0')} \tilde{a}(\vec{\gamma}_{(x_0', v_0')}(s))\, ds} \right| \leq \|\mathcal{A} - \tilde{\mathcal{A}}\|. \tag{5.3}$$

For $n \geq 3$, with $y(t) = \gamma_{(x_0', v_0')}(t)$,

$$\begin{aligned}
&\int_0^{\tau_+(x_0', v_0')} \int_{S_{y(t)}M} |k - \tilde{k}|(y(t), \dot{y}(t), w) F(x_0', y(t), w)\, d\omega_y(w)\, dt \\
&\qquad \leq \|\mathcal{A} - \tilde{\mathcal{A}}\| + \|F - \tilde{F}\|_\infty \int_0^{\tau_+(x_0', v_0')} \int_{S_{y(t)}M} \tilde{k}(y(t), \dot{y}(t), w)\, d\omega_y(w)\, dt.
\end{aligned} \tag{5.4}$$

Proof. Let $(x_0', v_0') \in \Gamma_-$ be arbitrarily fixed and let $\phi_{\varepsilon,x_0',v_0'} \in L^1(\Gamma_-)$ be defined as in Corollary 5.1. To simplify the formulas, since (x_0', v_0') is fixed, in the following we drop this dependence from the notation $\phi_\varepsilon = \phi_{\varepsilon,x_0',v_0'}$.

Let $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2$ be the decomposition of the albedo operator given by

$$\mathcal{A}_i f(x,v) = \int_{\Gamma_-} \alpha_i(x,v,x',v') f(x',v')\, d\mu(x',v'), \quad i = 0,1,2,$$

where α_i, $i = 0,1,2$ are the Schwartz kernels in Proposition 3.1.

Let $\phi \in L^\infty(\Gamma_+)$ with $\|\phi\|_\infty \le 1$. Since $\|\phi_\varepsilon\|_{L^1(\Gamma_-)} = 1$, the mapping properties of the albedo operator imply that

$$\left| \int_{\Gamma_+} \phi(x,v)[\mathcal{A} - \tilde{\mathcal{A}}]\phi_\varepsilon(x,v)\, d\mu(x,v) \right| \le \|\mathcal{A} - \tilde{\mathcal{A}}\|. \tag{5.5}$$

Next we evaluate each of the three terms in $\int_{\Gamma_+} \phi(x,v)[\mathcal{A} - \tilde{\mathcal{A}}]\phi_\varepsilon(x,v)\, d\mu(x,v)$ by using the decomposition in Proposition 3.1 and Fubini's theorem.

The first term is evaluated using the formula (3.6):

$$\begin{aligned} I_0(\phi,\varepsilon) &:= \int_{\Gamma_+} \phi(x,v)[\mathcal{A}_0 - \tilde{\mathcal{A}}_0]\phi_\varepsilon(x,v)\, d\mu(x,v) \\ &= \int_{\Gamma_-} \phi\big(\vec{\gamma}_{(x',v')}(\tau_+(x',v'))\big)\phi_\varepsilon(x',v') \\ &\qquad \times \Big[e^{-\int_0^{\tau_+(x',v')} a(\vec{\gamma}_{(x',v')}(s))\, ds} - e^{-\int_0^{\tau_+(x',v')} \tilde{a}(\vec{\gamma}_{(x',v')}(s))\, ds} \Big]\, d\mu(x',v'). \end{aligned}$$

Since the integrand above is in $L^\infty(\Gamma_-)$ by applying (5.1), we get for almost every $(x_0', v_0') \in \Gamma_-$

$$\begin{aligned} I_0(\phi)(x_0',v_0') &:= \lim_{\varepsilon\to 0} I_0(\phi,\varepsilon) = \phi\big(\vec{\gamma}_{(x_0',v_0')}(\tau_+(x_0',v_0'))\big) \\ &\qquad \times \Big(e^{-\int_0^{\tau_+(x_0',v_0')} a(\vec{\gamma}_{(x_0',v_0')}(s))\, ds} - e^{-\int_0^{\tau_+(x_0',v_0')} \tilde{a}(\vec{\gamma}_{(x_0',v_0')}(s))\, ds} \Big). \end{aligned} \tag{5.6}$$

To evaluate the second term we use the formula (3.7) and let $y = y(x',v',t) = \gamma_{(x',v')}(t)$:

$$\begin{aligned} I_1(\phi,\varepsilon) &:= \int_{\Gamma_+} \phi(x,v)[\mathcal{A}_1 - \tilde{\mathcal{A}}_1]\phi_\varepsilon(x,v)\, d\mu(x,v) \\ &= \int_{\Gamma_-} \phi_\varepsilon(x',v')\, d\mu(x',v') \Big\{ \int_0^{\tau_+(x',v')} \int_{S_yM} \phi\big(\vec{\gamma}_{(y,w)}(\tau_+(y,w))\big) \\ &\qquad \times \Big[F(x',y,w)k(y,\dot{y},w) - \tilde{F}(x',y,w)\tilde{k}(y,\dot{y},w) \Big]\, dw\, dt \Big\}. \end{aligned}$$

Apply again (5.1) for the continuous integrand above to obtain for almost every $(x_0', v_0') \in \Gamma_-$: with $y = y(x_0', v_0', t)$

$$\begin{aligned} I_1(\phi)(x_0',v_0') &:= \lim_{\varepsilon\to 0} I_1(\phi,\varepsilon) \\ &= \int_0^{\tau_+(x_0',v_0')} \int_{S_yM} \phi\big(\vec{\gamma}_{(y,w)}(\tau_+(y,w))\big) \\ &\qquad \times \big[F(x_0',y,w)k(y,\dot{y},w) - \tilde{F}(x_0',y,w)\tilde{k}(y,\dot{y},w) \big]\, d\omega_y(w)\, dt, \end{aligned} \tag{5.7}$$

or $I_1(\phi) = I_{1,1}(\phi) + I_{1,2}(\phi)$ with

(5.8)
$$I_{1,1}(\phi) = \int_0^{\tau_+(x_0',v_0')} \int_{S_yM} \phi\big(\vec{\gamma}_{(y,w)}(\tau_+(y,w))\big) F(x_0',y,w)(k-\tilde{k})(y,\dot{y},w)\, d\omega_y(w)\, dt,$$

(5.9)
$$|I_{1,2}(\phi)| \le \int_0^{\tau_+(x_0',v_0')} \int_{S_yM} |F-\tilde{F}|(x_0',y,w)\tilde{k}(y,\dot{y},w)\, d\omega_y(w)\, dt.$$

Consider the third term

$$\begin{aligned} I_2(\phi,\varepsilon) &= \int_{\Gamma_+} \phi(x,v)[\mathcal{A}_2 - \tilde{\mathcal{A}}_2]\phi_\varepsilon(x,v)\, d\mu(x,v) \\ &= \int_{\Gamma_-} \phi_\varepsilon(x',v')\, d\mu(x',v') \Big\{ \int_{\Gamma_+} \phi(x,v)(\alpha_2-\tilde{\alpha}_2)(x,v,x',v')\, d\mu(x,v) \Big\}. \end{aligned}$$

By (3.8), the map $(x',v') \mapsto \int_{\Gamma_+} \phi(x,v)(\alpha_2-\tilde{\alpha}_2)(x,v,x',v')\, d\mu(x,v)$ is in $L^\infty(\Gamma_-)$, and then, by (5.1), we get for almost every $(x_0',v_0') \in \Gamma_-$

$$I_2(\phi)(x_0',v_0') := \lim_{\varepsilon\to 0} I_2(\phi,\varepsilon) = \int_{\Gamma_+} \phi(x,v)(\alpha_2-\tilde{\alpha}_2)(x,v,x_0',v_0')\, d\mu(x,v). \tag{5.10}$$

The left hand side of (5.5) has three terms. We move the third term to the right hand side (with absolute values) and take the limit with $\varepsilon \to 0$ to get

$$|I_0(\phi) + I_1(\phi)|(x_0',v_0') \le \|\mathcal{A} - \tilde{\mathcal{A}}\| + I_2(|\phi|)(x_0',v_0'),\ \textit{a.e. } (x_0',v_0') \in \Gamma_-, \tag{5.11}$$

for any $\phi \in L^\infty(\Gamma_+)$ with $\|\phi\|_\infty = 1$.

We note that the negligible set on which the inequality above does not hold may depend on ϕ. We will consider a countable sequence of functions ϕ, and since the countable union of negligible sets is negligible, the inequality (5.11) holds almost everywhere on Γ_-, independently of the term in the sequence. This justifies the argument below for almost every (x_0',v_0') in Γ_-.

In (5.11), we shall choose two sequences of ϕ to conclude the two estimates of the lemma. First we show the estimate (5.3) by choosing $\phi_m \in L^\infty(\Gamma_+)$ which are 1 in a shrinking neighborhood of $(x_0,v_0) := \vec{\gamma}_{(x_0',v_0')}(\tau_+(x_0',v_0'))$. First, define $\phi_m(x_0,v)$ to be the indicator function for the set $\{v \in S_{x_0}M : \|v-v_0\|_{g(x_0)} < 1/m\}$, then extend ϕ_m by

$$\phi_m(x,v) = \begin{cases} 0 & \text{if } d_{\partial M}(x,x_0) \ge 1/m, \\ \phi_m(x,\mathcal{P}(v;x,x_0)) & \text{if } d_{\partial M}(x,x_0) < 1/m \end{cases}$$

where $\mathcal{P}(v;x,x_0)$ is the parallel transport of $v \in S_xM$ from x to x_0. Then (5.6) gives

$$I_0(\phi_m) = e^{-\int_0^{\tau_+(x_0',v_0')} a(\vec{\gamma}_{(x_0',v_0')}(s))\, ds} - e^{-\int_0^{\tau_+(x_0',v_0')} \tilde{a}(\vec{\gamma}_{(x_0',v_0')}(s))\, ds}$$

independently of m. From (5.7) we have $\lim_{m\to\infty} I_1(\phi_m) = 0$ since for any t, the support of $\phi\big(\vec{\gamma}_{(y(t),w)}(\tau_+(y,w))\big)$ in $w \in S_{y(t)}M$ shrinks to $\dot{y}(t)$. From (5.10) we also have $\lim_{m\to\infty} I_2(|\phi_m|) = 0$, since the support shrinks to one point. We use here the corollary of (3.8) that $(\alpha_2-\tilde{\alpha}_2)(\cdot,\cdot,x_0',v_0') \in L^1(\Gamma_+)$, for a.e. $(x_0',v_0') \in \Gamma_-$.

Next we prove the estimate (5.4). Recall that now $n \ge 3$. For $m > 0$, let $N_{(x_0',v_0'),q} \subset \overline{M}$ be the tubular neighborhood of the geodesic $y(t) = \gamma_{(x_0',v_0')}(t)$,

$0 \le t \le \tau_+(x_0', v_0')$, of radius $1/m$. We now define a sequence $\phi_m \in L^\infty(\Gamma_+)$. Set $\phi_m(x,v) = 0$ if $x \in N_{(x_0',v_0'),q}$; note that $I_0(\phi_m) = 0$ for all m. For $(x,v) \in \Gamma_+$ with $x \notin N_{(x_0',v_0'),q}$, $\phi_q(x,v) = 0$ if the geodesic $z(s) = \gamma_{(x,v)}(s)$, $-\tau_-(x,v) \le s \le 0$ does not intersect $N_{(x_0',v_0'),q}$. When $\gamma_{(x,v)}(\cdot)$ does intersect $N_{(x_0',v_0'),q}$, let $0 \le t(x,v) \le \tau_+(x_0',v_0')$ and $-\tau_-(x,v) \le s(x,v) \le 0$ be such that

$$d_g\big(y(t(x,v)), z(s(x,v))\big) = \min_{s,t}\{d_g(y(t), z(s)\}$$

and define

$$\phi_m(x,v) = \operatorname{sgn}(k-\tilde{k})\big(y(t(x,v)), \dot{y}(t(x,v)), \mathcal{P}\big(\dot{z}(s(x,v)); z(s(x,v)), y(t(x,v))\big)\big).$$

Notice that when (x,v) is of the form $\vec{\gamma}_{(y(t),w)}(\tau_+(y(t),w))$, $w \in S_{y(t)}M$, that is (x_0',v_0') and (x,v) are the beginning and end of a single-scattering broken geodesic, $\phi_m(x,v)$ takes the sign of $k-\tilde{k}$ at the point of scattering. Note also that the support of ϕ_m shrinks to a negligible set in Γ_+ as $m \to \infty$ since $n \ge 3$.

Now apply the estimate (5.11) to ϕ_m and use $I_0(\phi_m) = 0$ to get

$$|I_{1,1}(\phi_m)|(x_0',v_0') \le \|\mathcal{A} - \tilde{\mathcal{A}}\| + I_2(|\phi_m|)(x_0',v_0') + |I_{1,2}(\phi_m)|(x_0',v_0').$$

Since the support of ϕ_m shrinks to a set of measure zero in Γ_+ as $m \to \infty$, we get for almost every $(x_0',v_0') \in \Gamma_-$, $\lim_{m\to\infty} I_3(|\phi_m|)(x_0',v_0') = 0$. Finally, noting that $|I_{1,1}(\phi_m)| = I_{1,1}(\phi_m)$ and applying (5.9), from (5.8) we obtain for almost every $(x_0',v_0') \in \Gamma_-$

$$\begin{aligned}
\int_0^{\tau_+(x_0',v_0')} & \int_{S_yM} |k-\tilde{k}|(y,\dot{y},w) E(x_0',y) E\big(y, \gamma_{(y,w)}(\tau_+(y,w))\big)\, d\omega_y(w)\, dt \\
&= \lim_{m\to\infty} I_{1,1}(\phi_m) \\
&\le \|\mathcal{A} - \tilde{\mathcal{A}}\| + \int_0^{\tau_+(x_0',v_0')} \int_{S_yM} |F-\tilde{F}|(x_0',y,w)\tilde{k}(y,\dot{y},w)\, d\omega_y(w)\, dt.
\end{aligned}$$

The estimate (5.4) in the theorem follows. □

6. Stability modulo gauge transformations

In this section we prove Theorem 4.1.

We start with two pairs $(a,k), (\tilde{a},\tilde{k}) \in U_{\Sigma,\rho}$ and let

$$\varepsilon := \|\mathcal{A} - \tilde{\mathcal{A}}\|.$$

We shall find an intermediate pair $(a',k') \sim (a,k)$ such that (4.1) and (4.2) hold.

Define first the "trial" gauge transformation:

$$\varphi(x,v) := e^{-\int_0^{\tau_-(x,v)} (\tilde{a}-a)(\vec{\gamma}_{(x,v)}(s-\tau_-(x,v)))\, ds}, \quad a.e.\ (x,v) \in \overline{SM}. \tag{6.1}$$

Then $\varphi > 0$, $\varphi|_{\Gamma_-} = 1$, $\mathcal{D}\varphi(x,v) \in L^\infty(SM)$ and

$$\tilde{a}(x,v) = a(x,v) - \mathcal{D}\log\varphi(x,v). \tag{6.2}$$

Note, however, that $\varphi|_{\Gamma_+}$ is not equal to 1. We begin by estimating $\varphi|_{\Gamma_+}$. By (5.3), we have for almost every $(x_0',v_0') \in \Gamma_-$

$$\left| e^{-\int_0^{\tau_+(x_0',v_0')} a(\vec{\gamma}_{(x_0',v_0')}(s))\, ds} - e^{-\int_0^{\tau_+(x_0',v_0')} \tilde{a}(\vec{\gamma}_{(x_0',v_0')}(s))\, ds} \right| \le \varepsilon.$$

Changing variables $t = \tau_+(x_0', v_0') - s$ and denoting $(x_0, v_0) = \vec{\gamma}_{(x_0',v_0')}(\tau_+(x_0', v_0'))$ we get

$$\left| e^{-\int_0^{\tau_-(x_0,v_0)} a(\vec{\gamma}_{(x_0,v_0)}(t-\tau_-(x_0,v_0)))\,dt} - e^{-\int_0^{\tau_-(x_0,v_0)} \tilde{a}(\vec{\gamma}_{(x_0,v_0)}(t-\tau_-(x_0,v_0)))\,dt} \right| \le \varepsilon. \tag{6.3}$$

When (x_0', v_0') covers Γ_- almost everywhere we get (x_0, v_0) covers Γ_+ almost everywhere.

By the Mean Value theorem applied to $u \mapsto e^{-u}$ we obtain the lower bound

$$\begin{aligned} &\left| e^{-\int_0^{\tau_-(x_0,v_0)} a(\vec{\gamma}_{(x_0,v_0)}(t-\tau_-(x_0,v_0)))\,dt} - e^{-\int_0^{\tau_-(x_0,v_0)} \tilde{a}(\vec{\gamma}_{(x_0,v_0)}(t-\tau_-(x_0,v_0)))\,dt} \right| \\ &\quad = e^{-u_0} \left| \int_0^{\tau_-(x_0,v_0')} (\tilde{a} - a)\big(\vec{\gamma}_{(x_0,v_0)}(t - \tau_-(x_0,v_0))\big)\,dt \right| = e^{-u_0} |\log \varphi(x_0,v_0)| \\ &\quad \ge e^{-\operatorname{diam}(M)\Sigma} |\log \varphi(x_0,v_0)| \end{aligned} \tag{6.4}$$

where $u_0 = u_0(x_0, v_0, a, \tilde{a})$ is a value between the two integrals appearing in the exponents in the left hand side above, and φ is defined in (6.1).

From (6.3) and (6.4) we get the following estimate for the "trial" gauge φ:

$$|\log \varphi(x,v)| \le e^{\operatorname{diam}(M)\Sigma}\varepsilon, \ \ a.e.\ (x,v) \in \Gamma_+. \tag{6.5}$$

The "trial" gauge φ is not good enough since it does not equal 1 on Γ_+. We alter it to some $\tilde{\varphi} \in L^\infty(SM)$ with $\mathcal{D}\log\tilde{\varphi} \in L^\infty(SM)$ in such a way that $\tilde{\varphi}|_{\partial SM} = 1$. More precisely, for almost every $(x,\theta) \in \overline{SM}$, we define $\tilde{\varphi}(x,v)$ by

$$\log \tilde{\varphi}(x,v) := \log \varphi(x,v) - \frac{\tau_-(x,v)}{\tau(x,v)} \log \varphi\big(\vec{\gamma}_{(x,v)}(\tau_+(x,v))\big). \tag{6.6}$$

Since $0 \le \tau_-(x,v)/\tau(x,v) \le 1$ we get $\tilde{\varphi} \in L^\infty(SM)$, and clearly $\tilde{\varphi}|_{\partial SM} = 1$. Since $\mathcal{D}\tau(x,v) = \mathcal{D}\log\varphi\big(\vec{\gamma}_{(x,v)}(\tau_+(x,v))\big) = 0$ and $\mathcal{D}\tau_-(x,v) = 1$,

$$\mathcal{D}\log\tilde{\varphi}(x,v) = \mathcal{D}\log\varphi(x,v) - \frac{\log\varphi\big(\vec{\gamma}_{(x,v)}(\tau_+(x,v))\big)}{\tau(x,v)} \in L^\infty(SM). \tag{6.7}$$

Define now the pair (a', k') in the equivalence class of $\langle a, k\rangle$ by

$$a'(x,v) := a(x,v) - \mathcal{D}\log\tilde{\varphi}(x,v), \tag{6.8}$$

$$k'(x,v',v) := \frac{\tilde{\varphi}(x,v)}{\tilde{\varphi}(x',v')} k(x,v',v). \tag{6.9}$$

Now $\mathcal{A}'$, the albedo operator corresponding to (a', k'), satisfies $\mathcal{A}' = \mathcal{A}$, and

$$\|\mathcal{A}' - \tilde{\mathcal{A}}\| = \|\mathcal{A} - \tilde{\mathcal{A}}\| = \varepsilon.$$

Next we compare the pairs (a', k') with $(\tilde{a}, \tilde{k})$ and show them to satisfy (4.1) and (4.2). Using the definitions (6.2), (6.8), the relation (6.7), and the estimate (6.5) for φ on Γ_+, we have for almost every $(x,v) \in SM$:

$$\begin{aligned} |\tilde{a}(x,v) - a'(x,v)| &= \big|[\tilde{a} - a](x,v) + [a - a'](x,v)\big| \\ &= |\mathcal{D}\log\tilde{\varphi}(x,v) - \mathcal{D}\log\varphi(x,v)| \\ &= \frac{\big|\log\varphi\big(\vec{\gamma}_{(x,v)}(\tau_+(x,v))\big)\big|}{\tau(x,v)} \le \varepsilon \frac{e^{\operatorname{diam}(M)\Sigma}}{\tau(x,v)}. \end{aligned} \tag{6.10}$$

Since the coefficients are supported away from ∂M (by construction of M) such that (2.3) holds, following (6.10) we obtain the estimate (4.1) in the form

$$\|\tilde{a}-a'\|_\infty \le \varepsilon\frac{e^{\mathrm{diam}(M)\Sigma}}{c_0}, \tag{6.11}$$

with c_0 from (2.3).

Up to this point, all the arguments above also work for two dimensional domains. Next we prove the estimate (4.2). These arguments are specific to three or higher dimensions. Recall the formula (5.2) adapted to a': let $x'\in\partial M$, $y\in M$ and $w\in S_yM$ and let $v'\in S_{x'}M, t>0$ be such that $y=\gamma_{(x',v')}(t)$. Then from (6.11),

$$|a'(x,v)|\le\varepsilon\frac{e^{\mathrm{diam}(M)\Sigma}}{\tau(x,v)}+\Sigma,\qquad\text{and}\qquad \|a'\|_\infty\le\varepsilon\frac{e^{\mathrm{diam}(M)\Sigma}}{c_0}+\Sigma,$$

so (using the first, and the fact that τ is constant along geodesics),

$$\begin{aligned}|F'(x',y,w)| &= e^{-\int_0^t a'(\vec{\gamma}_{(x',v')}(s))\,ds}e^{-\int_0^{\tau_+(y,w)} a'(\vec{\gamma}_{(y,w)}(s))\,ds}\\ &\ge \exp\big(-2(\varepsilon e^{\mathrm{diam}(M)\Sigma}+\mathrm{diam}(M)\Sigma\big).\end{aligned} \tag{6.12}$$

Using the non-negativity of $\tilde{a}$ and a' we estimate

$$\begin{aligned}|[\tilde{F}-F'](x',y,w)| &\le \left|e^{-\int_0^t \tilde{a}(\vec{\gamma}_{(x',v')}(s))\,ds}-e^{-\int_0^t \tilde{a}(\vec{\gamma}_{(x',v')}(s))\,ds}\right|\\ &\qquad+\left|e^{-\int_0^{\tau_+(y,w)} a'(\vec{\gamma}_{(y,w)}(s))\,ds}-e^{-\int_0^{\tau_+(y,w)} a'(\vec{\gamma}_{(y,w)}(s))\,ds}\right|\\ &\le\left|\int_0^t[\tilde{a}-a'](\vec{\gamma}_{(x',v')}(s))\,ds\right|+\left|\int_0^{\tau_+(y,w)}[\tilde{a}-a'](\vec{\gamma}_{(y,w)}(s))\,ds\right|\\ &\le\varepsilon e^{\mathrm{diam}(M)\Sigma}\Big(\frac{t}{\tau(x',v')}+\frac{\tau_+(y,w)}{\tau(y,w)}\Big)\le 2\varepsilon e^{\mathrm{diam}(M)\Sigma}\end{aligned} \tag{6.13}$$

by (6.10). We now apply the lower bound for F' in (6.12), the upper bound for $\|\tilde{F}-F'\|_\infty$ from (6.13) and the hypothesis $\|\tilde{k}\|_{\infty,1}\le\rho$ to the estimate (5.4) with respect to the pairs (a',k') and $(\tilde{a},\tilde{k})$. With $y(t)=\gamma_{(x_0',v_0')}(t)$, we obtain

$$\begin{aligned}&\int_0^{\tau_+(x_0',v_0')}\int_{S_{y(t)}M}|k-\tilde{k}|(y(t),\dot{y}(t),w)\,d\omega_y(w)\,dt\\ &\qquad\le\varepsilon\big(1+2\,\mathrm{diam}(M)\rho\omega_{n-1}e^{\mathrm{diam}(M)\Sigma}\big)\exp\Big(2\,\mathrm{diam}(M)\big(\varepsilon\frac{e^{\mathrm{diam}(M)\Sigma}}{c_0}+\Sigma\big)\Big)\\ &\qquad=\varepsilon\, C_1,\quad\text{say.}\end{aligned}$$

Finally, integrating the formula above in $(x_0',v_0')\in\Gamma_-$ with the measure $d\mu(x_0',v_0')$, we get

$$\|\tilde{k}-k'\|_1\le\varepsilon\mathrm{Vol}(\partial M)\omega_{n-1}C_1.$$

Theorem 4.1 holds now with $C=\max\{\mathrm{Vol}(\partial M)\omega_{n-1}C_1, e^{\mathrm{diam}(M)\Sigma}/c_0\}$.

7. Stability of the equivalence classes in two dimensions

We prove here Theorem 4.2, making use of the results of [**14**].

Let $(a,k),(\tilde a,\tilde k)\in V_{\Sigma,\rho}$ be given with $\|\mathcal{A}-\tilde{\mathcal{A}}\|_*=\varepsilon$. As before, define the pair (a',k') in the equivalence class of $\langle a,k\rangle$ by (6.8) and (6.9). Then the corresponding albedo operator $\mathcal{A}'=\mathcal{A}$ and, thus,

$$\|\mathcal{A}'-\tilde{\mathcal{A}}\|_*=\|\mathcal{A}-\tilde{\mathcal{A}}\|_*=\varepsilon, \quad \text{and} \quad \|(\tilde\beta-\beta')|\sin\psi|\|_\infty\le\varepsilon. \tag{7.1}$$

The estimate (6.11) holds as in the case $n\ge 3$. Now

$$\frac{\tilde\varphi(x,v)}{\tilde\varphi(x',v')}=e^{-\int_0^{\tau_-(x,v)}(a'-a)(\vec\gamma_{(x,v)}(s-\tau_-(x,v)))\,ds+\int_0^{\tau_-(x',v')}(a'-a)(\vec\gamma_{(x',v')}(s-\tau_-(x',v')))\,ds}. \tag{7.2}$$

From (6.5) and (6.10), with $y(s)=\gamma_{(x,v)}(s-\tau_-(x,v))$,

$$\begin{aligned}\Big|\int_0^{\tau_-(x,v)}(a'-a)(y,\dot y)\,ds\Big| &\le \int_0^{\tau_-(x,v)}(|a'-\tilde a|+|\tilde a-a|)(y,\dot y)\,ds\\ &\le \int_0^{\tau_-(x,v)}\varepsilon\frac{e^{\operatorname{diam}(M)\Sigma}}{\tau(y,\dot y)}\,ds+\varepsilon e^{\operatorname{diam}(M)\Sigma}\le 2\varepsilon e^{\operatorname{diam}(M)\Sigma};\end{aligned}$$

the same estimate holds for the second exponent in (7.2). Thus, from the definition (6.9) and (7.2) we obtain

$$\frac{\tilde\varphi(x,v)}{\tilde\varphi(x',v')}\le\exp\big\{4\varepsilon e^{\operatorname{diam}(M)\Sigma}\big\} \implies \|k'\|_\infty\le\rho\exp\big\{4\varepsilon e^{\operatorname{diam}(M)\Sigma}\big\}. \tag{7.3}$$

Let

$$\tilde E_1(y,w',w):=\tilde E\big(\gamma_{(y,w')}(-\tau_-(y,w')),y\big)\tilde E\big(y,\gamma_{(y,w)}(\tau_+(y,w))\big)$$

be the total attenuation along the broken geodesic due to one scattering at $(y,w',w)\in S^2M$. Then (6.12) and (6.13) say

$$|E_1'(y,w',w)|\ge\exp\big(-2(\varepsilon e^{\operatorname{diam}(M)\Sigma}+\operatorname{diam}(M)\Sigma)\big)=C_1, \text{ say, and} \tag{7.4}$$

$$\|\tilde E_1-E_1'\|\le 2\varepsilon e^{\operatorname{diam}(M)\Sigma}. \tag{7.5}$$

The terms α_j in the expansion of the albedo kernel in Proposition 3.2 are the traces of distributions ϕ_j defined on $SM\times\Gamma_-$; see [**14**]. The ϕ_j are the kernels of the operators J, KJ and $(I-K)^{-1}K^2J$ ($j=0,1,2$, respectively) where

$$\begin{aligned}Jf_-(x,v)&=E(\gamma_{(x,v)}(-\tau_-(x,v)),x)f_-(\vec\gamma_{(x,v)}(\tau_-(x,v))),\\ Kf(x,v)&=\int_0^{\tau_-(x,v)}E\big(x,\gamma_{(x,v)}(t-\tau_-(x,v))\big)T_1f\big(\vec\gamma_{(x,v)}(t-\tau_-(x,v))\big)\,dt, \text{ with}\\ T_1f(x,v)&=\int_{S_xM}k(x,v',v)f(x,v')\,d\omega_x(v').\end{aligned}$$

Let γ be the trace operator on $L^\infty(\Gamma_+)$, which is shown in [**14**] to be well-defined.

Let $(x,v,x',v')\in\Gamma_+\times\Gamma_-$ be such that the geodesics $\gamma_{(x,v)}(\cdot)$ and $\gamma_{(x',v')}(\cdot)$ intersect at $(y,w',w)\in S^2M$. By Proposition 3.2 above,

$$\begin{aligned}E_1'(\tilde k-k')&=(E_1'-\tilde E_1)\tilde k+(\tilde E_1\tilde k-E_1'k')\\ &=(E_1'-\tilde E_1)\tilde k+(\tilde\beta-\beta')|\sin\psi|+(\gamma\phi_2'-\gamma\tilde\phi_2)|\sin\psi|,\end{aligned}$$

and so by (7.1), (7.3), (7.4) and (7.5),

$$C_1|\tilde k-k'|\le 2\varepsilon e^{\operatorname{diam}(M)\Sigma}\rho+\varepsilon+|\gamma\phi_2'-\gamma\tilde\phi_2||\sin\psi|. \tag{7.6}$$

Now

$$\begin{aligned}|\gamma\phi_2' - \gamma\phi_2| &= \gamma(I-K')^{-1}K'^2\phi_0' - \gamma(I-\tilde{K})^{-1}\tilde{K}^2\tilde{\phi}_0 \\ &= \gamma(I-K')^{-1}\big(K'^2\phi_0' - \tilde{K}^2\tilde{\phi}_0\big) + \gamma(I-\tilde{K})^{-1}(K'-\tilde{K})(I-K')^{-1}\tilde{K}^2\tilde{\phi}_0 \\ &= \gamma(I-K')^{-1}\big(K'^2-\tilde{K}^2\big)\phi_0' + \gamma(I-K')^{-1}\tilde{K}^2(\phi_0'-\tilde{\phi}_0) \\ &\qquad + \gamma(I-\tilde{K})^{-1}(K'-\tilde{K})(I-K')^{-1}\tilde{K}^2\tilde{\phi}_0. \end{aligned} \tag{7.7}$$

Lemma 9 of [**14**] estimates the first of these terms:

$$\|\gamma(I-K')^{-1}\big(K'^2-\tilde{K}^2\big)\phi_0'\|_\infty \le C_2\|k'-\tilde{k}\|_\infty(\|k'\|_\infty + \|\tilde{k}\|_\infty)(1-\log|\sin\psi|). \tag{7.8}$$

For the second we appeal to Proposition 7, and its proof, in [**14**]. Instead of using the estimate $\|E\|_\infty \le 1$, we use (6.13); together with the fact that $(I-K')^{-1}$ preserves L^∞, we readily obtain

$$\|\gamma(I-K')^{-1}\tilde{K}^2(\phi_0'-\tilde{\phi}_0) \le C_3\|\tilde{k}\|_\infty^2\|E'-\tilde{E}\|_\infty \le C_3\rho^2 2\varepsilon e^{\operatorname{diam}(M)\Sigma}. \tag{7.9}$$

The final term in (7.7) is estimated in [**14**, Lemma 10]:

$$\|\gamma(I-\tilde{K})^{-1}(K'-\tilde{K})(I-K')^{-1}\tilde{K}^2\tilde{\phi}_0\|_\infty \le C_4\|k'-\tilde{k}\|_\infty\rho^2(1+\rho). \tag{7.10}$$

Combining (7.6), (7.8), (7.9) and (7.10), for a new constant C, we obtain

$$\|k'-\tilde{k}\|_\infty \le C\varepsilon + C\rho\|k'-\tilde{k}\|_\infty$$

and so if $\rho < 1/C$, we obtain the final estimate

$$\|k'-\tilde{k}\|_\infty \le \frac{C}{1-C\rho}\varepsilon =: \tilde{C}\varepsilon.$$

Theorem 4.2 is now proven for a constant which is the maximum of the $\tilde{C}$ above and $\exp(\operatorname{diam}(M)\Sigma)/c_0$ (see (6.11)).

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