MICROLOCAL APPROACH TO TENSOR TOMOGRAPHY AND BOUNDARY AND LENS RIGIDITY

PLAMEN STEFANOV

Dedicated to Professor Vesselin Petkov on the occasion of his 65th birthday

Contents

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1. Introduction

This survey is based on a series of joint papers [SU4, SU5, SU6, SU7] by the author and Gunther Uhlmann. It is an extended version of the mini-course given by the author on the Symposium of Inverse Problems in Honor of Alberto Calderón in Rio de Janeiro, January 10–19, 2007.

We are not trying to give a full account on the progress in Tensor Tomography and Boundary and Lens Rigidity. While we will certainly acknowledge the contribution of other authors on this subject, our main goal is to present a microlocal point of view. Another powerful method, that we will not discuss, is the energy estimates method, initiated by Mukhometov (see [Mu1, Mu2, MuR] and the references there), and developed further by others; most recently by Pestov, Sharafutdinov, and Dairbekov, see [PS, Sh1, Sh2, Sh3, D].

We try to explain the ideas behind the proofs and skip details often. Clear references to where to find complete proofs are always given. We emphasize more on the analysis of simple manifolds in sections 3, 4 to make the presentation more accessible. The recent results on a more general class of manifolds, that we call regular, that in fact include all simple ones, will be only formulated and briefly discussed in section 5.

These notes target a graduate student audience. Basic knowledge of differential and Riemannian geometry is assumed. Knowledge of pseudo-differential operator (ΨDO) theory is also needed. In fact, we do not go beyond the construction of a parametrix of an elliptic ΨDO and the mapping properties of ΨDOs in Sobolev spaces. In section 3.10, we use analytic ΨDOs.

2. Formulation of the main problems

In what follows, M is a compact manifold of dimension $n \geq 2$ with boundary. We fix a finite analytic atlas on it. Thus the term real analytic function/metric on it makes sense. Moreover, for any function f (or more generally, a tensor field f) on M, the norm $\|f\|_{C^k(M)}$ is well defined as the maximum of the localized norm over all coordinate charts. In sections 3, 4, M will be diffeomorphic to a ball in $\mathbf{R}^n$. We keep M fixed and we will study different Riemannian metrics g on M. We freely use the Einstein summation convention and when g is fixed, we will use the convention of raising and lowering indices thus identifying covariant and contravariant tensor fields.

We will formulate below the three basic problems we are interested in: the linear tensor tomography problem, and the non-linear boundary rigidity and lens rigidity ones. We will show later that the tensor tomography problem is a linearization of the boundary rigidity.

2.1. **Tensor Tomography.** Informally speaking, tensor tomography tries to recover a tensor field from its integrals along geodesics connecting boundary points. We will make this more precise below.

Let M be as above, and let g be a smooth Riemannian metric on it that will be kept fixed in this section. We will parametrize the maximal geodesics in M with (at least one) endpoint on ∂M by their incoming points and directions.

Set

$$\partial_\pm SM := \{(x,\omega)\in TM;\ x\in\partial M,\ |\omega|=1,\ \pm\langle\omega,\nu\rangle>0\},$$

where $\nu(x)$ is the outer unit normal to ∂M (normal w.r.t. g, of course). Here and in what follows, we denote by $\langle\omega,\nu\rangle$ the inner product of the vectors ω, ν, and $|\omega|$ is meant w.r.t. g. Let $\gamma_{x,\omega}(t)$ be the (unit speed) geodesic through (x,ω), defined on its maximal interval contained in $[0,\infty)$. It may happen that $\gamma_{x,\omega}(t)$ is defined for all $t>0$; then we call the latter trapping, and we call (M,g) a trapping manifold. Otherwise, we call $\gamma_{x,\omega}$ non-trapping. In the latter case, the endpoint of $\gamma_{x,\omega}$ must be on ∂M. If all geodesics are non-trapping, then (M,g) is called a non-trapping manifold.

Let f be a covariant symmetric tensor field of order m, i.e., locally, f is given by its components $f_{i_1 i_2 \dots i_m}(x)$. As we mentioned above, we will freely raise indices if needed. Given a vector field v, introduce the notation $\langle f, v^m\rangle$ by writing in any local coordinates

$$\langle f, v^m\rangle = f_{i_1 i_2 \dots i_m} v^{i_1} v^{i_2} \dots v^{i_m}. \tag{2.1}$$

The superscript m is there to reminds us that $\langle f, v^m\rangle$ is non-linear w.r.t. v.

We define the geodesic ray transform of f by

$$If(\gamma) = \int \langle f(\gamma(t)), \dot\gamma^m(t)\rangle \, \mathrm{d}t, \tag{2.2}$$

where γ is any maximal geodesic in M. Unless otherwise stated, we assume that the geodesics are parametrized by an arc-length parameter. If is well defined at least when M is non-trapping and f is continuous. To emphasize on the dependence on the metric g, we sometimes denote I by I_g. Using the parametrization above, with some abuse of notation, we write

$$If(x,\omega) = \int \langle f(\gamma_{x,\omega}(t)), \dot\gamma^m_{x,\omega}(t)\rangle \, \mathrm{d}t, \quad (x,\omega) \in \partial_- SM. \tag{2.3}$$

Our main interest is in symmetric 2-tensors. Then

$$If(x,\omega) = \int f_{ij}(\gamma_{x,\omega}(t)) \dot\gamma^i_{x,\omega}(t) \dot\gamma^j_{x,\omega}(t) \, \mathrm{d}t, \quad (x,\omega) \in \partial_- SM, \tag{2.4}$$

where the integrand is written in local coordinates (somewhat incorrectly since this assumes existence of coordinates defined near the whole $\gamma_{x\omega}$; on the other hand, one can easily define such coordinates in a neighborhood of any non-trapping and non self-intersecting geodesic).

The natural question that arises is the following: is f uniquely determined by its ray transform If? Since I is a linear operator, this is equivalent to asking; does $If = 0$ imply $f = 0$? For now, f is continuous, but we will be more specific below. The answer is negative for any (M,g), if $m \geq 1$. To understand this better, start with the case $m = 1$. Let $\phi \in C^1(M)$, and consider the 1-form $f = \mathrm{d}\phi$ given locally by $f = \mathrm{d}\phi = \phi_{x^i} \mathrm{d}x^i$. Then

$$\langle f(\gamma), \dot\gamma\rangle = \phi_{x^i}(\gamma(t))\dot\gamma^i(t) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\gamma(t)). \tag{2.5}$$

Therefore, if $\phi = 0$ on ∂M, the fundamental theorem of calculus implies that

$$I(\mathrm{d}\phi) = 0. \tag{2.6}$$

On the other hand, $\mathrm{d}\phi$ does not need to vanish. Note that γ does not need to be a geodesic for (2.5) to hold, and therefore, (2.6) holds even if we integrate over any curve(s) connecting boundary points!

This generalizes to tensors of any order $m \geq 1$, but for geodesics only. Consider first the case $m = 2$. Then for any geodesic γ, (see the proof in section (3.2.2))

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v(\gamma(t)), \dot\gamma^{m-1}(t)\rangle = \langle dv, \dot\gamma^m(t)\rangle, \tag{2.7}$$

($m = 2$), where dv is the symmetric differential of f given in local coordinates by

$$(dv)_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}). \tag{2.8}$$

We use the notational convention $v_{i,j} = \nabla_j v_i$, where ∇ is the covariant derivative. For tensors of arbitrary order m, the symmetric differential dv is defined as the symmetrization of ∇v, i.e., as the mean of ∇v over all permutations of its indices. Then (2.7) still holds. Note that γ in (2.7) really has to be a geodesic, and in the proof, we use the geodesic equation. Then (2.7) implies that for any vector field v with $v = 0$ on ∂M, one has

$$I(dv) = 0. \tag{2.9}$$

It will become clear by the mapping properties of I that the regularity of v can be reduced to $v \in H_0^1(M)$.

Definition 2.1. *We call the tensor field f of order $m \geq 1$ potential, if $f = dv$ for some tensor field $v \in H_0^1(M)$ of order $m-1$.*

As we just saw, potential fields belong naturally to the kernel of I. We expect that at least for simple manifolds introduced below, this is the whole kernel of I.

Definition 2.2. *We say that I is s-injective ($m \geq 1$), if $If = 0$ for $f \in L^2(M)$ implies that f is potential, i.e., $f = dv$ with some $v \in H_0^1(M)$.*

If $m = 0$, i.e., if we integrate functions, then we study the injectivity of I in classical sense.

Solenoidal projections of tensor fields. Since I vanishes on potential tensors, it is quite reasonable to study I restricted on the orthogonal complement of all potential tensors. To this end, we have to define a certain scalar product of tensor fields. We will work in the L^2 space of symmetric tensor fields in M with scalar product

$$(f, h)_{L^2(M)} = \int_M f_{i_1 i_2 \dots i_m}(x) \bar{h}^{i_1 i_2 \dots i_m}(x) \, \mathrm{dVol}(x). \tag{2.10}$$

Here, $\mathrm{dVol}(x)$ is the volume measure given locally by $(\det g)^{1/2} \mathrm{d}x$. We hope that our choice of notation will not cause confusion with the L^2 space of functions (and tensor fields of different orders). It will be clear form the contest which space we mean. We define similarly Sobolev spaces. We define the *divergence* δf of a symmetric m-tensor field f ($m \geq 1$) as the formal adjoint of $-d$. In other words, δf is a symmetric $(m-1)$ tensor that in local coordinates is given by

$$(\delta f)_{i_1 \dots i_{m-1}} = \nabla^m f_{i_1 \dots i_{m-1} m}, \tag{2.11}$$

where $\nabla^m = g^{mi} \nabla_i$. In particular, if $m = 2$, then δf is a covector field, and locally, $(\delta f)_i = \nabla^j f_{ij}$.

Then we have the following (see [Sh1, SU4] and section 3.4).

Theorem 2.1. *In the space $L^2(M)$ of symmetric m-tensors, there exist a unique choice of orthogonal projections $\mathcal{P}$ and $\mathcal{S}$, $\mathcal{P} + \mathcal{S} = \mathrm{Id}$, so that any $f \in L^2(M)$ admits the orthogonal decomposition*

$$f = f^s + dv, \quad f^s = \mathcal{S} f, \quad dv = \mathcal{P} f \tag{2.12}$$

with some $v \in H_0^1(M)$, and $\delta f^s = 0$.

We call $f^s = \mathcal{S} f$ the solenoidal projection of f, and any tensor f with $\delta f = 0$ will be called solenoidal. The theorem above then states that any f admits unique decomposition into a potential and a solenoidal part. The s-injectivity of f can then be reformulated as follows.

Definition 2.3. *We say that I is s-injective ($m \geq 1$), if $If = 0$ for $f \in L^2(M)$ implies that $f^s = 0$.*

Clearly, Definitions 2.2 and 2.3 are equivalent.

We will briefly summarize some of the known results about the s-injectivity of I. Let us start with $m = 0$, i.e., integrals of functions. There are simple counter examples to injectivity in that case. Take the sphere S^2, and any function that is equal to 1, and respectively -1 in small neighborhoods of two symmetric neighborhoods of the North and the South pole, respectively. Here, symmetry is define by the antipodal map. Then f integrates to zero over any geodesic (grand circle). Now, to make this a manifold with boundary, cut a small neighborhood U of a point on the equator. Then $If = 0$ but $f \not\equiv 0$. More generally, take any odd f vanishing on U and remove U again. Therefore, some assumptions on (M, g) are needed, if we want to get an injective ray transform I. One such assumption is that (M, g) is simple.

Definition 2.4 (simple manifold)**.** *We say that (M, g) is a simple manifold, if ∂M is strictly convex w.r.t. g, and for any $x \in M$, the exponential map $\exp_x : \exp_x^{-1}(M) \to M$ is a diffeomorphism.*

Any metric g on M so that (M, g) is simple will be called a *simple metric* on M. The boundary ∂M is called strictly convex, if the second fundamental form on ∂M is strictly positive. The second condition above hides the requirement that any two points x, y in M are connected by a unique geodesic in M that depends smoothly on x, y. In particular, there are no conjugate points on any geodesic in M. Any simple M (w.r.t. some g) is necessarily diffeomorphic to a ball in $\mathbf{R}^n$, see e.g., [Sh1]. Therefore, in the analysis of simple manifolds, we can assume that M is a domain $\Omega \subset \mathbf{R}^n$.

If $(M, \partial M)$ is simple, then the geodesic ray transform I of functions and 1-forms is injective, respectively s-injective, see [Mu2, MuR, BG, AR]. The proof of this relies on energy estimates methods.

The case $m \geq 2$ is tougher, and $m = 2$ already possesses most, if not all of the difficulties. S-injectivity of I_g for $m \geq 2$ was previously proved in [PS] for metrics with negative curvature, in [Sh1] for metrics with small curvature. In the 2D case, it was proved in [Sh4] for all simple Riemannian surfaces with boundary, following the approach in [PU]. A conditional and non-sharp stability estimate for metrics with small curvature is also established in [Sh1]. Our main results about the Tensor Tomography problem are Theorems 3.1, 3.2 for simple manifolds, and Theorems 5.1, 5.2 about a more general class that we call regular manifolds.

2.2. **Boundary Rigidity.** Let M be as above. We equip M with different Riemannian metrics g. For any two points in M, let $\rho_g(x, y)$ be the distance between x and y measured in the metric g. In other words, $\rho_g(x, y)$ is the infimum of the lengths of all piecewise C^1 curves in M connecting x and y. We want to recall that the length of a curve $c : [0, 1] \mapsto M$ is given by

$$\text{length}(c) = \int_0^1 |\dot{c}(t)| \, \mathrm{d}t,$$

where, as always, $|\dot{c}(t)|$ is the length of the vector $\dot{c}$ in the metric, i.e., in local coordinates, $|\dot{c}(t)| = \sqrt{g_{ij}(c(t))\dot{c}^i(t)\dot{c}^i(t)}$. Then we ask whether one can determine g by knowledge of $\rho_g(x, y)$ restricted to all $x \in \partial M$, $y \in \partial M$. There is a clear obstruction to this. If $\psi : M \to M$ is any diffeomorphism so that $\psi = \mathrm{Id}$ on ∂M, one can easily show that $\rho_g = \rho_{\psi^* g}$, on $\partial M \times \partial M$, where ψ^* is the pull-back of g under ψ. We will call g and any such $\psi^* g$ isometric. The natural question then is the following:

The Boundary rigidity question. Given g and $\hat{g}$ on M, does

$$\rho_g = \rho_{\hat{g}} \quad \text{on } \partial M \times \partial M$$

imply that there is a diffeomorphism $\psi : M \to M$ so that $\psi|_\partial M = \mathrm{Id}$, and

$$\hat{g} = \psi^* g?$$

More generally, one can ask whether one can recover the topology of M as well from the boundary distance function, if only the boundary is given. We will assume however, that M is known.

Definition 2.5 (boundary rigidity)**.** *We say that (M, g) is boundary rigid, if for any metric $\hat{g}$ on M so that $\rho_g = \rho_{\hat{g}}$ on $\partial M \times \partial M$, one has $\hat{g} = \psi^* g$ with some diffeomorphism ψ fixing ∂M pointwise.*

It is not hard to find counter-examples to boundary rigidity. If there is an open set in M where g is very large, then all the minimizing curves will avoid that set. Therefore, ρ_g will not carry any information about g inside that set and we can modify g there (by keeping it large), and ρ_g on $\partial M \times \partial M$ will be the same. It is easy to see that those modifications do not need to be all isometric to g. A more specific example of this kind is the following. Let M be the northern closed hemisphere of S^2 with its natural metric that we will denote by g_0. Then $\rho_{g_0}(x, y)$ for any two boundary points is realized as the length of the shortest arc on ∂M connecting x and y. Let $0 \leq \phi$ be a smooth function supported in the interior of M, not identically zero. Then $\rho_{(1+\phi)g_0} = \rho_{g_0}$ on $\partial M \times \partial M$. On the other hand, g_0 and $(1 + \phi)g_0$ are not isometric because the volume of M in the second metric is strictly greater than that in the first one, if $\phi \not\equiv 0$.

Therefore, the boundary rigidity problem has to be considered on a class of manifolds in order to avoid counter-examples like those. One such class is the class of simple manifolds introduced above. A more general class of manifolds where one expects boundary rigidity is the class of SGM (strong geodesically minimizing) manifolds, see [C].

Unique recovery of g (up to an action of a diffeomorphism) is known for simple metrics conformal to each other [C, B, Mu1, Mu2, MuR, BG], for flat metrics [Gr], for simple locally symmetric spaces of negative curvature [BCG]. In two dimensions it was known for simple metrics with negative curvature [C2] and [O], and recently it was shown in [PU] for simple metrics with no restrictions on the curvature. In [SU3], the authors proved this for metrics in a small neighborhood of the Euclidean one. This result was used in [LSU] to prove a semiglobal solvability result. Our main results are local boundary rigidity near generic metrics, more precisely, near any metric with an s-injective ray transform I_g, see Theorems 4.1; and Theorem 4.3 about a conditional Hölder type of stability estimate.

The boundary rigidity problem arose in geophysics in an attempt to determine the inner structure of the Earth by measuring the travel times of seismic waves. It goes back to Herglotz [H] and Wiechert and Zoeppritz [WZ]. Although the emphasis has been in the case that the medium is isotropic, the anisotropic case has been of interest in geophysics since it has been found that the inner core of the Earth exhibits anisotropic behavior [Cr]. In differential geometry this inverse problem has been studied because of rigidity questions and is known as the boundary rigidity problem. In its present form, it was formulated by Michel [Mi].

2.3. **Lens Rigidity.** Let now M be a compact manifold with boundary, not necessarily diffeomorphic to a ball anymore. Let g be a Riemannian metric on it. As we saw above, such manifolds may fail to be boundary rigid. Instead of the boundary rigidity problem, we will study a closely related but a different one: the lens rigidity problem.

Let Φ^t be the geodesic flow on SM. We define the *scattering relation*

$$\Sigma : \partial_- SM \to \overline{\partial_+ SM}, \tag{2.13}$$

$\Sigma(x, \xi) = (y, \eta) = \Phi^{\mathcal{L}}(x, \xi)$, where $\mathcal{L} > 0$ is the first moment, at which the unit speed geodesic through (x, ξ) hits ∂M again. Note that at that point, the geodesic may touch ∂M tangentially, and may have an extension beyond $t = \mathcal{L}$; and eventually it may hit ∂M again or to remain trapping. This defines also $\mathcal{L}(x, \xi)$ as a function $\mathcal{L} : \partial_- SM \to [0, \infty]$. Note that Σ and $\mathcal{L}$ are not necessarily continuous.

It is convenient to think of Σ and $\mathcal{L}$ as defined on the whole ∂SM with $\Sigma = \mathrm{Id}$ and $\mathcal{L} = 0$ on $\overline{\partial_+ SM}$.

The lens rigidity question asks whether Σ, $\mathcal{L}$ determine g. Clearly, the way we posed this problem, one needs to know g on ∂M. Moreover, a diffeomorphism ψ fixing ∂M pointwise may not preserve $(x, \xi) \in \partial_\pm SM$; it only preserves the orthogonal projection of ξ on ∂M. If it does, then we have the same obstruction to uniqueness as in the boundary rigidity problem. So we have two options: either to require that $\psi = \mathrm{Id}$ on ∂M and $D\psi = \mathrm{Id}$ on ∂M, or to redefine the scattering relation in order to avoid the second condition. We will do the latter.

Since for $(x,\xi) \in \partial_- SM$, ξ is unit, it is determined by its orthogonal projection on the boundary. We will think of Σ as mapping x and the orthogonal projection of ξ onto a point $y \in \partial M$ and the orthogonal projection of the direction at y. More formally, let $\kappa_\pm : \partial_\pm SM \to B(\partial M)$ be the orthogonal projection onto the (open) unit ball tangent bundle. It extends continuously to the closure of $\partial_\pm SM$. Then $\kappa_\pm$ are homeomorphisms, and we set

$$\sigma = \kappa_+ \circ \Sigma \circ \kappa_-^{-1} : \overline{B(\partial M)} \longrightarrow \overline{B(\partial M)}, \quad \ell = \mathcal{L} \circ \kappa_-^{-1}. \tag{2.14}$$

According to our convention, $\sigma = \mathrm{Id}$, $\ell = 0$ on $\partial(\overline{B(\partial M)}) = S(\partial M)$. We equip $\overline{B(\partial M)}$ with the relative topology induced by $T(\partial M)$, where neighborhoods of boundary points (those in $S(\partial M)$) are given by half-neighborhoods.

We still need to know g on ∂M but only acting on tangent vectors to ∂M. The map σ however, that we still are going to call scattering relation, is invariant under isometric changes of g by $\psi^* g$, if ψ fixes ∂M pointwise. This justifies the following formulation.

The Lens Rigidity question. Given g and $\hat g$ on M, so that $g = \hat g$ on $T(\partial M)$, does

$$\sigma_g = \sigma_{\hat g}, \quad \ell_g = \ell_{\hat g} \quad \text{on } B(\partial M) \tag{2.15}$$

imply that there is a diffeomorphism $\psi : M \to M$ so that $\psi|_M = \mathrm{Id}$, and

$$\hat g = \psi^* g?$$

As before, one can ask whether one can recover the topology of M as well from σ_g, ℓ_g, if only the boundary is given. We will assume again that M is known.

Definition 2.6 (lens rigidity)**.** *We say that (M,g) is lens rigid, if for any metric $\hat g$ on M so that $g = \hat g$ on $T(\partial M)$, and (2.15) is fulfilled, one has $\hat g = \psi^* g$ with some diffeomorphism ψ fixing ∂M pointwise.*

The reason we call this lens rigidity is because of two manifolds are lens rigid, they act in the same way as lenses when viewed from outside. The scattering relation is encoded in the hyperbolic DN map for the wave equation $(\partial_t^2 - \Delta_g)u = 0$ or in the scattering operator.

There are very few results about this problem when the manifold is not simple. Croke [C2] has shown that if a manifold is lens rigid, a finite quotient of it is also lens rigid. A counter-example to lens rigidity is given in [CK].

2.4. **The Boundary Rigidity and the Lens Rigidity problems are equivalent on simple manifolds.** Assume now that M is simple. The following observation is due to Michel [Mi].

Lemma 2.1. *Let (M,g) be simple. Then, for any $(x,y) \in \partial M \times \partial M$,*

$$\begin{aligned} \Sigma(x, -\operatorname{grad}_x \rho(x,y)) &= (y, \operatorname{grad}_y \rho(x,y)), \\ \mathcal{L}(x, -\operatorname{grad}_x \rho(x,y)) &= \rho(x,y); \end{aligned}$$

and

$$\begin{aligned} \sigma(x, -\operatorname{grad}'_x \rho(x,y)) &= (y, \operatorname{grad}'_y \rho(x,y)), \\ \ell(x, -\operatorname{grad}'_x \rho(x,y)) &= \rho(x,y), \end{aligned}$$

where $\operatorname{grad}'\phi$ *stands for the tangential component of* $\operatorname{grad}\phi$ *on* $T(\partial M)$.

Proof. Recall that in Riemannian geometry, in local coordinates, $(\operatorname{grad} f)^i = g^{ij}\partial_j f$. Fix $(x,\xi) \in \partial_- SM$. Let $(y,\eta) = \Sigma(x,\xi) \in \partial_+ SM$.

We have $\operatorname{grad}_y \rho(x,y) = \eta$. This follows from the Jacobi theory of solving the eikonal equation but perhaps the shortest way to see this here is the following. By the Gauss lemma, η is orthogonal (in the metric) to the geodesic sphere $\rho(x,y) = \text{const}$. On the other hand, $\operatorname{grad}_y \rho(x,y)$ has the same property. Therefore, $\operatorname{grad}_y \rho(x,y)$ must be parallel to η. Since the directional derivative of $\rho(x,y)$ w.r.t. to y in the direction of η has length one (in the metric), and η has the same property, then $\operatorname{grad}_y \rho(x,y) = \eta$. Similarly, one gets $\operatorname{grad}_x \rho(x,y) = -\xi$. This proves the lemma. $\square$

Lemma 2.1 shows that the boundary rigidity and the lens rigidity problems are equivalent on simple manifolds. Actually, we get that a knowledge of the first component of σ is enough to recover ρ. More precisely, we have the following. The map π below is the natural projection, i.e., $\pi(x,\xi) = x$.

Proposition 2.1. *Let g, $\hat g$ be simple metrics on M. Then*
(a) If $\rho_g = \rho_{\hat g}$ on $\partial M \times \partial M$, then $g = \hat g$ on $T(\partial M)$ and $\sigma_g = \sigma_{\hat g}$, and $\ell_g = \ell_{\hat g}$.
(b) If $g = \hat g$ on $T(\partial M)$ and $\pi\circ\sigma_g = \pi\circ\sigma_{\hat g}$, then $\rho_g = \rho_{\hat g}$ on $\partial M \times \partial M$.

Proof. To prove (a), notice fist that if ρ is given on $\partial M \times \partial M$, one can easily recover g on $T(\partial M)$ by taking the limit $y \to x$. Then we can recover $\operatorname{grad}'_x \rho(x,y)$ and $\operatorname{grad}'_y \rho(x,y)$ because we can differentiate in tangential directions. Then by the lemma, we know $\sigma(x,\xi')$, and therefore $\Sigma(x,\xi)$, where $\xi = \operatorname{grad}_x \rho(x,y)$. We also know $\ell(x,\xi') = \rho(x,y)$. This implies that, under the conditions of the proposition, $\sigma_g = \sigma_{\hat g}$, $\ell = \ell_{\hat g}$ on that particular (x,ξ). Finally, by the simplicity assumption, given x, the map $y \mapsto \xi$ is a bijection, so those identities hold for all possible (x,ξ').

Next, we have $\rho(x,y) = \ell(x, \pi\circ\sigma(x,\xi))$, where ξ is determined by the equation $\pi\circ\Sigma(x,\xi) = y$, i.e, $\xi = \exp_x^{-1} y$. This easily implies (b). $\square$

3. Analysis of the linear Tensor Tomography problem for simple metrics

The purpose of this rather long section is to present the central ideas in [SU4, SU5] on the analysis of the linear operator I on simple manifolds. Those ideas also work on a class on non-simple manifolds with integrals over suitable subsets of geodesics, as shown in [SU6]. This is discussed in sections 4 and 5. We prefer however to emphasize on simple manifolds, and then to explain briefly how one can extend this approach as in [SU6].

3.1. **Main result: generic s-injectivity, and main ideas.** The purpose of this section is to sketch the proof of the following two theorems. We say that a function f defined on M is (real) analytic, if it extends as a real analytic one in a neighborhood of M, and we write $f \in \mathcal{A}(M)$. Similarly we define analytic functions on not necessarily open subsets of M.

Theorem 3.1. *Let g be a simple analytic metric in M. Then I_g is s-injective.*

We will introduce the norm $\|\cdot\|_{\tilde H^2(M_{\mathrm{e}})}$ later, see (3.43). Now, we will just mention that $M_{\mathrm{e}} \supset M$ and that $H^2 \subset \tilde H^2 \subset H^1$. Here and below, $M_{\mathrm{e}} \supset M$ is another simple manifold so that its interior contains M, see section 3.2.3. Instead of I, we will study the normal operator $N = I^*I$ in M_{e}, where the adjoint I^* is defined through a choice of a natural measure on $\partial_- SM$, see (3.11). We give a formal definition later. Then s-injectivity of I is equivalent to s-injectivity of $N : L^2(M) \to L^2(M_{\mathrm{e}})$, see Lemma 3.2. One can also replace $L^2(M)$ in this statement by $C^\infty(M)$, see Theorem 3.3.

Theorem 3.2. *There exists k_0 such that for each $k \geq k_0$, the set $\mathcal{G}^k(M)$ of simple $C^k(M)$ metrics in M for which I_g is s-injective is open and dense in the $C^k(M)$ topology. Moreover, for any $g \in \mathcal{G}^k$,*

$$\|f^s_M\|_{L^2(M)} \leq C\|N_g f\|_{\tilde{H}^2(M_e)}, \quad \forall f \in H^1(M), \tag{3.1}$$

with a constant $C > 0$ that can be chosen locally uniform in $\mathcal{G}^k$ in the $C^k(M)$ topology.

We will sketch the main ideas below.

We will show that N is a ΨDO of order -1 in the interior $M^{\rm int}$ of M. It cannot be elliptic, since it has an infinite dimensional kernel, but we will show that it is elliptic on solenoidal tensors. This will allow us to construct a parametrix Q so that QNf recovers f^s up to a smooth term in $M^{\rm int}$. Since we work in a manifold M with boundary, we will do this in the slightly larger manifold $M_{\rm e}$, and an additional step will be needed to reduce this to M.

The parametrix shows that solving $Nf = h$ (that also can be written as $Nf^s = h$) for f^s is reduced to a Fredholm equation $(\mathrm{Id} + K)f^s = Qh$. One can also arrange that K is self-adjoint. Therefore, if I, and therefore N, is s-injective, one gets that $\mathrm{Id} + K$ is injective on $\mathcal{S}L^2(M)$ (this requires a careful choice of Q so that QN is still injective there). On the other hand, if $\mathrm{Id} + K$ is injective, then it is invertible, and one can get the estimate (3.1).

So this estimate follows from the ellipticity of N on solenoidal tensors, and the assumption that I is s-injective.

So far g was fixed. Suppose now that I_{g_0} is s-injective. We want to show that (3.1) can be perturbed and remains true for g close to g_0. There is a lost of one derivative in the norm $\|\cdot\|_{\tilde{H}^2(M_{\rm e})}$, however. We have $\|Nf\|_{H^1(M_{\rm e})} \leq C\|f\|_{L^2(M)}$ but this does not hold for the $\tilde{H}^2$ norm of Nf. So (3.1) cannot be perturbed directly. On the other hand, the Fredholm equation $(\mathrm{Id}+K)f^s = Qh$ can, where $Q = Q_g$, $K = K_g$ (and $f^s = \mathcal{S}f$ with $\mathcal{S} = \mathcal{S}_g$). If $\mathrm{Id} + K_g$ is injective, it is also invertible (on the space of the solenoidal tensors) by the theory of compact operators, then it remains invertible under small perturbation of g. It remains to construct Q with more care to make sure that QN and N have the same kernel (i.e., Q does not increase the kernel). Those arguments will allow us to prove that the set of simple metrics $\mathcal{G}$ for which I_g is s-injective is open. Note that this argument alone does not show that this set is even non-empty, and the latter is guaranteed by Theorem 3.1. It was known previously that metric with small enough curvature belong to $\mathcal{G}$, see [Sh1].

To show that $\mathcal{G}$ is dense, we will show that all real analytic simple metrics belong to it, i.e., I_g is s-injective for any such g. We do that by using analytic microlocal ΨDOs. We show that $N : L^2(M) \to L^2(M_{\rm e})$ is such a ΨDO. Elliptic analytic ΨDOs have the nice property to recover the analytic singularities. Suppose for a moment that f is a function. Then N is elliptic, and $Nf = 0$ implies that f, extended as 0 outside M, is real analytic. Therefore, $f = 0$. Well, f is a tensor, N is elliptic only on solenoidal tensors, and the boundary causes some troubles. A modification of this argument still works, fortunately.

3.2. **Preliminaries.**

3.2.1. *Covariant derivatives.* We start with some preliminaries on tensor analysis. We refer to [Sh1, Sh2] for a more detailed exposition.

We want to recall first that a tensor field is defined invariantly as a multilinear map and that the component representation $f_{i_1\dots i_m}$ changes under a coordinate change according to the law

$$f'_{i_1\dots i_m} = f_{i_1\dots i_m}\frac{\partial x^{i_1}}{\partial x'^{i_1}}\cdots\frac{\partial x^{i_m}}{\partial x'^{i_m}}.$$

If we think of f as the form $f_{i_1\dots i_m}dx^{i_1}\dots dx^{i_m}$, then the formula above becomes self-evident. We will be interested mostly in symmetric tensor fields. Next, the operator ∇ of covariant differentiation sends m-tensors to $(m+1)$-tensors. If f is a function, then $(\nabla f)_i = \partial_{x^i} f$ locally, i.e., ∇f is just the usual gradient. For tensor fields of order $m \geq 1$, we want ∇ to satisfy the product rule, among other properties, which leads to the coordinate representation:

$$(\nabla f_{i_1\dots i_m})_k =: \nabla_k f_{i_1\dots i_m} = \partial_{x^k} f_{i_1\dots i_m} - \sum_{\alpha=1}^m \Gamma^p_{k i_\alpha} f_{i_1 \dots i_{\alpha-1} p i_{\alpha+1} i_m}. \tag{3.2}$$

Here Γ^k_{ij} are the Christofell symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{kp}\left(\frac{\partial g_{jp}}{\partial x^i} + \frac{\partial g_{ip}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^p}\right).$$

There is a similar formula for $\nabla_k f^{j_1\cdots j_p}$, and more generally, for $\nabla_k f^{j_1\dots j_p}_{i_1\dots i_m}$, see [Sh1]. The most interesting cases for us are

$$\nabla_k f_{ij} = \partial_{x^k} f_{ij} - \Gamma^p_{ki} f_{pj} - \Gamma^p_{kj} f_{ip}, \tag{3.3}$$

and

$$\nabla_k v_i = \partial_{x^k} v_i - \Gamma^p_{ki} v_p, \quad \nabla_k w^i = \partial_{x^k} w^i + \Gamma^i_{kp} w^p. \tag{3.4}$$

Note that the operation of lowering or raising an index commutes with taking a covariant derivative.

Given a vector field X, one denotes by ∇_X the covariant derivative along X given in local coordinates by $\nabla_X = X^i \nabla_i$. The geodesic equation then reads

$$\nabla_{\dot\gamma}\dot\gamma = 0,$$

i.e., in local coordinates,

$$\ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = 0.$$

3.2.2. *Proof of (2.7).* Using the rules of covariant differentiation, we write

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v(\gamma(t)), \dot\gamma^{m-1}(t)\rangle = \langle \nabla_{\dot\gamma} v, \dot\gamma^m(t)\rangle = \langle dv, \dot\gamma^m\rangle,$$

where $v = v(\gamma(t))$.

3.2.3. *Extension of M as a simple manifold.* One can check that the simplicity condition is an open one, i.e., it is preserved under a small C^2 perturbation of g. Using this, one can construct another manifold $M_{\mathrm{e}} \supset M$ of the same dimension, and extend g there so that (M_{e}, g) is still simple, and $M \Subset M_{\mathrm{e}}$. The later means that there is an open $U \subset M_{\mathrm{e}}$ so that $M \subset U \subset M_{\mathrm{e}}$.

3.2.4. *Semigeodesic (boundary normal) coordinates.* Given $x \in \mathbf{R}^n$, we write $x' = (x^1, \dots, x^{n-1})$.

Let $x' = x'(p)$ be local coordinates on ∂M, and set $x^n = \rho(p, \partial M)$. Then $x = (x', x^n)$ are called semigeodesic, or boundary normal coordinates. In those coordinates, $g_{in} = 0$, $\forall i$. This is easy to see on the boundary $x^n = 0$ because $\partial/\partial x^n$ is orthogonal to ∂M. For $x^n > 0$ (and $x^n \ll 1$), it follows from the Gauss Lemma that $\partial/\partial x^\alpha$ is orthogonal to $\partial/\partial x^n$ for $\alpha \neq n$, and this implies $g_{\alpha n} = 0$. This yields $\Gamma^i_{nn} = \Gamma^n_{in} = 0$, $\forall i$. Those coordinates cannot be extended to the whole M, of course. In those coordinates, the lines $x' = \text{const.}$ are geodesics, normal to the surfaces $x^n = \text{const.}$, and in particular to ∂M.

Let $p_0 \in M_{\mathrm{e}} \setminus M$. Consider all geodesics issued from p_0: $\gamma_{p_0,\theta}(t)$, where $\theta \in S_{p_0} M_{\mathrm{e}}$. Then, one can consider (θ, t) as polar coordinates on $T_{x_0} M$. One can easily see that θ runs over a closed subset of a hemisphere on $S_{p_0} M_{\mathrm{e}}$. Therefore, one can choose coordinates near p_0 so that θ' are coordinates on that subset. Considering (t, θ') as Cartesian coordinates, see also [SU4, sec. 9], one gets coordinates $(x', x^n) = (\theta', t)$ near γ_{x_0,θ_0} so that the latter is given by $\{(0, \dots, 0, t),\ 0 \leq t \leq l^+\}$. Moreover, those geodesics (lines in the x coordinates) are orthogonal to the geodesics spheres $x^n = \text{const.}$ by the

Gauss lemma. Therefore, $g_{in} = \delta_{in}$, and $\Gamma^i_{nn} = \Gamma^n_{in} = 0$, $\forall i$, as above. This should not be surprising — those coordinates are actually boundary normal coordinates as in the paragraph above, normal to (a part of) the geodesic sphere $\rho(p, p_0) = \varepsilon$, $0 < \varepsilon \ll 1$; and we parametrize that sphere by θ'.

3.3. **The linearization of the Boundary Rigidity problem is the tensor Tomography problem.** The boundary distance function ρ_g depends on g in a non-linear way. We will show now that the linearization of the boundary rigidity problem is reduced to the tensor tomography one.

Proposition 3.1. *Let (M, g) be simple, and let $\|\hat g - g\|_{C^2} \le \varepsilon$, $g = \hat g$ on ∂M. Then for $0 \le \varepsilon \ll 1$, $\hat g$ is still simple, and*

$$\rho_g(x,y) - \rho_{\hat g}(x,y) = \frac{1}{2}(If)(x, \exp_x^{-1} y) + R_{g,\hat g}(f), \tag{3.5}$$

where $f = \hat g - g$, and

$$\|R_{g,\hat g}(f)\|_{L^\infty(\partial M\times\partial M)} \le C\|f\|^2_{C^1(M)}. \tag{3.6}$$

The constant $C > 0$ depends on M and on an a priori bound on $\|g\|_{C^2}$.

Proof. See also [Sh1, Sh2, DPSU], and [E]. Set $g^\tau = g + \tau f$, $\tau \in [0,1]$. Fix x, y on ∂M. Set also

$$\phi(s,\tau) = \int_0^1 |\dot\gamma_s(t)|_{g^\tau}\,\mathrm{d}t,$$

where γ_s is the geodesic in the metric g^s connecting x and y. Here t is not an arc length parameter but is proportional to it. Note that $\phi(s,s) = \rho_{g^s}$. Then

$$\frac{\mathrm{d}\rho_{g^s}}{\mathrm{d}s} = \frac{\partial\phi}{\partial s}(s,s) + \frac{\partial\phi}{\partial\tau}(s,s).$$

Since γ_s minimizes the length functional related to g^s, for τ fixed, $(\partial\phi/\partial s)(s,s) = 0$, and we get

$$\frac{\mathrm{d}\rho_{g^s}}{\mathrm{d}s} = \frac{1}{2}\int_0^1 \frac{f_{ij}\dot\gamma_s^i\dot\gamma_s^j}{|\dot\gamma_s|_{g^s}}\mathrm{d}t = \frac{1}{2}(I_{g^s}f)(x, \exp_x^{-1} y).$$

In the last step, we used the fact that the integral in the middle is independent of the parametrization, so we can pass from t to an arc length parameter. Differentiate again to get

$$\frac{\mathrm{d}^2\rho_{g^s}}{\mathrm{d}s^2} = \int \left((\nabla_{\gamma_s'} f_{ij})\dot\gamma_s^i\dot\gamma_s^j + 2f_{ij}\dot\gamma_s^i(\nabla_{\gamma_s'}\dot\gamma_s^j)\right)\mathrm{d}t,$$

where $\gamma_s' = \mathrm{d}\gamma_s/\mathrm{d}s$, and t now is an arc length parameter. Therefore,

$$\left|\frac{\mathrm{d}^2\rho_{g^s}}{\mathrm{d}s^2}\right| \le C\|f\|_{C^1}\left(\|\gamma_s'\|_{C^0} + \|\nabla_{\gamma_s'}\dot\gamma_s^j\|_{C^0}\right).$$

Since γ^s solves the geodesic equation, it follows easily by differentiating w.r.t. a parameter that the term in the parentheses in the r.h.s. above is bounded by $C\|f\|_{C^1}$. ∎

The proof implies that (3.6) can be strengthened a bit to $|R_{g,\hat g}(f)(x,y)| \le C|x-y|\|f\|^2_{C^1(M)}$, see also [E].

3.4. **Decomposition into a solenoidal and a potential part.** We will prove Theorem 2.1 here. In fact, we will do something more — we will construct $\mathcal{S}, \mathcal{P}$ explicitly. We follow [Sh1, Sh2].

Proof of Theorem 2.1. Assume that Theorem 2.1 is true and such projections exist. Then for any f, $f = f^s + dv$, with $\delta f^s = 0$. Take divergence of both sides to get $\delta f = \delta dv$, and $v \in H_0^1(M)$, i.e., $v \in H^1(M)$, $v = 0$ on ∂M. Therefore, v solves

$$\begin{cases} \delta dv = \delta f & \text{in } M, \\ v|_{\partial M} = 0. \end{cases} \tag{3.7}$$

It is not hard to see that $-\delta d$ is an elliptic non-negative differential operator of order 2. We can think of symmetric tensors as vector-valued functions (if $m = 2$, the dimension is $n(n+1)/2$). Then $-\delta d$ can be thought of as a matrix-valued differential operator (a system). Note first that $-\delta d$ is formally self-adjoint, and clearly non-negative because $(-\delta dv, v) = \|dv\|^2$ for any $v \in H_0^1$. Here $(\cdot,\cdot)$ is the scalar product in the L^2 space of $(m-1)$-tensors. One can do the same thing, but without integrating to get the same for the principal symbols $\sigma_{\mathrm{p}}(\delta)$, $\sigma_{\mathrm{p}}(d)$ w.r.t. the scalar product as in (2.10) but without the integration. One could actually write down $\sigma_{\mathrm{p}}(\delta)$, $\sigma_{\mathrm{p}}(d)$ explicitly. In the case $m = 2$, we get

$$\frac{1}{\mathrm{i}}\big(\sigma_{\mathrm{p}}(\delta)f\big)_i = \xi^j f_{ij}, \quad \frac{1}{\mathrm{i}}\big(\sigma_{\mathrm{p}}(d)v\big)_{ij} = \frac{1}{2}\big(\xi^j v_i + \xi^i v_j\big). \tag{3.8}$$

Recall that $\xi^i = g^{ij}(x)\xi_j$, so in particular, those symbols depend on x in a "hidden" way. The ellipticity is then easy to check directly. In fact, we get that $-\delta d$ is strongly elliptic, i.e., not only $\sigma_{\mathrm{p}}(x,\xi)$ vanishes for $\xi = 0$ only, but it in fact, is a strictly positive tensor (matrix) for $\xi \neq 0$. The Dirichlet boundary conditions for such a strongly elliptic system are automatically coercive [Ta2]. Since the kernel and the cokernel of that system are trivial, we get that there is a unique solution satisfying the usual Sobolev estimates. We will denote the solution u to the system $\delta du = f$, $u = 0$ on ∂M by $u = (\delta d)_{\mathrm{D}}^{-1} u$. Then $(\delta d)_{\mathrm{D}}^{-1} : H^{-1} \to H_0^1$, see [Ta2, p. 307]. Its norm depends continuously on $g \in C^1$, see [SU6, Lemma 1]. Also, $(\delta d)_{\mathrm{D}}^{-1} : H^s \to H^{s+2} \cap H_0^1$, $s = 0, 1, \dots$ with a norm bounded by a constant depending on an upper bound of $\|g\|_{C^k}$, $k = k(m) \gg 1$. So we get from (3.7) that

$$v = (\delta d)_{\mathrm{D}}^{-1}\delta f. \tag{3.9}$$

This motivates the following definition

$$\mathcal{P} = d(\delta d)_{\mathrm{D}}^{-1}\delta, \quad \mathcal{S} = \mathrm{Id} - \mathcal{P}. \tag{3.10}$$

It is not hard now to see that those two operators indeed have the properties required.

Notice that the 1-form v so that $\mathcal{P}f = dv$, $v \in H_0^1(M)$, is uniquely determined. □

Remark 1. If $f = 0$ on ∂M (and if f is smooth enough so that the trace on ∂M makes sense), then we do not need to have the same for f^s! Moreover, even if $f = 0$ in a neighborhood of ∂M, we still may not have $f^s = 0$ on ∂M! The reason is that $f^s = \mathcal{S}f$ is obtained by applying the non-local operator $\mathcal{S}$ to f. This innocent fact is responsible for much of the difficulties in the analysis of I acting on tensors.

3.5. **An integral representation of the normal operator N.** Since M is diffeomorphic to a ball, we can think that $M = \bar{\Omega}$, where Ω is a bounded domain on $\mathbf{R}^n$ with smooth boundary. Therefore, we have global coordinates x on M. We can therefore freely use coordinate notation whenever needed.

On $\partial_- SM$, introduce the measure

$$\mathrm{d}\mu(x,\omega) = |\omega \cdot \nu(x)|\, \mathrm{d}S_x\, \mathrm{d}\sigma_x(\omega), \tag{3.11}$$

where $\mathrm{d}S_x$ and $\mathrm{d}\sigma_x(\omega)$ are the surface measures on ∂M and S_xM in the metric, respectively. Similarly, $\mathrm{d}\sigma$ is the induced measure on SM. In boundary normal coordinates, $\mathrm{d}S_x = (\det g)^{1/2}\mathrm{d}x^1 \dots \mathrm{d}x^{n-1}$, and $\mathrm{d}\sigma_x(\omega) = (\det g)^{1/2}\mathrm{d}\sigma_0(\omega)$, where $\mathrm{d}\sigma_0(\omega)$ is the measure on S^{n-1} induced by the Euclidean one. Denote by $\mathrm{d}\sigma$ the Liouville measure on SM. In the notation above, it is given by $\mathrm{d}\sigma = \mathrm{dVol}(x)\,\mathrm{d}\sigma_x(\omega) = (\det g)\,\mathrm{d}x'\,\mathrm{d}\sigma_0(\omega)$.

3.5.1. *Santaló's formula.* The following result, known as Santaló's formula, is useful in this analysis.

Lemma 3.1. *For every continuous function $\phi : SM \mapsto \mathbf{C}$, we have*

$$\int_{SM} \phi\,\mathrm{d}\sigma = \int_{\partial_- SM}\int_0^{\ell(z,\omega)} \phi\big(\gamma_{z,\omega}(t), \dot\gamma_{z,\omega}(t)\big)\,\mathrm{d}t\,\mathrm{d}\mu(z,\omega).$$

Sketch of the proof. The proof is based on Fubini's theorem. Note that (z,ω,t), where $z \in \partial M$, $\omega \in S_zM$, $t > 0$ are coordinates in SM, given by $x = \gamma_{z,\omega}(t)$, $\xi = \dot\gamma_{z,\omega}(t)$. Passing to those variables, in the l.h.s. above, we integrate first w.r.t. t, then w.r.t. (z,ω). The Jacobian of that change is 1 (w.r.t. the measures as in the lemma) because the geodesic flow preserves the Liouville measure. We refer to [Sh2] for more details. □

Lemma 3.1 easily implies that the map $I : L^2(M) \to L^2(\partial_- SM, \mathrm{d}\mu)$ is bounded, and therefore the *normal operator* $N := I^*I$ is a well defined bounded operator in $L^2(M)$.

3.5.2. *An expression for I^*.* Let $\psi(x,\xi) \in C(\partial_- SM)$, and assume for simplicity that $m = 2$. Then

$$(If,\psi) = \int_{\partial_- SM} \bar\psi(x,\xi)\int_0^{\ell(x,\xi)} f_{ij}(\gamma_{x,\xi}(t))\dot\gamma^i_{x,\xi}(t)\dot\gamma^j_{x,\xi}(t))\dot\gamma^i_{x,\xi}(t)\,\mathrm{d}t\,\mathrm{d}\mu(x,\xi).$$

By Lemma 3.1, we get

$$(If,\psi) = \int_{SM} f_{ij}(x)\xi^i\xi^j\bar\psi^\sharp(x,\xi)\,\mathrm{d}\sigma(x,\xi),$$

where $\psi^\sharp(x,\xi)$ is defined as the function that is constant along the orbits of the geodesic flow and that equals $\psi(x,\xi)$ on $\partial_- SM$. Then

$$(If,\psi) = \int_M f_{ij}(x)\int_{S_xM} \xi^i\xi^j\bar\psi^\sharp(x,\xi)\,\mathrm{d}\sigma_x(\xi)\,\mathrm{dVol}(x).$$

Therefore,

$$I^*\psi = \int_{S_xM} \xi^i\xi^j\psi^\sharp(x,\xi)\,\mathrm{d}\sigma_x(\xi). \tag{3.12}$$

3.5.3. *Two integral representations for N.* Using (3.12), we arrive at the following.

Proposition 3.2.

$$(Nf)^{i'j'}(x) = \int_{S_xM} \omega^{i'}\omega^{j'}\int f_{ij}(\gamma_{x,\omega}(t))\dot\gamma^i_{x,\omega}(t)\dot\gamma^j_{x,\omega}(t)\,\mathrm{d}t\,\mathrm{d}\sigma_x(\omega). \tag{3.13}$$

To simplify the notation, we assume that f is extended as zero outside M, and we integrate for all t. The generalization of the proposition for tensors of any order m is obvious.

Let us define N on $L^2(M_\mathrm{e})$ again by $\tilde N = I^*I$. A priori, this definition gives us a different operator, even if restricted to tensors supported in M. The reason is that the adjoint is in a different space. On the other hand, (3.13) shows that for such f, $\tilde Nf|_M = Nf$. Those remarks justify the notation N both for $\tilde N$ and N; we just think of N as the operator given by (3.13). Note that the Liouville theorem is the one responsible for this nice symmetry.

Lemma 3.2. *The following statements are equivalent:*
(a) I is s-injective on $L^2(M)$;
(b) $N : L^2(M) \to L^2(M)$ is s-injective;
(c) $N : L^2(M) \to L^2(M_e)$ is s-injective;

Proof. Let I be s-injective, and assume that $Nf = 0$ in M for some $f \in L^2(M)$. Then

$$0 = (Nf, f)_{L^2(M)} = \sum \|If\|^2_{L^2(\partial_- SM, \mathrm{d}\mu)} \quad \Longrightarrow \quad f^s = 0.$$

This proves the implication $(a) \Rightarrow (b)$. Next, $(b) \Rightarrow (c)$ is immediate. Assume (c) and let $f \in L^2(M)$ be such that $If = 0$. Then $Nf = 0$ in M_{e} by Proposition 3.2, therefore $f^s = 0$. Therefore, $(c) \Rightarrow (a)$. ☐

Remark 2. It follows from the proof above that if f is supported in M, then the equality $Nf = 0$ in M implies that $If = 0$ (on $\partial_- SM$), therefore $Nf = 0$ in M_{e}. This is not so clear from the integral representation below.

Split the integration in (3.13) w.r.t. t into two parts: for $t \geq 0$, and for $t \leq 0$. In the second integral, use the time-reversibility of the geodesic flow, i.e., the property $\gamma_{x,\xi}(t) = \gamma_{x,-\xi}(-t)$. Then we can write

$$(Nf)^{i'j'}(x) = 2\int_{S_x M} \omega^{i'}\omega^{j'} \int_0^\infty f_{ij}(\gamma_{x,\omega}(t))\dot\gamma^i_{x,\omega}(t)\dot\gamma^j_{x,\omega}(t)\,\mathrm{d}t\,\mathrm{d}\sigma_x(\omega).$$

Perform the change of variables $\xi = t\omega$ first, and then $y = \exp_x(\xi)$. The Jacobian of the first change is $t^{-n+1} = |\xi|^{-n+1}$. Note that here $|\xi|$ is considered in the metric, as always. Then $|\xi| = \rho(x, y)$. Moreover, $\omega = \xi/|\xi| = -\mathrm{grad}_x\rho(x, y)$, and $\xi = -\frac{1}{2}\mathrm{grad}_x\rho^2(x, y)$. Therefore the Jacobian of the second change is $|\det(\mathrm{d}\xi/\mathrm{d}y)| = \frac{1}{2}|\det(\partial^2\rho^2/\partial x\partial y)|/\det g(x)$ (the term $\det g(x)$ comes from the definition of grad). Since $\mathrm{d}\sigma_x = (\det g(x)^{1/2})\mathrm{d}\sigma_0$, we see that the measure after the change of variables is transformed into $|\det(\partial^2(\rho^2/2)/\partial x\partial y)|\,\mathrm{d}y$. Actually, by (3.29), that determinant is negative on the diagonal, and since it never vanishes, it is always negative; so the absolute value can be replaced by a negative sign. We therefore obtained the following.

Proposition 3.3.

$$(Nf)_{kl}(x) = \frac{2}{\sqrt{\det g(x)}}\int \frac{f^{ij}(y)}{\rho(x,y)^{n-1}}\frac{\partial\rho}{\partial y^i}\frac{\partial\rho}{\partial y^j}\frac{\partial\rho}{\partial x^k}\frac{\partial\rho}{\partial x^l}\left|\det\frac{\partial^2(\rho^2/2)}{\partial x\partial y}\right|\mathrm{d}y, \quad x \in M_e. \tag{3.14}$$

Let us recall that we always assume that g is extended as a simple metric in M_{e}. Also, we always extend functions or tensors defined in Ω, or similar domains, as 0 outside the domain.

3.6. **The Euclidean case.** In this section we explicitly compute the normal operator in the Euclidean case. Moreover, we show that then I is s-injective. We are going to prove much more general theorems below. The Euclidean case however, gives a deeper insight that one may think. We will show later that N is a ΨDO for any simple metric. It turns out, that the principal symbol of N in the general case is the same as in the Euclidean case, with a proper invariant interpretation of the formula! Moreover, the general procedure we are going to follow next for generic g is inspired by the Euclidean case. In this section, we use the notation Ω for the interior of M. Recall that $\Omega \subset \mathbf{R}^n$. We always extend functions or tensors fields supported in Ω as 0 outside Ω.

3.6.1. *The classical X-ray transform.* Let us start with the classical X-ray transform of functions

$$Xf(z,\omega) = \int f(z+t\omega)\,\mathrm{d}t, \quad z \in \mathbf{R}^n,\ \omega \in S^{n-1}.$$

Note that this is a partial case of I. If we parametrize Xf in the way we did before, we get

$$Nf(x) = X^*Xf(x) = 2\int \frac{f(y)}{|x-y|^{n-1}}\mathrm{d}y.$$

We now consider this in the whole $\mathbf{R}^n$ but applied to functions supported in Ω (or more generally, decaying fast enough). It is easy to see that

$$N = c_n \mathcal{F}^{-1}|\xi|^{-1}\mathcal{F},$$

with some $c_n > 0$, where $\mathcal{F}$ stands for the Fourier transform. In other words, $N = c_n|D|^{-1}$. The injectivity of N on $C_0(\mathbf{R}^n)$ is now immediate, and in fact, $f = c_n^{-1}|D|Nf$.

3.6.2. *Back to tensors, the Euclidean case.* We now turn our attention to I acting on symmetric 2-tensors. Instead of studying $g = e$, we will consider the equivalent case of a constant metric. Several of the calculations below can be found in [Sh1] for $g = e = \{\delta_{ij}\}$ and can be easily generalized to constant g by transforming g into e, for example by the symplectic transform $y = g^{1/2}x$, $\eta = g^{-1/2}\xi$, then $ds^2 = \sum(dy^i)^2$.

Let g be a constant coefficients metric. We will work in $\mathbf{R}^n$ first, assuming that f is compactly supported. Then we parameterize the geodesics (lines) by the direction ω and by the point z on the hyperplane $z^i\omega_i = 0$ where the line crosses that hyperplane. Then

$$I_g f(z,\omega) = \int f_{ij}(z+t\omega)\omega^i\omega^j\,\mathrm{d}t.$$

Any $f \in L^2(\mathbf{R}^n)$ can then be orthogonally decomposed uniquely into a solenoidal and potential part (different from the decomposition above!)

$$f = f^s_{\mathbf{R}^n} + dv_{\mathbf{R}^n} \quad \text{in } \mathbf{R}^n,$$

such that $\delta f^s_{\mathbf{R}^n} = 0$ in $\mathbf{R}^n$ and $f^s_{\mathbf{R}^n}$, $dv_{\mathbf{R}^n}$ are in $L^2(\mathbf{R}^n)$. Similarly to (3.10), we have

$$v_{\mathbf{R}^n} = (\delta d)^{-1}\,\delta f, \quad f^s_{\mathbf{R}^n} = f - d\,(\delta d)^{-1}\,\delta f, \tag{3.15}$$

with δd acting in the whole $\mathbf{R}^n$, and the notation $v_{\mathbf{R}^n}$ indicates that v is defined in the whole $\mathbf{R}^n$ and does not necessarily satisfy boundary conditions if f is supported in $\bar{\Omega}$. The inverse $(\delta d)^{-1}$ is defined through the Fourier transform. Actually, the latter provides a more detailed form of this decomposition. We have

$$(\hat{f}^s_{\mathbf{R}^n})_{kl} = \lambda^{ij}_{kl}(\xi)\hat{f}_{ij}(\xi), \tag{3.16}$$

where

$$\lambda^{ij}_{kl}(\xi) = \left(\delta^i_k - \frac{\xi_k\xi^i}{|\xi|^2}\right)\left(\delta^j_l - \frac{\xi_l\xi^j}{|\xi|^2}\right). \tag{3.17}$$

It is important to note that in general, $f^s_{\mathbf{R}^n}$ and $dv_{\mathbf{R}^n}$ are not compactly supported even if f is. It follows from Proposition 3.3, that for $f \in C_0$,

$$(N_e f)^{kl}(x) = 2f_{ij} * \frac{x^ix^jx^kx^l}{|x|^{n+3}}\sqrt{\det g}. \tag{3.18}$$

Taking into account that $\mathcal{F}|x|^\alpha = (c_n/2)(\det g)^{-1/2}|\xi|^{-\alpha-n}$ with c_n as below, and Fourier transforming the latter, we get

$$\mathcal{F}(N_e f)^{kl}(\xi) = c_n \hat{f}_{ij}(\xi) \frac{\partial^4}{\partial\xi_i\partial\xi_j\partial\xi_k\partial\xi_l}|\xi|^3, \quad c_n = \frac{\pi^{(n+1)/2}}{3\Gamma(n/2+3/2)}, \tag{3.19}$$

and

$$\partial^4|\xi|^3/\partial\xi_i\partial\xi_j\partial\xi_k\partial\xi_l = 3|\xi|^{-1}\sigma(\varepsilon^{ij}\varepsilon^{kl}), \quad \varepsilon^{ij}(\xi) = \delta^{ij} - \xi^i\xi^j/|\xi|^2. \tag{3.20}$$

Here $\sigma(\varepsilon^{ij}\varepsilon^{kl})$ is the symmetrization of $\varepsilon^{ij}\varepsilon^{kl}$, i.e., the mean of all similar products with all possible permutation of i, j, k, l, see [Sh1]. It is easy to see that $\delta N_e f = 0$ and that $f^s_{\mathbf{R}^n}$ can be recovered from $N_e f$ by the formula

$$[\hat{f}^s_{\mathbf{R}^n}]_{ij} = \left(\delta^{kl}_{ij} - \lambda^{kl}_{ij}\right)\hat{f}_{kl} = a_{ijkl}\mathcal{F}(N_e f)^{kl} = a^{kl}_{ij}\mathcal{F}(N_e f)_{kl}, \tag{3.21}$$

where $a_{ijkl}(\xi)$ is a rational function, homogeneous of order 1 singular only at $\xi = 0$ with explicit form

$$a_{ijkl} = |\xi|\left(c_1\delta_{ik}\delta_{jl} + c_2(\delta_{ij} - |\xi|^{-2}\xi_i\xi_j)\delta_{kl}\right). \tag{3.22}$$

The coefficients c_1 and c_2 depend on n only [Sh1]. So we get that given $N_e f$, one can recover $f^s_{\mathbf{R}^n}$ by

$$f^s_{\mathbf{R}^n} = AN_e f, \tag{3.23}$$

where $A = A(D)$ has the symbol in (3.22). In particular, $I_e f = 0 \Longrightarrow f^s_{\mathbf{R}^n} = 0 \Longrightarrow f = d^s v_{\mathbf{R}^n}$. We are halfway towards proving the following.

Proposition 3.4. *Let $\Omega \subset \mathbf{R}^n$ be convex, and let g be a constant metric, and let (Ω, g) be simple. Then I is s-injective.*

Proof. We claim that if $If = 0$ and $\operatorname{supp} f \subset \bar{\Omega}$, then $\operatorname{supp} v_{\mathbf{R}^n} \subset \bar{\Omega}$. Indeed, we already showed that $f = dv_{\mathbf{R}^n}$. Next, since v can be obtained from f by applying a ΨDO of order -1 with homogeneous constant (w.r.t. x) symbol, see (3.15), we easily get that $|v| = O(|x|^{-1})$, as $|x| \to \infty$. Now, $dv_{\mathbf{R}^n} = 0$ outside Ω. By (2.7), we get

$$v_{\mathbf{R}^n}(x)\cdot\xi = v_{\mathbf{R}^n}(x + s\xi)\cdot\xi, \quad \forall(x,\xi) \in \partial_+S\Omega,\ s > 0. \tag{3.24}$$

Take the limit $s \to \infty$ to conclude that $v_{\mathbf{R}^n}(x)\cdot\xi = 0$. Varying ξ, we get $v_{\mathbf{R}^n} = 0$ on $\partial\Omega$. This also holds if we extend $\partial\Omega$, then we get that $\operatorname{supp} v \subset \bar{\Omega}$. So we get that $v_{\mathbf{R}^n}$, restricted to Ω, coincides with v in the decomposition $f = f^s + dv$! Moreover, that restriction commutes with taking the symmetric differential d because $v = 0$ on $\partial\Omega$. So we get $f = dv$, i.e., f is potential field.

We want to emphasize that in general, given f and v, $v_{\mathbf{R}^n}$ related to f, we have $v_{\mathbf{R}^n} \neq v$ in Ω, and in particular, $v_{\mathbf{R}^n} \neq 0$ on $\partial\Omega$. We got an equality only under the assumption $If = 0$! □

Remark. It is worth mentioning, that if our goal is not a proof of s-injectivity of I_e but a recovery of f^s from $N_e f$, then we can proceed as above. Namely, since $f^s_{\mathbf{R}^n} = f - dv_{\mathbf{R}^n}$ in Ω, d commutes with the extension as zero, and $f = 0$ outside Ω, similarly to (3.24), we can write

$$v_{\mathbf{R}^n}(x)\cdot\xi - v_{\mathbf{R}^n}(x + s\xi)\cdot\xi = \int_0^s (AN_e f)(x + tv)\,\mathrm{d}t, \quad \forall(x,\xi) \in \partial_+S\Omega,\ s > 0. \tag{3.25}$$

Take the limit $s \to \infty$, to get

$$v_{\mathbf{R}^n}(x)\cdot\xi = \int_0^\infty (AN_e f)(x + tv)\,\mathrm{d}t, \quad \forall(x,\xi) \in \partial_+S\Omega,\ s > 0. \tag{3.26}$$

Choose $n-1$ linearly independent ξ's above, and we have recovered $h := v_{\mathbf{R}^n}(x)|_{\partial\Omega}$ in terms of $N_e f$. Now, let w be the solution w of the BVP

$$\delta dw = 0 \quad \text{in } \Omega, \quad w_{\mathbf{R}^n}|_{\partial\Omega} = h, \tag{3.27}$$

Then in Ω,

$$f^s = f^s_{\mathbf{R}^n} + dw = AN_e f + dw, \tag{3.28}$$

and w is expressible in terms of $N_e f$.

We would like to explicitly emphasize again that the decomposition of f in the whole $\mathbf{R}^n$ (in case $g = \text{const.}$) described in this section is different than the one in Ω described in section 3.4. Even if $g = e$, formulas (3.9) and (3.15) differ by the fact that the latter involves the resolvent $(\delta d)^{-1}$ in the whole space while (3.9) involves the solution of a boundary value problem $\delta dv = \delta f$ in Ω, $v = 0$ on $\partial\Omega$.

Explicit expressions of this kind for tensors of any order m can be found in [Sh1]. For our purposes however, it is important to know that N is a ΨDO elliptic on solenoidal tensors, and this can be done with a different representation of the principal symbol of N (different than (3.19)) that generalizes easily for any m, see (3.36).

3.7. N **is a pseudodifferential operator.** We show next that N is a ΨDO in the interior of M_{e}. Next proposition says that N is elliptic on solenoidal tensors in M_{e}. We want to warn the reader about hidden reefs here. Solenoidal tensors in M satisfy $\delta f = 0$ in M. The extension of f as zero to M_{e}, that we still denote by f, may not be solenoidal in M_{e}! Indeed, if f does not vanish on ∂M, then δf may produce non-zero delta type of terms and will then fail to be zero. For this reason, given $f \in L^2(M)$, f^s_M (extended as zero) and $f^s_{M_{\text{e}}}$ are different in general. This is the reason we study N in M_{e} first.

Proposition 3.5. *N is a classical ΨDO of order -1 in M_e^{int}. The principal symbol $\sigma_p(N)$ vanishes on tensors of the kind $f_{ij} = (\xi_i v_j + \xi_j v_i)/2$ and is non-negative on tensors satisfying $\xi^i f_{ij} = 0$.*

Proof. To express N as a pseudo-differential operator, we proceed as in [SU4, SU5], with a starting point (3.14). It is easy to see that for x close to y we have

$$\begin{aligned} \rho^2(x,y) &= G^{(1)}_{ij}(x,y)(x-y)^i(x-y)^j, \\ \frac{\partial \rho^2(x,y)}{\partial x^j} &= 2G^{(2)}_{ij}(x,y)(x-y)^i, \\ \frac{\partial^2 \rho^2(x,y)}{\partial x^j \partial y^j} &= -2G^{(3)}_{ij}(x,y), \end{aligned} \tag{3.29}$$

where $G^{(1)}_{ij}$, $G^{(2)}_{ij}$ $G^{(3)}_{ij}$ are smooth and on the diagonal and

$$G^{(1)}_{ij}(x,x) = G^{(2)}_{ij}(x,x) = G^{(3)}_{ij}(x,x) = g_{ij}(x).$$

Then N is a formal pseudo-differential operator with amplitude

$$\begin{aligned} M_{ijkl}(x,y,\xi) = 2\int e^{-\mathrm{i}\xi\cdot z} &\left(G^{(1)}z\cdot z\right)^{\frac{-n+1}{2}-2} \\ &\times \left[G^{(2)}z\right]_i \left[G^{(2)}z\right]_j \left[\tilde{G}^{(2)}z\right]_k \left[\tilde{G}^{(2)}z\right]_l \frac{\det G^{(3)}}{\sqrt{\det g}}\,\mathrm{d}z, \end{aligned} \tag{3.30}$$

where $\tilde{G}^{(2)}_{ij}(x,y) = G^{(2)}_{ij}(y,x)$. Note that M_{ijkl} is the Fourier transform of a positively homogeneous distribution in the z variable, of order $n-1$. Therefore, M_{ijkl} itself is positively homogeneous of order -1 in ξ. Write

$$M(x,y,\xi) = 2\int e^{-\mathrm{i}\xi\cdot z}|z|^{-n+1}m(x,y,\theta)\,\mathrm{d}z, \quad \theta = z/|z|, \tag{3.31}$$

where, contrary to our convention, $|\cdot|$ stands for the Euclidean norm, and

$$\begin{aligned} m_{ijkl}(x,y,\theta) =& 2\left(G^{(1)}\theta\cdot\theta\right)^{\frac{-n+1}{2}-2} \\ & \times \left[G^{(2)}\theta\right]_i\left[G^{(2)}\theta\right]_j\left[\tilde{G}^{(2)}\theta\right]_k\left[\tilde{G}^{(2)}\theta\right]_l\frac{\det G^{(3)}}{\sqrt{\det g(x)}}, \end{aligned} \tag{3.32}$$

and pass to polar coordinates $z = r\theta$. Since m is an even function of θ, smooth w.r.t. all variables, we get (see also [H, Theorem 7.1.24])

$$M(x,y,\xi) = 2\pi\int_{|\theta|=1} m(x,y,\theta)\delta(\theta\cdot\xi)\,\mathrm{d}\theta. \tag{3.33}$$

Again, $|\theta|$ is the Euclidean norm of θ. Now it is easy to see that M is an amplitude of order -1. Indeed, it is positively homogeneous of order -1 in ξ, and this makes it a classical amplitude.

To obtain the principal symbol, we set $x = y$ above to get

$$\sigma_p(N)(x,\xi) = M(x,x,\xi) = 2\pi\int_{|\theta|=1} m(x,x,\theta)\delta(\theta\cdot\xi)\,\mathrm{d}\theta, \tag{3.34}$$

where

$$m^{ijkl}(x,x,\theta) = 2\sqrt{\det g(x)}\left(g_{ij}(x)\theta^i\theta^j\right)^{\frac{-n+1}{2}-2}\theta^i\theta^j\theta^k\theta^l, \tag{3.35}$$

One can show that (3.34), (3.35) can be written in a more elegant way as

$$\sigma_p(N)^{ijkl}(x,\xi) = 2\pi\int_{S_xM_{\mathrm{e}}} \omega^i\omega^j\omega^k\omega^l\delta(\xi\cdot\omega)\,\mathrm{d}\sigma_x(\omega), \tag{3.36}$$

where $\xi\cdot\omega = \xi_i\omega^i$. Compare this with (3.19).

To prove ellipticity of $M(x,\xi)$ on solenoidal tensors at (x_0,ξ^0), notice that for any symmetric real f_{ij}, we have

$$m^{ijkl}(x_0,x_0,\theta)f_{ij}f_{kl} = 2\sqrt{\det g(x_0)}\left(g_{ij}(x_0)\theta^i\theta^j\right)^{\frac{-n+1}{2}-2}\left(f_{ij}\theta^i\theta^j\right)^2 \geq 0. \tag{3.37}$$

This and (3.34) imply that $M^{ijkl}(x_0,x_0,\xi^0)f_{ij}f_{kl} = 0$ yields $f_{ij}\theta^i\theta^j = 0$ for θ perpendicular to ξ^0, and close enough to θ_0. If in addition $(\xi^0)^jf_{ij} = 0$, then this implies $f_{ij}\theta^i\theta^j = 0$ for $\theta\in\text{neigh}(\theta_0)$, and that easily implies that it vanishes for all θ. Since f is symmetric, this means that $f = 0$.

The last statement of the lemma follows directly from (3.34), (3.35), (3.37).

Finally, we note that (3.35), (3.37) and the proof above generalizes easily for tensors of any order. □

3.8. **Construction of a parametrix for N.** Since N is not elliptic, we cannot construct a parametrix in the classical sense. What we can do however is to construct a parametrix Q, so that $QN = \mathcal{S} + K$, where K is smoothing, and $\mathcal{S}$ is the solenoidal projection. In the interior of M, this can be done as follows. Consider $W := N + |D|^{-1}\mathcal{P}$, where $|D|^{-1}$ is any properly supported parametrix of $(-\Delta_g)^{1/2}$. Then that operator is elliptic of order -1, and has a parametrix L of order

1 so that $LW = \mathrm{Id}+K$, K smoothing. Now, apply $\mathcal{S}$ to the left and right to get that $PN = \mathcal{S}+K_1$, where $P = \mathcal{S}L\mathcal{S}$ and K_1 is smoothing. Note that $\mathcal{S}$ is a ΨDO inside M but not near the boundary.

There is an essential problem with that construction. It holds for tensors supported in any compact inside M^{int} but not for general tensors. This is related to the following: when applied to such tensors, our operators are not ΨDOs near ∂M anymore (unless we want to use a specialized calculus), but the corresponding terms, for example $\mathcal{S}f$, are smooth near ∂M, up tp ∂M, by standard elliptic regularity for boundary value problems.

We will push ∂M a bit, in other words, we will work in M_{e}. Then we work with tensors f in M, extended as zero outside M. It seems that this resolves our problems, but not quite. For any such f, we have

$$PNf = f^s_{M_{\text{e}}} + K_1 f. \tag{3.38}$$

However, $f^s_{M_{\text{e}}}$ there is the solenoidal projection of f (extended as zero to $M_{\text{e}} \setminus M$) related to M_{e}, which explains the notation), not the one we want! This is similar to the need to work with two solenoidal projections of f in the Euclidean case:f^s in Ω and $f^s_{\mathbf{R}^n}$ in $\mathbf{R}^n$, see section 3.6.2. Let us denote the usual solenoidal projection f^s by f^s_M.

So, we have recovered $f^s_{M_{\text{e}}}$ from Nf, up to a smoothing term but it remains to recover f^s_M, given $f^s_{M_{\text{e}}}$.

Let us compare f^s_M and $f^s_{M_{\text{e}}}$ for $f \in L^2(M)$. We have $f^s_M = f - dv_M$, where $v_M = (\delta d)^{-1}_{\text{D}}\delta f$, similarly for $f^s_{M_{\text{e}}}$. Thus $f^s_M = f^s_{M_{\text{e}}} + dw$ in M, where the vector field $w = v_{M_{\text{e}}} - v_M \in H^1(M)$ solves

$$\delta dw = 0 \quad \text{in } M, \quad w|_{\partial M} = v_{M_{\text{e}}}. \tag{3.39}$$

We need to express $v_{M_{\text{e}}}|_{\partial M}$ in terms of Nf. This can be done as follows. Our inspiration comes from the Euclidean case, see the proof of Proposition 3.4 and the remark after it. By (3.38), and the fact that $f = 0$ outside M, one has

$$-dv_{M_{\text{e}}} = P_1 Nf - K_2 f \quad \text{in } M_{\text{e}} \setminus M. \tag{3.40}$$

For (x,ξ) in a one-sided neighborhood of $(x_0,\nu(x_0)) \in \Gamma_+$ in $T(M_{\text{e}} \setminus M)$, where $\nu(x_0)$ is the outer unit normal to ∂M, integrate the above along $\gamma_{x,\xi}$ until this geodesic hits ∂M_1, where $v_{M_{\text{e}}} = 0$; denote the corresponding time by $\tau(x,\xi)$. We therefore get

$$[v_{M_{\text{e}}}(x)]_i\,\xi^i = \int_0^{\tau(x,\xi)} [P_1Nf - K_2f]_{ij}(\gamma_{x,\xi}(t))\dot\gamma^i_{x,\xi}(t)\dot\gamma^j_{x,\xi}(t)\,\mathrm{d}t. \tag{3.41}$$

Compare this with (3.25), (3.26). Note that in the Euclidean case, $K_2 = 0$ because the parametrix is an exact inverse (but in the whole $\mathbf{R}^n$).

Clearly, for any fixed x, a set of n linearly independent ξ's in any neighborhood of $\nu(x_0)$ is enough to determine $v_{M_{\text{e}}}(x)$. This is done by solving a linear $n \times n$ system. We choose this set independent of x in a neighborhood of each $x_0 \in \partial M$, then by compactness argument we choose a finite covering and finite number of such sets. This allows us to construct an operator P_2, such that

$$v_{M_{\text{e}}}|_{\partial M} = P_2(P_1N - K_2)f. \tag{3.42}$$

To understand the mapping properties of P_2, consider first the case $m = 1$, i.e., f is an 1-form, and then v is just a function. Then P_2 is just antidifferentiation with zero initial conditions on ∂M_1. Let h be the r.h.s. of (3.40). Then one can express v through h as in (3.41), and this and (3.40) allows us easily to conclude that $P_2 : L^2(M_{\text{e}} \setminus M) \to H^1(M_{\text{e}} \setminus M)$. Therefore, $P_2P_1N : L^2 \to H^1$

(remember, P_1N is of order 0). Then we can take the trace on ∂M to get that $v_{M_{\rm e}} \in H^{1/2}(\partial M)$, and this is exactly what we need below.

Let us go back to the case $m=2$. If we try to do the same, there we face an essential difficulty: the symmetric differential d mapping 1-tensors into 2-tensors is elliptic, indeed, but $\mathrm{d}v$ (the usual differential of v) can be expressed through dv (the symmetric one) by a non-local operator, and we only have (3.40) on the exterior side of M. This does not allow us to use the arguments above to establish the same mapping properties of P_2. Instead, we do the following.

Let us denote again the r.h.s. of (3.40) by $h \in L^2$. Then express the r.h.s. of (3.41) as $\tilde P_2 h$. To estimate $\|P_2 h\|_{H^1}$ in $M_{\rm e}^{\rm int}\backslash M$, differentiate $\tilde P_2 h$. If we differentiate in the direction of ξ, this kills the integral and the result is in L^2. If we differentiate in any other direction, then the smoothing effect of the integral does not help and we need to differentiate h that is only in L^2. Let us assume now that actually, $h\in H^1(M)$. Then everything will be OK, but this would require that f, extended as zero is in H^1. In other words, f needs to be in $H^1(M)$, and in addition, we need to know that $f=0$ on ∂M. This is a requirement that we do not want to impose because we really want to work eventually with f^s instead of f and there are no reasonable assumptions on f that would guarantee that $f^s=0$ on ∂M.

By inspecting our argument carefully, we see that the derivative in any direction can be whiten as a tangential derivative plus a derivative in the direction of ξ. The latter one just kills the integral, as above. So we only need to worry about tangential derivatives. If x is not on ∂M, we work in local coordinates $x=(x',x^n)$, and “tangential” means tangent to $x^n = \text{const}$, i.e., $\partial_{x'}$. Any $f\in H^1(M)$, extended as zero outside M has such derivatives (in L^2). Moreover, if we apply any zero order ΨDO A to f, then the same applies to Af, because A and $\partial_{x'}$ commute up to an operator of order -1. Therefore, tangential derivatives of h exist.

Those arguments motivate the need to introduce the Hilbert space $\tilde H^2(M_{\rm e})$ below. Let $x=(x',x^n)$ be local coordinates in a neighborhood U of a point on ∂M such that $x^n=0$ defines ∂M. Then we set

$$\|f\|^2_{\tilde H^1(U)} = \int_U \Big(\sum_{j=1}^{n-1} |\partial_{x^j} f|^2 + |x^n\partial_{x^n} f|^2 + |f|^2\Big)\,\mathrm{d}x.$$

This can be extended to a small enough neighborhood V of ∂M contained in $M_{\rm e}$. Then we set

$$\|f\|_{\tilde H^2(M_{\rm e})} = \sum_{j=1}^n \|\partial_{x^j} f\|_{\tilde H^1(V)} + \|f\|_{\tilde H^1(M_{\rm e})}. \tag{3.43}$$

This norm defines a Hilbert space and $H^2(M_{\rm e})\subset \tilde H^2(M_{\rm e})\subset H^1(M_{\rm e})$. We also define the $\tilde H^2(M_{\rm e})$ space of symmetric 2-tensors and 1-forms. Note that it is “almost” H^2 but near ∂M, we take only tangential derivatives of ∇f to define the second order terms in the norm.

The space $\tilde H^2(M_{\rm e})$ has the property that for each $f\in H^1(M)$ (extended as zero outside M), we have $Nf\in \tilde H^2(M_{\rm e})$. This is not true if we replace $\tilde H^2(M_{\rm e})$ by $H^2(M_{\rm e})$.

We can return now to our parametrix construction. The arguments above show that

$$\|P_2P_1 h\|_{H^{1/2}(\partial M)} \le C\|h\|_{\tilde H^2(M_{\rm e})}, \quad \forall h\in \tilde H^2(M_{\rm e}),$$

and one can see that P_2K_2 depends continuously on $g\in C^k$, $k\gg 1$.

Let $R: H^{t-\frac12}(\partial M)\to H^t(M)$, be the solution operator $u=Rh$ of the boundary value problem

$$\delta d u = 0 \quad \text{in } M, \quad u|_{\partial M} = h. \tag{3.44}$$

Then R depends continuously on $g \in C^2$, see [SU5]. Then (3.39) and (3.42) show that $w|_M = RP_2(P_1N - K_2)f$. Therefore,

$$\begin{aligned} f_M^s &= f_{M_e}^s + dw = (P_1N - K_2)f + dRP_2(P_1N - K_2)f \\ &= (\mathrm{Id} + dRP_2)P_1Nf + Kf, \end{aligned}$$

where K is smoothing. Apply $\mathcal{S}_M$ to the identity above and set $Q = \mathcal{S}_M(\mathrm{Id} + dRP_2)P_1 = (\mathcal{S}_M + dRP_2)P_1$. This completes the sketch of the proof of the following.

Proposition 3.6. *Let $g \in C^k(M)$ be simple. Then for any $t = 1, 2, \dots,$ there exists $k > 0$ and a bounded linear operator*

$$Q : \tilde{H}^2(M_e) \longrightarrow \mathcal{S}L^2(M), \tag{3.45}$$

such that

$$QNf = f_M^s + Kf, \quad \forall f \in H^1(M), \tag{3.46}$$

where $K : H^1(M) \to \mathcal{S}H^{1+t}(M)$ extends to $K : L^2(M) \to \mathcal{S}H^t(M)$. If $t = \infty$, then $k = \infty$. Moreover, Q can be constructed so that K depends continuously on g in a small neighborhood of a fixed $g_0 \in C^k(M)$.

This proposition shows, that If, and therefore Nf, determine the singularities of f^s uniquely. In other words, we can recover f^s up to a term that is as smooth as we want. Moreover, it allows us to prove the first important result: finiteness and smoothness of $\mathrm{Ker}\, I$. This follows immediately from the fact that if $If = 0$, then f^s solves the Fredholm equation $(\mathrm{Id} + K)f = 0$.

Theorem 3.3. *Assume that g is simple metric in M and extend g as a simple metric in M_e.*

(a) The following estimate holds for each symmetric 2-tensor f in $H^1(M)$:

$$\|f_M^s\|_{L^2(M)} \le C\|N_g f\|_{\tilde{H}^2(M_e)} + C_s\|f\|_{H^{-s}(M_e)}, \quad \forall s > 0.$$

(b) $\mathrm{Ker}\, I_g \cap \mathcal{S}L^2(M)$ *is finite dimensional and included in $C^\infty(M)$.*

(c) Assume that I_g is s-injective in M, i.e., that $\mathrm{Ker}\, I_g \cap \mathcal{S}L^2(M) = \{0\}$. *Then for any symmetric 2-tensor f in $H^1(M)$ we have*

$$\|f^s\|_{L^2(M)} \le C\|N_g f\|_{\tilde{H}^2(M_e)}. \tag{3.47}$$

Part (b) follows from (a) [Ta1, Proposition V.3.1], and also can be deduced from the following argument: if K is compact and $\mathrm{Id} + K$ is injective, then it is invertible.

Remark 3. If $g \in C^k(M)$, then $C^\infty(M)$ in (b) should be replaced by $C^l(M)$ with $l = l(k) \to \infty$, as $k \to \infty$, by the arguments in section 3.9 below.

3.9. **Openness of the set of s-injective simple metrics. Proof of the a priori linear estimate.** We will now use the results of the previous section to show that the set of metrics with s-injective ray transform I_g is open in $C^k(M)$ for $k \gg 1$, and moreover, we have (3.1). In other words, we will prove Theorem 3.2 without the statement that $\mathcal{G}^k(M)$ is dense. As explained in the beginning of this long section, we start with the observation that K in (3.46) is a compact operator in $\mathcal{S}_g L^2(M)$. Therefore, if $\mathrm{Id} + K_g$ is injective for some $g = g_0$, then it is invertible, and remains so for g close to g. The later has to be understood in a topology that makes the maps $g \mapsto K_g$, $\mathcal{S}_g$ continuous. There is a small inconvenience here that the space $\mathcal{S}_g L^2(M)$ depends on g as well but this can be fixed by adding $\mathcal{P}_g$ to $\mathrm{Id} + K_g$. We claim that the $C^k(M)$ with $k \gg 1$ is one such topology. This can be justified as follows. Instead of working with ΨDOs with C^∞ symbols, we work with C^k symbols. ΨDOs of non-positive order are still bounded in any bounded domain, if the symbol is in C^{2n+1}, see [H, Theorem 18.1.11'] and [SU1]. In all basic operations with ΨDOs like composition, constructing a parametrix, etc., we work with finite symbol expansions, instead of infinite ones. Then the parametrix will invert the elliptic operator modulo an operator with a kernel that is C^l only, where $l = l(k) \to \infty$, as $k \to \infty$.

To make the argument above work, we have to resolve one more problem. Namely, we have to make sure that in (3.46), if N_{g_0} is s-injective, then so is $Q_{g_0} N_{g_0}$. Notice that any perturbation of Q by a finite rank operator Q_0 will contribute a finite rank term to K_{g_0}, so K_{g_0} will stay compact. So, if $\mathrm{Id} + K_{g_0}$ is not s-injective but N_{g_0} is, then $\mathrm{Ker}(\mathrm{Id} + K_{g_0})$ is finite. Then we construct Q_0 so that $(Q_{g_0} + Q_0)N_{g_0}$ has a trivial kernel. Roughly speaking, Q_0 maps $N_{g_0}\,\mathrm{Ker}(\mathrm{Id} + K_{g_0})$ into $\mathrm{Ker}(\mathrm{Id} + K_{g_0})$. We refer to [SU5, section 5] for more details.

3.10. **S-injectivity for analytic metrics.** We will sketch the proof of Theorem 3.1 here. Let g be real analytic in M. We can assume that ∂M is analytic, and that $g \in \mathcal{A}(M_{\mathrm{e}})$.

We will show first that then N is an analytic ΨDO in M^{int}. Our reference for analytic ΨDOs is [Tre]. Roughly speaking, those are ΨDOs with amplitudes $a(x, y, \xi)$, $(x, y, \xi) \in X \times X \times \mathbf{R}^n$ analytic in all variables and satisfying the usual symbol estimates (actually, only the one about the zero order derivatives is enough) in a complex neighborhood of $X \times X \times \mathbf{R}^n$. The negligible operators then are the ones that are analytic-regularizing, i.e., they send any distribution of compact support into a real analytic function. One can change the amplitude and therefore, destroy the analyticity in any compact in the ξ variable, and this will result in an analytically regularizing error. Next, one can ask that only (x, y) stay complex but ξ is real. Then the symbol estimates look like this:

$$\left|D_\xi^\alpha a(x, y, \xi)\right| \le C^{|\alpha|+1}\alpha!|\xi|^{m-|\alpha|}, \quad |\xi| \ge R_0 \sup(|\alpha|, 1).$$

Then the (x, y)-derivatives can be estimated by the Cauchy integral formula. Such an amplitude is called in [Tre] a *pseudoanalytic amplitude*, see [Tre, Definition V.2.1]. The corresponding ΨDO is called an analytic ΨDO.

An elliptic analytic ΨDO has the useful property that it has a parametrix that is a left inverse up to an analytic-regularizing operator. Now, suppose that $N : L^2(M) \to L^2(M_{\mathrm{e}})$ acts on functions, and we have already proved that N is an order -1 elliptic ΨDO. Then there is a parametrix Q so that $QN = \mathrm{Id} + K$ when acting of functions of compact support in M_{e}, and K is analytic-regularizing in M^{int}. If $Nf = 0$, then $f = -Kf$, where, as always, we extend f as zero outside M. Therefore, f is real analytic in $M^{\mathrm{int}} \setminus M$, and vanishes in $M^{\mathrm{int}} \setminus M$. Therefore, $f = 0$. So, N has a trivial kernel.

In case of tensors, N is an analytic ΨDO in M^{int} as follows from the representation (3.33), (3.32) of its amplitude. We have to choose the coordinates in (3.29) carefully however to make sure that (3.29) hold globally in M (other arguments can be applied here as well, see [SU5]). We work in a neighborhood of a fixed x_0, and then we choose x to be normal coordinates centered at x_0. Note that M in (3.33) has a singularity of the type $|\xi|^{-1}$ at $\xi = 0$ but it can be easily resolved.

Proposition 3.7. *Let $g \in \mathcal{A}(M_e)$ and assume that $If = 0$ with some $f \in L^2(M)$. Then $f^s_{M_e} \in \mathcal{A}(M_e)$*

Proof. Let us first work in M instead of working in M_{e} in order to see why M_{e} is needed. Let $If = 0$. Replace f by f^s, then we still have $If^s = 0$. We have $\delta f^s = 0$ in M. Since N is elliptic on solenoidal tensors, the pair $(|D|N, \delta)$ is an elliptic analytic ΨDO inside M so we get that f^s is analytic inside M. That does not tell us however what happens near ∂M, i.e., we do not know from those arguments that f^s extends as a real analytic tensor up to ∂M. The later means that f^s extends analytically to some neighborhood of ∂M.

We apply the same arguments to the extension of f to $M_{\rm e}$ as zero, in $M_{\rm e}$. Then we get that its solenoidal projection, that we denote by $f^s_{M_{\rm e}}$ is analytic inside $M_{\rm e}$ but perhaps not up to $\partial M_{\rm e}$. We can always assume that the latter is analytic. Then $f^s_{M_{\rm e}} = f - dv_{M_{\rm e}}$, and in $M_{\rm e} \setminus M$, we have $f^s_{M_{\rm e}} = -dv_{M_{\rm e}}$. On the other hand, $v_{M_{\rm e}}$ satisfies

$$\delta dv_{M_{\rm e}} = 0 \quad \text{in } M_{\rm e} \setminus M, \quad v|_{\partial M_{\rm e}} = 0.$$

Any solution to this equation is analytic up to the boundary $\partial M_{\rm e}$ [MN]. So we get the same for $f^s_{M_{\rm e}}$. ☐

Remark 4. We can quickly conclude that $f^s \in \mathcal{A}(M)$ as well, as in (3.39)–(3.42). This is not needed however because now we can simply replace M by $M_{\rm e}$. Eventually, we will show that $f^s = 0$, and $f^s_{M_{\rm e}} = 0$.

The next lemma is a boundary recovery result. We expect that $If = 0$ implies $f = dv$ with some $v \in H^1_0(M)$. We still cannot prove that for all simple metrics but the lemma below says that we can show that this is true at ∂M of infinite order.

Lemma 3.3. *Let $g \in C^k(M)$ be a simple metric. Then if $If = 0$ with $f \in L^2(M)$, then there exists a vector field $v \in C^l(M)$, with $v|_{\partial M} = 0$ and $l = l(k) \to \infty$, as $k \to \infty$, such that for $h := f - dv$ we have*

$$\partial^\alpha h|_{\partial M} = 0, \quad |\alpha| \le l, \tag{3.48}$$

and in boundary normal coordinates near any point on ∂M we have

$$h_{ni} = 0, \quad \forall i. \tag{3.49}$$

Proof. Without loss of generality, we may assume that $\partial M_{\rm e}$ is at distance $\epsilon > 0$ with $\epsilon > 0$ small enough, i.e., $\partial M_{\rm e} = \{x \in M_{\rm e} \setminus M,\ \rho(x, M) = \epsilon\}$. By Theorem 3.3, applied to $M_{\rm e}$,

$$f^s_{M_{\rm e}} \in C^l(M_{\rm e}), \tag{3.50}$$

where $l \gg 1$, if $k \gg 1$.

Let $x = (x', x^n)$ be boundary normal coordinates in a neighborhood of some boundary point. We recall how to construct v defined in M so that (3.49) holds, see [SU3] for a similar argument for the non-linear boundary rigidity problem, and [E, Sh3, SU4, SU5] for the present one. The condition $(f - dv)_{in} = 0$ is equivalent to

$$\nabla_n v_i + \nabla_i v_n = 2f_{in}, \quad v|_{x^n=0} = 0, \quad i = 1, \dots, n. \tag{3.51}$$

Recall that $\nabla_i v_j = \partial_i v_j - \Gamma^k_{ij} v_k$, and that in those coordinates, $\Gamma^k_{nn} = \Gamma^n_{kn} = 0$. If $i = n$, then (3.51) reduces to $\nabla_n v_n = \partial_n v_n = f_{nn}$, $v_n = 0$ for $x^n = 0$; we solve this by integration over $0 \le x^n \le \varepsilon \ll 1$; this gives us v_n. Next, we solve the remaining linear system of $n-1$ equations for $i = 1, \dots, n-1$ that is of the form $\nabla_n v_i = 2f_{in} - \nabla_i v_n$, or, equivalently,

$$\partial_n v_i - 2\Gamma^\alpha_{ni} v_\alpha = 2f_{in} - \partial_i v_n, \quad v_i|_{x^n=0} = 0, \quad i = 1, \dots, n-1, \tag{3.52}$$

(here $\alpha = 1, \dots, n-1$). Clearly, if g and f are smooth enough near ∂M, then so is v. If we set $f = f^s$ above (they both belong to $\operatorname{Ker} I$), then by (a) we get the statement about the smoothness of v. Since the condition (3.49) has an invariant meaning, this in fact defines a construction in some one-sided neighborhood of ∂M in M. One can cut v outside that neighborhood in a smooth way to define v globally in M. We also note that this can be done for tensors of any order m, see [Sh3], then we have to solve consecutively m ODEs.

Let $h = f - dv$, where v is as above. Then h satisfies (3.49), and let

$$h^s_{M_{\rm e}} = h - dw_{M_{\rm e}} \tag{3.53}$$

be the solenoidal projection of h in $M_{\rm e}$. Recall that h, according to our convention, is extended as zero in $M_{\rm e}\setminus M$ that in principle, could create jumps across ∂M. Clearly, $h^s_{M_{\rm e}} = f^s_{M_{\rm e}}$ because $f-h = dv$ in M with v as in the previous paragraph, and this is also true in $M_{\rm e}$ with h, f and v extended as zero (and then $v=0$ on $\partial M_{\rm e}$). In (3.53), the l.h.s. is smooth in $M_{\rm e}$ by (3.50), and h satisfies (3.49) even outside M, where it is zero. Then one can get $w_{M_{\rm e}}$ by solving (3.51) with M replaced by $M_{\rm e}$, and f there replaced by $h^s_{M_{\rm e}}\in C^l(M_{\rm e})$. Therefore, one gets that $w_{M_{\rm e}}$, and therefore h, is smooth enough across ∂M, if $g\in C^k$, $k\gg 1$, which proves (3.48).

One can give the following alternative proof of (3.48). One can easily check that N, restricted to tensors satisfying (3.49), is elliptic for $\xi_n\neq 0$. Since $Nh=0$ near M, with h extended as 0 outside M, as above, we get that this extension cannot have conormal singularities across ∂M. This implies (3.48), at least when $g\in C^\infty$. The case of g of finite smoothness can be treated by using parametrices of finite order in the conormal singularities calculus. $\square$

Proof of Theorem 3.1. To simplify the notation, we will replace M by $M_{\rm e}$. If we show that $f^s_{M_{\rm e}}=0$, we are done, because then we would get $f=dv$ with some v vanishing on $\partial M_{\rm e}$, and $f=0$ in $M_{\rm e}\setminus M$. This easily implies that $v=0$ in $M_{\rm e}\setminus M$, see (3.40), (3.41).

So, denote $M_{\rm e}$ by M. By Lemma 3.3 applied to f^s, there exists a smooth v_0 vanishing on ∂M, so that f^s-dv_0 has zero jet on ∂M. The proof of the lemma also implies that v_0 is real analytic near ∂M. On the other hand, f^s is analytic by Proposition 3.7. Therefore, $f^s=dv_0$ in a neighborhood of ∂M.

We need to show now that v_0 has an analytic extension everywhere in M. Consider

$$u_\pm(x,\xi)=\int_0^{\tau_\pm(x,\xi)} f^s_{ij}(\gamma_{x,\xi}(t))\dot\gamma_{x,\xi}(t)^i\dot\gamma^j_{x,\xi}(t)\,\mathrm{d}t,$$

where $\tau_\pm(x,\xi)$ is the time needed to reach ∂M from $(x,\pm\xi)$. We have

$$u_-+u_+=0 \tag{3.54}$$

because $If=0$. Next, $u_\pm$ is real analytic inside SM. For x close to ∂M, and ξ close to normal direction to the boundary (in other words, if $\partial M=\{x^n=0\}$ locally, we want $0<x^n\ll 1$, $|\xi'|\ll 1$), we have $u_\pm(x,\xi)=(v_0(x))_j\xi^j$, thus $\partial^\alpha_\xi u_\pm=0$ for such (x,ξ) and $|\alpha|=2$. This extends analytically to the whole SM. Therefore, u_+ is a linear function of ξ. Relation (3.54) shows that u_+ must be odd in ξ. Therefore, $u_+=v_j(x)\xi^j$ with v real analytic inside M, and near ∂M, $v=v_0$. So we showed that v_0 extends analytically. Since $f^s=dv$ near ∂M, by analytic extension, we get the same everywhere in M. That however implies $f^s=0$. $\square$

3.11. **End of the Proof of Theorem 3.2.** In section 3.9, we sketched the proof of Theorem 3.2 without the statement that $\mathcal{G}^k(M)$ is dense. Theorem 3.1 provides the missing part.

4. Generic Boundary Rigidity for simple metrics

We will formulate here a generic boundary rigidity result for simple manifolds, and will sketch its proof. We linearize near a metric with an s-injective I_g using the results in the previous section. For complete details, we refer to [SU5].

Theorem 4.1 ([SU5]). *Let k_0 and $\mathcal{G}^k(M)$ be as in Theorem 3.2. There exists $k\geq k_0$, such that for any $g_0\in\mathcal{G}^k$, there is $\varepsilon>0$, such that for any two metrics g_1, g_2 with $\|g_m-g_0\|_{C^k(M)}\leq\varepsilon$, $m=1,2$, we have the following:*

$$\rho_{g_1}=\rho_{g_2} \text{ on } (\partial M)^2 \quad \text{implies } g_2=\psi_*g_1 \tag{4.1}$$

with some $C^{k+1}(M)$-diffeomorphism $\psi: M\to M$ fixing the boundary pointwise.

We would like to note that if two metrics are isometric, i.e., $g_2 = \psi_* g_1$ with $\psi \in C^3$, and if $g_{1,2} \in C^k(M)$, $k \geq 2$, then ψ must be in C^{k+1}, and moreover, if $\|g_1\|_{C^k} + \|g_2\|_{C^k} \leq A$, then $\|\psi\|_{C^{k+1}} \leq C(A)$, see [SU5, Lemma 6].

4.1. **Recovery of the get of g in boundary normal coordinates.** We start with a boundary recover result. The next theorem can be considered as a non-linear version of Lemma 3.3.

Theorem 4.2. *Let g_1 and g_2 be two simple smooth metrics on M with the same boundary distance function. Then there exists a smooth diffeomorphism $\psi : M \to M$ fixing the boundary pointwise so that*

$$\partial^\alpha g_1 = \partial^\alpha (\psi^* g_2) \quad \text{on } \partial M \tag{4.2}$$

in any coordinate system, for any multiindex α.

Proof. We will prove something more specific. Choose boundary normal coordinates related to g_i, $i = 1, 2$. In principle, they depend on g_i. Identify them now. In other words, we consider a diffeomorphism ψ that maps the g_1 boundary normal coordinates to the g_2 boundary normal coordinates near ∂M, and then we extend it inside M. Then we set $\hat{g}_2 = \psi^* g_2$. Now g_1 and $\hat{g}_2$ have the same boundary normal coordinates. let us call them x. We will denote $\hat{g}_2$ again by g_2 and will show that g_1 and g_2 have the same jet at ∂M. It is enough to show that

$$\partial^k_{x^n} f = 0 \quad \text{for } x^n = 0, \forall k, \text{ where } f = g_1 - g_2. \tag{4.3}$$

We will sketch the proof in [LSU]. The equality (4.3) for $k = 0$ is immediate by studying the lengths of geodesics connecting x, y on ∂M and letting $y \to x$. Assume that (4.3) is wrong. Then there is an integer l so that $\partial^l_{x^n} f \not\equiv 0$ for $x^n = 0$ and let l be the least integer with that property. Then $\partial^l_{x^n} f(x_0) \neq 0$ for some $x_0 \in \partial M$. By studying the Taylor expansion of f w.r.t. x^n near $x^n = 0$, and with $x' = x'_0$ fixed, we see that there exists a unit vector ξ_0 tangent to ∂M so that either $f_{ij}(x)\xi^i\xi^j > 0$ or $f_{ij}(x)\xi^i\xi^j < 0$ for (x, ξ) near (x_0, ξ_0) and $x \notin \partial M$. We can assume the first inequality. Then we get $I_{g_j} f(x, \xi) > 0$, $j = 1, 2$ for all (x, ξ) close enough to (x_0, ξ_0), and ξ_0 not tangent to ∂M (we use the strict convexity here). Now, $I_{g_1} g_1 = \rho_{g_1}(x, y)$, where $y \in \partial M$ is the exit point of $\gamma^{g_1}_{x,\xi}$ (the superscript g_1 indicates that this is the geodesic in the metric g_1). So we get $\rho_{g_1}(x, y) > I_{g_2} f(x, \xi)$. On the other hand, $I_{g_2} f(x, \xi) \geq \rho_{g_2}(x, y)$ because the energy form for all smooth curves connecting x and y is minimized by $\gamma^{g_2}_{x,\xi}$. Those two inequalities contradict the given equality $\rho_{g_1} = \rho_{g_2}$ on $\partial M \times \partial M$. $\square$

If $g_{1,2}$ in Theorem 4.2 are of finite smoothness C^k, then (4.2) remains true for $|\alpha| \leq k - 2$.

A more general boundary recovery results was recently proved by the author and G. Uhlmann in [SU7], see Theorem 5.3.

4.2. **Proof of the generic boundary rigidity.**

Sketch of the Proof of Theorem 4.1. Let $g_0 \in \mathcal{G}^k$ with k large enough. Let g_1 and g_2 be two metrics such that $\rho_{g_1} = \rho_{g_2}$ on $\partial M \times \partial M$, and

$$g_1, g_2 \in \mathcal{B} = \left\{ g \in C^k(M);\ \|g - g_0\|_{C^k(M)} \leq \varepsilon \right\} \tag{4.4}$$

We will show that for $0 < \varepsilon \ll 1$, g_2 is isometric to g_1.

First, by Theorem 4.2, we may assume that g_1 and g_2 have the same boundary normal coordinates, and that (4.2) holds for $|\alpha|$ as large as needed, if $k \gg 1$. One can see that we still may assume that (4.4) holds. Using (4.2), we extend g_1 and g_2 in the same way to M_{e} by keeping those extensions C^k. Then we pass to semigeodesic coordinates as in the second paragraph of

Section 3.2.4, related to each metric. Each such coordinate system gives as a diffeomorphism ϕ_j from M to a domain $\Omega_j \subset \mathbf{R}^n$, $j = 1, 2$. A priori, Ω_1 may be different from Ω_2 but since g_1 and g_2 have the same scattering relation, we get that actually, $\phi_1 = \phi_2$ in $M_\mathrm{e} \setminus M$, and in particular, $\Omega_1 = \Omega_2$, that we will call just Ω. Denote also $\Omega_\mathrm{e} = \phi_1(M_\mathrm{e}) = \phi_2(M_\mathrm{e})$. Then we consider the push forwards $\phi_{1*}g_1$, $\phi_{2*}g_2$. It is important to note that the new metrics still agree at $\partial\Omega$ at any fixed order, if $k \gg 1$ because $\phi_1 = \phi_2$ in $M_\mathrm{e} \setminus M$. As above, we can still assume that the new metrics are in $\mathcal{B}$. This gives us that for $f := \phi_{1*}g_1 - \phi_{2*}g_2$ we have

$$f \in C^k(\Omega_\mathrm{e}), \quad \operatorname{supp} f \subset \Omega, \quad f_{in} = 0, \;\; i = 1, \dots, n. \tag{4.5}$$

We now use the fact that the linearization of $\rho^2_{g_1}(x, y)$ for $(x, y) \in (\partial\Omega)^2$ is $I_{g_1} f(x, \xi)$, see Proposition 3.1, with $\xi = \exp_x^{-1} y / |\exp_x^{-1} y|$, to get

$$\|N_{g_1} f\|_{L^\infty(\Omega_\mathrm{e})} \le C\|f\|^2_{C^1}, \tag{4.6}$$

with C uniform, if $k \ge 2$. Let $\varepsilon > 0$ be such that $\mathcal{B} \subset \mathcal{G}^k$, and the constant C in (3.1) is uniform in $\mathcal{B}$. Then using (3.1), (4.6), and interpolation estimates, we get that for any $0 < \mu < 1$,

$$\|f^s\|_{L^2} \le C\|f\|^{1+\mu}_{L^2}$$

with $C > 0$ uniform in $\mathcal{B}$, if $k = k(\mu) \gg 1$. The final step is to estimate f by f^s. There is no such estimate for generals f's, but we have the advantage here that f satisfies (4.5). Now, $f_{ni} = 0$ allows us to prove that $\|f\|_{L^2} \le C\|f^s\|_{H^2}$. Here is a brief sketch of that. Write $f = f^s + dv$. Then $(dv)_{in} = -f^s_{in}$. We can solve this equations for v, wee (3.51), and therefore estimate v and dv in terms of f^s. Therefore, we can estimate f in terms of f^s. For more details, see [E] and [SU5, Sec. 7.2]

Using interpolation estimates again, we get

$$\|f\|_{L^2} \le C\|f\|^{1+\mu}_{L^2}$$

with a new $\mu > 0$. This implies $f = 0$, if $\|f\|_{L^2} \ll 1$, and the latter condition is fulfilled, if $\varepsilon \ll 1$. This concludes the sketch of the proof of Theorem 4.1.

This sketch leaves hidden the need to know that f has zero derivatives across ∂M up to any fixed order, i.e., that the first condition in (4.5) holds. That is used in the interpolation estimates, to make sure that $N_{g_1} f$ is bounded in $H^{k+1}(\Omega_\mathrm{e})$ with some $k \gg 1$, if $f \in C^k$. $\square$

4.3. **Stability.** The linear stability estimate (3.1) in Theorem 3.2 and the "stable" proof of Theorem 4.1 above allow us to prove a Hölder type of conditional stability estimate.

Theorem 4.3 ([SU5]). *Let k_0 and $\mathcal{G}^k(M)$ be as in Theorem 3.2. Then for any $\mu < 1$, there exits $k \ge k_0$ such that for any $g_0 \in \mathcal{G}^k$, there are $\varepsilon_0 > 0$ and $C > 0$ with the property that that for any two metrics g_1, g_2 with $\|g_m - g_0\|_{C(M)} \le \varepsilon_0$, and $\|g_m\|_{C^k(M)} \le A$, $m = 1, 2$, with some $A > 0$, we have the following stability estimate*

$$\|g_2 - \psi_* g_1\|_{C^2(M)} \le C(A)\|\rho_{g_1} - \rho_{g_2}\|^\mu_{C(\partial M \times \partial M)}$$

with some diffeomorphism $\psi : M \to M$ fixing the boundary pointwise.

We will not present the proof here, see [SU5]. We basically follow the uniqueness proof above, and any time we use the fact that $\rho_{g_1} = \rho_{g_2}$ on $(\partial M)^2$, we replace it with the condition that $\rho_{g_1} - \rho_{g_2} = O(\delta)$, on $(\partial M)^2$ with $0 \le \delta \ll 1$, and we want to get eventually that $f = O(\delta^\mu)$. The proof is rather long and technical, although not really surprising. One of the important ingredients is the following stability at the boundary result, that is also of independent interest.

Theorem 4.4. *Let g_1 and g_2 be two simple metrics in M, and $\Gamma \subset\subset \Gamma' \subset \partial M$ be two sufficiently small open subsets of the boundary. Let ψ be as above. Then*

$$\left\|\partial_{x^n}^k(\psi_* g_2 - g_1)\right\|_{C^m(M)} \le C_{k,m}\left\|\rho_{g_2}^2 - \rho_{g_1}^2\right\|_{C^{m+2k+2}\left(\overline{\Gamma'\times\Gamma'}\right)},$$

where $C_{k,m}$ depends only on M and on a upper bound of g_1, g_2 in $C^{m+2k+5}(M)$.

The proof of Theorem 4.4 is actually the most difficult step in proving Theorem 4.3. It generalizes Theorem 4.2, but since the proof of the latter is not constructive, we could not just go over its steps and prove stability that way. On the other hand, one would expect that all derivatives of $\rho_g(x,y)$ at $y = x \in \partial M$ would recover recursively all derivatives of g in boundary normal coordinates at x. This is actually true, and done in [SU7], where the data is the scattering relation (determined uniquely by the boundary distance function). Having a constructive way to recover $\partial^\alpha g_{1,2}$, one could prove stability, too. We refer to [SU5] for the proof of Theorem 4.4, done before [SU7] that is still not constructive.

5. Generic Lens Rigidity for regular manifolds

We will describe here the results in [SU6, SU7] and we will be very sketchy about the proofs, even more than in the previous sections.

We study the lens rigidity question on M. Now, M is not necessarily diffeomorphic to a ball, and we may not have a global coordinate system anymore. The topology of M can be more complicated but we will still impose some topological condition. Next, we do not assume that ∂M is convex. We work with a subset of geodesics, i.e., we study the problem with incomplete data (under some conditions, of course). Finally, we do not assume lack of conjugate points anymore. We allow geodesics with conjugate points, but we need "enough" geodesics without conjugate points, and we use them only. Finally, (M,g) does not need to be non-trapping. The main results are of generic type, similarly to the ones above for simple metrics.

We start describing our assumptions.

Let $\mathcal{D}$ be an open subset of $\overline{B(\partial M)}$. Given $(x,\xi) \in \mathcal{D}$, let $\gamma_{\kappa_-^{-1}(x,\xi)}$ denote the geodesic issued from $\kappa_-^{-1}(x,\xi)$ with endpoint $\pi(\sigma(x,\xi))$, where π is the natural projection onto the base point. With some abuse of notation, we define

$$I_{\mathcal{D}}(x,\xi) = I(\gamma_{\kappa_-^{-1}(x,\xi)}), \quad (x,\xi) \in \mathcal{D}.$$

It is clear that one cannot hope to recover g from the scattering relation σ and the travel time ℓ restricted to $\mathcal{D}$, if (the closure of) the geodesics issued from $\mathcal{D}$ do not cover the whole M. The next condition is similar to that but it is in the phase space: we want the conormal bundle of those geodesics to cover T^*M so that we can recover the singularities. Moreover, we want those geodesics to be simple ones, since otherwise, one has examples where the singularities cannot be recovered.

Definition 5.1. *We say that $\mathcal{D}$ is **complete** for the metric g, if for any $(z,\zeta) \in T^*M$ there exists a maximal in M, finite length unit speed geodesic $\gamma : [0,l] \to M$ through z, normal to ζ, such that*

$$\{(\gamma(t),\dot\gamma(t));\ 0 \le t \le l\} \cap S(\partial M) \subset \mathcal{D}, \tag{5.1}$$

$$\text{there are no conjugate points on } \gamma. \tag{5.2}$$

*We call the C^k metric g **regular**, if a complete set $\mathcal{D}$ exists, i.e., if $\overline{B(\partial M)}$ is complete.*

If $z \in \partial M$ and ζ is conormal to ∂M, then γ may reduce to one point. Since (5.1) includes points where γ is tangent to ∂M, and $\sigma = \mathrm{Id}$, $\ell = 0$ there, knowing σ and ℓ on them provides no information about the metric g. On the other hand, we require below that $\mathcal{D}$ is open, so the purpose of (5.1) is to make sure that we know σ, ℓ near such tangent points.

Definition 5.2. *We say that (M,g) satisfies the **Topological Condition (T)** if any path in M connecting two boundary points is homotopic to a polygon $c_1\cup\gamma_1\cup c_2\cup\gamma_2\cup\dots\cup\gamma_k\cup c_{k+1}$ with the properties that for any j,*

(i) c_j is a path on ∂M;

(ii) $\gamma_j : [0,l_j]\to M$ is a geodesic lying in $M^{\rm int}$ with the exception of its endpoints and is transversal to ∂M at both ends; moreover, $\kappa_-(\gamma_j(0),\dot\gamma_j(0))\in\mathcal{D}$;

Notice that (T) is an open condition w.r.t. g, i.e., it is preserved under small C^2 perturbations of g.

5.1. **The Linear Problem for regular manifolds.** We will describe the results in [SU6] about the ray transform with incomplete data on regular manifolds.

To define the $C^K(M)$ norm in a unique way, and to make sense of real analytic g's, we choose and fix a finite real analytic atlas on M.

Theorem 5.1 ([SU6, SU7])**.** *Let $\mathcal{G}\subset C^k(M)$, with $k\gg 2$ depending on* $\dim(M)$ *only, be an open set of regular Riemannian metrics on M such that (T) is satisfied for each one of them. Let the set $\mathcal{D}\subset\overline{B(\partial M)}$ be open and complete for each $g\in\mathcal{G}$. Then there exists an open and dense subset $\mathcal{G}_s$ of $\mathcal{G}$ such that $I_{g,\mathcal{D}}$ is s-injective for any $g\in\mathcal{G}_s$.*

Moreover, there is a stability estimate similar to (3.1). The density in the theorem above is provided by the following result (compare with Theorem 3.1).

Theorem 5.2 ([SU6])**.** *Let g be an analytic, regular metric on M. Let $\mathcal{D}$ be complete. Then $I_{\mathcal{D}}$ is s-injective.*

The proof of Theorem 5.2 that we give in [SU6] is quite different from that of Theorem 3.1. The critical step is to show that $I_{\mathcal{D}}$ recovers the analytic wave front set of f^s inside T^*M. If one wants to recover the usual C^∞ wave front set of f^s inside T^*M, then this can be done by localizing near simple geodesics by standard cut-offs in the x and ξ variables. In the analytic case, however, such cut-offs would destroy the analyticity of the symbols. In the theory of the analytic ΨDOs, one works with special cut-offs $\chi_N(x)$ and $g^R(\xi)$ depending on large parameters with "good" control of the derivatives. We refer to [Tre] for details. Another approach based on complex deformation of the contour of the integration can be found in [Sj]. In our case, however, $I_{\mathcal{D}}$ is an FIO, and we need a cut-off before composing it with $I_{\mathcal{D}}^*$. This cannot be done, at least directly, with the pseudodifferential cut-offs χ_N and g^R. Instead, we apply the complex stationary phase method of [Sj]. As a result, we get that $I_{\mathcal{D}}f=0$ implies that the FBI transform of f^s inside T^*M, with analytic phase, and an analytic elliptic symbol, decays exponentially fast. This is one of the characterizations of absence of analytic wave front set. See [SU6] for details. We still have the same problems near the boundary, as before.

A new moment in the proof is the following. Using the microlocal analytic arguments above, we show that $f^s=dv_p$ locally, in a neighborhood U_p of any point on $p\in M$ with v_p that can depend on p. If $U_p\cap\partial M\neq\emptyset$, then we also have $v_p=0$ on ∂M. To complete the proof, we need to show that v_p can be chosen independently of p on the whole M. This is done by starting from a neighborhood of ∂M where one can uniquely define $v=v_0$, and showing that v_0 admits analytic continuation in the whole M. To show that this continuation is independent of the path, we need (T).

Having proved Theorem 5.2, we prove Theorem 5.1 by choosing an open subset of $\mathcal{D}$, still complete, so that the corresponding set of geodesics is a manifold. Using suitable smooth cot-off α on that manifold, we study $I_\alpha := \alpha I$ (another notation abuse), instead of $I_{\mathcal{D}}$, where the cut-off is a characteristic function. Then we follow the analysis of simple manifolds.

5.2. **The non-linear Lens Rigidity problem.** Here we sketch the results in [SU7]. We start with a boundary determination result that generalizes Theorem 4.2.

Theorem 5.3. *Let (M,g) be a compact Riemannian manifold with boundary and assume that we know $g|_{\partial M}$. Let $(x_0,\xi_0) \in S(\partial M)$ be such that the maximal geodesic γ_0 through it is of finite length, and assume that x_0 is not conjugate to any point in $\gamma_0 \cap \partial M$. If σ and ℓ are known on some neighborhood of (x_0,ξ_0), then the jet of g at x_0 in boundary normal coordinates is determined uniquely.*

Note that regularity of g is not needed here, nor (T) is needed. Also the boundary does not need to be convex, as in Theorem 4.2. The proof is based on analysis of the eikonal equation.

Theorem 5.4 below says, loosely speaking, that for the classes of manifolds and metrics we study, the uniqueness question for the non-linear lens rigidity problem can be answered locally by linearization. This is a non-trivial implicit function type of theorem however because our success heavily depends on the a priori stability estimate that the s-injectivity of $I_{\mathcal{D}}$ implies, see Theorem 5.1 and the remark after it. We work with two metrics g and $\hat g$; and will denote objects related to $\hat g$ by $\hat\sigma$, $\hat\ell$, etc. Note that (T) is not assumed in the first theorem.

Theorem 5.4. *Let $g_0 \in C^k(M)$ be a regular Riemannian metric on M with $k \gg 2$ depending on $\dim(M)$ only. Let $\mathcal{D}$ be open and complete for g_0, and assume that there exists $\mathcal{D}' \Subset \mathcal{D}$ so that $I_{g_0,\mathcal{D}'}$ is s-injective. Then there exists $\varepsilon > 0$, such that for any two metrics g, $\hat g$ satisfying*

$$\|g - g_0\|_{C^k(M)} + \|\hat g - g_0\|_{C^k(M)} \le \varepsilon, \tag{5.3}$$

the relations

$$\sigma = \hat\sigma, \quad \ell = \hat\ell \quad \text{on } \mathcal{D}$$

imply that there is a C^{k+1} diffeomorphism $\psi : M \to M$ fixing the boundary such that

$$\hat g = \psi^* g.$$

Next theorem is a version of [SU6, Theorem 3]. It states that the requirement that $I_{g_0,\mathcal{D}'}$ is s-injective is a generic one for g_0.

Theorem 5.5. *Let $\mathcal{G} \subset C^k(M)$, $k \gg 2$ depending on $\dim(M)$ only, be an open set of regular Riemannian metrics on M such that (T) is satisfied for each one of them. Let the set $\mathcal{D}' \subset \overline{B(\partial M)}$ be open and complete for each $g \in \mathcal{G}$. Then there exists an open and dense subset $\mathcal{G}_s$ of $\mathcal{G}$ such that $I_{g,\mathcal{D}'}$ is s-injective for any $g \in \mathcal{G}_s$.*

Theorems 5.4 and 5.5 combined imply that there is local uniqueness, up to isometry, near a generic set of regular metrics.

Corollary 5.1. *Let $\mathcal{D}' \Subset \mathcal{D}$, $\mathcal{G}$, $\mathcal{G}_s$ be as in Theorem 5.5. Then the conclusion of Theorem 5.4 holds for any $g_0 \in \mathcal{G}_s$.*

Remark 5. Condition (T) in Theorem 5.5, and Corollary 5.1 in some cases can be replaced by the assumption that (M,g) can be extended to $(\tilde M,\tilde g)$ that satisfies (T). One such case is if $(\tilde M,\tilde g)$ is a simple manifold, and we study σ, ℓ on its maximal domain, i.e., $\mathcal{D} = \overline{B(\partial M)}$. In particular, we get local generic lens rigidity for subdomains of simple manifolds when $\mathcal{D}$ is maximal.

6. Further results

The methods developed so far apply to other problems. In [FSU], B. Frigyik, G. Uhlmann and the author study the integral geometry problem of integrating functions over general family of curves, with a variable weight. We show that one has injectivity and stability for generic curves and weights.

In a joint work [DPSU] with N. Dairbekov, G. Paternain, and G. Uhlmann, we study boundary rigidity for magnetic systems. The dynamics there is described by the magnetic Hamiltonian $(D+\alpha)^2_g$, where g is a Riemannian metric, and α is an one-form. The corresponding Hamiltonian curves γ (in the base) are called magnetic geodesics. The linearized problem then is to integrate functions of the type

$$\phi(x,\xi) = h_{ij}(x)\xi^i\xi^j + \beta_i(x)\xi^i$$

over the magnetic flow in the phase space, i.e., when $(x,\xi) = (\gamma(t),\dot\gamma(t))$. The reason we have functions that are quadratic polynomials of ξ is that the Hamiltonian is of the same type. The non-linear problem is to recover g, α up to a gauge transformation, given either the scattering relation, or the magnetic action on the boundary that replaced the distance. Gauge transformations are given by $g \mapsto \psi^* g$, $\alpha \mapsto \psi^*\alpha + \mathrm{d}\phi$, where ψ is a diffeomorphism fixing ∂M as above, and ϕ is a function vanishing on ∂M. We prove generic uniqueness results of the type above.

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Department of Mathematics, Purdue University, West Lafayette, IN 47907