STABILITY OF COUPLED-PHYSICS INVERSE PROBLEMS WITH ONE INTERNAL MEASUREMENT

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Abstract. In this paper, we develop a general approach to prove stability for the non linear second step of hybrid inverse problems with one internal measurement. We work with general functionals of the form $F = \sigma|\nabla u|^p$, $0 < p \le 1$, where u is the solution of the elliptic partial differential equation $\nabla \cdot \sigma\nabla u = 0$ on a bounded domain Ω with boundary conditions $u|_{\partial\Omega} = f$. In the case $p = 1$ this problem has application to Current Density Impedance Imaging, where $F = \sigma|\nabla u|$ represents the magnitude of the current density field. We prove stability of the linearization and Hölder conditional stability for the non-linear problem of recovering σ from one internal measurement.

Keywords. current density impedance imaging, conductivity imaging, hybrid inverse problems.

1. Introduction

Couple-physics Inverse Problems or Hybrid Inverse Problems is a research area that is interested in developing the mathematical framework for medical imaging modalities that combine the best imaging properties of different types of waves (e.g., optical waves, electrical waves, pressure waves, magnetic waves, shear waves, etc) [4, 7, 8, 43]. In some applications of non-invasive medical imaging modalities (e.g., cancer detection) there is need for high contrast and high resolution images. High contrast discriminates between healthy and non-healthy tissue whereas high resolution is important to detect anomalies at and early stage [10]. In some situation current methodologies (e.g., electrical impedance tomography, optical tomography, ultrasound, magnetic resonance) focus only in a particular type of wave that can either recover high resolution or high contrast, but not both with the required accuracy. For instance, electrical impedance tomography (EIT) and optical tomography (OT) are high contrast modalities because they can detect small local variations in the electrical and optical properties of a tissue. However because of their high instability they are characterized by their low resolution images [13, 15]. On the other hand, ultrasound tomography and magnetic resonance imaging are modalities that provide high resolution but not necessarily high enough contrast since the difference between the index of refraction of the healthy and non-healthy tissue is very small [10].

The aim of hybrid inverse problems is to couple the physics of each wave to benefit from the imaging advantages of each one. Some examples of this physical coupling are: (i) ultrasound modulated electrical impedance tomography (UMEIT) also known as acoustic-electro tomography (AET) or electro acoustic tomography (EAT) [3,4,16,22,23]; (ii) current density impedance imaging (CDII) [18, 33–36]; and (iii) ultrasound modulated optical tomography (UMOT) also known as acoustic optical tomography (AOT) [2, 9, 11, 12, 37].

All of these hybrid inverse problems involve two steps. In the first step the high resolution modality takes an input boundary measurements f and provides an output internal functional of the form $F = \sigma|\nabla u|^p$ for $p > 0$, where u is the solution of the elliptic partial differential equation $\nabla \cdot \sigma\nabla u = 0$ on a bounded domain Ω with boundary conditions $u|_{\partial\Omega} = f$. Physically, σ is the

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unknown conductivity (or diffusion coefficient) and u is the electric potential (or photon-density) of the tissue, depending on whether we are looking for electrical (or optical) properties of the tissue. In the second step the high contrast modality recovers the conductivity (or diffusion coefficient) σ from the knowledge of the internal functional $\sigma|\nabla u|^p$ for $p > 0$. Different values of p represent different physical couplings, in the case of CDII, p equals 1, and in the case of UMEIT and UMOT, p equals 2. Other internal functionals have been studied as well [11].

In this paper we develop a general approach to prove stability for the non linear second step of these hybrid inverse problems. We work with general functionals of the form $\sigma|\nabla u|^p$, $0 < p \leq 1$. A unified manner of dealing with the linearization of this problem was proposed in [24], for the cases $0 < p < 1$ and $1 \leq p \leq 2$. For $0 < p < 1$ they consider the problem with one internal measurement and for $1 \leq p \leq 2$ they use two internal measurements. In both cases they prove that the linearization is elliptic in the interior of the domain. This implies stability of the linearized problem, up to a finite dimensional kernel, without necessarily having injectivity. The conductivity σ in [24] is perturbed by functions $\delta\sigma$ identically zero in a fixed neighborhood of the boundary.

As mentioned before, in the case $p = 1$ this problem has application to CDII. We denote by $F = |\mathbf{J}|$ the magnitude of the current density field $\mathbf{J} = -\sigma\nabla u$. In CDII we assumed that F is obtained by boundary measurements. An important observation is that the magnitude of the current density field does not necessarily depend on MRI [28, 36], as compared to Magnetic Resonance Electrical Impedance Tomography (MREIT), where one component of the magnetic field is obtained from boundary information by means of an MRI [25, 26, 29, 44]. This may lead to simpler methodologies to obtain F as suggested by [36].

The use of Current Density Imaging to image electrical conductivity goes back to [45]. Since then, there is an extensive literature dealing with this problem from different points of view. In [20] the authors reduced the MREIT conductivity imaging problem to the Neumann problem for the $1-$Laplacian. Similarly, but in a more geometrical approach, in [35] the authors reduced the CDII conductivity imaging problem to a variational problem associated to the Dirichlet problem for the $1-$Laplacian in the Riemannian metric $g = F^{2/(n-1)}I$. In both cases, in the process of transforming the initial inverse problem to the 1-Laplacian, infinitely many solutions are introduced and the problem suffers in general from non-uniqueness [21, 36].

To deal with these difficulties in [19], the authors proposed to assume knowledge of the magnitude of two current density fields F_1 and F_2, obtained from two boundary measurements. In [20] uniqueness for the Neumann problem for the $1-$Laplacian was proved in this case. To handle the non-uniqueness, for the conductivity problem associated with the Dirichlet problem for the $1-$Laplacian in the metric g when only one internal measurement is known, the authors introduced a variational approach. The key observation is that equipotencial surfaces minimize the surface area induced by the metric g. In [34] the authors proved that if if the data (f, F) is *admisible* then the solution of the inverse problem is unique. Conditional stability to recover the voltage potential u was established in [41] for the case of planar conductivities.

We take a different approach. We do not reduce the problem to the $p-$Laplacian, and hence we do not introduce additional solution to the inverse problem. For $0 < p \leq 1$, we show local Hölder stability, and hence injectivity, for the non-linear problem and its linearization. We use only one boundary measurement even in the case $p = 1$ (CDII). We allow perturbations in the whole domain, with appropriate boundary conditions. Our approach is based on a factorization of the linearization, see (1) below. Instead of analyzing the linearization using the pseudo-differential calculus, we analyze the only non-trivial factor in the factorization, which happens to be a second order differential operator. Even though the uniqueness of the non-linear problem has been established in [34], for the stability result, we need to prove injectivity (and stability) of the linearization which does not follow from that for the non-linear problem.

In the appendix, we generalize the abstract stability approach in [40] to transfer conditional stability of the linearization to conditional stability of the non-linear problem. The method of linearization to study stability of general inverse problems was previously discussed in [27, 38, 39], and since then it has been successfully used in the literature to investigate coefficient inverse problems. In our case, the behavior of the linearized problem depends on whether $0 < p < 1$, $p = 1$, or $p > 1$ as has been noted before, see, e.g., [10, 24]. The case $0 < p < 1$ is the simplest one since the linearized operator becomes elliptic and thus stable. When $p = 1$, the linearized operator can be considered as one parameter family of elliptic operators on a family of hypersurfaces allowing us to show stability by superposition of elliptic operators. For completeness in the exposition we analyze the case $0 < p < 1$ as well even though it does not appear in applications to medical imaging.

Our approach can also serve as a basis for numerical reconstruction. The inversion of the linearization is explicit, with the most intensive step being solving an elliptic boundary value problem ($p < 1$), or a family of elliptic ones on a family of surfaces ($p = 1$). In the latter case, one can regularize and reduce to a single elliptic PDE. The inversion of the linearization can be used in an iterative algorithm to solve the original non-linear equation.

There is an extensive bibliography in the case of m-multiple measurements ($m = n + 1$ for n odd and $m = n$ for n even) and under the assumption that the m gradients of the solutions have maximal rank in $\mathbb{R}^n$ at every point $x \in X$ the problem is well understood. A numerical approach was proposed in [14]. In [6] the authors showed that one can obtained $\sigma\nabla u_i$ with multiple measurements of the form $\sigma\nabla u_i \cdot \nabla u_j$. This previous simplification allowed them to proved stability for the case $\alpha = 1$. Under this same assumption over the gradients of the solutions and with general measurements of the form $\sigma^{2\alpha}\nabla u_i \cdot \nabla u_j$ the authors in [30] proved Lipschitz stability for the problem. Later in [31, 32] they extended this results for the anisotropic case.

1.1. **Main results.** Let Ω be a bounded simply connected open set of $\mathbb{R}^n$ with smooth boundary. Consider the strictly elliptic boundary value problem

$$\nabla \cdot \sigma\nabla u = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = f, \tag{1}$$

where σ is a function in $C^2(\overline{\Omega})$ such that $\sigma > 0$ in $\overline{\Omega}$ and $f \in C^{2,\alpha}(\partial\Omega)$, $0 < \alpha < 1$. By the Schauder estimates, $u \in C^2(\overline{\Omega})$. We say u is $\sigma-$harmonic if it satisfies equation (1). We address the question of whether we can determine σ, in a stable way, from the functional $F : C^2(\overline{\Omega}) \to C(\bar{\Omega})$ defined by

$$F(\sigma) = \sigma|\nabla u|^p,$$

with $p > 0$ is fixed. This problem has different behavior depending on whether $0 < p < 1$, $p = 1$ or $p > 1$.

We study stability of the non-linear problem by proving first stability for the linearization, see section 2, and then using Theorem A.1. The latter is a generalization of the main result in [40], that allows to obtain stability for the non-linear problem from stability of the linearized problem. Our main theorem about stability for the linearized problem is the following.

Theorem 1.1 (Stability of the linearization)**.** *Let u_0 be σ_0-harmonic with $\nabla u_0 \neq 0$ in $\overline{\Omega}$ and let $d_{\sigma_0}F$ be the differential of F at σ_0.*

- *Case $0 < p < 1$: there exist $C > 0$ such that*

$$\|h\| \leq C\|d_{\sigma_0}F(h)\|_{H^1(\Omega)} \quad \text{for every } h \in H_0^1(\Omega);$$

- *Case $p=1$: for any $\alpha_1 \in [0,1)$, there exist $C>0$ such that if $(1-\alpha_1)s_1 \geq 2$*

$$\|h\| \leq C\|d_{\sigma_0}F(h)\|^{\alpha_1}_{H^1(\Omega)}\|h\|^{1-\alpha_1}_{H^{s_1}(\Omega)} \quad \text{for every } h \in H^{s_1}(\Omega)\cap H^1_0(\Omega); \tag{2}$$

where $\nu(x)$ denotes the outer-normal vector to the boundary.

This together with Theorem A.1 gives our main result about stability for the non-linear problem.

Theorem 1.2 (Stability for the non-linear map F, case $0<p\leq 1$)**.** *Let $0<p\leq 1$. Let u_0 be σ_0-harmonic with $\nabla u_0 \neq 0$ in $\overline{\Omega}$. For any $0<\theta<1$, there exist $s>0$ so that if $\|\sigma\|_{H^s(\Omega)}<L$ for some $L>0$, there exist $\epsilon>0$ such that*

$$\|\sigma-\sigma_0\|_{C^2(\bar{\Omega})}<\epsilon$$

implies

$$\|\sigma-\sigma_0\|_{L^2(\Omega)}<C\|F(\sigma)-F(\sigma_0)\|^{\theta}_{L^2(\Omega)}. \tag{3}$$

Remark 1. In the case of $\mathbb{R}^2$ we can satisfy $\nabla u_0 \neq 0$ in $\overline{\Omega}$ by imposing conditions on f. For instance in [1] and [33] the authors showed if Ω is simply connected in $\mathbb{R}^2$, $\sigma_0 \in C^{\alpha}(\Omega)$ $0<\alpha<1$ and $u_0|_{\partial\Omega}$ is continuous and two-to-one map, except possibly at its maximum and minimum. Then $|\nabla u|>0$ in $\overline{\Omega}$.

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2. Linearization

We start by considering the linearized version of this problem. Denote by $\mathrm{d}F_{\sigma_0}$ the Gâteux derivative of F at some fixed σ_0. For σ in a C^2-neighborhood of σ_0 we get

$$F(\sigma)=F(\sigma_0)+\mathrm{d}F_{\sigma_0}(\sigma-\sigma_0)+\int_0^1(1-t)\mathrm{d}^2F_{\sigma_0+t(\sigma-\sigma_0)}(\sigma-\sigma_0,\sigma-\sigma_0)\mathrm{d}t \tag{4}$$

where $\mathrm{d}F_{\sigma_0}$ is given by

$$\mathrm{d}F_{\sigma_0}(h)=h|\nabla u_0|^p+p|\nabla u_0|^{p-2}\sigma_0\nabla u_0\cdot\nabla v_0(h) \tag{5}$$

and $\mathrm{d}^2F_{\sigma_t}$ by

$$\begin{aligned}\mathrm{d}^2F_{\sigma_t}(h,h)=p|\nabla u_t|^{p-2}\left(h\nabla u_t\cdot\nabla v_t(h)+\nabla v_t(h)\cdot\nabla v_t(h)+\nabla u_t\cdot\nabla w_t(h)\right)&\\ +p(p-2)|\nabla u_t|^{p-4}(\nabla u_t\cdot\nabla v_t(h))^2,&\end{aligned} \tag{6}$$

for $h=\sigma-\sigma_0\in C^2(\overline{\Omega})$ and $\sigma_t=\sigma_0+t(\sigma-\sigma_0)$ for $0\leq t\leq 1$ and u_t, v_t and w_t solving

$$\begin{aligned}\nabla\cdot\sigma_t\nabla u_t&=0 &&\text{in } \Omega, \quad u_t|_{\partial\Omega}=f;\\ \nabla\cdot\sigma_t\nabla v_t&=-\nabla\cdot h\nabla u_t &&\text{in } \Omega, \quad v_t|_{\partial\Omega}=0;\\ \nabla\cdot\sigma_t\nabla w_t&=-2\nabla\cdot h\nabla v_t &&\text{in } \Omega, \quad w_t|_{\partial\Omega}=0;\end{aligned} \tag{7}$$

for $0\leq t\leq 1$.

Let

$$R_{\sigma_0}(h)=\int_0^1(1-t)\mathrm{d}^2F_{\sigma_0+th}(h,h)\mathrm{d}t \quad \forall h\in C^2(\overline{\Omega}),$$

we claim that

$$\|R_{\sigma_0}(h)\|\leq C_{\sigma_0}\|h\|^2_{C^2(\Omega)} \tag{8}$$

where

$$C_{\sigma_0} = C \sup_{0\le t\le 1} \left((2p+1)\|\nabla u_t\|^p_{C^2(\overline{\Omega})} + p(p-2)\|\nabla u_t\|^{2p-2}_{C^2(\overline{\Omega})} \right)$$

with C depending only on Ω and the dimension n. Assuming the claim then dF_{σ_0} is a linearization of F at σ_0 with a quadratic remainder as in (24).

To show (8) we estimate (6) using inequalities (9) and (10). These last two inequalities are consequence of (7) and elliptic regularity [17]. Let $C > 0$ be a constant depending on Ω and the dimension n, using the convention that C can increase from step to step we have

$$\begin{aligned}\|\nabla v_t\|_{C^{1,\alpha}(\overline{\Omega})} &\le \|v_t\|_{C^{2,\alpha}(\overline{\Omega})} \quad \text{for} \quad \alpha\in(0,1)\\ &\le C\|\nabla\cdot h\nabla u_t\|_{C^{0,\alpha}(\overline{\Omega})} \le C\|h\nabla u_t\|_{C^{1,\alpha}(\overline{\Omega})} \quad \text{for} \quad \alpha\in(0,1)\\ &\le C\|h\|_{C^2(\overline{\Omega})}\cdot\|\nabla u_t\|_{C^2(\overline{\Omega})},\end{aligned} \tag{9}$$

and

$$\begin{aligned}\|\nabla w_t\| \le C\|\nabla w_t\|_{C^{1,\alpha}(\overline{\Omega})} &\le \|w_t\|_{C^{2,\alpha}(\overline{\Omega})} \quad \text{for} \quad \alpha\in(0,1)\\ &\le C\|\nabla\cdot h\nabla v_t\|_{C^{0,\alpha}(\overline{\Omega})} \le C\|h\nabla v_t\|_{C^{1,\alpha}(\overline{\Omega})} \quad \text{for} \quad \alpha\in(0,1)\\ &\le C\|h\|^2_{C^2(\overline{\Omega})}\cdot\|\nabla u_t\|_{C^2(\overline{\Omega})},\end{aligned} \tag{10}$$

where the last inequality follows by (9).

Decomposition of the Linearization. We decompose the linearization (4) and describe the geometry of dF_{σ_0} in more detail in the following two propositions. This analysis holds for any $p > 0$.

Proposition 2.1. *Let u_0 be σ_0-harmonic with $\nabla u_0 \neq 0$ in $\overline{\Omega}$, then*

$$\sigma_0 T_0 \frac{\mathrm{d}F_{\sigma_0}(\rho)}{\sigma_0|\nabla u_0|^p} = -L\Delta^{-1}_{\sigma_0,D}\sigma_0 T_0\rho \quad \textit{for} \quad \rho = (\sigma-\sigma_0)/\sigma_0 \in C^2(\overline{\Omega}), \tag{11}$$

where $T_0 = \nabla u_0\cdot\nabla$ is a transport operator along the gradient field of u_0, $\Delta_{\sigma,D}$ is the Dirichlet realization of $\Delta_\sigma := \nabla\cdot\sigma\nabla$ in Ω and L is a differential operator given by

$$Lv := -\nabla\cdot\sigma_0\nabla v + p\nabla\cdot\left(\sigma_0\frac{\nabla u_0\cdot\nabla v}{|\nabla u_0|^2}\nabla u_0\right).$$

Proof. Since $\nabla u_0 \neq 0$ in $\overline{\Omega}$ we can write (5) as

$$\mathrm{d}F_{\sigma_0}(\rho) = \sigma_0|\nabla u_0|^p\left(\rho + p\frac{\nabla u_0\cdot\nabla v_0(\rho)}{|\nabla u_0|^2}\right). \tag{12}$$

Solving (12) for the free ρ term and plugging that into the second equation in (7) we get

$$Lv_0 = \nabla\cdot\left(\frac{\mathrm{d}F_{\sigma_0}(\rho)}{|\nabla u_0|^p}\nabla u_0\right) \quad \text{in } \Omega, \qquad v_0|_{\partial\Omega} = 0.$$

The solution v_0 of the second equation in (7) satisfies

$$\nabla\cdot\sigma_0\nabla v_0 = -\nabla\cdot(\sigma-\sigma_0)\nabla u_0 = -\sigma_0\nabla u_0\cdot\nabla\rho$$

and is a linear operator in ρ that can be written as $v_0 = -\Delta^{-1}_{\sigma_0,D}\sigma_0 T_0\rho$. So we get

$$-L\Delta^{-1}_{\sigma_0,D}\sigma_0 T_0\rho = \nabla\cdot\left(\frac{\mathrm{d}F_{\sigma_0}(\rho)}{|\nabla u_0|^p}\nabla u_0\right) = \sigma_0\nabla u_0\cdot\nabla\left(\frac{\mathrm{d}F_{\sigma_0}(\rho)}{\sigma_0|\nabla u_0|^p}\right).$$

□

Notice that in the r.h.s. of (11), the only non-trivial operator in terms of injectivity is the second order differential operator L. We focus our attention on understanding this operator. Denote by $\Pi_0\omega = (\nabla u_0 \cdot \omega/|\nabla u_0|^2)\nabla u_0$ the orthogonal projection of the covector ω onto ∇u_0 in the Euclidean metric. Then $\Pi_\perp := \mathrm{Id} - \Pi_0$ is the orthogonal projection on the orthogonal complement of ∇u_0. Take a test function $\phi \in C_0^\infty(\Omega)$, and compute

$$\begin{aligned}(Lv, \phi) &= (\sigma_0 \nabla v, \nabla \phi) - p(\sigma_0 \Pi_0 \nabla v, \nabla \phi),\\ &= (\sigma_0 \Pi_\perp \nabla v, \Pi_\perp \nabla \phi) + (1-p)\left(\sigma_0 \Pi_0 \nabla v, \Pi_0 \nabla \phi\right).\end{aligned} \tag{13}$$

We therefore get

$$L = (\Pi_\perp \nabla)' \cdot \sigma_0 (\Pi_\perp \nabla) + (1-p)(\Pi_0 \nabla)' \cdot \sigma_0 (\Pi_0 \nabla),$$

where the prime stands for transpose in distribution sense.

Example 1. $\sigma_0 = 1$, $f = x^n$. Then $u_0 = x^n$ and $-L = \Delta_{x'} + (1-p)\partial_{x^n}^2$, where $x = (x', x^n)$. Notice that for $0 \le p < 1, L$ is an elliptic operator; for $p = 1, L$ becomes the restriction of the Laplacian on the planes $x^n = \text{const.}$; and for $p > 1, L$ is a hyperbolic operator.

Motivated by this example we find a local representation for L. We use the convention that Greek superscripts and subscripts run from 1 to $n-1$.

Proposition 2.2. *Let $u_0 \in C^2(\overline{\Omega})$ be σ_0-harmonic, with $\nabla u_0(x_0) \neq 0$ for $x_0 \in \Omega$. There exist local coordinates (y', y^n) near x_0 such that*

$$dx^2 = c^2 (\mathrm{d}y^n)^2 + g_{\alpha\beta}\mathrm{d}y^\alpha \mathrm{d}y^\beta, \quad g_{\alpha\beta} := \sum_i \frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^i}{\partial y^\beta}, \tag{14}$$

where $c = |\nabla u_0|^{-1}$. In this coordinates

$$L = -Q + (1-p)\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial y^n} c^{-2}\sigma_0 \sqrt{\det g}\frac{\partial}{\partial y^n}, \tag{15}$$

where Q is a second order elliptic positively defined differential operator in the variables y' smoothly dependent on y^n; in fact, Q is the restriction of Δ_{σ_0} on the level surfaces $u_0 = const.$

Proof. Notice first that u_0 trivially solves the eikonal equation $c^2|\nabla\phi|^2 = 1$ for the speed $c = |\nabla u_0|^{-1}$. Near some point x_0, we can assume that $u(x_0) = a$; then $u_0(x)$ is the signed distance from x to the level surface $u_0 = a$. Choose local coordinates y' on this level curve, and set $y^n = u_0(x)$. Then $y = (y', y^n)$ are boundary local coordinates to $u_0 = a$ and in those coordinates, the metric $c^{-2}\mathrm{d}x^2$ takes the form

$$g_{ij}\mathrm{d}x^i \mathrm{d}x^j = (\mathrm{d}y^n)^2 + c^{-2} g_{\alpha\beta}\mathrm{d}y^\alpha \mathrm{d}y^\beta, \quad g_{\alpha\beta} := \sum_{i=1}^n \frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^i}{\partial y^\beta}.$$

Then

$$\mathrm{d}x^2 = c^2(\mathrm{d}y^n)^2 + g_{\alpha\beta}\mathrm{d}y^\alpha \mathrm{d}y^\beta.$$

Let $\phi \in C_0^\infty(\Omega)$, using (13), we get that near x_0

$$\Pi_0 \nabla_x = c^{-1}\left(0, \ldots, \partial/\partial y^n\right).$$

Locally near x_0 we get,

$$\begin{aligned}(Lv, \phi) &= \int \sigma_0 \left(\sum_{i=1}^n \frac{\partial v}{\partial x^i}\frac{\partial \bar\phi}{\partial x^i} - p\frac{\partial v}{\partial y^n}\frac{\partial \bar\phi}{\partial y^n}\right)\mathrm{d}x\\ &= \int \sigma_0 \left(g^{\alpha\beta}\frac{\partial v}{\partial y^\alpha}\frac{\partial \bar\phi}{\partial y^\beta} + (1-p)c^{-2}\frac{\partial v}{\partial y^n}\frac{\partial \bar\phi}{\partial y^n}\right)|\det(\mathrm{d}x/\mathrm{d}y)|\,\mathrm{d}y.\end{aligned} \tag{16}$$

Hence

$$L = -\frac{1}{\sqrt{\det g}}\left(\frac{\partial}{\partial y^\beta}\sigma_0 g^{\alpha\beta}\sqrt{\det g}\frac{\partial}{\partial y^\alpha} + (1-p)\frac{\partial}{\partial y^n}c^{-2}\sigma_0\sqrt{\det g}\frac{\partial}{\partial y^n}\right),$$

which proves (15). ☐

Remark 2. In the two dimensional case we can get an explicit local coordinate system by taking $y^2 = u_0(x)$ and $y^1 = \tilde{u}_0$, with $\tilde{u}_0 \in H^1(\Omega)$ be any the σ_0-harmonic conjugate of u_0, that is $\nabla\tilde{u}_0 = (\sigma\nabla u_0)^\perp$, where $(a,b)^\perp = (b,-a)$. The level curves of v_0 (stream lines) are perpendicular to the level curves of u_0 (equipotential lines), see [5] for details.

Remark 3. Notice that if $p < 1$, L is elliptic (and positive); if $p > 1$, L is hyperbolic; and when $p = 1$, the operator $L = Q(y^n)$ can be considered as a one parameter family of elliptic operators on the level surfaces of u_0.

3. Stability estimates

We first provide a conditional stability estimate for the linearized problem of recovering σ from $\sigma|\nabla u|^p$ in (1) for $p > 0$. We address this question by using decomposition (11).

The proof of Theorem 1.1 is divided in some lemmas about the stability of the different operators in the decomposition (11), we start with the differential operator L

Lemma 3.1. *Let u_0 be σ_0-harmonic, with $\nabla u_0 \neq 0$ in $\overline{\Omega}$, then*

- *Case $0 < p < 1$: There exist $C > 0$ depending on σ, n, Ω and u_0 such that*

$$\|v\|_{H^2(\Omega)} \leq C\|Lv\|, \qquad \text{for } v \in H^1_0(\Omega)\cap H^2(\Omega). \tag{17}$$

- *Case $p = 1$: there exist $C > 0$ such that*

$$\|v\|^2_{L^2(\Omega)} \leq C(Lv, v), \qquad \text{for} \quad v \in C^\infty(\bar{\Omega}) \text{ with } v|_{\partial\Omega} = 0.$$

Proof. The proof for the elliptic case $0 < p < 1$ is an immediate consequence of elliptic theory (see for instance Theorem 8.12 in [17]) and injectivity of L with Dirichlet boundary conditions. The latter follows from integration by parts, see (13). We get that $Lv = 0$ with $v = 0$ on $\partial\Omega$ implies

$$\Pi_\perp\nabla v = \Pi_0\nabla v = 0 \quad\Longrightarrow\quad \nabla v = 0.$$

Then $v = 0$.

We now consider the case $p = 1$. There exists an open bounded Ω_1 containing $\overline{\Omega}$ and a C^2 extension of u_0 to $\overline{\Omega}_1$ denoted by u_1 such that $\nabla u_1 \neq 0$ on $\overline{\Omega}_1$. We extend v as zero in $\overline{\Omega}_1 \setminus \Omega$. Let $x_0 \in \overline{\Omega}$, and denote by Γ_0 the level surface of u_1 in $\overline{\Omega}_1$ containing x_0. Clearly Γ_0 is bounded and closed in $\overline{\Omega}_1$, hence a compact subset of $\mathbb{R}^n$. Its restriction to the interior is an open surface (locally given by $u_0 = \text{const.}$ with $\nabla u_0 \neq 0$). Note that any such level surface may have points on $\partial\Omega$, where it is not transversal to $\partial\Omega$.

Let $y = (y', y^n)$ be local boundary normal coordinates for $x_0 \in \Gamma_0$ as in (14). By compactness we can define these coordinates to an open neighborhood of $\Gamma_0 \cap \overline{\Omega}$ contained in Ω_1. In these coordinates we can write this open neighborhood as $\tilde{\Gamma}_0 \times (a_0 - \epsilon_0, a_0 + \epsilon_0)$, for $\tilde{\Gamma}_0 = \Gamma_0 \cap \tilde{\Omega}$, where $\Omega \Subset \tilde{\Omega} \Subset \Omega_1$; $a_0 = u_0(x_0)$; and $\epsilon_0 < \min\{\text{dist}(\partial\Omega, \partial\tilde{\Omega}), \text{dist}(\partial\tilde{\Omega}, \partial\Omega_1)\}$. Using representation (16), ellipticity of (1), and Poincaré inequality on $\tilde{\Gamma}_0$, we see that for each $x_0 \in \overline{\Omega}$ there exist ϵ_0 such that for all $0<\epsilon<\epsilon_0$

$$\begin{aligned}\int\limits_{a_0-\epsilon}^{a_0+\epsilon}\int\limits_{\tilde\Gamma_0} Lv\bar v|\det(\mathrm{d}x/\mathrm{d}y)|\,\mathrm{d}y'\mathrm{d}y^n &= \int\limits_{a_0-\epsilon}^{a_0+\epsilon}\int\limits_{\tilde\Gamma_0}\sigma_0 g^{\alpha\beta}\frac{\partial v}{\partial y^\alpha}\frac{\partial\bar v}{\partial y^\beta}|\det(\mathrm{d}x/\mathrm{d}y)|\,\mathrm{d}y'\mathrm{d}y^n\\ &\geq \frac{1}{C}\int\limits_{a_0-\epsilon}^{a_0+\epsilon}\int\limits_{\tilde\Gamma_0}|\nabla_{y'}v(y',y^n)|^2\mathrm{d}y'\,\mathrm{d}y^n\\ &\geq \frac{1}{C}\int\limits_{a_0-\epsilon}^{a_0+\epsilon}\int\limits_{\tilde\Gamma_0}|v(y',y^n)|^2\mathrm{d}y'\,\mathrm{d}y^n \geq \frac{1}{C}\|v\|_{L^2(\tilde\Gamma_0\times(a_0-\epsilon,a_0+\epsilon))}.\end{aligned}\tag{18}$$

By compactness of $\overline{\Omega}$ we can find finitely many neighborhoods of level curves of u_0, such that (18) holds in each of them and their union contains $\overline{\Omega}$, since (18) holds for all $0<\epsilon<\epsilon_0$ we can take them to be disjoint. Adding all this estimates we prove the lemma in the $p=1$ case as well. □

Lemma 3.2. *Let u_0 be σ_0-harmonic, with $\nabla u_0\neq 0$ in $\overline{\Omega}$, then there exist $C>0$ depending on u_0 and Ω such that*

$$\|h\|\leq C\|\nabla u_0\cdot\nabla h\| \quad \textit{for} \quad h|_{\partial\Omega}=0,\tag{19}$$

where $\nu(x)$ denotes the outer-normal vector to the boundary.

Proof. There exist an open bounded Ω_1 containing $\overline{\Omega}$ and a C^2 extension of u_0 to $\overline{\Omega}_1$ denoted by u_1 such that $\nabla u_1\neq 0$ on $\overline{\Omega}_1$. We extend h as zero in $\overline{\Omega}_1\setminus\Omega$. This extension commutes with the differential because $h=0$ on $\partial\Omega$. Let $x_0\in\overline{\Omega}$, denote by Γ_0 the level surface of u_1 in $\overline{\Omega_1}$ containing x_0. We work in $y=(y',y^n)$, local boundary normal coordinates for $x_0=(y_0',y_0^n)$ as in (14). Notice that since $\nabla u_1\neq 0$ in $\overline{\Omega_1}$, these coordinates can be extended through the integral curves of the gradient field of u_0.

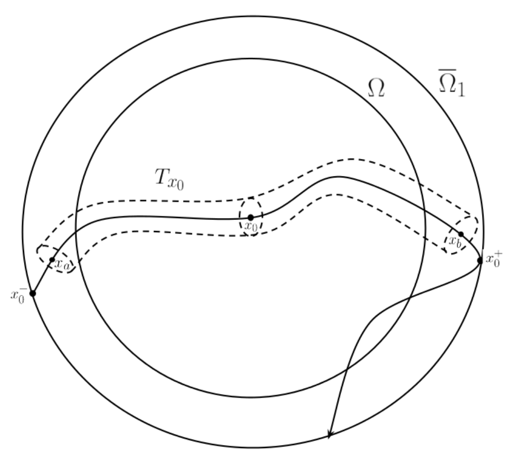

FIGURE 1. Tubular neighborhood T_{x_0} of integral curve of ∇u_0 from $x_a=x(a)$ to $x_b=x(b)$.

Let $x(t) : I \to \overline{\Omega}_1$ be a parametrization of the integral curve of ∇u_1 such that $x(0) = x_0$, $\dot{x}(t) = \nabla u_0(x(t))$, and I is the entire interval of definition of the integral curve. Denote by x_0^+ the first point on that the integral curve, starting from x_0 and traveling in the same direction of the flow, hits the boundary $\partial\Omega_1$. Similarly denote by x_0^- first point on that the integral curve, starting from x_0 and traveling in the opposite direction of the flow hits the boundary $\partial\Omega_1$. We know that $x_0^\pm$ exist because since

$$\frac{d}{dt}u(x(t)) = \nabla u_0(x(t)) \cdot \dot{x}(t) = \|\nabla u_0(x(t))\|^2 > 1/C > 0,$$

then $u(x(t))$ is strictly increasing along the integral curve $x(t)$; and u cannot grow indefinitely in $\overline{\Omega}_1$. This implies that the integral curve in $\overline{\Omega}_1$ cannot intersect themselves, and cannot be infinite.

Consider a tubular neighborhood of the integral curve $x(t)$ as $x_0^- < a \leq t \leq b < x_0^+$,

$$T_{x_0} = \{(y', y^n) \in \Omega_1 : |y' - y_0'| < \delta_0,\ a \leq t \leq b\},$$

where $\delta_0 > 0$ is small enough so that $T_{x_0} \cap \{y^n = a\}$ and $T_{x_0} \cap \{y^n = b\}$ are contained in $\Omega_1 \setminus \overline{\Omega}$ as shown in Figure 1. Since $h = 0$ in $\Omega_1 \setminus \Omega$, we can write

$$h(y', y^n) = \int_a^{y^n} (\nabla u_0 \cdot \nabla h)(y', t)\mathrm{d}t \quad \text{ for } \quad (y', y^n) \in T_{x_0}.$$

Using the Cauchy inequality we get that for $\delta_0 \geq \delta > 0$,

$$\begin{aligned} \|h(y)\|^2_{L^2(T_{x_0})} &= \int_{|y'-y_0'|<\delta} \int_a^b \left| \int_a^{y^n} (\nabla u_0 \cdot \nabla h)(y', t)\mathrm{d}t \right|^2 \mathrm{d}y^n \, \mathrm{d}y' \\ &\leq \int_{|y'-y_0'|<\delta} \int_a^b \int_a^{y^n} |(\nabla u_0 \cdot \nabla h)(y', t)|^2 \mathrm{d}t \, \mathrm{d}y^n, \mathrm{d}y' \\ &\leq (b-a)\|\nabla u_0 \cdot \nabla h\|_{L^2(T_{x_0})} \leq C\|\nabla u_0 \cdot \nabla h\|. \end{aligned} \tag{20}$$

We used here the $L^2(T_{x_0})$ norm in the y variables (without the Jacobian coming from the change of the variables) but that norm is equivalent to the original one. By the compactness of $\overline{\Omega}$, we can find $T_{x_0}, T_{x_1}, \dots, T_{x_m}$ such that their union covers $\overline{\Omega}$. Since the right-hand side in (20) does not depends on this collection of tubular neighborhoods we use a partition of unity subordinated to this covering to prove (19). □

We now present the proof for the theorem of conditional stability for the linearized problem.

Proof of Theorem 1.1. We first consider the case $p = 1$. Let $h \in C^2(\overline{\Omega})$ and denote $\rho = (\sigma - \sigma_0)/\sigma_0 = h/\sigma_0$. By Lemma 3.2, definition of $v_0 = u - u_0$, and interpolation estimate in section 4.3.1 in [42] we have

$$\|\rho\| \leq C\|\sigma_0 \nabla u_0 \cdot \nabla \rho\| \leq C\|v_0\|_{H^2(\Omega)} \leq C\|v_0\|^{\alpha_1} \cdot \|v_0\|^{1-\alpha_1}_{H^s(\Omega)}. \tag{21}$$

Using Proposition 2.1 and Lemma 3.1, we also obtain

$$\|v_0\| \leq C\|Lv_0\| \leq C \left\| \nabla u_0 \cdot \nabla \left(\frac{dF_{\sigma_0}(\rho)}{\sigma_0 \|\nabla u_0\|} \right) \right\| \leq C\|dF_{\sigma_0}(\rho)\|_{H^1(\Omega)}. \tag{22}$$

Finally, combining inequalities (21) and (22) we proof the theorem in the case $p = 1$. For the case $0 < p < 1$, we use the same reasoning and the better estimate (17) in Lemma 3.1 to conclude. □

We now present the proof of our main result as a consequence of Theorem A and Theorem 1.1

Proof of Theorem 1.2. Let $0 < \theta < 1$, $1 > \beta > \max\{\theta, 1/2\}$ and α_1 as in (2). We apply Theorem A taking

$$\mathcal{B}_1''' = H^s(\Omega), \quad \mathcal{B}_1'' = H^{s_1}(\Omega), \quad \mathcal{B}_1 = C^2(\Omega), \quad \mathcal{B}' = L^2(\Omega),$$
$$\mathcal{B}_2'' = \mathcal{B}_2' = \mathcal{B}_2 = H^1(\Omega),$$

with

$$(1-\mu_1)s_1 > \frac{n}{2} + 2, \quad (1-\mu_2)s_2 = 1, \quad (1-\mu_3)s = s_1, \quad \text{for} \quad \mu_1, \mu_2 \in (0,1). \tag{23}$$

We choose $0 < \mu = \alpha_1\mu_1\mu_2 < \min\{1/2, \beta\}$ by taking $\mu_1 = \alpha_1$ small enough, we then take μ_3 as

$$1 > \mu_3 = \frac{\beta - \mu}{\beta(1-\mu)} > \frac{1-2\mu}{1-\mu} > 0,$$

under the penalty of making s large enough.

First notice that as a consequence of (4) and (8) the differential of F and σ_0, $\mathrm{d}_{\sigma_0}F$, is a linearization with quadratic remainder as in (24). Second, conditional stability for the linearizion is consequence of Theorem 1.1, with $\alpha_1 = 1$ in the case $0 < p < 1$ and $0 < \alpha_1 < 1$ for $p = 1$. Notice that $s_1 = \frac{n+4}{2(1-\alpha_1)} > \frac{2}{1-\alpha_1}$. Third, interpolation estimates follow by (23). Finally, continuity of $\mathrm{d}F_{\sigma_0} : C^2(\overline{\Omega}) \to H^1(\Omega)$ follows by (5) and (9). Hence by Theorem A, for any $L > 0$ there exist $\epsilon > 0$ and $C > 0$, so that for any σ with

$$\|\sigma - \sigma_0\|_{C^2(\overline{\Omega})} < \epsilon, \qquad \|\sigma\|_{H^s(\overline{\Omega})} \le L,$$

one has

$$\|\sigma - \sigma_0\|_{C^2(\overline{\Omega})} \le C\|F(\sigma) - F(\sigma_0)\|^{\beta}_{H^1(\Omega)} < C\|F(\sigma) - F(\sigma_0)\|^{\theta}_{H^1(\Omega)}.$$

which proofs (3). □

Appendix A. Stability of non-linear inverse problems by linearization

The following conditional stability Theorem through linearization is a generalization of Theorem 2 in [40].

Theorem A.1. *Let $F : \mathcal{B}_1 \to \mathcal{B}_2$ be a continuous non-linear map between two Banach spaces. Assume the there exist Banach spaces $\mathcal{B}_1''' \subset \mathcal{B}_1'' \subset \mathcal{B}_1 \subset \mathcal{B}_1'$ and $\mathcal{B}_2'' \subset \mathcal{B}_2' \subset \mathcal{B}_2$ that satisfy the following:*

(1) ***α-order linearization**: for $\sigma_0 \in \mathcal{B}_1$ there exist $\mathrm{d}F_{\sigma_0} : \mathcal{B}_1 \to \mathcal{B}_2$ linear map and $\alpha > 1$ such that*

$$F(\sigma) = F(\sigma_0) + \mathrm{d}F_{\sigma_0}(\sigma - \sigma_0) + R_{\sigma_0}(\sigma - \sigma_0), \quad \text{with} \quad \|R_{\sigma_0},(\sigma - \sigma_0)\|_{\mathcal{B}_2} \le C_{\sigma_0}\|\sigma - \sigma_0\|^{\alpha}_{\mathcal{B}_1}, \tag{24}$$

for σ in some $\mathcal{B}_1$-neighborhood of σ_0. We say that $\mathrm{d}F_{\sigma_0}$ is the differential of F at σ_0 with remainder of order α.

(2) ***conditional stability of linearization**: there exist $C > 0$ such that*

$$\|h\|_{\mathcal{B}_1'} \le C\|\mathrm{d}F_{\sigma_0}h\|^{\alpha_1}_{\mathcal{B}_2'}\|h\|^{1-\alpha_1}_{\mathcal{B}_1''} \quad \text{for} \quad \alpha_1 \in (0,1].$$

(3) ***interpolation estimates**: there exist $C > 0$ such that*

$$\|g\|_{\mathcal{B}_2'} \le C\|g\|^{\mu_2}_{\mathcal{B}_2}\|g\|^{1-\mu_2}_{\mathcal{B}_2''}, \quad \|h\|_{\mathcal{B}_1} \le C\|h\|^{\mu_1}_{\mathcal{B}_1'}\|h\|^{1-\mu_1}_{\mathcal{B}_1''}, \quad \|h\|_{\mathcal{B}_1''} \le C\|h\|^{\mu_3}_{\mathcal{B}_1}\|h\|^{1-\mu_3}_{\mathcal{B}_1'''}$$

for $\mu_1, \mu_2 \in (0,1]$ and $1 \ge \mu_3 \ge \max\{0, (1-\alpha\mu)/(1-\mu)\}$ where $\mu = \alpha_1\mu_1\mu_2$.

(4) ***continuity of** $\mathrm{d}F_{\sigma_0}$: the differential $\mathrm{d}F_{\sigma_0}$ is continuous from $\mathcal{B}_1''$ to $\mathcal{B}_2''$.*

Then we have local conditional stability. For any $L > 0$ there exist $\epsilon > 0$ and $C > 0$, so that for any σ with

$$\|\sigma - \sigma_0\|_{\mathcal{B}_1} < \epsilon, \qquad \|\sigma\|_{\mathcal{B}_1'''} \leq L,$$

one has

$$\|\sigma - \sigma_0\|_{\mathcal{B}_1} \leq C\|F(\sigma) - F(\sigma_0)\|_{\mathcal{B}_2}^{\beta}. \tag{25}$$

where $\beta = \mu/(1 - \mu_3(1 - \mu))$. In particular one has Lipschitz stability (i.e., $\beta = 1$) when $\mu_3 = 1$, this happens for example when $\mathcal{B}_1''' = \mathcal{B}_1'$.

Proof. Let $L > 0$, we use the Hölder inequality $(a + b)^\eta \leq a^\eta + b^\eta$ for $a, b \geq 0$ and $0 < \eta < 1$. the following inequalities follow easily from the hypothesis

$$\begin{aligned}
\|\sigma - \sigma_0\|_{\mathcal{B}_1} &\leq C\|\sigma - \sigma_0\|_{\mathcal{B}_1'}^{\mu_1}\|\sigma - \sigma_0\|_{\mathcal{B}_1''}^{1-\mu_1} \\
&\leq C\|\mathrm{d}F_{\sigma_0}(\sigma - \sigma_0)\|_{\mathcal{B}_2'}^{\mu_1\alpha_1} \cdot \|\sigma - \sigma_0\|_{\mathcal{B}_1''}^{1-\alpha_1\mu_1} \\
&\leq C\|\mathrm{d}F_{\sigma_0}(\sigma - \sigma_0)\|_{\mathcal{B}_2}^{\mu} \cdot \|\mathrm{d}F_{\sigma_0}(\sigma - \sigma_0)\|_{\mathcal{B}_2''}^{\alpha_1\mu_1(1-\mu_2)} \cdot \|\sigma - \sigma_0\|_{\mathcal{B}_1''}^{1-\alpha_1\mu_1} \\
&\leq C\left(\|F(\sigma) - F(\sigma_0)\|_{\mathcal{B}_2} + C_{\sigma_0}\|\sigma - \sigma_0\|_{\mathcal{B}_1}^{\alpha}\right)^{\mu} \cdot \|\sigma - \sigma_0\|_{\mathcal{B}_1''}^{1-\mu} \\
&\leq C \cdot L^{(1-\mu_3)(1-\mu)}\left(\|F(\sigma) - F(\sigma_0)\|_{\mathcal{B}_2}^{\mu} + C_{\sigma_0}\|\sigma - \sigma_0\|_{\mathcal{B}_1}^{\alpha\mu}\right) \cdot \|\sigma - \sigma_0\|_{\mathcal{B}_1}^{\mu_3(1-\mu)}.
\end{aligned}$$

Hence we obtain

$$\|\sigma - \sigma_0\|_{\mathcal{B}_1}^{1-\mu_3(1-\mu)}(1 - C_{\sigma_0}\|\sigma - \sigma_0\|_{\mathcal{B}_1}^{\mu_3(\mu-1)+\alpha\mu-1}) \leq C\|F(\sigma) - F(\sigma_0)\|_{\mathcal{B}_2}^{\mu}$$

by hypothesis $\mu_3(1 - \mu) + \alpha\mu - 1 \geq 0$ then there exist $\epsilon > 0$ so that (25) holds. ☐

References

1. Giovanni Alessandrini, *An identification problem for an elliptic equation in two variables*, Annali di Matematica Pura ed Applicata **145** (1985), no. 1, 265–295.
2. Moritz Allmaras and Wolfgang Bangerth, *Reconstructions in ultrasound modulated optical tomography*, Journal of Inverse and Ill-Posed Problems **19** (2011), no. 6, 801–823, cited By (since 1996) 1.
3. H. Ammari, E. Bonnetier, Y. Capdeboscq, M. Tanter, and M. Fink, *Electrical Impedance Tomography by Elastic Deformation*, SIAM Journal on Applied Mathematics **68** (2008), no. 6, 1557–1573.
4. Habib Ammari, *An introduction to mathematics of emerging biomedical imaging*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 62, Springer, Berlin, 2008. MR 2440857 (2010j:44002)
5. Kari Astala, Tadeusz Iwaniec, and Gaven Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton Mathematical Series, vol. 48, Princeton University Press, Princeton, NJ, 2009. MR 2472875 (2010j:30040)
6. G. Bal, E. Bonnetier, F. Monard, and F. Triki, *Inverse diffusion from knowledge of power densities*, Inv. Probl. Imaging **7** (2013), no. 2.
7. Guillaume Bal, *Introduction to inverse problems*, lecture notes **9** (2004), 54.
8. ———, *Hybrid inverse problems and internal functionals*, Tech. report, 2011.
9. ———, *Cauchy problem for ultrasound modulated EIT*, Analysis and PDE (2012), to appear.
10. ———, *Hybrid inverse problems and redundant systems of partial differential equations*, arXiv preprint arXiv:1210.0265 (2012).
11. Guillaume Bal and Shari Moskow, *Local inversions in ultrasound modulated optical tomography*, arXiv preprint arXiv:1303.5178 (2013).
12. Guillaume Bal and John Schotland, *Inverse scattering and acousto-optic imaging*, Physical Review Letters **104** (2010), no. 4.
13. Liliana Borcea, *Electrical impedance tomography*, Inverse Problems **18** (2002), no. 6, R99–R136. MR 1955896
14. Y Capdeboscq, Jérôme Fehrenbach, Frédéric De Gournay, and Otared Kavian, *Imaging by modification: numerical reconstruction of local conductivities from corresponding power density measurements*, SIAM Journal on Imaging Sciences **2** (2009), 1003.

15. Margaret Cheney, David Isaacson, and Jonathan C Newell, *Electrical impedance tomography*, SIAM review **41** (1999), no. 1, 85–101.
16. Bastian Gebauer and Otmar Scherzer, *Impedance-acoustic tomography*, SIAM J. Appl. Math. **69** (2008), no. 2, 565–576. MR 2465856 (2009j:35381)
17. David Gilbarg and Neil S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 2001.
18. Karshi F. Hasanov, Angela W. Ma, Adrian I. Nachman, and Michael L. G. Joy, *Current density impedance imaging*, IEEE Trans. Med. Imaging **27** (2008), no. 9, 1301–1309.
19. Hyun Soo Khang, Byung Il Lee, Suk Hoon Oh, Eung Je Woo, Soo Yeol Lee, Min Hyoung Cho, Ohin Kwon, Jeong Rock Yoon, and Jin Keun Seo, *J-substitution algorithm in magnetic resonance electrical impedance tomography (mreit): phantom experiments for static resistivity images*, Medical Imaging, IEEE Transactions on **21** (2002), no. 6, 695–702.
20. Sungwhan Kim, Ohin Kwon, Jin Keun Seo, and Jeong-Rock Yoon, *On a nonlinear partial differential equation arising in magnetic resonance electrical impedance tomography*, SIAM journal on mathematical analysis **34** (2002), no. 3, 511–526.
21. Yong Jung Kim, Ohin Kwon, Jin Keun Seo, and Eung Je Woo, *Uniqueness and convergence of conductivity image reconstruction in magnetic resonance electrical impedance tomography*, Inverse Problems **19** (2003), no. 5, 1213.
22. Peter Kuchment and Leonid Kunyansky, *Synthetic focusing in ultrasound modulated tomography*, Inverse Probl. Imaging **4** (2010), no. 4, 665–673. MR 2726422 (2011g:44003)
23. ______, *2D and 3D reconstructions in acousto-electric tomography*, Inverse Problems **27** (2011), no. 5, 055013, 21. MR 2793832 (2012b:65166)
24. Peter Kuchment and Dustin Steinhauer, *Stabilizing inverse problems by internal data*, Inverse Problems **28** (2012), no. 8, 084007, 20. MR 2956563
25. Ohin Kwon, Chunjae Park, Eun-Jae Park, Jin Keun Seo, and Eung Je Woo, *Electrical conductivity imaging using a variational method in bz-based mreit*, Inverse Problems **21** (2005), no. 3, 969.
26. Ohin Kwon, Eung Je Woo, Jeong-Rock Yoon, and Jin Keun Seo, *Magnetic resonance electrical impedance tomography (mreit): simulation study of j-substitution algorithm*, Biomedical Engineering, IEEE Transactions on **49** (2002), no. 2, 160–167.
27. Mikhail Mikhaĭlovich Lavrentev, Vladimir Gavrilovich Romanov, and Sergeĭ Petrovich Shishatskiĭ, *Ill-posed problems of mathematical physics and analysis*, Translations of Mathematical Monographs, vol. 64, American Mathematical Society, Providence, RI, 198633, Translated from the Russian by J. R. Schulenberger, Translation edited by Lev J. Leifman. MR 847715 (87g:65003)
28. Xu Li, Yuan Xu, and Bin He, *Imaging electrical impedance from acoustic measurements by means of magnetoacoustic tomography with magnetic induction (mat-mi)*, Biomedical Engineering, IEEE Transactions on **54** (2007), no. 2, 323–330.
29. JJ Liu, HC Pyo, JK Seo, and EJ Woo, *Convergence properties and stability issues in mreit algorithm*, Contemporary Mathematics **408** (2006), 201.
30. Francois Monard and Guillaume Bal, *Inverse diffusion problems with redundant internal information*, arXiv preprint arXiv:1106.4277 (2011).
31. François Monard and Guillaume Bal, *Inverse anisotropic diffusion from power density measurements in two dimensions*, Inverse Problems **28** (2012), no. 8, 084001.
32. ______, *Inverse anisotropic conductivity from power densities in dimension n 3*, Communications in Partial Differential Equations (2013), no. just-accepted.
33. Adrian Nachman, Alexandru Tamasan, and Alexandre Timonov, *Conductivity imaging with a single measurement of boundary and interior data*, Inverse Problems **23** (2007), no. 6, 2551–2563.
34. ______, *Recovering the conductivity from a single measurement of interior data*, Inverse Problems **25** (2009), no. 3, 035014.
35. ______, *Reconstruction of planar conductivities in subdomains from incomplete data*, SIAM Journal on Applied Mathematics **70** (2010), no. 8, 3342–3362.
36. ______, *Current density impedance imaging*, Tomography and inverse transport theory, Contemp. Math., vol. 559, Amer. Math. Soc., Providence, RI, 2011, pp. 135–149. MR 2885199
37. Haewon Nam, *Ultrasound-modulated optical tomography*, Ph.D. thesis, Texas A&M University, 2002.
38. Carlo Domenico Pagani, *Questions of stability for inverse problems*, Rend. Sem. Mat. Fis. Milano **52** (1982), 599–608 (1985). MR MR802967 (87c:35156)
39. ______, *Stability of a surface determined from measures of potential*, SIAM J. Math. Anal. **17** (1986), no. 1, 169–181. MR MR819221 (87c:31005)

40. Plamen Stefanov and Gunther Uhlmann, *Linearizing non-linear inverse problems and an application to inverse backscattering*, Journal of Functional Analysis **256** (2009), no. 9, 2842–2866.
41. Alexandru Tamasan, Alexandre Timonov, and Johann Veras, *Numerical solution of the Dirichlet problem for the 1-Laplacian and Applications*, to appear.
42. Hans Triebel, *Interpolation theory, function spaces, differential operators*, second ed., Johann Ambrosius Barth, Heidelberg, 1995. MR 1328645 (96f:46001)
43. Lihong V Wang and Hsin-i Wu, *Biomedical optics: principles and imaging*, Wiley-Interscience, 2012.
44. Eung Je Woo and Jin Keun Seo, *Magnetic resonance electrical impedance tomography (mreit) for high-resolution conductivity imaging*, Physiological measurement **29** (2008), no. 10, R1.
45. Nanping Zhang, *Electrical impedance tomography based on current density imaging*, 1992.

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