Math. Z. 207, 461–480 (1991)

Mathematische Zeitschrift

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Inverse scattering problem for moving obstacles

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Received June 28, 1990; in final form November 26, 1990

1 Introduction

Let $Q \subset \mathbb{R}^{n+1}$, $n \geqq 3$ odd, be an open connected set with a smooth boundary ∂Q. Given $t \in \mathbb{R}$, we denote $\Omega(t) = \{x \in \mathbb{R}^n;\, (t, x) \in Q\}$, and $\mathcal{O}(t) = \mathbb{R}^n \setminus \Omega(t)$ is the obstacle at the time t. We impose the following conditions.

(i) There exists $\rho > 0$ such that $\mathcal{O}(t) \subset B_\rho = \{x;\, |x| < \rho\}$ for each t.
(ii) If $\nu = (\nu_t, \nu_x)$ is the inner unit normal to ∂Q, then $|\nu_t| < |\nu_x|$.

Conditions (i), (ii) mean that the obstacle remains within a fixed compact set and that the boundary moves with a speed less than 1. Consider the wave equation in Q with Dirichlet boundary conditions

$$
\begin{aligned}
u_{tt} - \Delta u &= 0 \quad \text{in } Q, \\
u &= 0 \quad \text{on } \partial Q.
\end{aligned}
\tag{1.1}
$$

This paper is devoted to the uniqueness of the inverse scattering problem for (1.1).

The scattering theory for moving obstacles has been developed by Cooper and Strauss [3–11, 23] (see also [1, 17–20, 26]). There are some essential differences between the stationary case (see [14]) and the case of moving obstacles. One of them is that the variables cannot be separated and in general it is not possible to reduce (1.1) to a stationary problem by using Fourier transform. Another phenomenon is that the local (global) energy may increase as the time tends to infinity [5, 20]. Thus to prove the existence of the scattering operator S we must impose some additional restrictions [7, 8, 11, 17, 18, 23]. Cooper and Strauss [8, 9] introduced the generalized scattering (echo) kernel $K^{\#}(s', \omega'; s, \omega)$ which makes sense whenever assumptions (i), (ii) are fulfilled. The distribution $K^{\#}$ is a natural generalization of the kernel of $S - \mathrm{Id}$ and $K^{\#}$ coincides with it when S exists. A similar generalized kernel can be introduced in the scattering theory for the wave equation with time-dependent potential [24, 25].

* Partially supported by Bulgarian Ministry of Culture, Science and Eduction, Grant 52

In this situation it is natural to study the following inverse problem:

(IP) Does $K^{\#}$ determine Q uniquely?

It is well-known that for stationary bodies S always exists and the answer to (IP) is affirmative [14] (see also [2, 21]). As it was shown by the author [24, 25] the same is true for time-dependent potentials. In the case of moving obstacles it is known that $K^{\#}$ determines uniquely the convex hull of $\mathcal{O}(t)$ for each t [9]. The proof of this result is based on the analysis of the leading singularity of $K^{\#}$ (see also [1, 17, 19]) in the spirit of the works of Majda [15, 16] and Soga [22] where stationary obstacles are studied. Nevertheless, to our knowledge there are no works dealing with (IP) for moving obstacles with arbitrary geometry. It looks a little surprising that for general moving obstacles the answer to (IP) is negative. This will be proved in Theorem 1.3 below. For this reason in order to investigate (IP) we need to impose some additional restrictions.

Definition 1.1 We say that $Q \in \mathcal{Q}$ if Q satisfies (i), (ii) and the following condition

(iii) There exists some $T>0$ such that $v_t=0$ for $|t|>T$.

In other words $\mathcal{Q}$ includes all obstacles which are stationary in the far past and in the far future. Let $U_0(t)$ be the unitary group related to the Cauchy problem for the wave equation [14] and denote by $U(t, s)$ the propagator associated with (1.1) (see Sect. 2). It is not hard to prove that for $Q \in \mathcal{Q}$ the scattering operator

$$S = s-\lim_{t \to \infty} U_0(-t)\, U(t, -t)\, U_0(-t)$$

exists (see Proposition 2.1). Our first result is the following.

Theorem 1.2 *Let $Q_i \in \mathcal{Q}$, $i=1, 2$ and let the scattering operators S_i, $i=1, 2$, associated with Q_i, coincide. Then $Q_1 = Q_2$.*

Next, we show that in general the answer to (IP) is negative, even for periodically moving obstacles. Before stating the corresponding theorem we need to recall the definition of $K^{\#}$ (see [8, 9]). Set $g(\xi) = -(2\pi)^{-(n-1)/2}\xi$ for $\xi \geqq 0$, $g(\xi)=0$ otherwise. Denote by $u(t, x; s, \omega)$ that solution of (1.1) which coincides with the plane wave $g(t+s-x\cdot\omega)$ for $t<-s-\rho$. Then the function $u_{sc}(t, x; s, \omega) = u(t, x; s, \omega) - g(t+s-x\cdot\omega)$ admits an *asymptotic wave profile* $u_{sc}^{\#}$, i.e. the limit

$$u_{sc}^{\#}(s', \omega'; s, \omega) = \lim_{t \to \infty} (t+s')^{(n-1)/2}\, \partial_t u_{sc}(t, (t+s')\,\omega'; s, \omega)$$

exists in the space $L^2_{\mathrm{loc}}(\mathbb{R}_{s'} \times S^{n-1}_{\omega'})$. The *generalized scattering kernel* $K^{\#}$ is then defined by the equality

$$K^{\#}(s', \omega'; s, \omega) = (-\partial/\partial s)^{(n+1)/2}\, u_{sc}^{\#}(s', \omega'; s, \omega)$$

and $K^{\#}$ is a continuous function of s, ω, ω' with values in $\mathcal{D}'(\mathbb{R}_{s'})$. In the case when S exists, $K^{\#}$ is the Schwartz kernel of the operator $\mathcal{R}(S-\mathrm{Id})\mathcal{R}^{-1}$ (see [8]), $\mathcal{R}$ being the translation representation of $U_0(t)$ introduced by Lax and Phillips [14]. Our second result is the following.

Theorem 1.3 *There exists a family of infinitely many distinct periodically moving obstacles satisfying* (i), (ii) *with the same generalized scattering (echo) kernel.*

Below we describe briefly our approach. Let Q_1 and Q_2 be two distinct domains in $\mathcal{Q}$ whose scattering operators S_i, $i=1,2$ coincide. Fix some $s\in\mathbb{R}$ and fix $\varphi\in[C_0^\infty(\mathbb{R}^n)\times C_0^\infty(\mathbb{R}^n)]\cap D_-^\rho$, D_-^ρ being the incoming space of Lax and Phillips [14]. We denote by u_i the solution of (1.1) related to Q_i having initial data φ for $t=s$. In Proposition 2.2 we prove that $S_1=S_2$ implies that $u=u_1-u_2$ vanishes for $|x|>\rho$ and all $t\in\mathbb{R}$, where ρ is chosen so that $\mathcal{O}_i(t)\subset B_\rho=\{x;\,|x|<\rho\}$ for all t, $i=1,2$. In the stationary case it is not hard to show that, in fact, u vanishes in the unbounded connected component of $\Omega_1\cap\Omega_2$ which easily yields $\Omega_1=\Omega_2$ [14] (see also [2, 21]). Indeed, in this case the function $v(k,x)=\int\exp(-itk)u(t,x)\,dt$ is a solution to the Helmholtz equation, consequently v is a real-analytic function with respect to x. Clearly, in the case of moving obstacles this argument is no more available. Nevertheless, assuming for definiteness that $\partial Q_2\cap Q_1\neq\emptyset$, we aim to prove that there exists $(t_0,x_0)\in\partial Q_2\cap Q_1$ such that $u(t_0,\cdot)$ vanishes in a neighborhood of x_0 in $\partial\Omega_2(t_0)$. This fact makes possible to obtain contradiction. Its proof is based essentially on two new ideas. First, we construct a two-parameter family $F_{t,s}\colon\Omega(s)\to\Omega(t)$ of diffeomorphisms related to a domain Q satisfying (i), (ii). We choose $F_{t,s}$ to be the flow associated with a (time-dependent) vector field $v\in C^\infty(\bar{Q};\mathbb{R}^n)$ which is chosen so that $|v|<1$, $v(t,x)=0$ for $|x|>\rho$ and $(1,v)$ is tangent to ∂Q. In some sense $v(t,x)$ can be considered as the velocity of $x\in\Omega(t)$ at the moment t. Thus we treat all points in the exterior of the obstacle as moving ones. Secondly, we formulate and prove a Holmgren type theorem exploiting the flow $F_{t,s}$ (see Theorem 3.1 below). It has a form suitable for the examination of noncylindric domains Q. By this theorem we conclude in Sect. 5 that there exists $(t_0,x_0)\in\partial Q_2\cap Q_1$ such that $u(t_0,\cdot)$ vanishes in some neighborhood of x_0 in $\partial\Omega_2(t_0)$, as required.

In Sect. 6 we prove Theorem 1.3. To this end we construct explicitly a family of periodically moving obstacles with the same generalized scattering kernel. First we build an obstacle $\mathcal{O}(t)$ that has a part of the boundary which cannot be reached by the waves coming from $\mathbb{R}^n\setminus B_\rho$. More precisely, there is an open set V, such that $V\cap\partial\Omega(t)$ is non-empty, stationary and the following property is satisfied: There is no curve $x(t)$ such that $x(0)\notin B_\rho$, $|x'(t)|\leq 1$, $x(t)\in\Omega(t)$ for any t, and $x(l)\in V\cap\Omega(l)$ for some l. This implies that every solution of (1.1) with initial data in D_-^ρ vanishes on $V\cap\Omega(t)$. Thus the generalized scattering kernel $K^\#$ does not contain any information about the shape of $V\cap\partial\Omega(t)$. So we may change this part of $\partial\Omega(t)$ and this will not reflect on $K^\#$.

2 Preliminary

Given $f=(f_1,f_2)\in C_0^\infty(\Omega(t))\times C_0^\infty(\Omega(t))$ define the energy norm of f by

$$\|f\|_{(t)}^2=\int_{\Omega(t)}(|\nabla f_1|^2+|f_2|^2)\,dx.$$

Denote by $\mathcal{H}(t)$ the closure of $C_0^\infty(\Omega(t))\times C_0^\infty(\Omega(t))$ with respect to the norm $\|\cdot\|_{(t)}$. $\mathcal{H}(t)$ is a Hilbert space with energy scalar product (see e.g. [8]). Given $f\in\mathcal{H}(s)$ we consider the mixed problem

$$\begin{aligned} u_{tt}-\Delta u&=0, && (t,x)\in Q,\\ u&=0, && (t,x)\in\partial Q,\\ (u,u_t)&=(f_1,f_2) && \text{for } t=s. \end{aligned}\tag{2.1}$$

The solution of (2.1) is given by $(u, u_t) = U(t, s)f$, where the propagator $U(t, s)$ is a two-parameter family of operators $U(t, s)$: $\mathscr{H}(s) \to \mathscr{H}(t)$ (see [8]).

Let $\mathscr{H}_0$ be the Hilbert space associated with the Cauchy problem for the free wave equation $\Box u = 0$ in $\mathbb{R}_t \times \mathbb{R}_x^n$. $\mathscr{H}_0$ has the same definition as $\mathscr{H}(t)$ with $\Omega(t) = \mathbb{R}^n$. We denote by $U_0(t)$ the unitary group in $\mathscr{H}_0$, related to the free wave equation [14]. Recall the definition of the scattering operator S associated with $U(t, s)$ and $U_0(t)$ given in the introduction. This definition is to be considered in the following sense (see also [14]). For any $f \in \mathscr{H}_0$ with compact support $U_0(-t)f$ vanishes in B_ρ for large t, therefore it can be considered as an element in $\mathscr{H}(-t)$ thus $U(t, -t)U_0(-t)f$ is well defined. Next, we regard $\mathscr{H}(t)$ as a subspace of $\mathscr{H}_0$, thus for large t the expression $U_0(-t)U(t, -t)U_0(-t)f$ is well defined. We say that S exists if the limit Sf exists for any $f \in \mathscr{H}_0$ with compact support and S is bounded. Then Sf can be defined for any $f \in \mathscr{H}_0$. In general S does not exists. Moreover, in some cases the local energy increases exponentially [5, 20]. Sufficient conditions for the existence of S can be found, for example, in [7, 8, 11, 17, 18, 23]. In our case the situation is simpler since we have assumed (iii).

Proposition 2.1 *Let $Q \in \mathscr{Q}$. Then the scattering operator S exists.*

Proof. Condition (iii) implies that $\Omega(t) = \Omega(T)$ for $t \geqq T$, and $\Omega(t) = \Omega(-T)$ for $t \leqq -T$. Denote by $U_\pm(t)$ the unitary groups in $\mathscr{H}(\pm T)$ related to the stationary obstacles $\mathscr{O}(\pm T)$. Then we have $U(t, s) = U_+(t-T)U(T, s)$ for $t \geqq T$ and $U(t, s) = U(t, -T)U_-(-T-s)$ for $s \leqq -T$. Using these relations and the fact that the wave operators associated with $U_\pm(t)$ and $U_0(t)$ exist [14], we complete the proof of the proposition. Note that in particular we obtain that the energy $\|U(t, s)\|$ is uniformly bounded in t, s.

In the remainder of this section we will proceed with the first step of the proof of Theorem 1.2. Let Q_i, $i = 1, 2$ be two domains satisfying (i), (ii), (iii) and denote by $U_i(t, s)$, S_i the corresponding propagators and scattering operators. Recall the definition of the spaces $D_\pm^\rho$ of Lax-Phillips [14]

$$D_\pm^\rho = \{f \in \mathscr{H}_0;\ [U_0(t)f](x) = 0 \quad \text{for } |x| < \pm t + \rho,\ \pm t > 0\}.$$

Fix $\rho > 0$ such that $\mathscr{O}_1(t) \cup \mathscr{O}_2(t) \subset B_{\rho/2}$ for each t.

Proposition 2.2 *Suppose $S_1 = S_2$. Then for all $\varphi \in [C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)] \cap D_-^\rho$ and all $t, s \in \mathbb{R}^2$ we have $[U_1(t, s)\varphi](x) = [U_2(t, s)\varphi](x)$ for $|x| > \rho$.*

Proof. Fix φ and s and set $u_i(t, x) = [U_i(t, s)\varphi]_1$, $u = u_1 - u_2$, $[v]_1$ being the first component of the vector v. Pick a cut-off function $\chi \in C^\infty$, such that $\chi(x) = 0$ in a neighborhood of $B_{\rho/2}$, $\chi(x) = 1$ for $|x| \geqq \rho$ and set $v = \chi u$. Then we have

$$\Box v = -(\Delta \chi)u - 2\nabla \chi \cdot \nabla u, \tag{2.2}$$

since $\Box u = 0$ for $|x| > \rho/2$. Moreover, $U_i(t, s)\varphi = U_0(t-s)\varphi$ for $t \leqq s$ [8], hence $u = 0$ for $t \leqq s$. Therefore,

$$(v, v_t) = \int_s^t U_0(t-\tau)(0, q(\tau, \cdot))\, d\tau \tag{2.3}$$

with $q \in C(\mathbb{R}_t; L^2(\mathbb{R}^n_x))$ given by the right hand side of (2.2). Note that $\operatorname{supp}_x q(t, x) \subset \{x; \rho/2 \leqq |x| \leqq \rho\}$ for all t. Given $a > s$, set

$$f^a = \int_s^a U_0(-\tau)(0, q(\tau, \cdot))\, d\tau, \qquad (v^a, \partial_t v^a) = U_0(t) f^a.$$

From (2.3) we find $v - v^a = \left[\int_a^t U_0(t-\tau)(0, q(\tau, \cdot))\, d\tau\right]_1$. The finite speed of propagations argument implies that

$$v = v^a = u \quad \text{for } |x| > t - a + \rho, \quad t \geqq a. \tag{2.4}$$

Furthermore, the equality $S_1 = S_2$ yields $(u, u_t) = U_1(t, s)\varphi - U_2(t, s)\varphi \to 0$ in $\mathscr{H}_0$, as $t \to \infty$. Combining this with (2.4) we get

$$\int_{t-a+\rho<|x|} |\partial_t v^a(t, x)|^2\, dx \to 0, \qquad \text{as } t \to \infty. \tag{2.5}$$

On the other hand, Theorem 2.4 in [14, Ch. IV.] (see also [8, Theorem 16]) says that

$$\int_{t-a+\rho<|x|} \left|\partial_t v^a - (-|x|)^{-(n-1)/2} (\mathscr{R} f^a)\left(|x| - t, \frac{x}{|x|}\right)\right|^2 dx \to 0, \tag{2.6}$$

as $t \to \infty$, with $\mathscr{R} f^a \in L^2(\mathbb{R} \times S^{n-1})$ being the translation representer of f^a [14]. Limits (2.5) and (2.6) together imply that $\mathscr{R} f^a(\sigma, \omega) = 0$ for $\sigma > \rho - a$, hence

$$\mathscr{R}(U_0(t) f^a)(\sigma, \omega) = 0 \qquad \text{for } \sigma > t - a + \rho. \tag{2.7}$$

Furthermore,

$$U_0(t) f^a = \int_s^a U_0(t-\tau)(0, q(\tau, \cdot))\, d\tau.$$

Huygens' principle implies that for $t \geqq a + \rho$ we have $U_0(t) f^a \in D^0_+$, thus $\mathscr{R}(U_0(t) f^a)(\sigma, \omega) = 0$ for $\sigma < 0$, $t \geqq a + \rho$ by Theorem 2.3 in [14, Ch. IV.]. Consequently,

$$\mathscr{R}(U_0(t) f^a)(\sigma, \omega) = 0 \qquad \text{for } \sigma < -\rho, \quad t \geqq a. \tag{2.8}$$

Combining (2.7) and (2.8) we find

$$\mathscr{R}(U_0(t) f^a)(\sigma, \omega) = 0 \qquad \text{for } |\sigma| > t - a + \rho, \quad t \geqq a.$$

On the other hand, $U_0(t) f^a$ has compact support for each t. An application of Corollary 3.3 of [14, Ch. IV.] implies that

$$\operatorname{supp}_x U_0(t) f^a \subset B_{t-a+\rho} \qquad \text{for } t \geqq a,$$

therefore $u(t, x)=0$ for $|x|>t-a+\rho$, $t \geqq a$ in view of (2.4). Setting $t=a$ we find $u(a, x)=0$ for $|x|>\rho$. Since $a>s$ may be chosen arbitrary, it follows that the assertion of the proposition is valid for $t>s$. As we have mentioned above, $u(t, x)$ vanishes for $t \leqq s$. The proof is complete.

3 The flow $F_{t,s}$ and the Holmgren type theorem

In the beginning of this section we will show that there exists a (not unique) family of diffeomorphisms $F_{t,s}$ associated with Q satisfying (i), (ii), with the following properties:

(a) $F_{t,s}$: $\overline{\Omega(s)} \to \overline{\Omega(t)}$ is a diffeomorphism smoothly depending on t, s.
(b) $F_{t,s} \circ F_{s,r} = F_{t,r}$ for all t, s, r; $F_{s,s}=\mathrm{Id}$.
(c) $|(d/dt) F_{t,s}(x)|<1$, $F_{t,s}(x)=x$ for $|x|>\rho$, all t, s.

Note that the plane $t=t_0$ intersects Q transversally in view of (ii), so $\partial\Omega(t)$ is a smooth manifold for each t.

We seek $F_{t,s}$ as a solution of the following ordinary differential equation

$$\text{(3.1)} \qquad \begin{aligned} &\frac{d}{dt} F_{t,s}(x)=v(t, F_{t,s}(x)), \qquad t\in\mathbb{R},\ x\in\Omega(s),\\ &F_{s,s}(x)=x, \qquad\qquad\qquad\qquad x\in\Omega(s), \end{aligned}$$

where $v(t, x)$ is a suitably chosen vector field. We interpret $v(t, x)$ as the velocity of $x\in\Omega(t)$ at the moment t. First we define the normal velocity $v_N(t, x)$ for $x\in\partial\Omega(t)$ by the equality $v_N=-v_t v_x/|v_x|^2$. Next, we peek a velocity field $v\in C^\infty(\bar{Q}; \mathbb{R}^n)$ satisfying the conditions

$$\text{(3.2)} \qquad v|_{\partial Q}=v_N, \qquad v|_{Q\setminus(\mathbb{R}\times B_\rho)}=0, \qquad |v|<1.$$

In other words, we assume that each $x\in\partial\Omega(t)$ moves with velocity v_N, each $x\in\Omega(t)$ moves with velocity v, while the points outside B_ρ remain stationary. Moreover, the last equality in (3.2) means that every point has a speed less than 1. Of course, there are many ways to construct a solution to (3.2). For example, we can proceed as follows. The collar neighborhood theorem says that there is a C^∞-embedding Φ: $\partial Q\times[0,\infty)\to Q$, such that $\Phi(t, x; 0)=(t, x)$ for $(t, x)\in\partial Q$. We set $\varphi(t, x; \alpha)=g(\alpha)v_N(t, x)$ for $(t, x)\in\partial Q$, $\alpha\geqq 0$, where g is a smooth cut-off function, such that $g(\alpha)=1$ for $0\leqq\alpha\leqq 1$, $g(\alpha)=0$ for $\alpha\geqq 2$ and $0\leqq g\leqq 1$. Put $\tilde{v}(t, x)=\varphi(\Phi^{-1}(t, x))$ for $(t, x)\in\Phi(\partial Q\times[0,\infty))$ and $\tilde{v}=0$ otherwise. Now, let χ be another smooth cut-off function, such that $\chi(x)=1$ for $|x|\leqq\rho/2$, $\chi(x)=0$ for $|x|\geqq\rho$ and $0\leqq\chi\leqq 1$. Then $v=\chi\tilde{v}$ is a solution to (3.2).

Given v satisfying (3.2), set $w=(1, v)\in C^\infty(\bar{Q}; \mathbb{R}^{n+1})$. We claim that the vector field w is tangent to the boundary $\partial\Omega$. Indeed, let $(t, x)\in\partial Q$. Then $w(t, x)=(1, v_N(t, x))$ and $w\cdot v=v_t+v_x\cdot v_N=0$. This property together with the estimate $|w|<\sqrt{2}$ imply that w defines a global flow on Q [12]. A standard argument shows that this flow induces a family of diffeomorphisms $F_{t,s}$ that solves (3.1). Clearly, $F_{t,s}$ satisfies (a), (b), (c).

Note that our construction of $F_{t,s}$ does not require (iii). However, if (iii) holds, we can choose v such that $v=0$ for $|t|>2T$. In this case, in addition to (a), (b), (c) we have

(d) $F_{t,s}=\mathrm{Id}$ for $t<-2T$, $s<-2T$ and for $t>2T$, $s>2T$.

Note that in the coordinates $(t, y)=(t, F_{t_0,t}(x))$ Q becomes cylindric domain. However, the wave operator $\square$ in these coordinates looks too complicated.

The next construction is necessary to formulate the Holmgren type theorem. Using $F_{t,s}$ one can easily show that $\Omega(t)$ is connected for each t. We omit the trivial proof of this assertion. However, it should be noted that this is not evident if we do not exploit $F_{t,s}$. Fix t_0 and let $x_0 \neq x_1$ be two points in $\Omega(t_0)$. Let

$$\gamma=\{x\in\mathbb{R}^n;\ x=x(\sigma),\ 0\leqq\sigma\leqq l\} \tag{3.3}$$

be a smooth curve in $\Omega(t_0)$, which is not self-intersecting, such that $x(0)=x_0$, $x(l)=x_1$, $|x'(\sigma)|\neq 0$. It is convenient to assume that the function $x(\sigma)$ is defined on some interval $I=[-\delta, l+\delta]$, $\delta>0$, larger than $[0, l]$. To the end of this section we fix a family $F_{t,s}$ with properties (a), (b), (c), (d), generated by a velocity field v satisfying (3.2). Consider the two-dimensional surface

$$\Gamma=\{(t, x);\ x=F_{t,t_0}(x(\sigma)),\ \sigma\in I,\ t\in\mathbb{R}\}\subset Q. \tag{3.4}$$

It is not hard to check that Γ is a smooth manifold and σ, t are global coordinates on Γ. We define two vector fields $A^\pm\in T\Gamma$ by the equality

$$A^\pm=\frac{\partial}{\partial\sigma}\pm a_\pm(\sigma, t)\frac{\partial}{\partial t}.$$

Here the functions $a_\pm>0$ are chosen so that for each (σ, t) the vector $A^\pm$, considered as a vector in $\mathbb{R}^{n+1}$ is characteristic. This condition leads us to the equation

$$|v(t, F_{t,t_0}(x(\sigma)))\pm a_\pm^{-1} D_x F_{t,t_0}(x(\sigma))\,x'(\sigma)|=1. \tag{3.5}$$

Since $|v|<1$, there exist two positive smooth functions $a_\pm$, satisfying (3.5). Moreover, we can find constants c_1, c_2, such that

$$0<c_1\leqq a_\pm(\sigma, t)\leqq c_2<\infty \quad \text{for all } t \text{ and } \sigma\in I. \tag{3.6}$$

Indeed, let $t>2T$. Then $D_xF_{t,t_0}=D_xF_{t,2T}D_xF_{2T,t_0}=D_xF_{2T,t_0}$ in view of (a), (b), (c), (d). Similarly $D_xF_{t,t_0}=D_xF_{-2T,t_0}$ for $t<-2T$. Hence $a_\pm$ does not depend on t for $|t|>2T$. This proves (3.6). Note that here we have used condition (iii) essentially.

Let $\sigma\to(t, x)=(t_\pm(\sigma), F_{t_\pm(\sigma),t_0}(x(\sigma)))$ be the integral curve of $A^\pm$, passing through (t_0, x_0) for $\sigma=0$. In other words, $t_\pm(\sigma)$ solves the following problem

$$\frac{d}{d\sigma}t_\pm(\sigma)=\pm a_\pm(\sigma, t_\pm), \qquad t_\pm(0)=t_0. \tag{3.7}$$

Condition (3.6) implies that the function $t_\pm$ is defined globally on I. As we will see in Sect. 6 in general the solutions of (3.7) cannot be continued on I.

Further, t_+ is monotonically increasing, while t_- is monotonically decreasing. We define a subset X of Γ by

(3.8) $$X=\{(t, x);\ x=F_{t,t_0}(x(\sigma)),\ 0\leqq\sigma\leqq l,\ t_-(\sigma)\leqq t\leqq t_+(\sigma)\}.$$

We especially note that X depends on t_0, x_0, x_1, γ and $F_{t,s}$.

Now we are ready to state and prove the following Holmgren type theorem.

Theorem 3.1 *Let u be a distribution satisfying the wave equation $\square u=0$ in an open subset of $\mathbb{R}^{n+1}$ containing X. Suppose that u vanishes in a neighborhood of the curve*

(3.9) $$\{(t, x);\ x=F_{t,t_0}(x(l)),\ t_-(l)\leqq t\leqq t_+(l)\}.$$

Then $\operatorname{supp} u\cap X=\emptyset$.

Before giving the proof we make the following observation. Consider the special case when the obstacle Q is stationary and $F_{t,s}=\mathrm{Id}$. If $|x'(\sigma)|=1$, then Γ is a cylindric set and

$$X=\{(t, x);\ x=x(\sigma),\ 0\leqq\sigma\leqq l,\ t_0-\sigma\leqq t\leqq t_0+\sigma\}.$$

Then Theorem 3.1 says that if u solves the wave equation near X and $u(t, x)=0$ for $t_0-l-\varepsilon<t<t_0+l+\varepsilon$, $|x-x_1|<\varepsilon$ with some $\varepsilon>0$, then u vanishes in a neighborhood of X. This assertion can be easily proved by approximating γ with polygons and by applying the Holmgren theorem for the wave equation (see [14, Ch. IV., Theorem 1.5], or [13, Theorem 8.6.8].

4 Proof of Theorem 3.1

The idea of the proof is the following. We construct a sequence $X_\varepsilon(m)$, $m\in[0, l)$ of open subsets of $\mathbb{R}^{n+1}$ with non-characteristic C^1-boundary depending continuously on m, such that $\square u=0$ in a neighborhood of $X_\varepsilon(m)$ for any m, $X\subset X_\varepsilon(0)$ and $u=0$ in $X_\varepsilon(m)$ for m close to l. Then we prove that $X_\varepsilon(m)\cap\operatorname{supp} u=\emptyset$ for any m by applying the local Holmgren theorem ([13, Theorem 8.6.5]).

According to the assumptions, we have $\square u=0$ in A, $u=0$ in A_0, where A and A_0 are open sets such that $A\supset X$, A_0 contains (3.9). We can assume that $A_0\subset A$. Denote

$$\tilde{W}=\{(\sigma, t);\ -\delta<\sigma<l+\delta,\ t_-(l)-\delta<t<t_+(l)+\delta\}\times(-1, 1)^{n-1}.$$

By a standard argument we can find a diffeomorphism $G_1\colon \tilde{W}\to G_1(\tilde{W})\supset X$, such that $G_1(\sigma, t, 0)=(t, F_{t,t_0}(x(\sigma)))$. We can assume that $A\subset G_1(\tilde{W})$. Set $W=G_1^{-1}(A)$. Then $G_1\colon W\to A$ is a diffeomorphism, too. Without loss of generality we can assume that $W_0:=G_1^{-1}(A_0)$ is given by the conditions $(\sigma, t, q)\in W$, $l-\mu<\sigma$ with some $\mu>0$, where $q=(q_1, \ldots, q_{n-1})$.

Given any $m\in[0, l-\mu/2]$ we search a number $\alpha>1$ and a function $t(\sigma)$ such that

$$(4.1_+)\qquad \frac{d}{d\sigma}t(\sigma)=\alpha a_+(\sigma, t), \qquad \sigma\in I,$$
$$t(m)=t_0,$$
$$t(l)=t_+(l)+v.$$

Here v is a positive number which will be specified later. To solve $(4.1)_+$ denote by $t_\alpha(\sigma)$ the function $t(\sigma)$ satisfying the first and the second equation in (4.1_+). To deal with the third equation, observe that $t_1(l)\leqq t_+(l)$ (see (3.7)). On the other hand, $t_\alpha(l)$ depends continuously and monotonically on α and $t_\alpha(l)\geqq t_0+\alpha c_0(l-m)\to\infty$, as $\alpha\to\infty$. Thus there exists a unique $\alpha>1$ such that $t=t_\alpha$ satisfies the third equation in (4.1_+). Denote by $\alpha_{+,m}$, $t_{+,m}(\sigma)$ the solution of (4.1_+). Similarly, let $\alpha_{-,m}$, $t_{-,m}(\sigma)$ solve the problem

$$(4.1_-)\qquad \frac{d}{d\sigma}t(\sigma)=-\alpha a_-(\sigma, t), \qquad \sigma\in I,$$
$$t(m)=t_0,$$
$$t(l)=t_-(l)-v.$$

Given $m\in[0, l-\mu/2]$ we define the following analog of X:

$$X(m)=\{(t, x);\, x=F_{t,t_0}(x(\sigma)),\, m\leqq\sigma\leqq l,\, t_{-,m}(\sigma)\leqq t\leqq t_{+,m}(\sigma)\}\subset\Gamma.$$

Clearly, $X(m_1)\subset X(m_2)$ for $m_1>m_2$, $X\subset X(0)$. We choose $v>0$ small enough to ensure the inclusion $X(0)\subset A$. We aim to show that $X(0)\cap\operatorname{supp}u=\emptyset$. Observe that for $m\in(l-\mu, l-\mu/2]$ we have $X(m)\subset A_0$, thus $X(m)\cap\operatorname{supp}u=\emptyset$.

Given $m\in[0, l-\mu/2]$ consider the map $G_{2,m}\colon W\to\mathbb{R}^{n+1}$, defined by

$$(4.2)\qquad \begin{aligned} y_1&=(t_{+,m}(\sigma)-t_{-,m}(\sigma))/(t_{+,m}(l)-t_{-,m}(l)),\\ y_2&=(2t-t_{+,m}(\sigma)-t_{-,m}(\sigma))/(t_{+,m}(l)-t_{-,m}(l)),\\ y_k&=q_{k-2},\ k=3, \dots, n+1,\end{aligned}$$

$(\sigma, t, q)\in W$. Using the fact that $t_{+,m}-t_{-,m}$ is monotonically increasing function, we deduce that $G_{2,m}\colon W\to G_{2,m}(W)$ is a diffeomorphism. Consider the composition $G_m=G_1\circ G_{2,m}^{-1}$, $G_m\colon G_{2,m}(W)\to A$. Observe that if we consider the triangle

$$Y=\{y\in\mathbb{R}^{n+1};\, 0\leqq y_1\leqq 1,\, |y_2|\leqq y_1,\, y_3=\dots=y_{n+1}=0\},$$

then we have $X(m)=G_m(Y)$. Since $G_{2,m}$ depends continuously on m, we deduce that

$$U_0:=\bigcap_{0\leqq m\leqq l-\mu/2} G_{2,m}(W)$$

is open. Let U be another open set, such that $Y\subset U\Subset U_0$. So, we restricted G_m to a diffeomorphism $G_m\colon U\to G_m(U)$, where $Y\subset U$, $X(m)\subset G_m(U)\subset A$.

Next we introduce a family of open sets $X_\varepsilon(m)$ in the following way. Set

$$Y_\varepsilon=\bigcup_{y\in Y}B_\varepsilon(y), \qquad X_\varepsilon(m)=G_m(Y_\varepsilon).$$

Here $B_\varepsilon(y)=\{x; |x-y|<\varepsilon\}$ and $\varepsilon>0$ is chosen so that $Y_\varepsilon \Subset U$. Clearly, $X_\varepsilon(m)$ is open, $X(m)\subset X_\varepsilon(m)\subset A$ for any m. Furthermore, if ε is small enough, then $X_\varepsilon(m)\Subset A_0$ for $m=l-\mu/2$ and therefore for m sufficiently close to $l-\mu/2$. Hence, for such m we have $\overline{X_\varepsilon(m)}\cap \operatorname{supp} u=\emptyset$. We shall prove that in fact

$$\overline{X_\varepsilon(m)}\cap \operatorname{supp} u=\emptyset \quad \text{for all } m\in[0, l-\mu/2]. \tag{4.3}$$

Note that (4.3) yields Theorem 3.1 because $X\subset X(0)\subset X_\varepsilon(0)$.

Assume that (4.3) is not true. Set $m_0=\sup\{m; \overline{X_\varepsilon(m)}\cap \operatorname{supp} u \neq \emptyset\}<l-\mu/2$. Then one can easily prove that

$$\partial X_\varepsilon(m_0)\cap \operatorname{supp} u\neq\emptyset, \qquad X_\varepsilon(m_0)\cap \operatorname{supp} u=\emptyset. \tag{4.4}$$

Since $A_0\cap \operatorname{supp} u=\emptyset$, we can replace $\partial X_\varepsilon(m_0)$ in (4.4) by $\partial X_\varepsilon(m_0)\setminus A_0$. Below we will show that for any m the surface $\partial X_\varepsilon(m)\setminus A_0$ is non-characteristic. According to the local Holmgren theorem [13, Theorem 8.6.8], (4.4) implies that u must vanish in a neighborhood of $\partial X_\varepsilon(m_0)\setminus A_0$ which contradicts the choice of m_0.

It remains to prove that ε can be chosen so that for any $m\in[0, l-\mu/2]$ the surface $\partial X_\varepsilon(m)\setminus A_0$ is non-characteristic. In order to simplify the proof first we will show that this can be done for a fixed m. Let the outer normal ξ to $\partial X_\varepsilon(m)$ at some point $(\tilde t, \tilde x)\notin A_0$ be a characteristic vector (i.e. $|\xi_t|=|\xi_x|$). Without loss of generality we can assume $\xi=(1, \xi_x)$, $|\xi_x|=1$. We interpret $(\tilde t, \tilde x, \xi)$ as a covector. After a change of coordinates $y=G_m^{-1}(t, x)$, $(\tilde t, \tilde x, \xi)$ is transformed into $(\tilde y, \eta)$, where $\eta=(DG_m)^*(\tilde y)\,\xi$, $\tilde y=G_m^{-1}(\tilde t, \tilde x)$. Then η is normal to ∂Y_ε at $\tilde y\in\partial Y_\varepsilon$. On the other hand, $\tilde y\in\partial B_\varepsilon(\hat y)$ with some $\hat y\in Y$ and η is normal to $\partial B_\varepsilon(\hat y)$ at $\tilde y$. Therefore, $\varepsilon\eta=\tilde y-\hat y$. Since $\tilde y$ is a boundary element, then $B_\varepsilon(\tilde y)\cap Y=\emptyset$. By the convexity of Y we get

$$(y-\hat y)\cdot\eta\leqq 0 \quad \text{for all } y\in Y. \tag{4.5}$$

Passing to the coordinates $(t, x)=G_m(y)$ we transform the covector $(\hat y, \eta)$ into $(\hat t, \hat x, \hat\xi)$, where $(\hat t, \hat x)\in X(m)$. Since $|\hat y-\tilde y|=\varepsilon$, we have $|(\hat t, \hat x)-(\tilde t, \tilde x)|<C\varepsilon$, $|\hat\xi-\xi|<C\varepsilon$. Since $(\tilde t, \tilde x)\notin A_0$, changing the size of A_0 a little bit we may assume that $(\hat t, \hat x)\notin A_0$. Relation (4.5) implies that for any curve $t(s)$, $x(s)$ such that $t(0)=\hat t$, $x(0)=\hat x$, $(t(s), x(s))\in X(m)$ for $s\geqq 0$ sufficiently small, we have $\hat\xi\cdot(t'(0), x'(0))\leqq 0$, therefore $\xi\cdot(t'(0), x'(0))/|(t'(0), x'(0))|\leqq C\varepsilon$. To get a contradiction it suffices to show that there exists $\beta>0$, such that for any $(\hat t, \hat x)\in X(m)\setminus A_0$ we can find a curve of the type described above, such that

$$\xi\cdot(t'(0), x'(0))/|(t'(0), x'(0))|\geqq\beta. \tag{4.6}$$

We have to consider two cases. Recall that $\hat y=G_m^{-1}(\hat t, \hat x)$, $0\leqq\hat y_1<1$, $|\hat y_2|\leqq\hat y_1$.

Case A. $0<\hat y_1<1$, $|\hat y_2|<\hat y_1$. Consider the line $y'=(0, 1, 0, \dots, 0)$, $y(0)=\hat y$. Then the tangent to the curve $(t(s), x(s))=G_m(y(s))$ at $s=0$ has the form $C(1, v)$, where $v=v(\hat t, \hat x)$, $C>0$. Then $(1, v)\cdot(1, \xi_x)=1+v\cdot\xi_x>\beta$, where $\beta=\min\{1-|v(t, x)|; (t, x)\in X(m)\}>0$. Thus (4.6) holds.

Case B. $0\leqq\hat y_1<1$, $|\hat y_2|=\hat y_1$. First assume $\hat y_2=\hat y_1$. Then $\hat t=t_{+,m}(\sigma)$ for some $\sigma\in[0, l)$ and $x=F_{\hat t, t_0}(x(\sigma))$. Consider the line $y'=(1, 1, 0, \dots, 0)$, $y(0)=\hat y$. The tangent to the curve $G_m(y(s))$ at $s=0$ is given by $C\lambda$, where $C>0$, $\lambda=(1, \lambda_x)$, λ_x

$=v(\hat{t}, \hat{x})+\alpha_{+,m}^{-1}a_+(\sigma, \hat{t})D_x F_{\hat{t},t_0}(x(\sigma))x'(\sigma)$ and $\alpha_{+,m}>1$. According to (3.5) we have $|\lambda_x|<1$ and moreover $|\lambda_x|\leqq\lambda_0<1$, where λ_0 does not depend on $(\hat{t}, \hat{x})\in X(m)$. Therefore $\xi\cdot\lambda>|\lambda|\beta$, with some constant $\beta>0$, which proves (4.6) in this case. If $-\hat{y}_2=\hat{y}_1>0$ we can proceed in a similar manner considering the curve $y'=(-1, 1, 0, \ldots, 0)$, $y(0)=\hat{y}$.

Therefore, (4.6) holds and for any fixed $m\in[0, l-\mu/2]$ the surface $\partial X_\varepsilon(m)\setminus A_0$ is non-characteristic for $0<\varepsilon<\varepsilon(m)$. Since m runs over a compact interval and $\varepsilon(m)$ depends on the behavior of G_m on a compact set, we deduce that $\varepsilon>0$ can be chosen so that $\partial X_\varepsilon(m)\setminus A_0$ is non-characteristic for all $m\in[0, l-\mu/2]$. This completes the proof of Theorem 3.1.

Remark. In the proof we have several times used estimate (3.6) which is implied by (iii). Nevertheless, we note that Theorem 3.1 remains true without condition (iii) with minor changes in the proof. However then we must assume that the set X is well-defined, i.e. the solutions $t_\pm$ of (3.7) can be continued on $[0, l]$.

5 Proof of Theorem 1.2

In this section we prove Theorem 1.2 by means of Theorem 3.1. Let $Q_i\in\mathcal{Q}$, $i=1, 2$ and suppose that $S_1=S_2$. Denote by $U_i(t, s)$ the corresponding propagators. We fix $s\in\mathbb{R}$ and $\varphi\in[C_0^\infty(\mathbb{R}^n)\times C_0^\infty(\mathbb{R}^n)]\cap D_-^\rho$ and set

$$u_i(t, x)=[U_i(t, s)\varphi]_1, \qquad u=u_1-u_2. \tag{5.1}$$

The number $\rho>0$ is chosen so that $\mathcal{O}_1(t)\cup\mathcal{O}_2(t)\subset B_{\rho/2}$ for all t. Fix $T>0$ satisfying (iii) both for Q_1 and Q_2. By Proposition 2.2 we have

$$u(t, x)=0 \quad \text{for } |x|\geqq\rho, \quad \text{all } t. \tag{5.2}$$

Suppose that $Q_1\neq Q_2$. Then without loss of generality we can assume that $Q_1\cap\partial Q_2\neq\emptyset$. Further we will show that u vanishes on a part of ∂Q_2. For this reason we prepare the following.

Proposition 5.1 *Suppose that $Q_1\cap\partial Q_2\neq\emptyset$. Then there exists $(t_0, x_0)\in Q_1\cap\partial Q_2$ and a neighborhood $U\Subset\Omega_1(t_0)$ of x_0, such that every solution u of the wave equation in $Q_1\cap Q_2$ which is smooth in $\bar{Q}_1\cap\bar{Q}_2$ and satisfies* (5.2), *vanishes on $\{t_0\}\times\partial\Omega_2(t_0)\cap U$.*

Proof. Pick $(t_0, x_0)\in Q_1\cap\partial Q_2$ and fix some $x_1\notin B_{2\rho}$. Let $\gamma\subset\Omega_1(t_0)$ be a curve of the kind (3.3) joining x_0 and x_1. Choose some family $F_{t,s}^{(1)}$ of diffeomorphisms related to Q_1 with properties (a), (b), (c), (d) listed in Sect. 3. Denote by Γ and X the sets (3.4) and (3.8) respectively, associated with t_0, x_0, x_1, γ and $F_{t,s}^{(1)}$. We have $X\subset Q_1$, $(t_0, x_0)\in\partial Q_2$. Clearly, $\sigma=\sigma(t, x)$ is a well-defined function on Γ. Set

$$\tilde{\sigma}=\sup_{(t,x)\in\Gamma\cap(\mathbb{R}^{n+1}\setminus Q_2)}\sigma(t, x). \tag{5.3}$$

According to (d), the supremum above could be taken only for $|t|\leqq 2T$. Hence, there exists $(\tilde{t}, \tilde{x})\in\Gamma\cap(\mathbb{R}^{n+1}\setminus Q_2)$, such that $\tilde{\sigma}=\sigma(\tilde{t}, \tilde{x})$, i.e. $\tilde{x}=F_{\tilde{t},t_0}(x(\tilde{\sigma}))$. By (5.3), $(\tilde{t}, \tilde{x})\in\partial Q_2$. Moreover, $\tilde{\sigma}<l$ because otherwise we would have $\tilde{x}=x_1\in B_{2\rho}$. Set

$$\tilde{X}=\{(t, x);\ x=F_{t,\tilde{t}}^{(1)}(\tilde{x}(\sigma)),\ \tilde{\sigma}\leqq\sigma\leqq l,\ \tilde{t}_-(\sigma)\leqq t\leqq\tilde{t}_+(\sigma)\},$$

where $\tilde{t}_\pm$ solve the problem $\dfrac{d}{d\sigma}\tilde{t}_\pm=\pm a_\pm(\sigma, \tilde{t}_\pm)$, $\tilde{t}_\pm(\tilde{\sigma})=\tilde{t}$, and $\tilde{x}(\sigma)=F_{\tilde{t},t_0}(x(\sigma))$ (compare with (3.7)). These arguments show that without loss of generality we can assume that $(\tilde{t}, \tilde{x})=(t_0, x_0)$ and that for the set X defined by (3.8) (with $F_{t,s}^{(1)}$ instead of $F_{t,s}$), we have

(5.4)
$$(t, F_{t,t_0}(x(\sigma)))\in Q_1\cap Q_2 \quad \text{for } \sigma>0 \quad \text{and}$$
$$X\subset Q_1\cap\bar{Q}_2, \qquad X\cap\partial Q_2=(t_0, x_0).$$

Clearly, $|x_0|\leqq\rho/2$. We will first show that

(5.5)
$$u(t_0, x_0)=0.$$

Given $\varepsilon\in(0, l)$, denote by X_ε the set (3.8) associated with $t_0, x(\varepsilon), x_1, \gamma$ and $F_{t,s}^{(1)}$. More precisely, let $t_\pm^\varepsilon(\sigma)$ solve the problem

(5.6)
$$\frac{d}{d\sigma}t_\pm^\varepsilon=\pm a_\pm(\sigma, t_\pm^\varepsilon), \qquad t_\pm^\varepsilon(\varepsilon)=t_0.$$

Then X_ε is given by

$$X_\varepsilon=\{(t, x);\ x=F_{t,t_0}^{(1)}(x(\sigma)),\ \varepsilon\leqq\sigma\leqq l,\ t_-^\varepsilon(\sigma)\leqq t\leqq t_+^\varepsilon(\sigma)\}.$$

Comparing (3.7) and (5.6) we see that $t_-(\sigma)<t_-^\varepsilon(\sigma)$ and $t_+^\varepsilon(\sigma)<t_+(\sigma)$, thus $X_\varepsilon\subset X$ and from (5.3) we get $X_\varepsilon\subset Q_1\cap Q_2$. Now we are in position to apply Theorem 3.1. Note that $F_{t,t_0}^{(1)}(x(l))=x(l)=x_1\notin B_{2\rho}$, hence u vanishes for x close to x_1 and all t in view of (5.2). Thus Theorem 3.1 yields $u=0$ on X_ε. In particular we get $u(t_0, x(\varepsilon))=0$. Taking $\varepsilon\to 0$ we obtain (5.5).

Arguing as before we see that in order to prove the proposition it suffices to find a neighborhood U of x_0 with the following property.

(*) For any $\tilde{x}\in\partial\Omega_2(t_2)\cap U$ we can construct a curve $\tilde{\gamma}$ joining $\tilde{x}$ and x_1 such that the set $\tilde{X}$ (see (3.8)) associated with $t_0, \tilde{x}, x_1$, the curve $\tilde{\gamma}$ and $F_{t,s}^{(1)}$, satisfies $\tilde{X}\subset Q_1\cap\bar{Q}_2$, $\tilde{X}\cap\partial Q_2=(t_0, \tilde{x})$.

We will construct $\tilde{\gamma}$ as a small perturbation of γ. Using a standard argument we find an open set $V\Subset\Omega_1(t_0)\cap B_\rho$ and a diffeomorphism $M\colon V\to B_1=\{y;\ |y|<1\}$ such that $x_0\in V$ and $M(\partial\Omega_2(t_0)\cap V)$ is given by $y_n=0$. Pick a smooth function ψ so that $\psi(y)=1$ for $|y|<1/3$ and $\psi(y)=0$ for $|y|>2/3$. Given $\xi\in\mathbb{R}^{n-1}$ consider the map $P_\xi(y)=y+\psi(y)(\xi, 0)$. Here $(\xi, 0)=(\xi_1, \ldots, \xi_{n-1}, 0)$. Clearly, $P_\xi\colon B_1\to B_1$ is a diffeomorphism provided $|\xi|$ is sufficiently small and so is $R_\xi=M^{-1}\circ P_\xi\circ M\colon V\to V$. We choose $\mu>0$ so that $F_{t,t_0}^{(2)}(V)\Subset\Omega_1(t)\cap B_\rho$ for $|t-t_0|<\mu$. Here $F_{t,s}^{(2)}$ is a family of the kind constructed in Sect. 3 associated with Q_2.

Further, let $\phi(t)$ be a smooth cut-off function such that $\phi(t)=1$ for $|t-t_0|<\mu/3$, $\phi(t)=0$ for $|t-t_0|>2\mu/3$. We set

$$R_\xi^t(x)=\begin{cases}[F_{t,t_0}^{(2)}\circ R_{\phi(t)\xi}\circ F_{t_0,t}^{(2)}](x) & \text{for } x\in F_{t,t_0}^{(2)}(V),\\ x, \text{ otherwise.}\end{cases}$$

Clearly, R_ξ^t is a global diffeomorphism in $\mathbb{R}^n$ and $\tilde{R}_\xi(t,x):=(t,R_\xi^t(x))$ is a diffeomorphism in $\mathbb{R}^{n+1}$. Note that $\tilde{R}_\xi$ preserves Q_1, Q_2 and any plane $t=\text{const}$. In fact, $\tilde{R}_\xi\neq \mathrm{Id}$ only in a small neighborhood of (t_0,x_0). Moreover, $\tilde{R}_\xi\to\mathrm{Id}$ together with its derivatives, as $|\xi|\to 0$. Set $x_\xi=R_\xi^{t_0}(x_0)\in\partial\Omega_2(t_0)$.

Let $X(0)\subset Q_1$ be the same as in Sect. 4 (with $F_{t,s}^{(1)}$ instead of $F_{t,s}$). According to (5.4) we have $X(0)\subset Q_1\cap\bar{Q}_2$, $X(0)\cap\partial Q_2=(t_0,x_0)$. Set $\Gamma_\xi=\tilde{R}_\xi(\Gamma)$, $X_\xi(0)=\tilde{R}_\xi(X(0))\subset\Gamma_\xi$. Clearly,

$$\text{(5.7)}\qquad \begin{aligned}\Gamma_\xi&=\{(t,x);\,x=F_{\xi;t,t_0}^{(1)}(x_\xi(\sigma)),\,\sigma\in I,\,t\in\mathbb{R}\},\\ X_\xi(0)&=\{(t,x);\,x=F_{\xi;t,t_0}^{(1)}(x_\xi(\sigma));\,\sigma\in[0,l],\,t_{-,0}(\sigma)\leqq t\leqq t_{+,0}(\sigma)\},\end{aligned}$$

where $F_{\xi;t,s}^{(1)}=R_\xi^t\circ F_{t,s}^{(1)}\circ(R_\xi^s)^{-1}$, $x_\xi(\sigma)=R_\xi^{t_0}(x(\sigma))$. In other words, Γ_ξ is defined as Γ with the same t_0, x_1 and with x_0, γ, and $F_{t,s}^{(1)}$ replaced by x_ξ, $\gamma_\xi=\{x;\,x=x_\xi(\sigma)\}$ and $F_{\xi;t,s}^{(1)}$. Note that $F_{\xi;t,s}^{(1)}$ is generated by the vector field $w_\xi=R_{\xi*}^t w$, where w_ξ is of the kind $w_\xi=(1,v_\xi)$. Since $v_\xi=v$ for $\xi=0$ we can choose $|\xi|$ small enough to arrange the inequality $|v_\xi|<1$. Thus v_ξ satisfies (3.2) and $F_{\xi;t,t_0}^{(1)}$ satisfies conditions (a), (b), (c), (d) of Sect. 2.

Next we construct a set $X_\xi\subset\Gamma_\xi$ of the kind (3.8) associated with t_0, x_ξ, x_1, γ_ξ and $F_{\xi;t,s}^{(1)}$. In other words,

$$\text{(5.8)}\qquad X_\xi=\{(t,x);\,x=F_{\xi;t,t_0}^{(1)}(x_\xi(\sigma));\,\sigma\in[0,l],\,t_-^\xi(\sigma)\leqq t\leqq t_+^\xi(\sigma)\}.$$

Here $t_\pm^\xi$ solves an equation of the kind (3.7) with $a_\pm=a_\pm^\xi(\sigma,t)$ defined as in (3.5). Consider (5.7). We have $(d/d\sigma)t_{\pm,0}=\pm\alpha_{\pm,0}a_\pm(\sigma,t_{\pm,0})$, where $\alpha_{\pm,0}>1$. Since $a_\pm^\xi=a_\pm$ for $\xi=0$, we see that $a_\pm^\xi\leqq\alpha_{\pm,0}a_\pm$ for $|\xi|<\varepsilon$ with some $\varepsilon>0$. Therefore we get $t_{-,0}(\sigma)\leqq t_-^\xi(\sigma)$, $t_+^\xi(\sigma)\leqq t_{+,0}(\sigma)$. Compare (5.7) and (5.8) to deduce $X_\xi\subset X_\xi(0)$. Hence, $X_\xi\subset Q_1\cap\bar{Q}_2$, $X_\xi\cap\partial Q_2=(t_0,x_\xi)$ for $|\xi|<\varepsilon$. This verifies (*). Arguing as in the proof of (5.5) we complete the proof of Proposition 5.1.

Now, let u be as in (5.1). By Proposition 5.1, $u_1(t_0,\cdot)-u_2(t_0,\cdot)$ vanishes on $\partial\Omega_2(t_0)\cap U$, where $U\Subset\Omega_1(t_0)$. Since u_2 satisfies the Dirichlet boundary condition on ∂Q_2, we get

$$\text{(5.9)}\qquad [U_1(t_0,s)\varphi]_1|_{\partial\Omega_2(t_0)\cap U}=0$$

for all $s\in\mathbb{R}$, $\varphi\in[C_0^\infty(\mathbb{R}^n)\times C_0^\infty(\mathbb{R}^n)]\cap D_-^\varrho$.

Choose a smooth $f=(f_1,f_2)\in\mathscr{H}_1(t_0)$, such that $f_1\neq 0$ on $\partial\Omega_2(t_0)\cap U$. We claim that there exist sequences $s_k\in\mathbb{R}$, $\varphi_k\in[C_0^\infty(\mathbb{R}^n)\times C_0^\infty(\mathbb{R}^n)]\cap D_-^\varrho$, with the property

$$\text{(5.10)}\qquad U_1(t_0,s_k)\varphi_k\to f\quad\text{in } \mathscr{H}_1(t_0),\quad\text{as } k\to\infty.$$

According to the proof of Proposition 2.1, we have $U_1(s,t_0)f=U_-(s+T)\cdot U_1(-T,t_0)f$ for $s<-T$. Here $U_-(t)$ is the propagator related to the stationary obstacle $\mathcal{O}_1(-T)$. Since the wave operator $s-\lim_{t\to-\infty} U_0(-t)U_-(t)$ exists [14], there is a $g\in\mathcal{H}_0$, such that

(5.11) $$\|U_0(s)g-U_1(s,t_0)f\|_{\mathcal{H}_0}\to 0, \quad \text{as } s\to-\infty.$$

Given $k\in\mathbb{N}$, choose $g_k\in C_0^\infty\times C_0^\infty$ with $\|g-g_k\|_{\mathcal{H}_0}<1/k$. Then pick $s_k< -\max\{|x|;\, x\in\operatorname{supp} g_k\}-\rho$ such that $\|U_0(s)g-U_1(s,t_0)f\|_{\mathcal{H}_0}<1/k$ for $s\leqq s_k$. Thus

$$\|U_0(s_k)g_k-U_1(s_k,t_0)f\|_{\mathcal{H}_0}<\frac{2}{k}.$$

Setting $\varphi_k=U_0(s_k)g_k$, we see that for $t\leqq 0$ $U_0(t)\varphi_k=0$ in $B_{\rho+t}$ by Huygens' principle, therefore $\varphi_k\in D_-^\rho$. In particular, $\varphi_k\in\mathcal{H}_1(s_k)$ and

$$\|U_1(t_0,s_k)\varphi_k-f\|_{\mathcal{H}_1(t_0)}<\frac{2C}{k},$$

where $C=\sup_{s,t}\|U(t,s)\|$. Thus (5.10) is verified.

Poincaré inequality combined with (5.10) implies that $[U_1(t_0,s_k)\varphi_k]_1\to f_1$ in $H^1_{\mathrm{loc}}(\Omega_1(t_0))$. Therefore,

$$[U_1(t_0,s_k)\varphi_k]_1|_{\partial\Omega_2(t_0)\cap U}\to f_1|_{\partial\Omega_2(t_0)\cap U}$$

in $H^{1/2}(\partial\Omega_2(t_0)\cap U)$, as $k\to\infty$. Taking into account (5.9) we deduce that $f_1=0$ on $\partial\Omega_2(t_0)\cap U$, which contradicts our choice of f_1.

The proof of Theorem 1.2 is complete.

Remark. It should be mentioned that condition (iii) has been used twice in the proof of Theorem 1.2. First, to show that the solutions of (3.7) can be continued in the whole interval $[0,l]$ (which is not true in general) and, secondly, to show that the set $\{U(t_0,s)\varphi;\, s\in\mathbb{R},\, \varphi\in[C_0^\infty(\mathbb{R}^n)\times C_0^\infty(\mathbb{R}^n)]\cap D_-^\rho\}$ is dense in $\mathcal{H}(t_0)$ (see (5.10)). The second condition is related to the local energy decay at $t\to-\infty$ (see (5.11)).

6 The example

In this section we prove Theorem 1.3. To this end we construct explicitly a family of periodically moving obstacles satisfying (i), (ii) with the same generalized scattering kernel. For the sake of simplicity we assume $n=3$. To avoid confusions in the notations we note that for any vector x the subscript j in x_j will always denote the j-th component of x. For example, x_1^k, $x_1(t)$ denote the first component of x^k and $x(t)$, respectively.

It is not hard to see that there exists a function $\psi\in C_0^\infty(\mathbb{R})$ with the properties

(6.1) $$\psi(\sigma)=\begin{cases}1, & \text{for } |\sigma|\leqq 2,\\ 0, & \text{for } |\sigma|\geqq 11/4,\end{cases}$$

$$|\psi|\leqq 1, \quad |\psi'|\leqq 3/2.$$

Next we set $\phi(t, \sigma)=\psi(\sigma)\sin(k(\sigma-t/2))$. Here k is a large parameter. In what follows we assume k fixed so that

(6.2) $$k\geqq 2\pi.$$

Consider the hypersurface $\Sigma\subset\mathbb{R}^4$ and the domain $Q_0\subset\mathbb{R}^4$ given by

$$\Sigma=\{(t, x)\in\mathbb{R}^4;\ (x_2-\phi(t, x_1))^2+x_3^2=r^2,\ t\in\mathbb{R},\ |x_1|<3\},$$
$$Q_0=\{(t, x)\in\mathbb{R}^4;\ (x_2-\phi(t, x_1))^2+x_3^2<r^2,\ t\in\mathbb{R},\ |x_1|<3\}.$$

Here $r\in(0, 1/2)$ is a small parameter which will be specified later. Clearly, Σ is a smooth manifold and $\Sigma\subset\partial Q_0$. We set $\Omega_0(t)=\{x;\ (t, x)\in Q_0\}$. The domain $\Omega_0(t)$ can be considered as a small neighborhood of the curve

(6.3) $$\gamma_t=\{x;\ x_2=\phi(t, x_1),\ x_3=0,\ |x_1|\leqq 3\}.$$

We shall prove below that Σ is time-like, i.e. that the normal to Σ satisfies (ii).

Let us define a velocity field v associated with Σ. Further we denote by K_α the cube

$$K_\alpha=\{x\in\mathbb{R}^3;\ |x_i|\leqq\alpha,\ i=1, 2, 3\}.$$

Pick a function $\chi\in C_0^\infty(\mathbb{R}^3)$, such that $|\chi|\leqq 1$, $\chi(x)=1$ for $x\in K_{11/4}$ and $\chi(x)=0$ for $x\notin K_3$. Given $(t, x)\in\mathbb{R}^4$ we set

$$v(t, x)=\tfrac{1}{2}\chi(x)(1,\ \psi'(x_1)\sin(k(x_1-t/2)),\ 0)$$

Clearly, $v(t, x)=(1/2, 0, 0)$ for $x\in K_2$ while $v=0$ for $x\notin K_3$. Let us estimate $|v|$. We have

$$|v|^2\leqq\tfrac{1}{4}(1+[\psi'(x_1)\sin(k(x_1-t/2))]^2)\leqq\tfrac{1}{4}(1+\tfrac{9}{4})<1,$$

by virtue of (6.1). Next, we claim that $w=(1, v)$ is tangent to Σ. Indeed, the vector

$$\nu=((\phi-x_2)\phi_t,\ (\phi-x_2)\phi_{x_1},\ -(\phi-x_2),\ x_3),$$

where $x\in\Sigma$, $\phi=\phi(t, x_1)$, is normal to Σ. Since $x_1\in\operatorname{supp}\psi$ and $x\in\Sigma$ yields $x\in K_{11/4}$ and therefore $\chi(x)=1$, we find

$$\nu\cdot w=(\phi-x_2)[\phi_t+\tfrac{1}{2}\chi(x)(\phi_{x_1}-\psi'(x_1)\sin(k(x_1-t/2)))]=0.$$

In particular, it follows that $|\nu_t|<|\nu_x|$. Indeed, $0=\nu_t+\nu_x\cdot v$ implies $|\nu_t|=|\nu_x\cdot v|<|\nu_x|$. Note that v satisfies only the second and the third condition in (3.2) while instead of $v=v_N$ on ∂Q_0 we have that $w=(1, v)$ is tangent to ∂Q_0. Nevertheless, this is sufficient to conclude that $v(t, x)$ generates a flow $F_{t,s}\colon\mathbb{R}^3\to\mathbb{R}^3$ that maps $\Omega_0(s)$ onto $\Omega_0(t)$. Note that $F_{t,s}=\mathrm{Id}$ on $\mathbb{R}^3\setminus K_3$. Denote $v_0=(1/2, 0, 0)$.

Lemma 6.1 *Let $x\in K_2$. Then $F_{t,s}(x)=x+(t-s)v_0$ for $|t-s|\leqq 2\operatorname{dist}(x, \partial K_2)$.*

Proof. Put $G_{t,s}(x)=x+(t-s)v_0$. Then $(d/dt)G_{t,s}(x)=v_0$. Furthermore, if $|t-s| \leqq 2\,\mathrm{dist}(x, \partial K_2)$, then $G_{t,s}(x)\in \bar{K}_2$ thus $v_0=v(t, G_{t,s}(x))$. Therefore, both $y=F_{t,s}(x)$ and $y=G_{t,s}(x)$ solve the problem

$$\frac{d}{dt}y=v(t, y), \qquad |t-s|\leqq 2\,\mathrm{dist}(x, \partial K_2),$$
$$y(s)=x.$$

Consequently, $F_{t,s}=G_{t,s}$ for such s, t which completes the proof.

Proposition 6.2 *There is no piecewise smooth curve* $\{x;\, x=x(t),\ a\leqq t\leqq b\}$ *such that* $x_1(a)=3$, $x(t)\in\overline{\Omega_0(t)}$ *for* $a\leqq t\leqq b$, $x_1(b)=-3$, *and* $|x'(t)|\leqq 1$.

Piecewise smooth here means that $x(t)$ is continuous and there exists a partition $a=t_0<t_1<\ldots<t_N=b$ of $[a, b]$, such that $x(t)$ is C^∞-smooth on each subinterval $[t_{j-1}, t_j]$, $j=1, \ldots, N$.

Proof. Suppose the contrary. Let $p=\{x=x(t),\ a\leqq t\leqq b\}$ be a curve with properties given above. Set

$$a_0=\sup\{t\in[a, b];\ x_1(t)\geqq 0\}.$$

Clearly, $x_1(a_0)=0$. Since $x_1(b)=-3$ and $|x'|\leqq 1$, it follows that $b-a_0\geqq 3$. Thus setting

$$b_0=a_0+2\pi/k,$$

we conclude that $b_0-a_0\leqq 1$ (see (6.2)) hence $[a_0, b_0]\subset[a, b]$. Consider the curves

$$p_0=\{x;\ x=x(t),\ a_0\leqq t\leqq b_0\}\subset p,$$
$$\tilde{p}_0=\{x;\ x=\tilde{x}(t),\ a_0\leqq t\leqq b_0\}\subset\overline{\Omega_0(a_0)},$$

where $\tilde{x}(t)=F_{a_0,t}(x(t))$. Assume in what follows that $t\in[a_0, b_0]$. Since $x_1(a_0)=0$, $|x'|\leqq 1$, $0<b_0-a_0\leqq 1$, we have $|x_1(t)|\leqq 1$. Further, since $x(t)\in\overline{\Omega_0(t)}$ it follows that $|x_2(t)|\leqq 1+r$, $|x_3(t)|\leqq r$. Hence $\mathrm{dist}(x(t), \partial K_2)\geqq 1-r>1/2$. By Lemma 6.1, $\tilde{x}(t)=F_{a_0,t}(x(t))=x(t)-(t-a_0)v_0\in K_2$. Therefore $\tilde{p}_0$ is given by

$$\tilde{p}_0=\{x;\ x=x(t)-(t-a_0)v_0,\ a_0\leqq t\leqq b_0\}.$$

From the inequalities $|\tilde{x}'|\leqq 3/2$, $0<b_0-a_0\leqq 1$ we deduce that the length $l(\tilde{p}_0)$ of $\tilde{p}_0$ admits the estimate $l(\tilde{p}_0)\leqq 3/2$. On the other hand, $\tilde{p}_0\subset\overline{\Omega_0(a_0)}\cap K_2$, which can be considered as a neighborhood of the curve

$$\gamma'=\{x;\ x_2=\sin(k(x_1-a_0/2)),\ x_3=0,\ |x_1|\leqq 2\}\subset\gamma_{a_0},$$

(compare with (6.3)) with a size depending on r. It is not hard to see that the length $l(\gamma'')$ of the path

$$\gamma''=\{x;\ x_2=\sin(k(x_1-a_0/2)),\ x_3=0,\ -\pi/k\leqq x_1\leqq 0\}\subset\gamma'$$

is greater than 2. Therefore, the length of the shortest curve $\{x=y(t),\ A\leqq t\leqq B\}$ in $\overline{\Omega_0(a_0)}\cap\{x;\ -\pi/k\leqq x_1\leqq 0\}$ with the properties $y_1(A)=0$, $y_1(B)=-\pi/k$ tends to $l(\gamma'')>2$, as $r\to 0$. Since $l(\tilde{p}_0)\leqq 3/2$, we conclude that for r small enough $\tilde{x}_1(b_0)>-\pi/k$. Thus $x_1(b_0)=\tilde{x}_1(b_0)+(b_0-a_0)/2=\tilde{x}_1(b_0)+\pi/k>0$. This fact contradicts the choice of a_0. This completes the proof of the proposition.

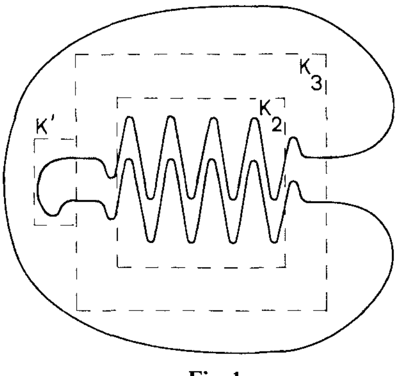

Fig. 1.

Let $Q \subset \mathbb{R}^4$ be a domain with smooth boundary such that $\Omega(t) = \{x; (t, x) \in Q\}$ satisfies the conditions

(A1) $\Omega(t) \cap K_3 = \Omega_0(t)$;
(A2) $\Omega(t) \cap (\mathbb{R}^3 \setminus K_3)$ is stationary (time-independent);
(A3) if $K' = \{x; -4 \leqq x_1 \leqq -3, |x_2| \leqq 1, |x_3| \leqq 1\}$, then
$\partial\Omega(t) \cap \partial K' = \partial\Omega_0(t) \cap \partial K' = \{x; x_1 = -3, x_2^2 + x_3^2 = r^2\}$;
(A4) $\Omega(t) \subset K_5$;

(see Fig. 1). Clearly, Q satisfies (i), (ii) with $\rho > 5\sqrt{3} = \frac{1}{2} \operatorname{diam} K_5$. Moreover, $\Omega(t + 4\pi/k) = \Omega(t)$ thus the motion is periodic with period $4\pi/k$. Below we denote $\Omega' = \Omega(t) \cap K'$, which does not depend on t by virtue of (A2).

As a consequence of Proposition 6.2 we obtain.

Corollary 6.3 *There is no piecewise smooth curve* $\{x = x(t),\ a \leqq t \leqq b\}$ *such that* $x(a) \notin B_\rho$, $x(t) \in \overline{\Omega(t)}$ *for any* $t \in [a, b]$, $x(b) \in \Omega'$ *and* $|x'| \leqq 1$.

In other words, if we travel in the exterior of the obstacle with a speed not greater than 1, starting from $\mathbb{R}^3 \setminus B_\rho$, we will never reach Ω'. Let us compare Corollary 6.3 with the approach in Sect. 3. It is not hard to see that for any choice of t_0, $x_0 \in \Omega'$, $F_{t,s}$ and γ joining x_0 and some $x_1 \notin B_\rho$ the solution $t_-(\sigma)$ of problem (3.7) cannot be continued on I. Indeed, supposing the contrary we would obtain a contradiction with Corollary 6.3. Moreover, following the construction given above one can find a moving obstacle for which the unique continuation property fails, i.e. there exists a non-trivial solution of (1.1) vanishing for large $|x|$, all t.

Using the above proposition we shall show that every solution $U(b, a)f$, $b > a$, with $\operatorname{supp} f \cap B_\rho = \emptyset$ vanishes in Ω'. To this end we are going to apply the principle of causality. It is known (see [9, Lemma 2] or [1]) that if $f = 0$ in $B_R \cap \overline{\Omega(a)}$, then $U(b, a)f = 0$ in $B_{R-|b-a|} \cap \Omega(b)$. Therefore, given $y \in \operatorname{supp} U(b, a)f \subset \overline{\Omega(b)}$ there exists $x \in \operatorname{supp} f \subset \overline{\Omega(a)}$, such that $|y - x| \leqq |b - a|$. Iterating this argument N times we obtain the following.

Lemma 6.4 (causality) *Let Q satisfy* (i), (ii). *Let* $y \in \operatorname{supp} U(b, a)f$, $f \in \mathscr{H}(a)$, $b \geqq a$. *Then for any* $N \in \mathbb{N}$ *and for any partition* $a = t_0 < t_1 < \ldots < t_N = b$ *of the interval* $[a, b]$, *there exists* $x^0 \in \operatorname{supp} f$ *and a polygon*

$$p = \{x; x = x(t), a \leqq t \leqq b\}$$

with vertices $x^0, \ldots, x^N$, *of the kind*

$$x(t)=x^{j-1}+\frac{x^j-x^{j-1}}{t_j-t_{j-1}}(t-t_{j-1}) \quad \text{for } t_{j-1}\leqq t\leqq t_j,$$

$j=1,\ldots,N$, *such that* $x^0=x(t_0)$, $y=x^N=x(t_N)$ *and* $x^j=x(t_j)\in\overline{\Omega(t_j)}$, $|x^j-x^{j-1}| \leqq |t_j-t_{j-1}|$ *for* $j=1,\ldots,N$.

Remark. It is natural to conjecture that the principle of causality for (1.1) admits the following formulation: Under the assumptions of Lemma 6.4 there exists a piecewise smooth curve $x=x(t)$, $a\leqq t\leqq b$ joining $y=x(b)$ with some $x^0 = x(a)\in\operatorname{supp} f\subset\overline{\Omega(a)}$, such that $|x'(t)|\leqq 1$, $x(t)\in\overline{\Omega(t)}$ for any t. The proof of this assertion however seems to be very technical. Nevertheless, the above lemma is sufficient for our purposes. Note that we have $|x'(t)|\leqq 1$ in Lemma 6.4, but in general $x(t)$ may leave $\overline{\Omega(t)}$ for $t\neq t_j$, $j=1,\ldots,N$.

Combining Proposition 6.2 and Lemma 6.4 we are going to prove the following.

Proposition 6.5 *Let* Q *be a domain with properties* (A1)–(A4). *Then for any* $b\geqq a$ *and for any* $f\in\mathscr{H}(a)$ *with* $\operatorname{supp} f\cap B_\rho=\emptyset$ *we have* $U(b,a)f=0$ *in* Ω'.

Proof. Suppose the contrary. Then given $N\in\mathbb{N}$ there exists a polygon of the kind given in Lemma 6.4 joining some $x^0=x(a)\notin B_\rho$ with some $x^N=x(b)\in\Omega'$. As mentioned above, in general $x(t)$ may leave $\overline{\Omega(t)}$ for some t, thus we cannot apply directly Corollary 6.3 to obtain a contradiction. Below it is convenient to assume that $t_j=a+j(b-a)/N$, $j=1,\ldots,N$. The integer N will be specified later.

By (A1)–(A4) we see that if $\max_j |t_j-t_{j-1}|$ is sufficiently small (smaller than 1/2 is sufficient), then p must pass through K_3. In particular, there exist $\hat{a}, \hat{b}$ such that $x_1(\hat{a})=3$, $x_1(\hat{b})=-3$. Set

$$a'=\sup\{t;\, x_1(t)\geqq 3\}, \qquad b'=\inf\{t;\, x_1(t)\leqq -3\}.$$

Clearly, $x_1(a')=3$, $x_1(b')=-3$ and $x_1(t)\in[-3,3]$ for $t\in[a',b']\subset[a,b]$.

Next we observe that $\overline{\Omega_0(t)}$ admits the following characterization. Put $y(t,\sigma)=(\sigma,\phi(t,\sigma),0)$. Then γ_t is given by $x=y(t,\sigma)$, $|\sigma|\leqq 3$ (see (6.3)); and $x\in\overline{\Omega_0(t)}$ if and only if $|x_1|\leqq 3$ and $|x-y(t,x_1)|\leqq r$. Let $t\in[a',b']$ and pick $t_j\in[a',b']$ such that $|t-t_j|<(b-a)/N$. Then

$$\begin{aligned}|x(t)-y(t,x_1(t))| &\leqq |x(t)-x^j|+|x^j-y(t_j,x_1^j)|+|y(t_j,x_1^j)-y(t,x_1(t))| \\ &\leqq |t-t_j|+r+C_\phi|t-t_j|\leqq(1+C_\phi)\frac{b-a}{N}+r,\end{aligned}$$

where $C_\phi>0$ depends only on the derivatives of ϕ. We choose N so that

$$(1+C_\phi)\frac{b-a}{N}<r.$$

Then we have

$$|x(t)-y(t,x_1(t))|<2r,$$

which shows that if we replace r with $2r$ in the definition of $\Omega_0(t)$, then $x(t)\in\overline{\Omega_0(t)}$ for any $t\in[a', b']$. Recall that $x_1(a')=3$, $x_1(b')=-3$, $|x'|\leqq 1$. An application of Proposition 6.2 yields a contradiction and the proof is complete.

Now consider a family $\mathscr{F}$ of domains Q with the properties:

(B1) each $Q\in\mathscr{F}$ satisfies (A1)–(A4);
(B2) $\Omega_1(t)\cap(\mathbb{R}^3\backslash K')=\Omega_2(t)\cap(\mathbb{R}^3\backslash K')$ for any $Q_1\in\mathscr{F}$, $Q_2\in\mathscr{F}$.

Note that we do not impose any restriction on $\Omega(t)\cap K'$, thus the geometry of $\partial\Omega(t)\cap K'$ may be arbitrary, provided that (A3) holds. Hence $\mathscr{F}$ consists of infinitely many distinct domains Q and the corresponding obstacles $\mathscr{O}(t)$ move with the same period $4\pi/k$. Below we shall prove that all obstacles in $\mathscr{F}$ have the same generalized scattering kernel.

Let $Q_1\in\mathscr{F}$, $Q_2\in\mathscr{F}$. Choose $\varphi\in C_0^\infty(\mathbb{R}^3)\times C_0^\infty(\mathbb{R}^3)$ and set $f=U_0(-R-\rho)\varphi$, where $R>\max\{|x|;\ x\in\operatorname{supp}\varphi\}$. Huygens' principle implies $f\in D_-^\rho$. Set $u_i=[U_i(t,-R-\rho)f]_1$, where $U_i(t,s)$ is the propagator related to Q_i, $i=1,2$. By Proposition 6.5, $u_i=0$ in $\Omega_i'=\Omega_i(t)\cap K'$, $i=1,2$. On the other hand, $\Omega_1(t)\cap(\mathbb{R}^3\backslash K')=\Omega_2(t)\cap(\mathbb{R}^3\backslash K')$ by (B2). Therefore, both u_1 and u_2 solve the problem

$$\begin{aligned} \Box u&=0 && \text{in } Q_1,\\ u&=0 && \text{on } \partial Q_1,\\ (u,u_t)&=f && \text{for } t=-R-\rho. \end{aligned}$$

Hence $u_1=u_2$. Now let $K_i^\#$ be the generalized scattering kernels related to Q_i, $i=1,2$. By [9, Sect. 3] (see also [8, Theorem 9]) u_i has an asymptotic wave profile $u_i^\#(s,\omega)$ and

$$u_i^\#(s',\omega')=-\mathscr{R}\varphi(s',\omega')-\int_{|\omega|=1}\int_{-\infty}^{\infty} K_i^\#(s',\omega';s,\omega)\,\mathscr{R}\varphi(s,\omega)\,ds\,d\omega,$$

$\mathscr{R}\varphi$ being the translation representer of φ. Since $u_1^\#=u_2^\#$ we get

$$\int_{|\omega|=1}\int_{-\infty}^{\infty}(K_1^\#-K_2^\#)(s',\omega';s,\omega)\,\mathscr{R}\varphi(s,\omega)\,ds\,d\omega=0$$

for any $\varphi\in C_0^\infty(\mathbb{R}^3)\times C_0^\infty(\mathbb{R}^3)$. Therefore, $K_1^\#=K_2^\#$.

This completes the proof of Theorem 1.3.

Acknowledgements. The author would like to thank Vesselin Petkov for his helpful advice during the preparation of this work. Thanks are also due to the referee for his valuable remarks.

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