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INVERSE SCATTERING AND INVERSE BOUNDARY VALUE PROBLEMS FOR THE LINEAR BOLTZMANN EQUATION

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1 Introduction

Consider the Boltzmann equation

$$\frac{\partial}{\partial t}u(t,x,v) = -v\cdot\nabla_x u(t,x,v) - \sigma_a(x,v)u(t,x,v) + \int_V k(x,v',v)u(t,x,v')\,dv' \tag{1.1}$$

in $\mathbf{R}^n \times V \ni (x,v)$, V being an open subset of $\mathbf{R}^n$, $n\geq 2$. Equation (1.1) describes the dynamics of a flow of particles in $\mathbf{R}^n$ under the assumption that the interaction between them is neglectable (no non-linear terms). This is the case for example for a low-density flow of neutrons. The term involving σ_a describes the loss of particles from $(x,v)\in\mathbf{R}^n\times V$ due to absorption or scattering into another point (x,v'), while the last term in (1.1) involving k

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represents the production at $x \in \mathbf{R}^n$ of particles with velocity v form particles with velocity v'. The total rate of this production at (x, v') is given by

$$\sigma_p(x, v') = \int_V k(x, v', v)\, dv.$$

Following [RS] we say that the pair (σ_a, k) is *admissible*, if

(i) $0 \le \sigma_a \in L^\infty(\mathbf{R}^n \times V)$,

(ii) $0 \le k(x, v', \cdot) \in L^1(V)$ for a.e. $(x, v') \in \mathbf{R}^n \times V$ and $\sigma_p \in L^\infty(\mathbf{R}^n \times V)$,

(iii) There is an open bounded set $X \subset \mathbf{R}^n$, such that $k(x, v', v)$ and $\sigma_a(x, v)$ vanish if $x \notin \bar{X}$.

Denote $T_0 = -v \cdot \nabla_x$ with domain $D(T_0) = \{f \in L^1(\mathbf{R}^n \times V);\ v \cdot \nabla_x f \in L^1(\mathbf{R}^n \times V)\}$. It is well-known that T_0 is a generator of a strongly continuous group $U_0(t)f = f(x - tv, v)$ of isometries on $L^1(\mathbf{R}^n \times V)$ preserving the non-negative functions. Following the widely accepted notations, let us introduce the operators

$$\begin{array}{rclr} [A_1 f](x,v) &=& -\sigma_a(x,v) f(x,v), & T_1 = T_0 + A_1,\ D(T_1) = D(T_0), \\ [A_2 f](x,v) &=& \int_V k(x,v',v) f(x,v')\, dv', & T = T_0 + A_1 + A_2 = T_1 + A_2,\ D(T) = D(T_0) \end{array}$$

and set $A = A_1 + A_2$. Operators A_1 and A_2 are bounded on $L^1(\mathbf{R}^n \times V)$ and T_1, T are generators of strongly continuous groups $U_1(t) = e^{tT_1}$, $U(t) = e^{tT}$, respectively [RS]. For $U_1(t)$ we have an explicit formula

$$[U_1(t)f](x,v) = e^{-\int_0^t \sigma_a(x - sv, v) ds} f(x - tv, v), \tag{1.2}$$

while for $U(t)$ we have

$$\|U(t)\| \le e^{Ct}, \quad C = \|\sigma_p\|_{L^\infty}. \tag{1.3}$$

We work in the Banach space $L^1(\mathbf{R}^n \times V)$, so here $\|U(t)\|$ is the operator norm of $U(t)$ in $\mathcal{L}(L^1(\mathbf{R}^n \times V))$. It should be mentioned also that $U(t)$ preserves the cone of non-negative functions for $t \ge 0$.

One can define the wave operators associated with T, T_0 by

$$W_- = \operatorname*{s-lim}_{t \to \infty} U(t) U_0(-t), \tag{1.4}$$

$$\tilde{W}_+ = \operatorname*{s-lim}_{t \to \infty} U_0(-t) U(t). \tag{1.5}$$

If W_-, $\tilde{W}_+$ exist, then one can define the scattering operator

$$S = \tilde{W}_+ W_-$$

as a bounded operator in $L^1(\mathbf{R}^n \times V)$. Scattering theory for (1.1) has been developed in [Hej], [S], [V1] and we refer to these papers (see aslo [RS]) for sufficient conditions guaranteeing the existence of S. We would like to mention here also [P1], [U], [E], [St], [V2]. An abstract approach based on the Limiting Absorption Principle has been proposed in [M]. We will show in Section 2 however that S can always be defined as an operator $S : L^1_{\mathrm{c}}(\mathbf{R}^n \times V \setminus \{0\}) \to L^1_{\mathrm{loc}}(\mathbf{R}^n \times V \setminus \{0\})$. The first inverse problem we are interested in is the following: Does S determine uniquely σ_a, k? We show that the answer is affirmative if σ_a is independent of v.

Theorem 1.1 *Let (σ_a, k), $(\hat\sigma_a, \hat k)$ be two admissible pairs such that σ_a, $\hat\sigma_a$ do not depend on v and denote by S, $\hat S$ the corresponding scattering operators. Then, if $S = \hat S$, we have $\sigma_a = \hat\sigma_a$, $k = \hat k$.*

One can relax a little bit the assumption that σ_a, $\hat\sigma_a$ do not depend on v. For example, assume that $\sigma_a = \sigma_a(x, |v|)$, $\hat\sigma_a = \hat\sigma_a(x, |v|)$. Then it is clear from the proof (see also (1.6) below) that the uniqueness result in Theorem 1.1 still holds. However, it is important to note that Theorem 1.1 fails to be true for general σ_a. Consider for example the pairs $(\sigma_a, 0)$, $(\hat\sigma_a, 0)$, where $\hat\sigma_a = \sigma_a(x + p(x,v)v, v)$, with p some nontrivial continuous function such that $p(x,v)v$ is bounded on $\mathbf{R}^n \times V$. Then, if $(\sigma_a, 0)$ is admissible, so is $(\hat\sigma_a, 0)$. Since $k = 0$, we have

$$Sf = e^{-\int_{-\infty}^{\infty}\sigma_a(x-sv,v)ds} f, \quad (k = 0) \tag{1.6}$$

and it is easy to see that $S = \hat S$ although $\sigma_a \neq \hat\sigma_a$. Note that if $k = 0$, and σ_a does not depend on v, it follows from (1.6) that S determines uniquely the X-ray transform of σ_a and therefore σ_a.

The proof of Theorem 1.1 is constructive, it implies an explicit procedure for recovering σ_a and k from S. It turns out that all the information necessary to recover σ_a, k is contained in the behavior of the Schwartz kernel $\mathcal{S}(x, v, x', v')$ of S near the singularities $(x, v) = (x', v')$ and $x = x'$, $v \neq v'$, respectively.

Next object we consider is the so-called *albedo* operator $\mathcal{A}$. Assume that X is convex and has C^1-smooth boundary ∂X. We propose the following definition of $\mathcal{A}$ which generalizes that given in [AE], [EP], [P2]. Denote $\Gamma_\pm = \{(x,v) \in \partial X \times V;\ \pm n(x)\cdot v > 0\}$, where $n(x)$ is the outer normal to ∂X at $x \in \partial X$. Consider the measure $d\xi = |n(x)\cdot v| d\mu(x) dv$ on $\Gamma_\pm$, where $d\mu(x)$ is the measure on ∂X. Let us solve the problem

$$\begin{cases} (\partial_t - T)u &= 0 \quad \text{in } \mathbf{R}\times X \times V, \\ u|_{\mathbf{R}\times\Gamma_-} &= g, \\ u|_{t\ll 0} &= 0, \end{cases} \tag{1.7}$$

for $u(t, x, v)$, where $g \in L^1_{\mathrm{c}}(\mathbf{R};\ L^1(\Gamma_-, d\xi))$ and T is considered as a differential operator in $X \times V$. We will see in Section 4 that (1.7) has a unique solution $u \in C(\mathbf{R}; L^1(X\times V))$ and one can define the albedo operator $\mathcal{A}$ by

$$\mathcal{A}g = u|_{\mathbf{R}\times\Gamma_+} \in L^1_{\mathrm{loc}}(\mathbf{R}; L^1(\Gamma_+, d\xi)). \tag{1.8}$$

Operator $\mathcal{A} : L^1_{\mathrm{c}}(\mathbf{R};\ L^1(\Gamma_-, d\xi)) \to L^1_{\mathrm{loc}}(\mathbf{R}; L^1(\Gamma_+, d\xi))$ maps the incoming flux on ∂X to the outgoing flux on ∂X. It can be seen that $\mathcal{A}g$ can be defined more generally for $g \in L^1(\mathbf{R}\times\Gamma_-, dt\, d\xi)$ with $g = 0$ for $t \ll 0$. It has been shown in [AE], [EP], [P2] that there is a relationship between S and $\mathcal{A}$. We show below that in fact $\mathcal{A}$ determines S uniquely and conversely, S determines $\mathcal{A}$ uniquely by means of explicit formulae. To this end, let us define the extension operators $E_\pm$ and the restriction (trace) operators $R_\pm$ as follows. Set

$$\Omega = \{(x,v) \in \mathbf{R}^n \times V\backslash\{0\};\ \exists t \in \mathbf{R}, \text{ such that } x - tv \in X\}, \tag{1.9}$$

and define the functions

$$\tau_\pm(x,v) = \max\{t \in \mathbf{R};\ x \pm tv \in \partial X\}, \quad (x,v) \in \Omega.$$

Given $g \in L^1(\mathbf{R} \times \Gamma_\pm,\, dt\, d\xi)$, consider the following operators of extension:

$$(E_\pm g)(x,v) = \begin{cases} g(\pm\tau_\pm(x,v), x \pm \tau_\pm(x,v)v, v), & (x,v) \in \Omega \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that $E_\pm : L^1(\mathbf{R} \times \Gamma_\pm,\, dt\, d\xi) \to L^1(\mathbf{R}^n \times V)$ are isometric. Denote by $R_\pm$ the operator of restriction

$$R_\pm f = f|_{\Gamma_\pm}, \quad f \in C(\mathbf{R}^n \times V).$$

Although $R_\pm$ is not a bounded operator on $L^1(\mathbf{R}^n \times V)$ (see [C1], [C2]), $R_\pm U_0(t) f \in L^1(\mathbf{R} \times \Gamma_\pm,\, dt\, d\xi)$ is well defined for any $f \in L^1(\mathbf{R}^n \times V)$ (see (4.5)). Denote by χ_Ω the characteristic function of Ω. We establish the following relationships between S and $\mathcal{A}$.

Theorem 1.2 *Assume that X is convex. Then*
(a) $\mathcal{A}g = R_+ U_0(t) S E_- g$, $g \in L^1_{\rm c}(\mathbf{R} \times \Gamma_-, dt\, d\xi)$,
(b) $Sf = E_+ \mathcal{A} R_- U_0(t) f + (1-\chi_\Omega) f$, $f \in L^1_{\rm c}(\mathbf{R}^n \times V \backslash \{0\})$,
(c) $\mathcal{A}$ extends to a bounded operator

$$\mathcal{A} : L^1(\mathbf{R} \times \Gamma_-,\, dt\, d\xi) \to L^1(\mathbf{R} \times \Gamma_+,\, dt\, d\xi)$$

if and only if S extends to a bounded operator on $L^1(\mathbf{R}^n \times V)$.

Remark 1 Let us decompose $L^1(\mathbf{R}^n \times V) = L^1(\Omega) \oplus L^1((\mathbf{R}^n \times V) \setminus \Omega)$. A similar decomposition of course holds for $L^1_{\rm c}(\mathbf{R}^n \times V \backslash \{0\})$. Then S leaves invariant both spaces, moreover $S|_{L^1((\mathbf{R}^n \times V)\setminus\Omega)} = Id$, so S can be decomposed as a direct sum $S = S_1 \oplus Id$. Denote $\mathcal{R}_\pm = R_\pm U_0(\,\cdot\,) : L^1(\Omega) \to L^1(\mathbf{R} \times \Gamma_\pm,\, dt\, d\xi)$. We will see in Section 4 that $\mathcal{R}_\pm$ are isometric and invertible and $\mathcal{R}_\pm^{-1} = \mathcal{E}_\pm$ with $\mathcal{E}_\pm : L^1(\mathbf{R} \times \Gamma_\pm,\, dt\, d\xi) \to L^1(\Omega)$, $\mathcal{E}_\pm f := E_\pm f|_{L^1(\Omega)}$. Then we can rewrite Theorem 1.2 (a), (b) in the following way

$$\begin{array}{rcll} \mathcal{A} & = & \mathcal{R}_+ S_1 \mathcal{E}_- & \text{on } L^1_{\rm c}(\mathbf{R} \times \Gamma_-, dt\, d\xi) \\ S & = & \mathcal{E}_+ \mathcal{A} \mathcal{R}_- \oplus Id & \text{on } L^1_{\rm c}(\mathbf{R}^n \times V \backslash \{0\}), \end{array}$$

or even more simply as

$$\mathcal{A} = \mathcal{R}_+ S_1 \mathcal{E}_-, \qquad S_1 = \mathcal{E}_+ \mathcal{A} \mathcal{R}_-$$

with $S_1 = S|_{L^1_{\rm c}(\Omega)}$ as above. Thus $\mathcal{A}$ can be obtained from S_1 by a conjugation with invertible isometric operators and vice-versa.

Remark 2 The albedo operator is defined in [AE] in somewhat different manner by (1.8), provided that u solves (1.7) for $t > 0$ and u satisfies $u|_{t=0} = 0$ instead of $u|_{t \ll 0} = 0$ (in fact, it is assumed that u satisfies a non-zero initial condition, but this can be easily reduced to the case of zero initial condition). The relationship between $\mathcal{A}$ and S established in [AE] can be written as $R_+ S U_0(t) = \mathcal{A} R_- U_0(t)$, which can be obtained as a consequence of Theorem 1.2(b) (or (a)).

Remark 3 Some of the notations above are a little bit ambiguous. Namely, the expression $E_+\mathcal{A}R_-U_0(t)f$ in Theorem 1.2(b) seems to depend on t, while the left-hand side of the equality in which it is involved is independent of t. In fact, here $U_0(t)f$ is a function of x, v and t is a parameter, the same applies to $R_-U_0(t)f$. Since the operator $\mathcal{A}$ acts on functions $g = g(t, x, v)$ depending on t as well, we consider now t as a variable and apply $\mathcal{A}$ to the function $(t, x, v) \mapsto R_-U_0(t)f$. The result is a function of t, x and v. Next we apply E_- and obtain a function of x and v only. Perhaps a more precise notation in Theorem 1.2(b) would be $E_+\mathcal{A}R_-U_0(\,\cdot\,)f$.

An immediate consequence of Theorem 1.2 is that $\mathcal{A}$ determines uniquely σ_a, k for σ_a independent of v and X convex. However, we can prove this for not necessarily convex domains as well independently of Theorem 1.2.

Theorem 1.3 *Let (σ_a, k), $(\hat{\sigma}_a, \hat{k})$ be two admissible pairs with σ_a, $\hat{\sigma}_a$ independent of v and denote by X any open bounded set with C^1-smooth boundary with the property that σ_a, k, $\hat{\sigma}_a$, $\hat{k}$ vanish outside $\bar{X}$. Then, if the albedo operators $\mathcal{A}$, $\hat{\mathcal{A}}$ on ∂X coincide, we have $\sigma_a = \hat{\sigma}_a$, $k = \hat{k}$.*

It should be mentioned that in the case where k is of the form $k = RK + Q$ with R, Q known and $K = K(x, v)$, the uniqueness of the inverse problem studied in Theorem 1.3 has been investigated in [PV] for convex X under some smallness assumptions that guarantee that the corresponding integral equations can be solved by successful approximations. We would like to mention here also [D], where the problem of determination of k from the stationary albedo operator in the one-dimensional case is considered.

The proof of Theorem 1.3 is constructive as well. We study the Schwartz kernel of $\mathcal{A}$ and describe the first two singular terms in it. We show that σ_a can be recovered from the first term, while the second one determines k, similarly to the proof of Theorem 1.1.

Finally, we would like to mention that we have found some analogy between the albedo operator $\mathcal{A}$ and the Dirichlet-to-Neumann map Λ related to the boundary value problem for the Schrödinger operator or for the conductivity operator $\nabla \cdot \gamma \nabla$, $\gamma = \gamma(x) > 0$ [SU1]. More precisely, denote by Λ the operator acting on the boundary of a bounded domain mapping the Dirichlet data of the solution to $(-\Delta + q)v = 0$ (respectively $\nabla \cdot \gamma \nabla v = 0$) to its Neumann data. As proven in [SU1], Λ determines uniquely q (respectively γ). We found that $\mathcal{A}$ in our case is in some sense an analogue to Λ or more precisely to the time-dependent Dirichlet-to-Neumann map associated with the wave equation $(\partial_t^2 - \Delta_x + q)v = 0$. It is well-known that there is a close relation between the scattering operator for the Schrödinger equation and Λ. Theorem 1.2 we prove can be considered as an analogue of this result in transport theory. We would like to mention however, that the Schrödinger equation and the Boltzmann equation have quite different properties.

Some of the results of this paper have been announced in [CS]. It should be mentioned that the main theorems can be generalized to the case where σ_a, k depend on t as well.

2 The special solution and the scattering operator

An important role in our analysis is played by the following special solutions. Given $(x', v') \in \mathbf{R}^n \times V\backslash\{0\}$ consider the following problem

$$\left\{\begin{array}{rcl} (\partial_t - T)u & = & 0 \quad \text{in } \mathbf{R}\times\mathbf{R}^n\times V \\ u|_{t\ll 0} & = & \delta(x - x' - tv)\delta(v - v'), \end{array}\right. \tag{2.1}$$

δ being the Dirac delta function. We will show that (2.1) has unique solution $u^\#(t, x, v, x', v')$, with $u^\#$ depending continuously on t with values in $\mathcal{D}'(\mathbf{R}^n_x \times V_v \times \mathbf{R}^n_{x'} \times V_{v'}\backslash\{0\})$. Moreover, we have the following singular expansion of $u^\#$.

Theorem 2.1 *Problem (2.1) has unique solution $u^\# = u_0^\# + u_1^\# + u_2^\#$, where*

$$\begin{array}{rcl} u_0^\# & = & e^{-\int_0^\infty \sigma_a(x-sv,v)ds}\delta(x - x' - tv)\delta(v - v') \\ u_1^\# & = & \displaystyle\int_0^\infty e^{-\int_0^s \sigma_a(x-\tau v,v)d\tau} e^{-\int_0^\infty \sigma_a(x-sv-\tau v',v')d\tau} k(x - sv, v', v)\delta(x - sv - (t - s)v' - x')\, ds, \\ u_2^\# & \in & C\left(\mathbf{R};\, L^\infty_{\mathrm{loc}}(\mathbf{R}^n{}_{x'} \times V_{v'};\, L^1(\mathbf{R}^n{}_x \times V_v))\right). \end{array}$$

Proof. Pick $\varphi \in C^\infty_{\mathrm{c}}(\mathbf{R}^n \times V\backslash\{0\})$ and consider the problem

$$\left\{\begin{array}{rcl} (\partial_t - T)w & = & 0 \quad \text{in } \mathbf{R}\times\mathbf{R}^n\times V \\ w|_{t\ll 0} & = & \varphi(x - tv, v), \end{array}\right. \tag{2.2}$$

Since $\min\{|v|;\, (x, v) \in \operatorname{supp}\varphi \text{ for some } x\} > 0$, there exists $t_0 = t_0(\varphi)$, such that $\varphi(x - tv, v) = 0$ for $(x, v) \in X \times V$, $t < -t_0$. Then

$$w := U(t + t_0)U_0(-t_0)\varphi \tag{2.3}$$

solves (2.2) and it is easy to see that w does not depend on the particular choice of t_0. Applying Duhamel's principle

$$U(t - r) = U_1(t - r) + \int_r^t U(t - s)A_2U_1(s - r)\, ds, \tag{2.4}$$

we get

$$w = U_1(t + t_0)U_0(-t_0)\varphi + \int_{-t_0}^t U(t - s)A_2U_1(s + t_0)U_0(-t_0)\varphi\, ds.$$

Applying Duhamel's formula one more time, we obtain

$$w = w_0 + w_1 + w_2,$$

where

$$\begin{array}{rcl} w_0 & = & U_1(t + t_0)U_0(-t_0)\varphi \\ w_1 & = & \displaystyle\int_{-t_0}^t U_1(t - s)A_2U_1(s + t_0)U_0(-t_0)\varphi\, ds \\ w_2 & = & \displaystyle\int_{-t_0}^t \int_{s_1}^t U(t - s_2)A_2U_1(s_2 - s_1)A_2U_1(s_1 + t_0)U_0(-t_0)\varphi\, ds_2\, ds_1. \end{array}$$

For the first term w_0 we have

$$w_0 = e^{-\int_0^{t+t_0} \sigma_a(x-sv,v)ds}\varphi(x-tv,v) = (u_0^{\#}(t,x,v,\,\cdot\,,\,\cdot\,),\varphi), \tag{2.5}$$

where $u_0^{\#}$ is as stated above and $(u_0^{\#}(t,x,v,\,\cdot\,,\,\cdot\,),\varphi)$ is the action of the distribution $u_0^{\#}$ (with t, x, v considered as parameters) on the test function φ, i.e. formally $(u_0^{\#}(t,x,v,\,\cdot\,,\,\cdot\,),\varphi) = \int\int u_0^{\#}(t,x,v,x',v')\varphi(x',v')\,dx'\,dv'$. For w_1 we have

$$\begin{aligned}
w_1 &= \int_{-t_0}^{t} U_1(t-s)A_2 w_0(s,\,\cdot\,,\,\cdot\,)\,ds \\
&= \int_0^{t+t_0} U_1(s)A_2 w_0(t-s,\,\cdot\,,\,\cdot\,)\,ds \\
&= \int_0^{t+t_0}\int_V e^{-\int_0^s \sigma_a(x-\tau v,v)d\tau} k(x-sv,v',v) w_0(t-s,x-sv,v')\,dv'\,ds \\
&= \int_0^{t+t_0}\int_V e^{-\int_0^s \sigma_a(x-\tau v,v)d\tau} e^{-\int_0^\infty \sigma_a(x-sv-\tau v',v')d\tau} \\
&\qquad\qquad \times k(x-sv,v',v)\varphi(x-(t-s)v'-sv,v')\,dv'\,ds \\
&= (u_1^{\#}(t,x,v,\,\cdot\,,\,\cdot\,),\varphi),
\end{aligned} \tag{2.6}$$

where $u_1^{\#}$ is as stated above.

Finally, for w_2 we get by changing the order of integration

$$w_2 = \int_{-t_0}^{t} U(t-s_2)A_2 w_1(s_2,\,\cdot\,,\,\cdot\,)\,ds_2. \tag{2.7}$$

Substituting the formula for w_1, we obtain

$$\begin{aligned}
(A_2 w_1)(s_2,x,v) &= \int_V\int_0^\infty\int_V E(s_1,x,v'',v')k(x,v'',v)k(x-s_1v'',v',v'') \\
&\qquad \times\varphi(x-(s_2-s_1)v'-s_1v'',v')\,dv'\,ds_1\,dv'',
\end{aligned} \tag{2.8}$$

where

$$E(s,x,v,v') = e^{-\int_0^s \sigma_a(x-\tau v,v)d\tau} e^{-\int_0^\infty \sigma_a(x-sv-\tau v',v')d\tau}.$$

The second integral in (2.8) is in fact over a bounded interval. Since the integrals in (2.8) are absolutely convergent, we can change the order of integration freely. Let us make the following change $x' = x-(s_2-s_1)v'-s_1v''$ in the first integral in (2.8). We get

$$\begin{aligned}
&(A_2 w_1)(s_2,x,v) \\
&= \int_0^\infty\int_V\int_{\mathbf{R}^n} s_1^{-n} E(s_1,x,\frac{x-x'-(s_2-s_1)v'}{s_1},v')k(x,\frac{x-x'-(s_2-s_1)v'}{s_1},v) \\
&\qquad \times k(x'+(s_2-s_1)v',v',\frac{x-x'-(s_2-s_1)v'}{s_1})\varphi(x',v')dx'dv'ds_1.
\end{aligned} \tag{2.9}$$

Here we suppose that $k(x,v',v)$ is prolonged as 0 for v or v' outside V. There is a singularity above in s_1, but the integral (2.9) converge because we obtained it from the convergent integral (2.8). Denote

$$\begin{aligned} M(s_1,s_2,x,v,x',v') &= s_1^{-n} E(s_1,x,\frac{x-x'-(s_2-s_1)v'}{s_1},v')k(x,\frac{x-x'-(s_2-s_1)v'}{s_1},v) \\ &\quad \times k(x'+(s_2-s_1)v',v',\frac{x-x'-(s_2-s_1)v'}{s_1}). \end{aligned}$$

Then we can rewrite (2.9) as

$$(A_2 w_1)(s_2,x,v) = \int_0^\infty \int_V \int_{\mathbf{R}^n} M(s_1,s_2,x,v,x',v')\varphi(x',v')dx'\,dv'\,ds_1.$$

By performing the change $x = x' + (s_2 - s_1)v' + s_1 v''$, we get for $s_1 > 0$

$$\begin{aligned} &\int_V \int_{\mathbf{R}^n} M(s_1,s_2,x,v,x',v')\,dx\,dv \\ &= \int_V \int_V E(s_1,x'+(s_2-s_1)v'+s_1v'',v'',v')k(x'+(s_2-s_1)v'+s_1v'',v'',v) \\ &\qquad \times k(x'+(s_2-s_1)v',v',v'')\,dv''\,dv \\ &\leq \|\sigma_p\|^2_{L^\infty}, \end{aligned}$$

because $0 < E(s,x,v,v') \leq 1$ for $s \geq 0$. Therefore,

$$M \in L^\infty((\mathbf{R}_+)_{s_1} \times \mathbf{R}_{s_2} \times \mathbf{R}^n{}_{x'} \times V_{v'};\, L^1(\mathbf{R}^n{}_x \times V_v)) \tag{2.10}$$

and moreover, for each compact $K \subset \mathbf{R}^n \times V\backslash\{0\}$ there exists $t_0 = t_0(K)$, such that if $(x',v') \in K$, then M vanishes for $s_1 > s_2 + t_0$ and for $s_2 < -t_0$ (provided that $s_1 > 0$). Therefore, the following integral is well-defined

$$u_2^\# := \int_{-\infty}^t \int_0^\infty U(t-s_2)M(s_1,s_2,\,\cdot\,,\,\cdot\,,x',v')\,ds_1\,ds_2,$$

and we have

$$u_2^\# \in C\left(\mathbf{R}_t;\; L^\infty_{\mathrm{loc}}(\mathbf{R}^n{}_{x'} \times V_{v'}\backslash\{0\},\, L^1(\mathbf{R}^n{}_x \times V_v))\right).$$

On the other hand, by (2.7)

$$w_2(t,x,v) = \int_{\mathbf{R}^n \times V} u_2^\#(t,x,v,x',v')\varphi(x',v')\,dx'\,dv'. \tag{2.11}$$

We are ready now to conclude the proof of Theorem 2.1. We found (see (2.5), (2.6), (2.11)) that the unique solution to (2.2) has the form $w = (u^\#(t,x,v,\,\cdot\,,\,\cdot\,),\varphi)$, where $u^\# = u_0^\# + u_1^\# + u_2^\#$ is a distribution with properties as stated above. It is clear now that u solves the transport equation in distributional sense and satisfies the initial condition in (2.1) as well, therefore u solves (2.1). Moreover, this solution is unique because the solution to (2.2) is unique. $\square$

We will prove next that the wave operators W_-, $\tilde{W}_+$ (see (1.4), (1.5)) always exist as operators between suitably chosen spaces.

Proposition 2.1 *The limits W_-, $\tilde{W}_+$ exist as operators between the spaces*

$$W_- : L^1_{\rm c}(\mathbf{R}^n \times V\backslash\{0\}) \longrightarrow L^1(\mathbf{R}^n \times V)$$
$$\tilde{W}_+ : L^1(\mathbf{R}^n \times V) \longrightarrow L^1_{\rm loc}(\mathbf{R}^n \times V\backslash\{0\})$$

Proof. Pick $f \in L^1_{\rm c}(\mathbf{R}^n \times V\backslash\{0\})$. Since $\min\{|v|;\, (x,v) \in \operatorname{supp} f$ for some $x\} > 0$, for some $t_0 = t_0(f)$ we have $U_0(-t)f = 0$ in X for $t > t_0$. Moreover, $U(t)U_0(-t)f = U(t_0)U_0(-t_0)f$ for $t \geq t_0$ and therefore $W_- f = U(t_0)U_0(-t_0)f$. This proves in particular that the limit $W_- : L^1_{\rm c}(\mathbf{R}^n \times V\backslash\{0\}) \to L^1(\mathbf{R}^n \times V)$ (see (1.4)) exists as an operator between these two spaces. Next, let us fix $g \in L^1(\mathbf{R}^n \times V)$ and a compact set $K \subset \mathbf{R}^n \times V\backslash\{0\}$ and consider $[U_0(-t)U(t)g](x,v)$ for large t and $(x,v) \in K$. We claim that this is independent of t for $t > t_1$ with some $t_1 = t_1(K)$. In particular, this would prove that the limit $\tilde{W}_+$ (see (1.5)) exists as an operator $\tilde{W}_+ : L^1(\mathbf{R}^n \times V) \longrightarrow L^1_{\rm loc}(\mathbf{R}^n \times V\backslash\{0\})$ and $\tilde{W}_+ g|_K = U_0(-t_1)U(t_1)g|_K$. Indeed, the Duhamel's principle

$$U(t) = U_0(t) + \int_0^t U_0(t-s)AU(s)\,ds \tag{2.12}$$

implies

$$U_0(-t)U(t)g = g + \int_0^t U_0(-s)AU(s)g\,ds. \tag{2.13}$$

Since $AU(s)g = 0$ for $x \notin X$, we have $U_0(-s)AU(s)g = (AU(s)g)(x+sv, v) = 0$ for $(x,v) \in K$, $s > t_1 = t_1(K)$. Therefore, $U_0(-t)U(t)g|_K$ does not depend on t for $t > t_1$ and our claim is proved. $\square$

Now we are in position to define the scattering operator. Set

$$S = \tilde{W}_+ W_- : L^1_{\rm c}(\mathbf{R}^n \times V\backslash\{0\}) \longrightarrow L^1_{\rm loc}(\mathbf{R}^n \times V\backslash\{0\}), \tag{2.14}$$

where $\tilde{W}_+$, W_- are as in Proposition 2.1. In fact, as can be seen from the proof of the proposition above, S is well defined on the wider subset $\{f;\, \exists t_0 = t_0(f)$, such that $U_0(t)f = 0$ for $x \in X$, $t < -t_0\}$ (the incoming space). Now it is clear that $\tilde{W}_+$, W_-, S exist in classical sense if and only if these operators given in Proposition 2.1 and (2.14) can be extended as bounded operators on $L^1(\mathbf{R}^n \times V)$.

Proposition 2.2 *$u^\#(0,x,v,x',v')$ is the Schwartz kernel of W_-.*

Proof. We will prove something more — that $u^\#(t,x,v,x',v')$ is the Schwartz kernel of $U(t)W_-$. Pick $\varphi \in C_{\rm c}^\infty(\mathbf{R}^n \times V\backslash\{0\})$. From the proof of Proposition 2.1 it follows that $W_-\varphi = U(t_0)U_0(-t_0)\varphi$ for some large $t_0 = t_0(\varphi)$. Therefore, $U(t)W_-\varphi = U(t+t_0)U_0(-t_0)\varphi$ and by (2.3) we deduce that $U(t)W_-\varphi = w$, where w solves (2.2), i.e. $w = (u^\#(t,x,v,\cdot\,,\cdot\,),\varphi)$. This completes the proof of the proposition. $\square$

Denote by $\mathcal{S}(x,v,x',v') \in \mathcal{D}'(\mathbf{R}^n \times V\backslash\{0\} \times \mathbf{R}^n \times V\backslash\{0\})$ the Schwartz kernel of the scattering operator S.

Proposition 2.3 *$\mathcal{S}(x,v,x',v') = \lim_{t\to\infty} u^\#(t, x+tv, v, x', v')$.*

Proof. Let $\varphi \in C_c^\infty(\mathbf{R}^n\times V\backslash\{0\})$. Then $U(t)W_-\varphi = (u^\#(t,x,v,\,\cdot\,,\,\cdot\,),\varphi)$ and $U_0(-t)U(t)W_-\varphi = (u^\#(t,x+tv,v,\,\cdot\,,\,\cdot\,),\varphi)$. Therefore, for any compact $K\subset \mathbf{R}^n\times V\backslash\{0\}$ we have $S\varphi|_K = \lim_{t\to\infty}(u^\#(t,x+tv,v,\,\cdot\,,\,\cdot\,),\varphi)|_K$. According to the final part of the proof of Proposition 2.1, $U_0(-t)U(t)W_-\varphi|_K$ does not depend on t for large t. Therefore, in the last limit it suffices to take $t>t_0(K)$, so the limit exists trivially. $\square$

Proposition 2.4

$$\begin{aligned}\mathcal{S}(x,v,x',v') &= e^{-\int_{-\infty}^{\infty}\sigma_a(x-\tau v,v)d\tau}\delta(x-x')\delta(v-v')\\ &\quad+\int_{-\infty}^{\infty}e^{-\int_s^\infty \sigma_a(x+\tau v,v)d\tau}(A_2u^\#)(s,x+sv,v,x',v')\,ds.\end{aligned}$$

Proof. It follows from (2.4) that

$$U(2t)-U_1(2t)=\int_{-t}^{t}U_1(t-s)A_2U(s+t)\,ds.$$

Pick up $f\in L^1_c(\mathbf{R}^n\times V\backslash\{0\})$ and fix a compact $K\subset\mathbf{R}^n\times V\backslash\{0\}$. Then there exists $t_0=t_0(f,K)$, such that

$$\begin{aligned}Sf|_K &= U_0(-t_0)U(2t_0)U_0(-t_0)f|_K\\ &= U_0(-t_0)U_1(2t_0)U_0(-t_0)f|_K+U_0(-t_0)\int_{-t_0}^{t_0}U_1(t_0-s)A_2U(s+t_0)U_0(-t_0)f|_K ds\\ &= e^{-\int_{-\infty}^{\infty}\sigma_a(x-\tau v,v)d\tau}f|_K+\int_{-t_0}^{t_0}U_0(-t_0)U_1(t_0-s)A_2w(s,\,\cdot\,,\,\cdot\,)|_K ds\\ &= e^{-\int_{-\infty}^{\infty}\sigma_a(x-sv,v)ds}f|_K+\int_{-t_0}^{t_0}e^{-\int_s^{t_0}\sigma_a(x+\tau v,v)d\tau}(A_2w)(s,x+sv,v)|_K ds,\end{aligned}$$

where w solves (2.2) with $\varphi=f$. $\square$

3 Reconstruction of σ_a, k from S.

Assume that we are given the scattering operator S corresponding to an admissible pair (σ_a,k). We will show in this section how one can recover σ_a, k constructively. In particular, this will prove Theorem 1.1.

We will show next that the singular expansion of the special solution $u^\#(t,x,v,x',v')$ established in Theorem 2.1 implies a similar expansion of the scattering kernel $\mathcal{S}$.

Theorem 3.1 *We have $S=S_0+S_1+S_2$, where the Schwartz kernels $\mathcal{S}_j(x,v,x',v')$ of the operators S_j, $j=0,1,2$ satisfy*

$$\begin{aligned}\mathcal{S}_0 &= e^{-\int_{-\infty}^{\infty}\sigma_a(x-\tau v,v)d\tau}\delta(x-x')\delta(v-v')\\ \mathcal{S}_1 &= \int_{-\infty}^{\infty}e^{-\int_s^\infty\sigma_a(x+\tau v,v)d\tau}e^{-\int_0^\infty\sigma_a(x+sv-\tau v',v')d\tau}k(x+sv,v',v)\delta(x-x'+s(v-v'))\,ds\\ \mathcal{S}_2 &\in L^\infty_{\rm loc}(\mathbf{R}^n{}_{x'}\times V_{v'}\backslash\{0\};\,L^1_{\rm loc}(\mathbf{R}^n{}_x\times V_v\backslash\{0\})).\end{aligned}$$

Proof. The proof follows by substituting $u^\# = u_0^\# + u_1^\# + u_2^\#$ from Theorem 2.1 into the limit in Proposition 2.3 or the integral representation of $\mathcal{S}$ found in Proposition 2.4. As already mentioned above, for $(x,v,x',v') \in U \subset\subset \mathbf{R}^n \times V\backslash\{0\} \times \mathbf{R}^n \times V\backslash\{0\}$ the limit in Proposition 2.3 trivially exists and the integrals in Proposition 2.4 are taken over bounded intervals. $\square$

We are ready now to complete the proof of Theorem 1.1. The idea of the proof is the following. Suppose we are given the scattering operator S corresponding to a unknown admissible pair (σ_a, k). Then we know the kernel $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2$. It follows from Theorem 2.1 and Theorem 3.1 that $\mathcal{S}_0$ is a singular distribution supported on the hyperplane $x = x'$, $v = v'$ of dimension $2n$, $\mathcal{S}_1$ is supported on a $(3n+1)$-dimensional surface (for $v \neq v'$), while $\mathcal{S}_2$ is a function. Therefore, $\mathcal{S}_j$, $j = 0,1,2$ have different degrees of singularities and given $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2$, one can always recover $\mathcal{S}_0$ and $\mathcal{S}_1$. From $\mathcal{S}_0$ one can recover the X-ray transform of σ_a and therefore σ_a itself, provided that σ_a is independent of v. Next, suppose for simplicity that σ_a, k are continuous. Then for fixed x, v, v' with $v \neq v'$, $\mathcal{S}_1$ is a delta-function supported on the line $x' = x + s(v - v')$, $s \in \mathbf{R}$ with density a multiple of $k(x + sv, v', v)$. Therefore, one can recover that density for each s and in particular setting $s = 0$ we get $k(x, v', v)$.

Pick a function $\varphi \in C_c^\infty(\mathbf{R}^n)$ with $\varphi(0) = 1$, $\varphi(x) = 0$ for $|x| > 1$ and $0 \leq \varphi \leq 1$. Fix a compact set $K \subset \mathbf{R}^n \times V\backslash\{0\}$ and let $\chi \in C_c^\infty(\mathbf{R}^n \times V\backslash\{0\})$ be such that $\chi = 1$ on K and $0 \leq \chi \leq 1$. For $\varepsilon > 0$ sufficiently small set

$$\phi_\varepsilon(x,v,x',v') = \varphi\Big(\frac{x - x'}{\varepsilon}\Big)\varphi\Big(\frac{v - v'}{\varepsilon}\Big)\chi(x,v). \tag{3.1}$$

Note that $\phi_\varepsilon = 0$ for (x', v') outside some other compact subset K' of $\mathbf{R}^n \times V\backslash\{0\}$ for ε sufficiently small.

Proposition 3.1 *With ϕ_ε as above we have*

$$\lim_{\varepsilon \to 0} \int\int \mathcal{S}(x,v,x',v')\phi_\varepsilon(x,v,x',v')\, dx'\, dv' = e^{-\int_\infty^\infty \sigma_a(x - \tau v, v) d\tau}\chi(x,v),$$

in $L^1(\mathbf{R}^n \times V)$, where the integral is to be considered in distribution sense.

Proof. Note that a priori the formal integral above is a distribution in $\mathcal{D}'(\mathbf{R}^n \times V\backslash\{0\})$, but we will show that by Theorem 3.1 in fact it belongs to $L^1(\mathbf{R}^n \times V)$ and the limit above holds in the same space. For $\mathcal{S}_0$ we have

$$\int\int \mathcal{S}_0(x,v,x',v')\phi_\varepsilon(x,v,x',v')\, dx'\, dv' = e^{-\int_\infty^\infty \sigma_a(x - \tau v, v) d\tau}\chi(x,v). \tag{3.2}$$

Next,

$$\begin{aligned} 0 &\leq \int\int\int\int \mathcal{S}_1(x,v,x',v')\phi_\varepsilon(x,v,x',v')\, dx'\, dv'\, dx\, dv \\ &\leq \int\int\int\int \chi(x,v)\, \varphi\left(\frac{-s(v - v')}{\varepsilon}\right)\varphi\left(\frac{v - v'}{\varepsilon}\right)\, k(x + sv, v', v)\, ds\, dv'\, dx\, dv \end{aligned}$$

$$
\begin{aligned}
&\le \iiiint \chi(x,v)\,\varphi\Big(\frac{v-v'}{\varepsilon}\Big)\,k(x+sv,v',v)\,ds\,dv'\,dx\,dv \\
&= \iiiint \chi(x-sv,v)\,\varphi\Big(\frac{v-v'}{\varepsilon}\Big)\,k(x,v',v)\,ds\,dv'\,dx\,dv \\
&\le 2A\int_W \iint \varphi\Big(\frac{v-v'}{\varepsilon}\Big)\;k(x,v',v)\,dv'\,dx\,dv \\
&\le 2A\int_{D_\varepsilon} k(x,v',v)\,dx\,dv'\,dv. \qquad (3.3)
\end{aligned}
$$

We have used above the fact that the integral in s is taken over some bounded interval $[-A,A]$ with $A>0$ depending on χ and X, and we denoted $W=\{v;\ \exists x, \text{ such that } (x,v)\in\operatorname{supp}\chi\}$. The last integral is taken over the bounded set

$$
D_\varepsilon=\{(x,v',v);\ x\in X,\ v\in W,\ |v-v'|<\varepsilon\}.
$$

Let us estimate the measure $\operatorname{meas}(D_\varepsilon)$. We have

$$
\operatorname{meas}(D_\varepsilon)=\int_W\int_X\int_{|v-v'|<\varepsilon} dv'\,dx\,dv = C\varepsilon^n\int_W\int_X dx\,dv = C'\varepsilon^n.
$$

Therefore, in (3.3) we have a locally integrable function (see (ii), (iii) in section 1) and the integration is performed over a set D_ε with $\operatorname{meas}(D_\varepsilon)\to 0$. Since the Lebesgue integral is absolutely continuous withe respect to the Lebesgue measure, we get that

$$
\lim_{\varepsilon\to 0}\int\!\!\int \mathcal{S}_1(x,v,x',v')\phi_\varepsilon(x,v,x',v')\,dx'\,dv' = 0 \quad \text{in } L^1(\mathbf{R}^n_x\times V_v). \qquad (3.4)
$$

Finally, we have

$$
\begin{aligned}
&\iint\Big|\iint \mathcal{S}_2(x,v,x',v')\phi_\varepsilon(x,v,x',v')\,dx'\,dv'\Big|\,dx\,dv \\
&\qquad\le \int_{E_\varepsilon} |\mathcal{S}_2(x,v,x',v')|\;dx\,dv\,dx'\,dv' \qquad (3.5)
\end{aligned}
$$

with $E_\varepsilon=\{(x,v,x',v');\ (x,v)\in\operatorname{supp}\chi,\ |x-x'|\le\varepsilon,\ |v-v'|\le\varepsilon\}$. There exists $\varepsilon_0>0$ such that for $0<\varepsilon<\varepsilon_0$ we have $E_\varepsilon\subset E_{\varepsilon_0}\subset\mathbf{R}^n\times V\backslash\{0\}\times\mathbf{R}^n\times V\backslash\{0\}$ and $\mathcal{S}_2$ is an integrable function on E_{ε_0} by Theorem 3.1. As before, it is easy to see that $\operatorname{meas}(E_\varepsilon)=O(\varepsilon^{2n})\to 0$ as $\varepsilon\to 0$. Therefore, (3.5) tends to 0 and we obtain

$$
\lim_{\varepsilon\to 0}\int\!\!\int \mathcal{S}_2(x,v,x',v')\phi_\varepsilon(x,v,x',v')\,dx'\,dv' = 0 \quad \text{in } L^1(\mathbf{R}^n_x\times V_v). \qquad (3.6)
$$

Now, (3.2), (3.4) and (3.6) together complete the proof of Proposition 3.1. $\square$

Assume now that σ_a is independent of v. We deduce from Proposition 3.1 that one can recover

$$
\int_{-\infty}^{\infty}\sigma_a(x-\tau v)d\tau \qquad (3.7)
$$

for a.e. $(x, v) \in \mathbf{R}^n \times V$. Since V is an open set, we see that we know (3.7) for a.e. $(x, v) \in \mathbf{R}^n \times U$, where U is a small neighborhood of some $v_0 \in V \setminus \{0\}$. Thus we can recover the X-ray transform

$$\int_{-\infty}^{\infty} \sigma_a(x - \tau\omega) d\tau \tag{3.8}$$

of $\sigma_a(x)$ for a.e. $(x, \omega) \in \mathbf{R}^n \times \tilde{U}$, where $\tilde{U}$ is a small neighborhood of $v_0/|v_0|$ in S^{n-1}. The latter is sufficient to recover σ_a (see e.g. [Hel]). We note that in the particular case where for any $\omega \in S^{n-1}$, the velocity space V contains some v of the kind $v = r\omega$ with $r = r(\omega) > 0$, we can recover the X-ray transform (3.8) for a.e. $(x, \omega) \in \mathbf{R}^n \times S^{n-1}$ and therefore we can write an explicit formula [Hel] for $\sigma_a(x)$. As mentioned in the Introduction, this argument works also for $\sigma_a = \sigma_a(x, |v|)$.

We proceed next with the reconstruction of $k(x, v', v)$.

Choose two functions $\varphi \in C_c^\infty(\mathbf{R}^n)$ with $0 \le \varphi \le 1$, $\varphi(0) = 1$, $\varphi(x) = 0$ for $|x| > 1$ and $\varphi_1 \in C_c^\infty(\mathbf{R})$ with $\int \varphi_1(s) ds = 1$, $0 \le \varphi_1$. Set $\mathcal{W} = \{(v', v) \in V \times V;\ v \ne 0,\ v' \ne 0,\ v \ne v'\}$. For $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ set

$$\psi_{\varepsilon_1,\varepsilon_2} = \frac{1}{\varepsilon_1} \varphi_1\left(\frac{(x - x') \cdot (v - v')}{\varepsilon_1 |v - v'|^2} \right) \varphi\left(\frac{1}{\varepsilon_2} \left(x - x' - \frac{(x - x') \cdot (v - v')}{|v - v'|^2} (v - v') \right) \right). \tag{3.9}$$

Proposition 3.2 *With $\psi_{\varepsilon_1,\varepsilon_2}$ as above we have*

$$\lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \int \mathcal{S}(x, v, x', v') \psi_{\varepsilon_1,\varepsilon_2}(x, v, x', v')\, dx' = e^{-\int_0^\infty \sigma_a(x + \tau v, v) d\tau} e^{-\int_0^\infty \sigma_a(x - \tau v', v') d\tau} k(x, v', v),$$

where the integral is to be considered in distribution sense and the limit holds in $L^1_{\mathrm{loc}}(\mathbf{R}^n \times \mathcal{W})$.

Proof. First, note that

$$\int \mathcal{S}_0(x, v, x', v') \psi_{\varepsilon_1,\varepsilon_2}(x, v, x', v')\, dx' = 0 \quad \text{in } \mathbf{R}^n \times \mathcal{W}, \tag{3.10}$$

because $\mathcal{S}_0|_{v \ne v'} = 0$. Next, for $\mathcal{S}_1$ we get

$$\int \mathcal{S}_1(x, v, x', v') \psi_{\varepsilon_1,\varepsilon_2}(x, v, x', v')\, dx' = \int \tilde{E}(s, x, v', v) k(x + sv, v', v) \frac{1}{\varepsilon_1} \varphi_1\Big(-\frac{s}{\varepsilon_1} \Big)\, ds.$$

Here $\tilde{E}(s, x, v', v) = e^{-\int_s^\infty \sigma_a(x + \tau v, v) d\tau} e^{-\int_0^\infty \sigma_a(x + sv - \tau v', v') d\tau}$. It is easy to see that the mapping $s \to E(s, x, v', v) k(x + sv, v', v) \in L^1_{\mathrm{loc}}(\mathbf{R}^n \times V \times V)$ is continuous. Therefore,

$$\begin{aligned} \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} & \int \mathcal{S}_1(x, v, x', v') \psi_{\varepsilon_1,\varepsilon_2}(x, v, x', v')\, dx' \\ &= \lim_{\varepsilon_1 \to 0} \int \tilde{E}(s, x, v', v) k(x + sv, v', v) \frac{1}{\varepsilon_1} \varphi_1\Big(-\frac{s}{\varepsilon_1} \Big)\, ds \\ &= \tilde{E}(0, x, v', v) k(x, v', v) \qquad \text{in } L^1_{\mathrm{loc}}(\mathbf{R}^n \times \mathcal{W}). \end{aligned} \tag{3.11}$$

Finally, choose a compact set $K \subset \mathbf{R}^n \times \mathcal{W}$ and consider

$$\begin{aligned} \int_K \Big| \int \mathcal{S}_2(x, v, x', v') \psi_{\varepsilon_1,\varepsilon_2}(x, v, x', v')\, dx' \Big| dx\, dv'\, dv &\le \frac{\|\varphi_1\|_{L^\infty}}{\varepsilon_1} \int_K \int_{X'} |\mathcal{S}_2(x, v, x', v')| \\ \times \varphi\left(\frac{1}{\varepsilon_2} \left(x - x' - \frac{(x - x') \cdot (v - v')}{|v - v'|^2} (v - v') \right) \right) & dx'\ dx dv' dv. \end{aligned} \tag{3.12}$$

Note that on $\operatorname{supp}\psi_{\varepsilon_1,\varepsilon_2}$ we have $|x-x'| \le C\max\{\varepsilon_1,\varepsilon_2\}$, so for ε_1 and ε_2 bounded, x' belongs to some compact set X'. The integration in (3.12) is taken over the set

$$(x',x,v',v) \in F_{\varepsilon_2} := (X'\times K) \cap \left\{ \left| x - x' - \frac{(x-x')\cdot(v-v')}{|v-v'|^2}(v-v') \right| < \varepsilon_2 \right\}.$$

For fixed x, v', v the last set is a cylinder with radius ε_2, thus we conclude that $\operatorname{meas}(F_{\varepsilon_2}) = O(\varepsilon_2^{n-1})$ and therefore

$$\begin{aligned}\int_K \left| \int \mathcal{S}_2(x,v,x',v')\psi_{\varepsilon_1,\varepsilon_2}(x,v,x',v')\,dx' \right| dx\,dv'\,dv \\ \le \frac{\|\varphi_1\|_{L^\infty}}{\varepsilon_1}\int_{F_{\varepsilon_2}} |\mathcal{S}_2(x,v,x',v')|dx'dxdv'dv \to 0 \quad \text{as } \varepsilon_2\to 0, \end{aligned} \tag{3.13}$$

because the Lebesgue integral is absolutely continuous with respect to the Lebesgue measure. Combining (3.10), (3.11) and (3.13) we complete the proof of the proposition. □

Assume now that we are given the scattering operator corresponding to an admissible pair (σ_a,k) with $\sigma_a=\sigma_a(x)$. By Proposition 3.1, one can recover $\sigma_a(x)$. Next, by Proposition 3.2 we can explicitly recover

$$e^{-\int_0^\infty \sigma_a(x+\tau v)d\tau} e^{-\int_0^\infty \sigma_a(x-\tau v')d\tau} k(x,v',v)$$

almost everywhere in $\mathbf{R}^n\times V\times V$. Since $\sigma_a(x)$ is already known, we get $k(x,v',v)$ for a.e. (x,v',v).

Finally, we would like to mention that Propositions 3.1, 3.2 can be written in terms of the operator S itself rather than in terms of its distribution kernel. Let ϕ_ε be the same as in (3.1) and consider the function $\phi_\varepsilon(y,w,x,v)$, where y, w are regarded as parameters (i.e. in (3.1) we replace (x,v,x',v') by (y,w,x,v)). Then Proposition 3.1 is equivalent to

$$\lim_{\varepsilon\to 0} S\phi_\varepsilon(y,w,\,\cdot\,,\,\cdot\,)|_{y=x,\,w=v} = e^{-\int_\infty^\infty \sigma_a(x-\tau v,v)d\tau}\chi(x,v)$$

in $L^1(\mathbf{R}^n\times V)$. Similarly, Proposition 3.2 can be rewritten as

$$\begin{aligned}\lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0} S[\psi_{\varepsilon_1,\varepsilon_2}(y,w,\,\cdot\,,\,\cdot\,)\rho(\,\cdot\,)]|_{y=x,\,w=v} \\ = \int_V e^{-\int_0^\infty \sigma_a(x+\tau v,v)d\tau} e^{-\int_0^\infty \sigma_a(x-\tau v',v')d\tau} k(x,v',v)\rho(v')\,dv'\end{aligned}$$

in $L^1_{\mathrm{loc}}(\mathbf{R}^n\times V\setminus\{\{0\}\cup\operatorname{supp}\rho\})$ for any $\rho=\rho(v')\in C_c^\infty(V\setminus\{0\})$.

4 The albedo operator

Assume that X is convex and ∂X is C^1-smooth. Consider the functions $\tau_\pm(x,v)$ and the operators $E_\pm$, $R_\pm$ defined in the Introduction. It should be noted that $\tau_\pm$ have the properties $\tau_\pm(x+tv,v)=\tau_\pm(x,v)\mp t$ and $(x\pm\tau_\pm(x,v)v,v)\in\Gamma_\pm$ for any (x,v). Using this property, we can show that $E_\pm$ are closely connected to the solution of the following boundary value problem

$$\left\{\begin{array}{rcl}(\partial_t - T_0)v & = & 0 \quad \text{in } \mathbf{R}\times X\times V,\\ v|_{\mathbf{R}\times\Gamma_\pm} & = & g.\end{array}\right. \tag{4.1}$$

Indeed, taking into account that

$$R_\pm U_0(t)E_\pm g = g, \tag{4.2}$$

we see that the solution to (4.1) is given by $v = U_0(t)E_\pm g|_{X\times V}$. We consider in (4.1) T_0 as a differential operator in $X\times V$. As mentioned in the Introduction, $E_\pm : L^1(\mathbf{R}\times\Gamma_\pm,\, dt\, d\xi) \to L^1(\mathbf{R}^n\times V)$ is isometric, i.e.

$$\|E_\pm g\|_{L^1(\mathbf{R}^n\times V)} = \|g\|_{L^1(\mathbf{R}\times\Gamma_\pm,\, dt\, d\xi)}. \tag{4.3}$$

Equality (4.3) follows easily by making a change of variables in the corresponding integral. Indeed,

$$\|E_\pm g\|_{L^1(\mathbf{R}^n\times V)} = \int_\Omega |g(\pm\tau_\pm(x,v), x\pm\tau_\pm(x,v)v, v)|\, dx\, dv.$$

Let us choose new variables $t = \pm\tau_\pm(x,v)$, $y = x\pm\tau_\pm(x,v)$. Then $(y,v)\in\Gamma_\pm$ and $dx = dt|v\cdot n(y)|d\mu(y)$, thus we get

$$\|E_\pm g\|_{L^1(\mathbf{R}^n\times V)} = \int_{\Gamma_\pm}\int |g(t,y,v)|\, dt\, |v\cdot n(y)|d\mu(y)dv,$$

which proves (4.3).

Denote by χ_Ω the characteristic function of Ω (see (1.9)). Then the following property holds

$$E_\pm R_\pm U_0(t)f = \chi_\Omega f. \tag{4.4}$$

We recall (see also Remark 3 in the Introduction) that $E_\pm$ acts on functions depending both on t and (x,v) and the result is a function independent of t. Thus in the left-hand side of (4.4) t is one of the variables, not a parameter. Taking into account (4.3), we conclude that

$$\|R_\pm U_0(\,\cdot\,)f\|_{L^1(\mathbf{R}\times\Gamma_\pm,\, dt\, d\xi)} = \|\chi_\Omega f\|_{L^1(\mathbf{R}^n\times V)}. \tag{4.5}$$

Note that $R_\pm$ are unbounded as operators from $L^1(\mathbf{R}^n\times V)$ into $L^1(\Gamma_\pm, d\xi)$. We refer to [C1], [C2] for more precise results and trace theorems. We are not going to make use of these trace theorems however (except in the proof of Proposition 5.2), because we will always apply $R_\pm$ to time dependent functions like $U_0(t)f$ or $U(t)f$ and will consider the result as a function of both variables x and t belonging (locally) to $L^1(\mathbf{R}\times\Gamma_\pm,\, dt\, d\xi)$. Then $R_\pm U_0(t)f$ is well defined according to (4.5) and for $R_\pm U(t)f$ we have:

Lemma 4.1 $R_\pm U(\,\cdot\,) : L^1(\mathbf{R}^n\times V) \longrightarrow L^1_{\mathrm{loc}}(\mathbf{R};\, L^1(\Gamma_\pm,\, d\xi))$ *is continuous. More precisely, for each $a>0$ we have*

$$\int_{-a}^{a}\|R_\pm U(t)f\|_{L^1(\Gamma_\pm,d\xi)}dt \le C(a)\|\chi_\Omega f\|_{L^1(\mathbf{R}^n\times V)}, \quad f\in L^1(\mathbf{R}^n\times V).$$

Proof. Given $f \in L^1(\mathbf{R}^n \times V)$, set $f = f_1 + f_2$ with $f_1 = \chi_\Omega f$, $f_2 = (1-\chi_\Omega)f$. Using Duhamel's principle (2.4), we see that $U(t)f_2 = U_0(t)f_2$ and thus by (4.5), $R_\pm U(t)f_2 = 0$. For f_1 we have by using (2.12) and (4.5)

$$
\begin{aligned}
&\int_{-a}^{a} \|R_\pm U(t)f_1\|_{L^1(\Gamma_\pm, d\xi)} dt \\
&\qquad \le \|f_1\|_{L^1(\mathbf{R}^n\times V)} + \int_{-a}^{a} \left\| \int_0^t R_\pm U_0(t-s)AU(s)f_1\, ds \right\|_{L^1(\Gamma_\pm,d\xi)} dt \\
&\qquad \le \|f_1\|_{L^1(\mathbf{R}^n\times V)} + \int_{-a}^{a}\int_{-a}^{a} \|R_\pm U_0(t-s)AU(s)f_1\|_{L^1(\Gamma_\pm,d\xi)} ds\, dt \\
&\qquad = \|f_1\|_{L^1(\mathbf{R}^n\times V)} + \int_{-a}^{a}\int_{-a}^{a} \|R_\pm U_0(t-s)AU(s)f_1\|_{L^1(\Gamma_\pm,d\xi)} dt\, ds \\
&\qquad \le \|f_1\|_{L^1(\mathbf{R}^n\times V)} + \int_{-a}^{a} \|U_0(-s)AU(s)f_1\|_{L^1(\mathbf{R}^n\times V)} ds \\
&\qquad \le \left(1 + 2a\|\sigma_p\|_{L^\infty} e^{a\|\sigma_p\|_{L^\infty}}\right) \|f_1\|_{L^1(\mathbf{R}^n\times V)}.
\end{aligned}
$$

□

Lemma 4.2 *$R_- U(t)f|_{\mathbf{R}_+\times\Gamma_-} = R_- U_0(t)f|_{\mathbf{R}_+\times\Gamma_-}$ for any $f \in L^1(\mathbf{R}^n \times V)$.*

Proof. We have to show that

$$\int_0^\infty \|R_- U(t)f - R_- U_0(t)f\|_{L^1(\Gamma_-,d\xi)} dt = 0.$$

By inspecting the proof of Lemma 4.1 we see that it suffices to prove that

$$\int_s^\infty \|R_- U_0(t-s)AU(s)f\|_{L^1(\Gamma_-,d\xi)} dt = 0, \quad s \ge 0,$$

which is equivalent to

$$\int_0^\infty \|R_- U_0(t)AU(s)f\|_{L^1(\Gamma_-,d\xi)} dt = 0, \quad s \ge 0.$$

In order to complete the proof, it is enough to observe, that $R_- U_0(t)h|_{\mathbf{R}_+\times\Gamma_-} = 0$ for any h with $h(x,v) = 0$ for $x \notin \bar{X}$. □

Given $g \in L^1_{\mathrm{c}}(\mathbf{R};\, L^1(\Gamma_-,\, d\xi))$, consider the problem (1.7).

Proposition 4.1 *Problem (1.7) has unique solution in $C(\mathbf{R}; L^1(X \times V))$ given by $u = U(t)W_- E_- g|_{X\times V}$.*

Proof. Note first that the uniqueness follows from the fact that the homogeneous problem (with $g = 0$) has only a trivial solution, because the transport operator with boundary conditions $u|_{\mathbf{R}\times\Gamma_-} = 0$ generates a continuous semigroup of solution operators. Next, note that if t_0 is such that $g = 0$ for $t < -t_0$, we have $U_0(t)E_- g = 0$ in $X \times V$ for $t < -t_0$ and moreover, $U(t)U_0(-t)E_- g = U(t_0)U_0(-t_0)E_- g$ for $t > t_0$, so although $E_- g$ does not necessarily belong to $L^1_{\mathrm{c}}(\mathbf{R}^n \times V\backslash\{0\})$ (see (1.4) and Proposition 2.1), the limit $W_- E_- g$

trivially exists. Set $w = U(t)W_-E_-g = U(t+t_0)U_0(-t_0)E_-g$. Then w clearly solves the Boltzmann equation in $\mathbf{R}\times X\times V$. We have that $R_-w|_{t<-t_0} = 0$ and by Lemma 4.2 and (4.2), $R_-w|_{t>-t_0} = R_-U_0(t)E_-g|_{t>-t_0} = g|_{t>-t_0} = g$. Therefore, w satisfies the boundary condition as well. Thus setting $u = w|_{X\times V}$, we get a solution to (4.1). $\square$

We see now that the definition of $\mathcal{A}$, given in (1.8)

$$\mathcal{A}g = R_+u, \quad \mathcal{A}: L^1_{\mathrm{c}}(\mathbf{R};\, L^1(\Gamma_-,\, d\xi)) \longrightarrow L^1_{\mathrm{loc}}(\mathbf{R}; L^1(\Gamma_+, d\xi)), \tag{4.6}$$

u being the solution to (1.7), is correct. Indeed, by Proposition 4.1, $\mathcal{A}g = R_+U(t)\,W_-E_-g$ and by Lemma 4.1, $\mathcal{A}g \in L^1_{\mathrm{loc}}(\mathbf{R};\, L^1(\Gamma_+, d\xi))$. We note that in fact, $\mathcal{A}g$ is well defined also for $g \in L^1(\mathbf{R}\times\Gamma_-,\, dt\,d\xi)$ with $g=0$ for $t \ll 0$, and then $\mathcal{A}g \in L^1((a,\infty)\times\Gamma_+,\, dt\,d\xi)$ for any $a\in\mathbf{R}$, but as Theorem 1.2(c) shows (see the proof below), $\mathcal{A}$ extends as an operator $\mathcal{A}: L^1(\mathbf{R}\times\Gamma_-,\, dt\,d\xi) \to L^1(\mathbf{R}\times\Gamma_+,\, dt\,d\xi)$ if and only if the scattering operator exists as a bounded operator on $L^1(\mathbf{R}^n\times V)$.

Proof of Theorem 1.2. Consider first (a). Pick $g\in L^1_{\mathrm{c}}(\mathbf{R}\times\Gamma_-,\, dt\,d\xi)$. Then $E_-g \in L^1_{\mathrm{c}}(\mathbf{R}^n\times V\backslash\{0\})$ and SE_-g is well defined. By (4.6) and Proposition 4.1 we have $\mathcal{A}g = R_+U(t)W_-E_-g$. Denote

$$\Omega_+ = \{(x,v)\in\Omega;\; x+tv\notin X \text{ for any } t\geq 0\}.$$

We claim that

$$U_0(t)SE_-g|_{\Omega_+\cap K} = U(t)W_-E_-g|_{\Omega_+\cap K} \tag{4.7}$$

for any compact $K\subset\mathbf{R}^n\times V\backslash\{0\}$. Indeed,

$$U_0(t)SE_-g = \lim_{s\to\infty} U_0(t-s)U(s)W_-E_-g \tag{4.8}$$

and the limit exists in $L^1_{\mathrm{c}}(\mathbf{R}^n\times V\backslash\{0\})$. We have (see (2.13))

$$U_0(t-s)U(s)W_-E_-g|_{\Omega_+} = U_0(t)W_-E_-g|_{\Omega_+} + \int_0^s U_0(t-\tau)AU(\tau)W_-E_-g|_{\Omega_+}d\tau \tag{4.9}$$

with $U_0(t-\tau)AU(\tau)W_-E_-g = (AU(\tau)W_-E_-g)(x-(t-\tau)v, v)$.For $(x,v)\in\Omega_+$ this function vanishes provided that $t-\tau<0$. So in fact the integral in (4.9) is taken over the the interval $0\le\tau\le t$ only and therefore (4.9) is independent of s for $s\geq t$. Therefore, we can put $s=t$ in (4.9) in order to get the limit (4.8) which implies immediately (4.7). Let us now apply R_+ to both sides of (4.7) to get

$$R_+U_0(t)SE_-g = R_+U(t)W_-E_-g = \mathcal{A}g.$$

Consider (b). Pick $f\in L^1_{\mathrm{c}}(\mathbf{R}^n\times V\backslash\{0\})$ and set $g = R_-U_0(t)f$. Then we have $g\in L^1_{\mathrm{c}}(\mathbf{R};\, L^1(\Gamma_-,\, d\xi))$ and $E_-g = \chi_\Omega f$ (which is true whenever $E_-g\in L^1_{\mathrm{c}}(\mathbf{R}^n\times V\backslash\{0\})$). An application of (a) yields $\mathcal{A}R_-U_0(t)f = R_+U_0(t)S\chi_\Omega f$ by (4.4). Applying E_+ to both sides and using again (4.4), we get

$$\chi_\Omega S\chi_\Omega f = E_+\mathcal{A}R_-U_0(t)f. \tag{4.10}$$

Now, since for the special solution $u^{\#}$ we have $u^{\#} = \delta(x-x'-tv)\delta(v-v')$ in $(\mathbf{R}^n\times V\setminus\{0\})\setminus\Omega$, we get $S(1-\chi_\Omega)f = (1-\chi_\Omega)f$ for any f. On the other hand, by Proposition 2.4, $(1-\chi_\Omega)Sf = (1-\chi_\Omega)f$ for any f. In other words, S leaves $L^1_c(\Omega)$, $L^1_c((\mathbf{R}^n\times V\setminus\{0\})\setminus\Omega)$ invariant and $\chi_\Omega S\chi_\Omega f = Sf - (1-\chi_\Omega)f$. Substituting this into (4.10), we complete the proof of (b).

Finally, (c) is an immediate consequence of (a), (b), (4.3) and (4.5). $\square$

5 Reconstruction of σ_a, k from $\mathcal{A}$. The non-convex case

In this section we prove Theorem 1.3. In the case where X is convex, the uniqueness result in Theorem 1.3 is an immediate consequence of Theorem 1.1 and Theorem 1.2. If X is not convex, then one can still deduce Theorem 1.3 from the previous two theorems using an argument from [SU2], where the Dirichlet-to-Neumann map is considered (see Proposition 5.2 below). Namely, one can show that $\mathcal{A} = \hat{\mathcal{A}}$ entails $u^{\#} = \hat{u}^{\#}$ outside $X\times V$ and therefore, by Proposition 2.3 we could conclude that $\sigma_a = \hat{\sigma}_a$, $k = \hat{k}$. We will give however another proof of Theorem 1.3 as well, that implies a constructive procedure for recovering σ_a, k and describes the Schwartz kernel of the albedo operator $\mathcal{A}$.

We assume in this section that X is an open bounded set with C^1-smooth boundary, not necessarily convex. First we will show that (1.7) still has unique solution in this more general situation. In what follows we need the semigroups $\tilde{U}_0(t)$, $\tilde{U}_1(t)$, $\tilde{U}(t)$ (see e.g. [Vi], [V2]), related to the solution of the following problem

$$\left\{\begin{array}{rcl} (\partial_t - T_i)u &=& 0 \quad \text{in } \mathbf{R}\times X\times V, \\ u|_{\mathbf{R}_+\times\Gamma_-} &=& 0, \\ u|_{t=0} &=& f, \end{array}\right. \tag{5.1}$$

T_i being T_0, T_1 and T, respectively (regarded as differential operators). More precisely, T_0, T_1 and T, acting on functions vanishing on Γ_-, extend to generators of strongly continuous semigroups $\tilde{U}_0(t)$, $\tilde{U}_1(t)$, $\tilde{U}(t)$ on $L^1(X\times V)$ and the solution to (5.1) is given by $u = \tilde{U}_i(t)f$. It is easy to check that we have the following explicit formulae

$$\tilde{U}_0(t)f = f(x-tv,v)\theta(x,x-tv) \tag{5.2}$$

$$\tilde{U}_1(t)f = e^{-\int_0^t \sigma_a(x-sv,v)ds} f(x-tv,v)\theta(x,x-tv) \tag{5.3}$$

where

$$\theta(x,y) = \left\{\begin{array}{ll} 1, & \text{if } px+(1-p)y\in X \text{ for each } p\in[0,1], \\ 0, & \text{otherwise.}\end{array}\right.$$

Let us modify a little bit the definition of $\tau_\pm$ given in the Introduction. Set

$$\tau_\pm(x,v) = \min\{t\ge 0;\ x\pm tv\in\partial X\},\quad (x,v)\in X\times V\setminus\{0\}.$$

If X is convex, then the definition given above agrees with that proposed in the Introduction. Using τ_-, we can write explicitly the solution of (1.7) in the case where $T = T_0$ or $T = T_1$. For the case $T = T_1$ we have that the solution of (1.7) reads $u = G_-(t)g$, where

$$G_\pm(t)g := e^{\pm\int_0^{\tau_\pm(x,v)}\sigma_a(x\pm sv,v)ds} g(t\pm\tau_\pm(x,v), x\pm\tau_\pm(x,v)v, v). \tag{5.4}$$

For X convex and $\sigma_a = 0$, we have $G_\pm(t)g = U_0(t)E_\pm g$. It is not hard to see that the following generalization of (4.3) holds

$$\sup_t \|G_-(t)g\|_{L^1(X\times V)} \le \|g\|_{L^1(\mathbf{R}\times\Gamma_\pm,\, dt\, d\xi)} \tag{5.5}$$

and moreover, $G_-(t) : L^1(\mathbf{R}\times\Gamma_-,\, dt\, d\xi) \to L^1(X\times V)$ is strongly continuous in t. Similar statements hold for $G_+(t)$ as well if we restrict our considerations outside a small neighborhood of $v = 0$, because the exponential in (5.4) may not bounded in this case as $v \to 0$. We have the following generalization of Proposition 4.1 to the case where X is not necessarily convex.

Proposition 5.1 *Given $g \in L^1_{\mathrm{c}}(\mathbf{R};\, L^1(\Gamma_-,\, d\xi))$, problem (1.7) has unique solution $u \in C(\mathbf{R};\, L^1(X\times V))$ given by*

$$u = G_-(t)g + \int_{-\infty}^t \tilde U(t-s)A_2 G_-(s)g\, ds. \tag{5.6}$$

Proof. First, observe that the integral above is taken over a finite interval $[-t_0, t]$, where t_0 is such that $g = 0$ for $t < -t_0$. It is easy to see that u, given by the formula above, satisfies the Boltzmann equation in $X\times V$ in distribution sense and belongs to $C(\mathbf{R};\, L^1(X\times V))$. For $t < -t_0$ we have that $G_-(t)g$ and the integral above vanishes, thus $u|_{t<-t_0} = 0$. Finally, $u|_{\mathbf{R}\times\Gamma_-} = G_-(t)g|_{\mathbf{R}\times\Gamma_-} = g$, because $\tilde U(t)$ satisfies homogeneous boundary conditions on $\mathbf{R}\times\Gamma_-$. Notice that the requirement $g \in L^1_{\mathrm{c}}(\mathbf{R};\, L^1(\Gamma_-,\, d\xi))$ can be relaxed to $g = 0$ for $t \ll 0$. $\square$

Following the proof of Lemma 4.1 and using (5.2), we can prove that $R_+\tilde U(\,\cdot\,) : L^1(X\times V) \to L^1_{\mathrm{loc}}(\mathbf{R};\, L^1(\Gamma_+, d\xi))$ is continuous. Thus, using this fact and Proposition 5.1 we can define the albedo operator in this case as well by (1.8).

Next we prove Theorem 1.3 by showing that $\mathcal{A}$ determines uniquely the special solution $u^\#$ outside $X\times V$. Although the reconstruction procedure described after this proposition implies Theorem 1.3 as well, we include Proposition 5.2 because it suggests much shorter way of demonstrating Theorem 1.3.

Proposition 5.2 *$\mathcal{A}$ determines uniquely the special solution $u^\#$ for x outside X.*

Proof. Here we follow essentially [SU2], where the Dirichlet-to-Neumann map related to a second order elliptic equation is considered. Let (σ_a, k), $(\hat\sigma_a, \hat k)$ be two admissible pairs supported (with respect to x) in X and denote by T, $\hat T$, $u^\#$, $\hat u^\#$, etc. the operators T, the special solutions $u^\#$, etc., related to (σ_a, k) and $(\hat\sigma_a, \hat k)$, respectively. Choose $\varphi \in C^\infty_{\mathrm{c}}(\mathbf{R}^n \times V\backslash\{0\})$ and set $w := (u^\#(t,x,v,\,\cdot\,,\,\cdot\,), \varphi)$, $\hat w := (\hat u^\#(t,x,v,\,\cdot\,,\,\cdot\,), \varphi)$, i.e. w, $\hat w$ solve (2.2) with $T = T$, $T = \hat T$, respectively. Since $\varphi \in D(T) = D(\hat T)$, we have $w \in D(T)$, $\hat w \in D(T)$ for each t (see (2.3)), therefore (see [C1], [C2]) the traces $w|_{\Gamma_\pm}$, $\hat w|_{\Gamma_\pm}$ are well defined as elements in $L^1_{\mathrm{loc}}(\Gamma_\pm, d\xi)$ depending continuously on t. Let v solve

$$\begin{cases} (\partial_t - T)v &= 0 \qquad \text{in } \mathbf{R}\times X\times V \\ v|_{\mathbf{R}\times\Gamma_-} &= \hat w|_{\mathbf{R}\times\Gamma_-} \\ v|_{t\ll 0} &= 0. \end{cases} \tag{5.7}$$

Set

$$u =: \begin{cases} v, & x \in X \\ \hat{w}, & x \notin X. \end{cases} \tag{5.8}$$

Assume that $\mathcal{A} = \hat{\mathcal{A}}$. Since $\hat{w}$ clearly solves the problem

$$\begin{cases} (\partial_t - \hat{T})\hat{w} &= 0 \qquad \text{in } \mathbf{R}\times X \times V \\ \hat{w}|_{\mathbf{R}\times\Gamma_-} &= \hat{w}|_{\mathbf{R}\times\Gamma_-} \\ \hat{w}|_{t\ll 0} &= 0. \end{cases} \tag{5.9}$$

and $\mathcal{A} = \hat{a}$, we get from (5.7), (5.9), that $v|_{\mathbf{R}\times\Gamma_+} = \hat{w}|_{\mathbf{R}\times\Gamma_+}$, therefore

$$v|_{\mathbf{R}\times\Gamma_\pm} = \hat{w}|_{\mathbf{R}\times\Gamma_\pm}. \tag{5.10}$$

Combining (5.8) and (5.10) we deduce that u, which is absolutely continuous function along the rays $s \mapsto (x+sv, v)$ with possible jumps on $\Gamma_- \cup \Gamma_+$, in fact has no jumps on these rays. Since both v and $\hat{w}$ solve the Boltzmann equation (1.1) in $X\times V$ and outside $X\times V$, and there is no jump at the boundary, we conclude that u satisfies (1.1) everywhere. Therefore, $u = w$, because the solution to (2.2) is unique. In particular (see (5.8)) we get $w = \hat{w}$ for $x \notin X$. $\square$

Theorem 1.3 is now an immediate consequence of Proposition 5.2 and Proposition 2.3.

Note that the proof of Theorem 1.3 provided above is not constructive. Below we will give explicit formulae for the reconstruction of σ_a, k which in particular provides another proof of Theorem 1.3. This proof is based on an analysis of the Schwartz kernel of the operator $\mathcal{A}$. A priori this kernel it is a distribution in $\mathcal{D}'(\mathbf{R}\times\Gamma_+\times\mathbf{R}\times\Gamma_-)$. Denote by δ_1 the Dirac delta function on $\mathbf{R}^1$ and by $\delta_y(x)$ the Delta function on ∂X defined by $(\delta_y, \varphi) = \varphi(y)$.

Theorem 5.1 *The Schwartz kernel of $\mathcal{A}$ has the form $\alpha(t-t', x, v, x', v')$, i.e. formally $(\mathcal{A}g)(t,x,v) = \int_{\mathbf{R}\times\Gamma_-} \alpha(t-t', x, v, x', v') g(t', x', v') dt' d\mu(x') dv'$ with $\alpha = \alpha_0 + \alpha_1 + \alpha_2$, where $\alpha_j(\tau, x, v, x', v')$ ($(x,v)\in\Gamma_+$, $(x',v')\in\Gamma_-$) satisfy*

$$\begin{aligned} \alpha_0 &= e^{-\int_0^{\tau_-(x,v)} \sigma_a(x-sv,v)ds} \delta_{\{x-\tau_-(x,v)v\}}(x')\delta(v-v')\delta_1(\tau-\tau_-(x,v)) \\ \alpha_1 &= \int e^{-\int_0^s \sigma_a(x-pv,v)dp} e^{-\int_0^{\tau_-(x-sv,v')} \sigma_a(x-sv-pv',v')dp} \delta_1(\tau - s - \tau_-(x-sv,v')) \\ &\qquad \times k(x-sv, v', v)\delta_{\{x-sv-\tau_-(x-sv,v')v'\}}(x')\theta(x-sv,x)ds \\ |n(x')\cdot v'|^{-1}\alpha_2 &\in L^\infty\Big(\Gamma_-;\, L^1_{\mathrm{loc}}(\mathbf{R}_\tau;\, L^1(\Gamma_+, d\xi))\Big). \end{aligned}$$

Proof. The proof is similar to that of Theorem 2.1. Fix $g \in C_c^\infty(\mathbf{R}\times\Gamma_-)$ and let u solve (1.7). Combining (5.6) with Duhamel's formula, we get $u = u_0 + u_1 + u_2$ with

$$\begin{aligned} u_0 &= G_-(t)g, \\ u_1 &= \int_0^\infty \tilde{U}_1(s) A_2 G_-(t-s) g\, ds \\ u_2 &= \int_{-\infty}^t \int_0^\infty \tilde{U}(t-s_2) A_2 \tilde{U}_1(s_1) A_2 G_-(s_2 - s_1) g\, ds_1\, ds_2. \end{aligned}$$

By (5.4),

$$R_+u_0 = \int_{\mathbf{R}\times\Gamma_-} \alpha_0(t-t',x,v,x',v')g(t',x',v')\,dt'\,d\mu(x')\,dv',$$

where the integral is to be considered in distribution sense. For u_1 we have by (5.3), (5.4),

$$\begin{aligned} u_1 &= \int_V\int_0^\infty e^{-\int_0^s \sigma_a(x-pv,v)dp} e^{-\int_0^{\tau_-(x-sv,v')}\sigma_a(x-sv-pv',v')dp}\theta(x-sv,x)k(x-sv,v',v) \\ &\quad \times g(t-s-\tau_-(x-sv,v'),x-sv-\tau_-(x-sv,v')v',v')\,ds\,dv', \end{aligned} \tag{5.11}$$

thus

$$R_+u_1 = \int_{\mathbf{R}\times\Gamma_-} \alpha_1(t-t',x,v,x',v')g(t',x',v')\,dt'\,d\mu(x')\,dv'.$$

Next,

$$u_2 = \int_{-\infty}^t \tilde{U}(t-s_2)A_2u_1(s_2,\,\cdot\,,\,\cdot\,)\,ds_2. \tag{5.12}$$

Using (5.11), we get

$$\begin{aligned} (A_2u_1)(s_2,x,v) &= \int_V\int_0^\infty\int_V E(s_1,x,v'',v')k(x,v'',v)k(x-s_1v'',v',v'')\theta(x,x-s_1v'') \\ &\quad \times g(s_2-s_1-\tau_-(x-s_1v'',v'),x-s_1v''-\tau_-(x-s_1v'',v')v',v')\,dv'ds_1dv'' \end{aligned}$$

with

$$E(s,x,v,v') = e^{-\int_0^s \sigma_a(x-pv,v)dp} e^{-\int_0^{\tau_-(x-sv,v')}\sigma_a(x-sv-pv',v')dp}.$$

Set $y' = x - s_1v''$. Then

$$\begin{aligned} (A_2u_1)(s_2,x,v) &= \int_0^\infty\int_V\int_X s_1^{-n}E(s_1,x,\frac{x-y'}{s_1},v')k(x,\frac{x-y'}{s_1},v)k(y',v',\frac{x-y'}{s_1})\theta(x,y') \\ &\quad \times g(s_2-s_1-\tau_-(y',v'),y'-\tau_-(y',v')v',v')\,dy'dv'ds_1 \end{aligned}$$

Let us make the change $y' \mapsto (x',x_1)$, where $x' = y' - \tau_-(y',v')v' \in \partial X$, $x_1 = \tau_-(y',v')$. This change is smooth except on a closed set of measure zero corresponding to y' such that the ray $\{y'-pv',\, p\in(0,\tau_-(y',v'))\}$ is tangent to ∂X at some point. One can first integrate outside a neighborhood of the singular set with measure $\varepsilon > 0$, where we have $dy' = |n(y')\cdot v'|d\mu(x')dx_1$, and then let $\varepsilon \to 0$. Thus we get

$$\begin{aligned} &(A_2u_1)(s_2,x,v) \\ &\quad = \int_0^\infty\int\int_{\Gamma_-} s_1^{-n}\theta(x,x'+x_1v')E(s_1,x,\frac{x-x'-x_1v'}{s_1},v')k(x,\frac{x-x'-x_1v'}{s_1},v) \\ &\quad\quad \times k(x'+x_1v',v',\frac{x-x'-x_1v'}{s_1})g(s_2-s_1-x_1,x',v')\,d\xi'\,dx_1\,ds_1. \end{aligned} \tag{5.13}$$

Here $d\xi' := |n(x')\cdot v'|d\mu(x')dv'$. Denote

$$M(s_1,\tilde{s}_2,x,v,x',v') = s_1^{-n}E(s_1,x,v'',v')k(x,v'',v)k(x'+s_1v',v',v'')\theta(x,x'+(\tilde{s}_2-s_1)v'),$$

where we have set $v'' = (x - x' - (\tilde{s}_2 - s_1)v')/s_1$. It is easy to see that

$$\int_V \int_{\mathbf{R}^n} M(s_1, \tilde{s}_2, x, v, x', v') dx\, dv \le \|\sigma_p\|^2_{L^\infty}. \tag{5.14}$$

By (5.12) and (5.13) we have

$$\begin{aligned} u_2 &= \int_{-\infty}^{t} \int_0^\infty \int_{\Gamma_-} \int \tilde{U}(t - s_2) M(s_1, s_2 - t', \cdot, \cdot, x', v') g(t', x', v') dt'\, d\xi'\, ds_1\, ds_2 \\ &= \int_{\Gamma_-} \int \tilde{\alpha}_2(t - t', x, v, x', v') g(t', x', v') dt'\, d\mu(x') dv' \end{aligned}$$

with

$$\tilde{\alpha}_2(\tau, x, v, x', v') = |n(x') \cdot v'| \int_{-\infty}^{\tau} \int_0^\infty \tilde{U}(\tau - \tilde{s}_2) M(s_1, \tilde{s}_2, \cdot, \cdot, x', v') ds_1 d\tilde{s}_2. \tag{5.15}$$

By (5.14),

$$|n(x') \cdot v'|^{-1} \tilde{\alpha}_2 \in C\left(\mathbf{R}_\tau;\, L^\infty(\Gamma_-;\, L^1(X_x \times V_v))\right). \tag{5.16}$$

By (5.16) and the remark after the proof of Proposition 5.1 we obtain

$$|n(x') \cdot v'|^{-1} \int_{-a}^{a} \|R_+ \tilde{\alpha}_2(\tau, \cdot, \cdot, x', v')\|_{L^1(\Gamma_+, d\xi)} d\tau \le C(a)$$

for any $a > 0$ and all $(x', v') \in \Gamma_-$. Setting $\alpha(\tau, \cdot, \cdot, x', v') = R_+ \tilde{\alpha}(\tau, \cdot, \cdot, x', v')$ we complete the proof of the theorem. □

We proceed with explicit formulae relating the kernel of $\mathcal{A}$ and σ_a, k. Next two propositions are analogues of Proposition 3.1 and Proposition 3.2. Recall again that for each v the set of x, for which $\tau_-(x, v)$ has a jump, is of measure zero. Thus the function τ_- is smooth on Γ_+ outside of a closed set of measure zero. Choose $0 \le \chi \in C_c^\infty(\Gamma_+)$ supported outside that singular set. Let φ, φ_1 be as in (3.9) and for $\varepsilon > 0$ sufficiently small set

$$\phi_\varepsilon(\tau, x, v, x', v') = \varphi\left(\frac{x - \tau_-(x, v)v - x'}{\varepsilon}\right) \varphi\left(\frac{v - v'}{\varepsilon}\right) \varphi_1(\tau - \tau_-(x, v)) \chi(x, v).$$

Proposition 5.3 *With ϕ_ε as above we have*

$$\lim_{\varepsilon \to 0} \int_{\Gamma_+} \int \alpha(\tau, x, v, x', v') \phi_\varepsilon(\tau, x, v, x', v')\, d\tau\, d\mu(x') dv' = e^{-\int_0^{\tau_-(x,v)} \sigma_a(x - sv, v) ds} \chi(x, v) \tag{5.17}$$

in $L^1(\Gamma_+, d\xi)$, where the integral is to be considered in distribution sense.

Proof. We have $\alpha = \alpha_0 + \alpha_1 + \alpha_2$ with α_j as in Theorem 5.1. It is clear that α_0 satisfies (5.17). For α_1 we have (compare with (3.3))

$$0 \le \int_{\Gamma_+} \int_{\Gamma_-} \int \alpha_1(\tau, x, v, x', v') \phi_\varepsilon(\tau, x, v, x', v')\, d\tau\, d\mu(x') dv'\, d\xi$$

$$
\begin{aligned}
&\le \int_{\Gamma_+}\int_V\int \varphi\Big(\frac{sv+\tau_-(x-sv,v')v'-\tau_-(x,v)v}{\varepsilon}\Big)\varphi\Big(\frac{v-v'}{\varepsilon}\Big)\\
&\qquad \times\varphi_1(s+\tau_-(x-sv,v')-\tau_-(x,v))k(x-sv,v',v)\chi(x,v)\,ds\,dv'\,d\xi\\
&\le C\int_V\int_{\Gamma_+}\int \chi(x,v)\varphi\Big(\frac{v-v'}{\varepsilon}\Big)k(x-sv,v',v)\,ds\,|n(x)\cdot v|d\mu(x)dv\,dv'\\
&\le C'\int_W\int_V\int_X \varphi\Big(\frac{v-v'}{\varepsilon}\Big)k(x,v',v)\,dx\,dv'\,dv\\
&\to 0, \quad \text{as } \varepsilon\to 0, \qquad (5.18)
\end{aligned}
$$

by the same arguments as in the proof of Proposition 3.1. Finally,

$$
\begin{aligned}
&\int_{\Gamma_+}\Big|\int_{\Gamma_-}\int \alpha_2(\tau,x,v,x',v')\phi_\varepsilon(\tau,x,v,x',v')\,d\tau\,d\mu(x')dv'\Big|d\xi\\
&\qquad\le \int_{\Gamma_+}\int_{\Gamma_-}\int |n(x')\cdot v'|^{-1}|\alpha_2(\tau,x,v,x',v')|\phi_\varepsilon(\tau,x,v,x',v')\,d\tau\,d\xi'\,d\xi\\
&\qquad\le \int_{E_\varepsilon}|n(x')\cdot v'|^{-1}|\alpha_2(\tau,x,v,x',v')|\,d\tau\,d\xi'\,d\xi\\
&\qquad\to 0, \quad \text{as } \varepsilon\to 0, \qquad (5.19)
\end{aligned}
$$

where $E_\varepsilon=\{(\tau,x,v,x',v')\in\mathbf{R}\times\Gamma_+\times\Gamma_-;\ |\tau|\le A,\ (x,v)\in\operatorname{supp}\chi,\ |v-v'|\le\varepsilon\}$ with some $A=A(\chi,\varphi_1)>0$. Clearly, $\operatorname{meas}(E_\varepsilon)\to 0$, as $\varepsilon\to 0$, where $\operatorname{meas}(E_\varepsilon)$ is associated with $d\tau\,d\xi'\,d\xi$. On the other hand, by Theorem 5.1 the integrand in the last integral is a L^1-function. As before, we conclude from this that the limit in (5.19) is zero, as stated. Combining (5.18), (5.19) we complete the proof. $\square$

By Proposition 5.3 one can recover the X-ray transform of $\sigma_a(x)$, provided that σ_a is independent of v and therefore one can recover σ_a itself.

We proceed with the recovery of k. Next proposition is an analogue of Proposition 3.2. Let $\psi_{\varepsilon_1,\varepsilon_2}$ and $\mathcal{W}$ be the same as in Proposition 3.2.

Proposition 5.4 *We have*

$$
\begin{aligned}
&\lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0} G_+(0)\int_{\mathbf{R}}\int_{\partial X}\alpha(t-t',x,v,x',v')\psi_{\varepsilon_1,\varepsilon_2}(x-tv,v,x'-t'v',v')\,d\mu(x')\,dt'\\
&\qquad = e^{-\int_0^{\tau_-(x,v')}\sigma_a(x-pv',v')dp}k(x,v',v), \qquad (5.20)
\end{aligned}
$$

where the integral is to be considered in distribution sense and the limit holds in $L^1_{\rm loc}(X\times\mathcal{W})$.

Remark. The restriction of $\psi_{\varepsilon_1,\varepsilon_2}(x-tv,v,x'-t'v',v')$ on $\mathbf{R}_t\times\Gamma_+\times\mathbf{R}_{t'}\times\Gamma_-\setminus\{v=v'\}$ is not necessary a function of compact support on that variety, but as will be seen from the proof of Proposition 5.4, the formal integral above is well defined. Operator $G_+(0)$ above (see (5.4)) is applied to the formal integral considered as a function of t, x, v.

Proof. Note first that for $v\ne v'$ we have $\alpha_0=0$. Next, for α_1 we get

$$
\begin{aligned}
&\int_{\mathbf{R}}\int_{\partial X}\alpha_1(t-t',x,v,x',v')\psi_{\varepsilon_1,\varepsilon_2}(x-tv,v,x'-t'v',v')\,d\mu(x')\,dt'\\
&\qquad = \int E(s,x,v',v)k(x-sv,v',v)\theta(x-sv,x)\frac{1}{\varepsilon_1}\varphi_1\Big(\frac{t-s}{\varepsilon_1}\Big)\,ds,
\end{aligned}
$$

where $E(s,x,v',v) = e^{-\int_0^s \sigma_a(x-pv,v)dp} e^{-\int_0^{\tau_-(x-sv,v')} \sigma_a(x-sv-pv',v')dp}$. Function $s \to E(s,x,v',v)$ $k(x-sv,v',v)\theta(x-sv,x)$ is integrable with values in $L^1(\Gamma_+ \times V_{v'},\, d\xi\, dv')$. Therefore, as $\varepsilon_1 \to 0$, the limit above exists in $L^1(\mathbf{R}_s \times \Gamma_+ \times V_{v'},\, ds\, d\xi\, dv')$ and we have

$$\begin{aligned} \lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0} \int_{\mathbf{R}}\int_{\partial X} \alpha_1(t-t',x,v,x',v')\psi_{\varepsilon_1,\varepsilon_2}(x-tv,v,x'-t'v',v')\, d\mu(x')\, dt' \\ = \quad E(t,x,v',v)k(x-tv,v',v)\theta(x-tv,x). \end{aligned} \tag{5.21}$$

By applying $G_+(0)$ to both sides of the equality above we get that (5.20) holds with $\alpha = \alpha_1$.

Finally, let us fix a compact set $K \subset \Gamma_+ \times V_{v'}$ that does not intersect the varieties $v = v'$, $v' = 0$. Then for any $a > 0$ we have

$$\begin{aligned} \int_{-a}^{a}\int_K \left| \int_{\mathbf{R}}\int_{\partial X} \alpha_2(t-t',x,v,x',v')\psi_{\varepsilon_1,\varepsilon_2}(x-tv,v,x'-t'v',v')\, d\mu(x')\, dt' \right| d\xi\, dv'\, dt \\ \le \quad \int_{-a}^{a}\int_{F_{\varepsilon_2}} |n(x')\cdot v'|^{-1}|\alpha_2(t-t',x,v,x',v')|\, dt'\, d\xi\, d\xi'\, dt, \end{aligned} \tag{5.22}$$

with F_{ε_2} a set of measure tending to 0, as $\varepsilon_2 \to 0$ (compare with (3.13)). By performing the change $\tau = t - t'$ and using Theorem 5.1, we see that (5.22) tends to 0, as $\varepsilon_2 \to 0$. Therefore, (5.20) with $\alpha = \alpha_2$ converges to 0 in $L^1_{\rm loc}(\mathbf{R}_+;\, L^1(K))$. A straightforward generalization of (5.5) implies that $G_+(0) : L^1_{\rm loc}(\mathbf{R} \times \Gamma_+,\, dt\, d\xi) \to L^1_{\rm loc}(X \times V)$ is continuous. Thus, applying $G_+(0)$ we get from (5.22)

$$\lim_{\varepsilon_2\to 0} G_+(0) \int_{\mathbf{R}}\int_{\partial X} \alpha(t-t',x,v,x',v')\psi_{\varepsilon_1,\varepsilon_2}(x-tv,v,x'-t'v',v')\, d\mu(x')\, dt' = 0 \tag{5.23}$$

in $L^1_{\rm loc}(X \times \mathcal{W})$. Combining (5.21) and (5.23), we complete the proof of Proposition 5.4. □

Now the reconstruction of σ_a and k goes along the following lines. Given the albedo operator $\mathcal{A}$, we first recover $\sigma_a(x)$ (provided that σ_a depends on x only) by using Proposition 5.3. Next, since σ_a is already known, we know $G_+(0)$ and the exponential factor in (5.20), so by Proposition 5.4 we can recover $k(x,v',v)$ almost everywhere in $X \times V \times V$.

Finally, we note that one can rewrite Propositions 5.3, 5.4 in terms of the operator $\mathcal{A}$ rather than in terms of its distribution kernel in a manner similar to that at the end of Section 3.

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