Microlocal Analysis and Integral Geometry
(working title)

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March 29, 2018

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CHAPTER I

Preliminaries

This draft represents the current state of a book project by Plamen Stefanov and Gunther Uhlmann. The book is intended to be accessible to beginning graduate students. We intend to demonstrate the power of microlocal methods in Integral Geometry, through the geodesic X-ray transform of functions and tensor fields. At the beginning we introduce the reader to the Euclidean X-ray and Radon transforms not because we think the world needs another exposition of this type besides the classical books by Helgason and Natterer but because on microlocal (principal symbol) level, the analysis starts to look Euclidean; and knowing well the Euclidean theory helps to understand the general case.

[...]

A few words about the notation and the conventions used. When we say property A holds "near x" (or "near the set K"), we mean that there exists an open set $U \ni x$, or an open set $U \supset K$, respectively, such that property A holds there. It is equivalent to saying "in a neighborhood of". The expression

$$\|Af\|_1 \le C\|f\|_2$$

means that there exists a constant $C > 0$, independent of f, of course, so that this inequality holds for all f with finite $\|\cdot\|_2$ norm, i.e., in the space defined by that norm. We often prove estimates of this sort for f in some dense subspace, like C_0^∞ or $\mathcal{S}$; then they can be extended by continuity.

CHAPTER II

Basic properties of the X-ray transform and the Radon transform in the Euclidean space

1. Definition, the Fourier Slice Theorem

1.1. The X-ray transform.

1.1.1. *Definition.* We define the X-ray transform of a function f in $\mathbf{R}^n$, as a map that associates to f its integral, assuming that it exists,

$$Xf(\ell) = \int_\ell f \, \mathrm{d}s \tag{1.1}$$

along any given (undirected) line ℓ in $\mathbf{R}^n$. Here $\mathrm{d}s$ is the unit length measure on ℓ. Lines in $\mathbf{R}^n$ can be parameterized by initial points $x \in \mathbf{R}^n$ and directions $\theta \in S^{n-1}$, thus we can write, without changing the notation,

$$Xf(x,\theta) = \int_{\mathbf{R}} f(x+s\theta)\, \mathrm{d}s, \quad (x,\theta) \in \mathbf{R}^n \times S^{n-1}. \tag{1.2}$$

That parameterization is not unique because for any x, θ, t,

$$Xf(x,\theta) = Xf(x+t\theta,\theta), \quad Xf(x,\theta) = Xf(x,-\theta). \tag{1.3}$$

The latter identity reflects the fact that we consider the lines as undirected ones.

The Fubini Theorem allows us to define Xf for any $f \in L^1(\mathbf{R}^n)$, see also Proposition 1.3 below. In Section 1.3, we will extend the definition to distributions in $\mathcal{E}'(\mathbf{R}^n)$.

We will count the number of variables that we used to parameterize Xf. For any θ, it is enough to restrict x to a hyperplane perpendicular to θ, that takes away one dimension. One such hyperplane is

$$\theta^\perp := \{x|\ x\cdot\theta = 0\}. \tag{1.4}$$

Then $Xf(x,\theta)$ is an even (w.r.t. θ) function of $2n-2$ variables, while f depends on n variables. Therefore, if $n=2$, Xf and f depends on the same number of variables, 2. We say that the problem of inverting X is then a *formally determined* problem. If $n \geq 3$, then Xf depends on more variables, making the problem *formally overdetermined*. On the other hand, in dimensions $n \geq 3$, if we know $Xf(\ell)$ for all lines, we also know $Xf(\ell)$ for the n-dimensional family of lines that consists of all ℓ parallel to a fixed 2-dimensional plane, say the one spanned by $(1,0,\dots,0)$ and $(0,1,0,\dots,0)$. It is then enough to solve the 2-dimensional problem of inverting R on each such plane. This is one way one can reduce the problem of inverting X to a formally determined one (that we can solve, as we will see later) using partial data. For this reason, very often the X-ray transform in analyzed in two dimensions only.

1.1.2. *Motivation.*

X-ray Computed Tomography (CT). A motivating example is X-ray medical imaging. An X-ray source is placed at different positions around patient's body, and the intensity I of the rays is measured after the rays go through the body. The intensity depends on the position x and the direction θ of the rays. It solves the transport equation

$$(\theta \cdot \nabla_x + \sigma(x))\, I(x,\theta) = 0, \tag{1.5}$$

where σ is the absorption of the body. Equation (1.5) simply says that the directional derivative of I in the direction θ equals $-\sigma I$. A natural initial/boundary condition is to require that

$$\lim_{s\to-\infty} I(x+s\theta,\theta) = I_0,$$

where I_0 is the source intensity, that may depend on the line. Since f is of compact support in this case, the limit above is trivial. Then (1.5) has the explicit solution

$$I(x,\theta) = e^{-\int_{-\infty}^0 \sigma(x+s\theta)\,\mathrm{d}s} I_0.$$

The measurement outside patent's body is modeled by

$$\lim_{s\to\infty} I(x+s\theta,\theta) =: I_1,$$

and this limit is trivial as well. Since both I_1 and I_0 are known, we may form the quantity

$$-\log(I_1/I_0) = \int_{-\infty}^{\infty} \sigma(x+s\theta)\,\mathrm{d}s \tag{1.6}$$

That is exactly $X\sigma(x,\theta)$. The problem then reduces to recovery of σ given $X\sigma$.

One may think if I/I_0 as the scattering operator for (1.5). Then finding f from I/I_0 is an inverse scattering problem.

Relation to the transport equation and Single-Photon Emission Computed Tomography (SPECT). We already saw that X is related to the transport equation (1.5); then Xf is given by (1.6). There is another, more direct connection. Since in (1.5) we took a logarithm, we can set $u = -\log I$ (after the normalization $I_0 = 1$) and plug in $I = e^{-u}$ in (1.5) to get $\theta\cdot\nabla_x u = \sigma$ with the "initial condition" $u = 0$ for $x\cdot\theta \ll 0$. Replacing σ by the more conventional notation f, we get the following transport equation for u with a source term f

$$\theta\cdot\nabla_x u = f, \quad u|_{x\cdot\theta\ll 0} = 0, \tag{1.7}$$

with f compactly supported. If $\operatorname{supp} f \subset B(0,R)$, then the initial condition above is equivalent to $u = 0$ on the plane $x\cdot\theta = -R$. Then

$$Xf = u|_{x\cdot\theta\gg 0}, \tag{1.8}$$

which can also be written as $Xf = u|_{x\cdot\theta=R}$. The reason we can replace the "$\ll$" and the "$\gg$" conditions with the ones on the planes $x\cdot = \pm R$ is that the transport equation is simply the ODE $\mathrm{d}u/\mathrm{d}t = f$ on every line $t\mapsto x = x_0 + t\theta$; and once that line leaves the support of f, the function u is just a constant there.

The transport equation (1.7) is the model of SPECT. This is a medical imaging technique based on delivering radioisotopes to a patient's body, for example by injecting them into the blood stream. The radioisotopes emit gamma rays which are detected outside the body. If the concentration of the radioisotopes is modeled by a function $f(x)$, then we have the problem of determining a source from external measurements, and the transport equation (1.7) is a good model for the radiance at a given point x and direction θ. Then Xf models the measurements. One can include attenuation in the model, see .... ref?

1.2. The Radon Transform. The Radon transform Rf of a function f is defined as integrals of f over all hyperplanes π in $\mathbf{R}^n$:

$$Rf(\pi) = \int_\pi f \, \mathrm{d}S. \tag{1.9}$$

Here $\mathrm{d}S$ is the Euclidean surface measure on each such hyperplane. The transform R is well defined on $L^1(\mathbf{R}^n)$, see also Problem 1.5 below. Each such hyperplane can be written in exactly two different ways in the form

$$\pi = \{x|\ x\cdot\omega = p\} = \{x|\ x\cdot(-\omega) = -p\}$$

with $p \in \mathbf{R}$, $\omega \in S^{n-1}$. We then write

$$Rf(p,\omega) = \int_{x\cdot\omega=p} f \, \mathrm{d}S_x. \tag{1.10}$$

Then Rf is an even function on $\mathbf{R}\times S^{n-1}$.

If we consider the hyperplanes in $\mathbf{R}^n$ as oriented ones, then (p,ω) and $(-p,-\omega)$, with $\pm\omega$ reflecting the choice of the orientation, correspond to different planes.

The problem of finding f given Rf is always a formally determined one since both f and Rf are functions of n variables.

More generally, one can define a Radon transform of f, sometimes called a generalized Radon transform, as integrals of f over all k-dimensional linear subspaces with the natural Euclidean measure on each one of them. Then $k=1$ corresponds to the X-ray transform; $k=n-1$ corresponds to the Radon transform defined in (1.9).

In $\mathbf{R}^2$, the two transforms coincide. Indeed, every hyperplane in $\mathbf{R}^2$ is a line; and the induced Euclidean measure is the arc-length one.

1.3. The transpose X' and extending X to $\mathcal{E}'(\mathbf{R}^n)$.

1.3.1. *Two ways to parameterize the lines through a domain.* Since (x,θ) and $(x+s\theta,\theta)$ define the same line, we will will choose a parameterization of Xf as follows. For any $\theta\in S^{n-1}$, we restrict x to $\theta^\perp$, see (1.4). Then we set

$$\Sigma = \left\{(z,\theta)|\ \theta\in S^{n-1},\ z\in\theta^\perp\right\}.$$

We can think about Σ as the tangent bundle TS^{n-1} without the transformation laws under coordinate changes. In particular, this defines a differentiable structure on Σ. Locally, Σ is diffeomorphic to $\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}$, and local coordinate charts can be constructed by projecting $\theta^\perp$ to $\theta_0^\perp$ for θ in some neighborhood of a fixed θ_0; and by choosing a local chart on S^{n-1} near θ_0. We define a measure $\mathrm{d}\sigma$ on Σ by setting

$$\mathrm{d}\sigma(z,\theta) = \mathrm{d}S_z\,\mathrm{d}\theta,$$

where, somewhat incorrectly, $\mathrm{d}\theta$ denotes the standard measure on S^{n-1}, and $\mathrm{d}S_z$ is the Euclidean measure on the hyperplane $\theta^\perp$. Note that in this parameterization, each directed line has unique coordinates but each undirected one has two pairs of coordinates.

We introduce now another parameterization of Xf that is more convenient when considering later integrals over geodesics or more general curves. Let us assume that we will apply X only to functions supported in some bounded domain Ω with a strictly convex smooth boundary. The strict convexity assumption is not restrictive since we can always enlarge the domain to a strictly convex one, for example a ball, that contains the domain of interest. Set

$$\partial_\pm S\Omega = \left\{(x,\theta)\in\partial\Omega\times S^{n-1}|\ \pm\nu(x)\cdot\theta\geq 0\right\}, \tag{1.11}$$

where ν is the exterior unit normal to $\partial\Omega$. On $\partial_\pm S\Omega$, define the measure

$$\mathrm{d}\mu(x,\theta) = |\nu(x)\cdot\theta|\,\mathrm{d}S_x\,\mathrm{d}\theta, \tag{1.12}$$

where $\mathrm{d}S_x$ is the surface measure on $\partial\Omega$. There is a natural map

$$\partial_\pm S\Omega \ni (x,\theta)\longmapsto (z,\theta)\in\Sigma, \tag{1.13}$$

where z is the intersection of the ray $\{x+s\theta|\ s\in\mathbf{R}\}$ with $\theta^\perp$, see Figure II.1. The map (1.13) is invertible on its range. Given (z,θ), x is the intersection of the ray $\{z+s\theta|\ s\in\mathbf{R}\}$ with $\partial\Omega$ having the property that at x, the vector θ points into Ω; when we have the negative sign in (1.13); and θ points away from Ω otherwise.

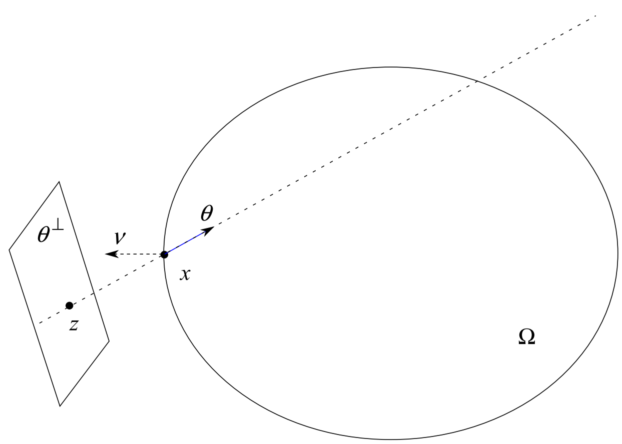

FIGURE II.1. Two ways to parameterize a line

PROPOSITION 1.1. *The maps (1.13) and their inverses are isometries.*

PROOF. The proof is immediate. Fix θ, and project locally $\partial\Omega$ on $\theta^\perp$, in the direction of θ, near some point x so that $(x,\theta)\in\partial_\pm S\Omega$. The Jacobian of that projection is $1/|\nu(x)\cdot\theta|$. $\square$

COROLLARY 1.2. *Let $\Omega_0\subset\mathbf{R}^n$ be a bounded domain and let $\Omega_{1,2}\supset\Omega_0$ be two other domains with strictly convex boundaries. For each line ℓ through Ω_0, let $(x_{1,2},\theta)\in\partial_+S\Omega_{1,2}$ be the corresponding coordinates of ℓ defined as above. Then the map $(x_1,\theta)\mapsto(x_2,\theta)$ is an invertible isometry between the subsets of $(\partial_+S\Omega_1,\ \mathrm{d}\mu_1)$ and $(\partial_+S\Omega_2,\ \mathrm{d}\mu_2)$ corresponding to the lines through Ω_0. The same statement holds for $\partial_+S\Omega_{1,2}$.*

Here, $\mathrm{d}\mu_{1,2}$ are the measures defined un (1.12) for each Ω_i, $i = 1, 2$. The corollary says that it does not matter how we parameterize the lines through Ω_0 using the second approach above; each choice of a strictly convex domain larger (or equal) to Ω_0 gives as "the same" parameterization. The proof follows directly from Proposition 1.1 since the map $(z_1, \theta) \mapsto (z_2, \theta)$ is simply a composition of the map (1.13) for one of the domains with the inverse for the other one.

1.3.2. *Relation to the transport equation, revisited.* The relation to the transport equation (1.7) in the new parameterization is the following. Let u solve

$$\theta \cdot \nabla_x u = f, \quad u|_{\partial_- S\Omega} = 0. \tag{1.14}$$

Then

$$Xf = u|_{\partial_+ S\Omega}. \tag{1.15}$$

This makes Xf a function on $\partial_+ S\Omega$. And alternative definition is to define $\tilde{u}$ as the solution of

$$\theta \cdot \nabla_x \tilde{u} = f, \quad u|_{\partial_+ S\Omega} = 0. \tag{1.16}$$

Then

$$Xf = -u|_{\partial_- S\Omega}. \tag{1.17}$$

Definition (1.14), (1.15) is more intuitive since Xf can be interpreted as the observed signal exiting Ω due to the source f. On the othet hand, parameterizing geodesics with incoming points abd directions might be considered more natural; then to write Xf as an integral, we integrate for postive value of the arc-length parameter. We will use both.

1.3.3. *Extension of X to larger classes of functions or distributions.* We already indicated that Xf can be defined for any $f \in L^1(\mathbf{R}^n)$, and the next proposition show that X is actually a bounded map there.

PROPOSITION 1.3. *The operator $X : C_0^\infty(\mathbf{R}^n) \to C_0^\infty(\Sigma)$ extends to a bounded map*

$$X : L^1(\mathbf{R}^n) \to L^1(\Sigma, \mathrm{d}\sigma)$$

with norm $|S^{n-1}|$.

PROOF.

$$\begin{aligned} \|Xf\|_{L^1(\Sigma, \mathrm{d}\sigma)} &= \int_{S^{n-1}} \int_{\theta^\perp} \left| \int_{\mathbf{R}} f(z + s\theta)\, \mathrm{d}s \right| \mathrm{d}S_z \, \mathrm{d}\theta \\ &\le \int_{S^{n-1}} \int_{\theta^\perp} \int_{\mathbf{R}} |f(z + s\theta)|\, \mathrm{d}s \, \mathrm{d}S_z \, \mathrm{d}\theta \\ &= |S^{n-1}| \|f\|_{L^1(\mathbf{R}^n)}. \end{aligned}$$

It remains to notice that this is an equality if $f \ge 0$. □

PROBLEM 1.1. Let $\Omega \subset \mathbf{R}^n$ be as above. Using Proposition 1.3, prove that X extends to a bounded map

$$X : L^1(\Omega) \to L^1(\partial_- S\Omega, \mathrm{d}\mu),$$

and also to a bounded map

$$X : L^2(\Omega) \to L^2(\partial_- S\Omega, \mathrm{d}\mu). \tag{1.18}$$

As a first step towards extending X to distributions, let us view now X as the map

$$X : C_0^\infty(\mathbf{R}^n) \longrightarrow C_0^\infty(\Sigma) \tag{1.19}$$

or

$$X : C_0^\infty(\Omega) \longrightarrow C_0^\infty(\partial_\pm S\Omega) \tag{1.20}$$

in case of functions with support in a fixed domain Ω.

Problem 1.2. Prove that the maps (1.19), (1.20) are linear and continuous, see section A.4.

We will compute now the transpose X' of X in (1.19) with respect to the measure $\mathrm{d}\sigma$. Let $\phi \in C_0^\infty(\mathbf{R}^n)$, $\psi \in C^\infty(\Sigma)$. We have

$$\int_\Sigma (X\phi)\psi\,\mathrm{d}\sigma = \int_\Sigma \int_{\mathbf{R}} \phi(z + s\theta)\,\psi(z,\theta)\,\mathrm{d}s\,\mathrm{d}S_z\,\mathrm{d}\theta. \tag{1.21}$$

Set $x = z + s\theta$, where $z \in \theta^\perp$. For a fixed $\theta \in S^{n-1}$, $(z, s) \mapsto x$ is an isomorphism with a Jacobian equal to 1. The inverse is given by

$$z = x - (x\cdot\theta)\theta, \quad s = x \cdot \theta.$$

We therefore have

$$\int_\Sigma (X\phi)\psi\,\mathrm{d}\sigma = \int_{S^{n-1}} \int_{\mathbf{R}^n} \phi(x)\,\psi(x - (x\cdot\theta)\theta, \theta)\,\mathrm{d}x\,\mathrm{d}\theta.$$

Therefore, for $\psi \in C^\infty(\Sigma)$,

$$X'\psi(x) = \int_{S^{n-1}} \psi(x - (x\cdot\theta)\theta, \theta)\,\mathrm{d}\theta. \tag{1.22}$$

We can interpret this formula in the following way. The function ψ is a function on the set (that we made a manifold) of lines. Given $x \in \mathbf{R}^n$, for any $\theta \in S^{n-1}$ we evaluate ψ on the line through x in the direction of θ, and then integrate over θ. In other words, $X'\psi(x)$ is an integral of $\psi = \psi(\ell)$ over all lines ℓ through x

$$X'\psi(x) = \int_{\ell \ni x} \psi(\ell)\,\mathrm{d}\ell_x,$$

where $\mathrm{d}\ell_x$ is the unique measure on $\{\ell \ni x\}$ that is invariant under orthogonal transformations, with total measure $|S^{n-1}|$, i.e., $\mathrm{d}\ell_x = \mathrm{d}\theta$ in the parameterization that we use. Compare this to (1.1) which can also be written in the form

$$Xf(\ell) = \int_{x\in\ell} f(x)\,\mathrm{d}s \tag{1.23}$$

The transform X' is often called a backprojection — it takes a function defined on lines to a function defined on the "x-space" $\mathbf{R}^n$.

Problem 1.3. Show that

$$X' : C_0^\infty(\Sigma) \longrightarrow C^\infty(\mathbf{R}^n) \tag{1.24}$$

is continuous, see Definition A.4.2 but $X'\psi$ may not be of compact support, if ψ is. In other words, X' does not satisfy the assumptions of Definition A.4.1.

This makes it impossible to define X on $\mathcal{D}'(\mathbf{R}^n)$, as it could be expected (even for smooth functions we need a certain decay at infinity), but we can define it on the space $\mathcal{E}'(\mathbf{R}^n)$ of compactly supported distributions, see section A.4.

DEFINITION 1.4. *Let $f \in \mathcal{E}'(\mathbf{R}^n)$. Then we define $Xf \in \mathcal{D}'(\Sigma)$ as the linear functional*

$$\langle Xf, \psi \rangle = \langle f, X'\psi \rangle, \quad \forall \psi \in C^\infty(\Sigma). \tag{1.25}$$

By the results in section A.4, $X' : \mathcal{E}'(\mathbf{R}^n) \to \mathcal{D}'(\Sigma)$ is correctly defined and is sequentially continuous. In fact, $X' : \mathcal{E}'(\mathbf{R}^n) \to \mathcal{E}'(\Sigma)$, that also follows from the fact that one can replace $C_0^\infty(\Sigma)$ in (1.24) by $C^\infty(\Sigma)$.

It is worth noticing that we computed X' with respect to the measures $\mathrm{d}\sigma$ on Σ on the left in (1.25); and with respect to the standard measure $\mathrm{d}x$ in $\mathbf{R}^n$ on the right. Therefore, if Xf is locally L^1, we have

$$\langle Xf, \psi \rangle = \int_\Sigma (Xf)\psi \, \mathrm{d}\sigma, \tag{1.26}$$

and if $X'\psi$ is locally L^1, we have

$$\langle f, X'\psi \rangle = \int_{\mathbf{R}^n} f X'\psi \, \mathrm{d}x.$$

This gives us a way to identify locally $L^1(\Sigma)$ functions with distributions in $\mathcal{E}'(\Sigma)$.

We can also use (1.25) and the property that X maps continuously $C_0^\infty(\mathbf{R}^n)$ into $C_0^\infty(\Sigma)$, see Problem 1.2, (and in particular preserves the compactness of the support), to define $X'g \in \mathcal{D}'(\mathbf{R}^n)$ for any $g \in \mathcal{D}'(\Sigma)$.

DEFINITION 1.5. *Let $g \in \mathcal{D}'(\Sigma)$. Then $X'g \in \mathcal{D}'(\mathbf{R}^n)$ is defined by*

$$\langle X'g, \phi \rangle = \langle g, X\phi \rangle, \quad \forall \phi \in C^\infty(\mathbf{R}^n). \tag{1.27}$$

By the results in section A.4, $X' : \mathcal{D}'(\Sigma) \to \mathcal{D}'(\mathbf{R}^n)$ is correctly defined and is sequentially continuous.

Let us assume now that Xf, for f supported in Ω, is parameterized by points in $\partial_- S\Omega$, as above, i.e., we view X as the map (1.20). Then we choose the pairing according to the measure $\mathrm{d}\mu$, i.e., for Xf locally in L^1,

$$\langle Xf, \psi \rangle = \int_{\partial_- S\Omega} (Xf)\psi \, \mathrm{d}\mu. \tag{1.28}$$

By Proposition 1.1, the right-hand sides of (1.26) and (1.28) are the same, therefore $\langle \cdot, \cdot \rangle$ does not change. Also, this defines $Xf \in \mathcal{E}'(\partial_- S\Omega)$ for any $f \in \mathcal{E}'(\Omega)$ by (1.25). Note that the pairing (1.28) yields "essentially the same" transpose X' as before, i.e., the transpose under (1.28) composed with the isometry (1.13), equals X'. The two operators are actually two different parameterizations of X' defined by duality. The operator $X'X$ that we consider later, will remain the the same if we consider X' given by (1.28), by Proposition 1.1.

By (1.18), the adjoint X^* is well defined and on $L^2(\partial_- S\Omega, \mathrm{d}\mu)$, we have $X^* = X'$.

1.4. The transpose R' and extending R to $\mathcal{E}'(\mathbf{R}^n)$. Formula (1.10) defines R as an operator

$$R : C_0^\infty(\mathbf{R}^n) \longrightarrow C_0^\infty(\mathbf{R} \times S^{n-1}). \tag{1.29}$$

We fix the standard measure $\mathrm{d}s \, \mathrm{d}\omega$ on $\mathbf{R} \times S^{n-1}$.

PROBLEM 1.4. Show that R in (1.29) is linear and continuous.

PROBLEM 1.5. Show that R extends to a bounded operator
$$R : L^1(\mathbf{R}^n) \to L^1(\mathbf{R} \times S^{n-1}).$$

We will compute now the transpose R' of R in (1.10). Let $\phi \in C_0^\infty(\mathbf{R}^n)$, $\psi \in C_0^\infty(\mathbf{R} \times S^{n-1})$. We have
$$\int_{\mathbf{R}\times S^{n-1}} (R\phi)\psi \,\mathrm{d}p\,\mathrm{d}\omega = \int_{\mathbf{R}\times S^{n-1}} \int_{x\cdot\omega=p} \phi(x)\,\psi(p,\omega)\,\mathrm{d}S_x\,\mathrm{d}p\,\mathrm{d}\omega.$$
For a fixed ω, $\int_{\mathbf{R}} \int_{x\cdot\omega=p} f\,\mathrm{d}S_x\,\mathrm{d}p$ is just an integral of f over $\mathbf{R}^n$, by Fubini's theorem. Therefore,
$$\int_{\mathbf{R}\times S^{n-1}} (R\phi)\psi \,\mathrm{d}s\,\mathrm{d}\omega = \int_{S^{n-1}} \int_{\mathbf{R}^n} \phi(x)\psi(x\cdot\omega,\omega)\,\mathrm{d}x\,\mathrm{d}\omega.$$
So we get for $\psi \in C_0^\infty(\mathbf{R} \times S^{n-1})$,

$$R'\psi(x) = \int_{S^{n-1}} \psi(x\cdot\omega,\omega)\,\mathrm{d}\omega. \tag{1.30}$$

Similarly to what we had before, ψ is a function on the set of oriented hyperplanes (and on the set of hyperplanes when ψ is even). Then we can think of $\mathbf{R}'\psi$ as an integral of $\psi = \psi(\pi)$ over the set of all hyperplanes π through x. Similarly to X', R' is also called sometimes a backprojection.

PROBLEM 1.6. Show that
$$R' : C_0^\infty(\mathbf{R} \times S^{n-1}) \longrightarrow C^\infty(\mathbf{R}^n)$$
is continuous but it does not preserve the compactness of the support in general.

As above, using the procedure described in section A.4, we extend the definition of R to compactly supported distributions as follows.

DEFINITION 1.6. *Let $f \in \mathcal{E}'(\mathbf{R}^n)$. Then we define $Rf \in \mathcal{D}'(\mathbf{R} \times S^{n-1})$ as the linear functional*

$$\langle Rf, \psi\rangle = \langle f, R'\psi\rangle, \quad \forall \psi \in C_0^\infty(\mathbf{R} \times S^{n-1}). \tag{1.31}$$

It is easy to see again that $R : \mathcal{E}'(\mathbf{R}^n) \to \mathcal{D}'(\mathbf{R} \times S^{n-1})$ is sequentially continuous, and that actually, $R : \mathcal{E}'(\mathbf{R}^n) \to \mathcal{E}'(\mathbf{R} \times S^{n-1})$.

When $f \in L^1$, we choose the form $\langle\cdot,\cdot\rangle$ on the l.h.s. of (1.31) to be defined as an integral w.r.t. the measure $\mathrm{d}s\,\mathrm{d}\theta$. That is the same form that we used to compute R' and it justifies calling the new operator R and extension of the original one, defined on $C_0^\infty(\mathbf{R}^n)$.

1.5. The Fourier Slice Theorem. The transforms X and R are closely related to the Fourier transform.

THEOREM 1.7. *For any $f \in L^1(\mathbf{R}^n)$,*
$$\hat{f}(\zeta) = \int_{\theta^\perp} e^{-\mathrm{i}z\cdot\zeta} Xf(z,\theta)\,\mathrm{d}S_z, \quad \forall \theta \perp \zeta,\ \theta \in S^{n-1}.$$

We denote by $\mathcal{F}_{\theta^\perp}$ the Fourier transform in the z variable on $\theta^\perp$. With this notation, the Fourier Slice Theorem reads: for any θ, $\hat{f}|_{\theta^\perp} = \mathcal{F}_{\theta^\perp} Xf$.

PROOF. The integral on the right equals

$$\int_{\theta^\perp}\int_{\mathbf{R}} e^{-\mathrm{i}z\cdot\zeta} f(z+s\theta)\,\mathrm{d}s\,\mathrm{d}S_z.$$

Set $x = z + s\theta$ and note that $x\cdot\zeta = z\cdot\zeta$ when $\zeta \perp \theta$. Then we see that the integral above equals $\hat f(\zeta)$. □

Therefore, taking the Fourier transform of $Xf(z,\theta)$ w.r.t. z on the hyperplane $\theta^\perp$ (then the dual variable ζ will belong to $\theta^\perp$ as well) gives us the Fourier transform $\hat f(\zeta)$.

Theorem 1.7 immediately implies injectivity of X on $L^1(\mathbf{R}^n)$, and one can also see that it implies injectivity of X on $\mathcal{E}'(\mathbf{R}^n)$. The latter also follows from the reconstruction formulas in section 2, see Problem 2.1.

In fact, for compactly supported functions, it implies a bit more. The decisive argument in the proof is the analyticity of the Fourier transform of compactly supported functions.

COROLLARY 1.8. *Let $f \in L^1(\mathbf{R}^n)$ have compact support and let $Xf(\cdot,\theta) = 0$ for θ in an infinite set of (distinct) unit vectors, then $f = 0$.*

PROOF. Note first that $\hat f$ is analytic. If $Xf(\cdot,\theta) = 0$ for a fixed θ and f compactly supported, then $\hat f(\xi) = 0$ for $\xi \perp \theta$, therefore $\hat f(\xi) = (\theta\cdot\xi)g_\theta(\xi)$ with g_θ analytic, as well. Repeating this with θ in the set $\{\theta_j\}_{j=1}^\infty$, where $Xf(\cdot,\theta)$ vanishes, we get $\hat f(\xi) = g_k(\xi)\,\Pi_{j=1}^k(\theta_j\cdot\xi)$ with g_k analytic, for any k. Therefore, for any $\omega \in S^{n-1}$, $\hat f(t\omega) = O_k(|t|^k)$, as $t \to 0$. Since $t \mapsto \hat f(t\omega)$ is analytic, we get $\hat f(t\omega) = 0$ for any unit ω, thus $\hat f = 0$. □

Finitely many "roentgenograms" however are not enough to recover f.

PROPOSITION 1.9. *Let $\{\theta_j\}_{j=1}^k$ be a finite set of unit vectors. Then there exists an infinite dimensional linear space of functions f with supports in a fixed compact so that $Xf(\cdot,\theta_j) = 0$, $j = 1,\dots k$.*

PROOF. Motivated by the proof of the preceding theorem, for a fixed $0 \neq \phi \in C_0^\infty(\mathbf{R}^n)$, set $f = \left(\Pi_{j=1}^k\theta_k\cdot\nabla_x\right)\phi$. Then f has the desired property. It is easy to see that the operator in the parentheses is injective on $C_0^\infty(\mathbf{R}^n)$. This implies that any finitely dimensional linear space of ϕ's under that operation will produce a linear space of f's with the same dimension. Therefore the so constructed space of f's is infinite dimensional. □

Theorem 1.7 provides a constructive way to recover f from Xf. Another consequence worth mentioning is the following.

COROLLARY 1.10. *Let $f \in L^1(\mathbf{R}^n)$, and let*

$$Xf(z,\theta_0) = 0 \quad \textit{for a fixed } \theta_0 \textit{ and all } z \in \theta_0^\perp.$$

Then

$$\hat f(\xi) = 0 \quad \textit{for all } \xi \perp \theta_0.$$

This corollary has a microlocal generalization that we will formulate later. We will see in ??? that knowing locally an X-ray type of transform over geodesic-like curves with no conjugate points near a single curve γ_0 recovers the conormal singularities at that curve.

We proceed with the Fourier Slice Theorem for R .

THEOREM 1.11. *For any $f \in L^1(\mathbf{R}^n)$,*

$$\hat{f}(r\omega) = \int_{\mathbf{R}} e^{-\mathrm{i}pr} Rf(p,\omega)\,\mathrm{d}p, \quad \forall r \in \mathbf{R},\ \forall \omega \in S^{n-1}.$$

Introduce the notation $\mathcal{F}_p$ for the Fourier transform in the p variable. Then the equality above can be written in the form: for any ω, $\hat{f} = \mathcal{F}_p Rf$ on the ray $\xi = r\omega$.

PROOF. The integral on the right equals

$$\int_{\mathbf{R}} \int_{x\cdot\omega=p} e^{-\mathrm{i}pr} f(x)\,\mathrm{d}S_x\,\mathrm{d}p = \int_{\mathbf{R}} \int_{x\cdot\omega=p} e^{-\mathrm{i}r\omega\cdot x} f(x)\,\mathrm{d}S_x\,\mathrm{d}p = \hat{f}(r\omega).$$

□

Similarly, Theorem 1.11 implies injectivity of R on $L^1(\mathbf{R}^n)$ and an explicit inversion. We also have the following.

COROLLARY 1.12. *Let $f \in L^1(\mathbf{R}^n)$, and let*

(1.32) $$Rf(p,\omega_0) = 0 \quad \textit{for a fixed } \omega_0 \textit{ and all } p.$$

Then

$$\hat{f}(\xi) = 0 \quad \textit{for all } \xi \textit{ parallel to } \omega_0.$$

Corollary 1.12 can be formulated in a way similar to Corollary 1.10. Under the condition (1.32), $\hat{f}(\xi) = 0$ on vectors ξ perpendicular to the planes over which we integrate. Again, there is a microlocal generalization of this statement for more general Radon type of transforms localized near a single hypersurface — under some conditions, we can recover the conormal singularities to that surface.

We conclude this section with a property that states that the action of X or R on a convolution is again a convolution in the z, and respectively the p variable.

THEOREM 1.13. *For f, g in $\mathcal{S}(\mathbf{R}^n)$,*

(1.33) $$\begin{aligned} X(f*g)(z,\theta) &= \int_{\theta^\perp} Xf(z-y,\theta)Xg(y,\theta)\,\mathrm{d}S_y, \\ R(f*g)(p,\omega) &= \int_{\mathbf{R}} Rf(p-q,\omega)Rg(q,\omega)\,\mathrm{d}q. \end{aligned}$$

The proof follows from the Fourier Slice theorem, or by a direct calculation.

PROBLEM 1.7. Let $g \in C_0^\infty(\mathbf{R}^n)$ be fixed. Show that (1.33) extend to any $f \in L^1(\mathbf{R}^n)$ or any $f \in \mathcal{E}'(\mathbf{R}^n)$ with the convolutions on the right considered in distribution sense.

1.6. Intertwining properties. The following intertwining formulas take place

(1.34) $$R\Delta = d_p^2 R, \quad R' d_p^2 = \Delta R',$$

on $C_0^\infty(\mathbf{R}^n)$ and on $C_0^\infty(\mathbf{R}\times S^{n-1})$, respectively. The proof is straightforward, either by direct computations or by using the Fourier Slice Theorem.

Let Δ_z denote the Laplacian in the z variable on each $\theta^\perp$. Note that Δ_z is independent on the way we choose a Cartesian coordinate system on each $\theta^\perp$. We set $|D_z| = (-\Delta_z)^{1/2}$. Similarly to the proposition above, we have

(1.35) $$X\Delta = \Delta_z X, \quad X'\Delta_z = \Delta X',$$

on $C_0^\infty(\mathbf{R}^n)$, and $C_0^\infty(\Sigma)$, respectively.

2. Inversion formulas

Theorems 1.7, 1.11 already imply inversion formulas on suitable spaces but we will provide below more direct formulas.

We start with a useful lemma.

LEMMA 2.1. *For any* $f \in \mathcal{S}(\mathbf{R}^n)$,

$$\int_{S^{n-1}} \int_{\omega^\perp} f(x)\, \mathrm{d}S_x\, \mathrm{d}\omega = |S^{n-2}| \int_{\mathbf{R}^n} \frac{f(x)}{|x|}\, \mathrm{d}x.$$

PROOF. By the Fourier Slice Theorem,

$$\begin{aligned}\int_{\omega^\perp} f(x)\, \mathrm{d}S_x = Rf(0,\omega) &= \frac{1}{2\pi} \int_{\mathbf{R}} \hat{f}(r\omega)\, \mathrm{d}r \\ &= \frac{1}{2\pi} \left(\int_{\mathbf{R}_+} \hat{f}(r\omega)\, \mathrm{d}r + \int_{\mathbf{R}_+} \hat{f}(-r\omega)\, \mathrm{d}r \right).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{S^{n-1}} \int_{\omega^\perp} f(x)\, \mathrm{d}S_x\, \mathrm{d}\omega &= \frac{1}{\pi} \int_{S^{n-1}} \int_{\mathbf{R}} \hat{f}(r\omega)\, \mathrm{d}r\, \mathrm{d}\omega \\ &= \frac{1}{\pi} \int \hat{f}(\xi)|\xi|^{1-n}\, \mathrm{d}\xi \\ &= |S^{n-2}| \int f(x)|x|^{-1}\, \mathrm{d}x,\end{aligned}$$

and we used the Parseval's equality together with (A.2.2) □

2.1. The Schwartz kernel of $X'X$ and inversion formulas for X. We start with computing the Schwartz kernel of $X'X$ first.

PROPOSITION 2.2. *For any* $f \in \mathcal{S}(\mathbf{R}^n)$,

$$X'Xf(x) = 2 \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-1}} \mathrm{d}y. \tag{2.1}$$

PROOF. By (1.2), (1.22),

$$\begin{aligned}X'Xf(x) &= \int_{S^{n-1}} Xf(x - (x\cdot\theta)\theta, \theta)\, \mathrm{d}\theta \\ &= \int_{S^{n-1}} \int_{\mathbf{R}} f(x + s\theta - (x\cdot\theta)\theta, \theta)\, \mathrm{d}s\, \mathrm{d}\theta \\ &= \int_{S^{n-1}} \int_{\mathbf{R}} f(x+s\theta)\, \mathrm{d}s\, \mathrm{d}\theta.\end{aligned} \tag{2.2}$$

We split the s-integral in two parts: over $s>0$ and $s<0$. Then we make the change of variables $(s,\theta) \mapsto (-s,-\theta)$ in the second one. Thus we get

$$\begin{aligned}X'Xf(x) &= 2\int_{S^{n-1}} \int_0^\infty f(x+s\theta)\, \mathrm{d}s\, \mathrm{d}\theta \\ &= 2\int_{\mathbf{R}^n} \frac{f(x+z)}{|z|^{n-1}} \mathrm{d}z = 2\int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-1}} \mathrm{d}y.\end{aligned}$$

□

In next theorem, if $n = 2$, S^{n-2} reduces to a two point set: $\{-1, 1\}$; and then $|S^{n-2}| = 2$.

THEOREM 2.3. *For any* $f \in \mathcal{S}(\mathbf{R}^n)$,

$$f = c_n |D| X' X f, \quad c_n = \left(2\pi |S^{n-2}|\right)^{-1}. \tag{2.3}$$

PROOF. By Proposition 2.2, $X'Xf = 2f * h$, where $h(x) = |x|^{-n+1}$. The function h is locally in L^1 and positively homogeneous of order $-n+1$. Its Fourier transform is given by

$$\hat{h} = \frac{\pi |S^{n-2}|}{|\xi|},$$

see (A.2.2). Therefore, $X'Xf = (1/c_n)|D|^{-1} f$. This yields (2.3) immediately. □

counter-example?

The Fourier multiplier $|D|$ is a non-local operator. Therefore, if we want to recover f only in a neighborhood of some x_0 by means of formula (2.1), it is not enough to know Xf for all lines ℓ that intersect that neighborhood.

A different inversion formula, with the non-local operator between X' and X follows below. Formula (2.4) belongs to the class of the "filtered back-projection" formulas, with the operator $|D_z|$ playing the role of a "filter" before back-projecting with X'. In applications, D_z is often combined with a cut-off for high frequencies (high values of the dual variable of z) to clean up noise. In the latter case, the reconstruction is not exact, of course.

THEOREM 2.4. *For any* $f \in \mathcal{S}(\mathbf{R}^n)$,

$$f = c_n X' |D_z| X f, \quad c_n = \left(2\pi |S^{n-2}|\right)^{-1}. \tag{2.4}$$

PROOF. One might attempt to prove (2.4) by extending the intertwining formulas (1.35) to the non-local operator $|D_z|$. That would require however extending X to some class of non-compactly supported distributions.

Let f, g be in $\mathcal{S}(\mathbf{R}^n)$. Note first that $|D_z|Xf$ is well defined in $L^2(\Sigma)$. Then

$$(X'|D_z|Xf, g)_{L^2(\mathbf{R}^n)} = (|D_z|Xf, Xg)_{L^2(\Sigma)} = (2\pi)^{1-n} (\mathcal{F}_z |D_z| Xf, \mathcal{F}_z Xg)_{L^2(\Sigma)},$$

where $\mathcal{F}_z$ is the Fourier transform in the z variable. For any fixed θ, let ρ be the dual variable to z. Then $\mathcal{F}_z |D_z| = |\rho| \mathcal{F}_z$, and the latter expression is coordinate independent. Combining this with the Fourier Slice Theorem (Theorem 1.7),

$$\begin{aligned}(X'|D_z|Xf, g)_{L^2(\mathbf{R}^n)} &= (2\pi)^{1-n} (|\rho| \mathcal{F}_z Xf, \mathcal{F}_z Xg)_{L^2(\Sigma)} \\ &= (2\pi)^{1-n} \int_{S^{n-1}} \int_{\theta^\perp} \hat{f}(\rho) \bar{\hat{g}}(\rho) |\rho| \, \mathrm{d}S_\rho \, \mathrm{d}\theta.\end{aligned}$$

By Lemma 2.1,

$$(X'|D_z|Xf, g)_{L^2(\mathbf{R}^n)} = (2\pi)^{1-n} |S^{n-2}| (\hat{f}, \hat{g})_{L^2(\mathbf{R}^n)} = 2\pi |S^{n-2}| (f, g)_{L^2(\mathbf{R}^n)}. \tag{2.5}$$

This completes the proof. □

PROBLEM 2.1. Prove that the inversion formulas of Theorem 2.3 and Theorem 2.4 remain true for any $f \in \mathcal{E}'(\mathbf{R}^n)$, where X and X' are the extensions to distribution spaces as explained in section 1.1.

In Figure II.2, we present a reconstruction based on Theorem 2.4. The third image on the right is $X'Xf$, and we will see in Chapter IV that $X'Xf$ is one degree smoother than f.

FIGURE II.2. Reconstruction of the Shepp-Logan phantom. Left: original; Center reconstruction with the filtered backprojection formula (2.4); Right: the "unfiltered" backprojection $X'Xf$.

2.2. The Schwartz kernel of $R'R$ and inversion formulas for R. We have similar results for the Radon transform R.

PROPOSITION 2.5. *For any $f \in \mathcal{S}(\mathbf{R}^n)$,*

$$R'Rf(x) = |S^{n-2}| \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|}\,\mathrm{d}y. \tag{2.6}$$

PROOF. By (1.10), (1.30),

$$\begin{aligned} R'Rf(x) &= \int_{S^{n-1}} Rf(x\cdot\omega,\omega)\,\mathrm{d}\omega \\ &= \int_{S^{n-1}} \int_{y\cdot\omega = x\cdot\omega} f(y)\,\mathrm{d}S_y\,\mathrm{d}\omega \\ &= \int_{S^{n-1}} \int_{z\cdot\omega=0} f(z+x)\,\mathrm{d}S_z\,\mathrm{d}\omega. \end{aligned} \tag{2.7}$$

An application of Lemma 2.1 completes the proof. □

THEOREM 2.6. *For any $f \in \mathcal{S}(\mathbf{R}^n)$,*

$$f = C_n |D|^{n-1} R'Rf, \quad C_n = \frac{1}{2}(2\pi)^{1-n}. \tag{2.8}$$

PROOF. By Proposition 2.5, $R'R$ is a convolution with $|S^{n-2}||x|^{-1}$. Its Fourier transform is given by

$$2(2\pi)^{n-1}|\xi|^{1-n},$$

see (A.2.2). Therefore,

$$(R'Rf)\hat{} = 2(2\pi)^{n-1}|\xi|^{1-n}\hat{f}.$$

Solve for $\hat{f}$ to complete the proof. □

Note that $|D|^{n-1} = (-\Delta)^{(n-1)/2}$. If n is odd, then $|D|^{n-1}$ is a local (differential) operator, and knowledge of Rf for all hyperplanes passing through any fixed neighborhood of some x_0 is enough to recover f in that neighborhood by formula (2.8). If n is even, then $|D|^{n-1}$ is a non-local pseudo-differential operator. In particular, if we use formula (2.8) for reconstruction and n is even, for a compactly supported f we need to compute $R'Rf(x)$ for all values of x, including those far from $\operatorname{supp} f$. This does not prove however that integrals of f over all lines through some open set (region of interest) do not determine uniquely f there — it just shows that this cannot be done with that formula. Actually, for n even, local information about Xf is insufficient, indeed, see Section IV.4.

needs to be introduced elsewhere

Let H be the Hilbert transform, see ...

$$Hg(p) = \frac{1}{\pi}\,\mathrm{pv}\int_{\mathbf{R}} \frac{g(s)}{p-s}\,\mathrm{d}s, \tag{2.9}$$

where "pv$\int$" stands for an integral in a principal value sense.

THEOREM 2.7. *For any* $f \in \mathcal{S}(\mathbf{R}^n)$,

$$f = \begin{cases} C_n' R' d_p^{n-1} Rf, & n \text{ odd},\\ C_n' R' H d_p^{n-1} Rf, & n \text{ even}, \end{cases} \tag{2.10}$$

where d_p *stands for the derivative of* $Rf(p,\omega)$ *w.r.t.* p, H *is the Hilbert transform w.r.t.* p *and*

$$C_n' = \begin{cases} (-1)^{(n-1)/2} C_n, & n \text{ odd},\\ (-1)^{(n-2)/2} C_n, & n \text{ even}, \end{cases}$$

with $C_n = \frac{1}{2}(2\pi)^{1-n}$ *is as in Theorem 2.6.*

In the even dimensional case, formulas (2.10) have the following advantage to (2.6). If f is compactly supported, we only need the reconstruction in a compact set. As mentioned above, with (2.6), we still have to compute $R'Rf$ in the whole $\mathbf{R}^n$, because the Hilbert transform H is a non-local operator. On the other hand, when we use (2.4), and restrict the result to a compact set, then we need to know HRf in a compact set, as well, as it can be easily seen from (1.30). On the other hand, Rf is compactly supported because f is compactly supported, too. Therefore, in the integral (2.9), where $g = Rf$, both p and s belong to bounded intervals, and no computations are done in infinite domains.

should we call that Remark?

REMARK 2.1. The appearance of the Hilbert transform H for n even, and the different constants for n odd/even may look strange at first glance, especially when compared to the inversion formula (2.6) in Theorem 2.6, that looks the same for all $n \ge 2$. For n even, note first that $H = -\mathrm{i}\,\mathrm{sgn}(D_p)$, $D_p = -\mathrm{i}d_p$, therefore,

$$(-1)^{(n-2)/2} H d_p^{n-1} = |D_p|^{n-1}, \quad n \text{ even}.$$

On the other hand,

$$(-1)^{(n-1)/2} d_p^{n-1} = |D_p|^{n-1}, \quad n \text{ odd}.$$

Therefore, in both cases, (2.10) can be written as

$$f = C_n R' |D_p|^{n-1} Rf \tag{2.11}$$

REMARK 2.2. Comparing (2.11) with the inversion formula (2.6) in Theorem 2.6, we may ask ourselves whether we can prove Theorem 2.7 directly from Theorem 2.6, using the intertwining property (1.34). When n is odd, this can be done without problems since $|D|^{n-1} = (-\Delta)^{(n-1)/2}$ is an integer power of the Laplacian then. If we could justify the intertwining property (1.34) for fractional powers of the Laplacian, then one would have $|D|^{n-1}R' = R'|D_p|^{n-1}$ on a certain distribution space, where Xf belongs; and that would prove the theorem for n even, as well. Proving the latter identity by duality, for example, would require extending R to a class of non-compactly supported distributions, by the means of the Fourier Slice Theorem, for example. That poses some technical challenges, Instead of doing this, we will give a more direct proof.

Proof of Theorem 2.7. Let f, g be in $\mathcal{S}(\mathbf{R}^n)$, and let n be even. Then

$$\begin{aligned}(R'Hd_p^{n-1}Rf,g)_{L^2(\mathbf{R}^n)} &= \left(Hd_p^{n-1}Rf,Rg\right)_{L^2(\mathbf{R}\times S^{n-1})}\\ &= (2\pi)^{-1}\left(\mathcal{F}_pHd_p^{n-1}Rf,\mathcal{F}_pRg\right)_{L^2(\mathbf{R}\times S^{n-1})}\\ &= (2\pi)^{-1}\left((\mathrm{i}\rho)^{n-1}(-\mathrm{i})\mathrm{sgn}(\rho)\mathcal{F}_pRf,\mathcal{F}_pRg\right)_{L^2(\mathbf{R}\times S^{n-1})},\end{aligned}$$

where $\mathcal{F}_p$ is the Fourier transform w.r.t. p, and ρ is the dual variable of p.

By the Fourier Slice Theorem (Theorem 1.11),

$$\mathcal{F}_pRg(\rho,\omega)=\hat{g}(\rho\omega),\quad \mathcal{F}_pRf(\rho,\omega)=\hat{f}(\rho\omega),$$

and they are both even functions of ρ. Moreover, the factor

$$(\mathrm{i}\rho)^{n-1}(-\mathrm{i})\mathrm{sgn}(\rho)=(-1)^{(n-2)/2}|\rho|^{n-1} \tag{2.12}$$

is even as well. Therefore,

$$\begin{aligned}&\left(R'Hd_p^{n-1}Rf,g\right)_{L^2(\mathbf{R}^n)}\\ &\quad= (2\pi)^{-1}(-1)^{(n-2)/2}\int_{\mathbf{R}\times S^{n-1}}\hat{f}(\rho\omega)\bar{\hat{g}}(\rho\omega)|\rho|^{n-1}\,\mathrm{d}\rho\,\mathrm{d}\omega\\ &\quad= 2(2\pi)^{-1}(-1)^{(n-2)/2}\int_{\mathbf{R}_+\times S^{n-1}}\hat{f}(\rho\omega)\bar{\hat{g}}(\rho\omega)\rho^{n-1}\,\mathrm{d}\rho\,\mathrm{d}\omega\\ &\quad= 2(2\pi)^{n-1}(-1)^{(n-2)/2}(f,g)_{L^2(\mathbf{R}^n)}.\end{aligned} \tag{2.13}$$

That proves the theorem for n even.

Even though we proved the theorem for n odd by using (1.34), note that the proof above carries over to that case, as well. We compute $(R'd_p^{n-1}Rf,g)_{L^2(\mathbf{R}^n)}$ in the same way, and the only difference is that the factor $-\mathrm{i}\,\mathrm{sgn}(\rho)$ will be missing. Then (2.12) is replaced by $(-1)^{(n-1)/2}|\rho|^{n-1}$, that is still even. A sa result we get

$$\begin{aligned}&\left(R'd_p^{n-1}Rf,g\right)_{L^2(\mathbf{R}^n)}\\ &\quad= (2\pi)^{-1}(-1)^{(n-1)/2}\int_{\mathbf{R}\times S^{n-1}}\hat{f}(\rho\omega)\bar{\hat{g}}(\rho\omega)|\rho|^{n-1}\,\mathrm{d}\rho\,\mathrm{d}\omega\\ &\quad= 2(2\pi)^{-1}(-1)^{(n-1)/2}\int_{\mathbf{R}_+\times S^{n-1}}\hat{f}(\rho\omega)\bar{\hat{g}}(\rho\omega)\rho^{n-1}\,\mathrm{d}\rho\,\mathrm{d}\omega\\ &\quad= 2(2\pi)^{n-1}(-1)^{(n-1)/2}(f,g)_{L^2(\mathbf{R}^n)}.\end{aligned} \tag{2.14}$$

□

Problem 2.2. Prove that the inversion formulas of Theorem 2.6 and Theorem 2.7 remain true for any $f\in\mathcal{E}'(\mathbf{R}^n)$, where R and R' are the extensions to distribution spaces as explained in section 1.1.

3. Stability estimates

We already established some mapping properties of X and R in L^1 and L^2 spaces, see Problem 1.1, Problem 1.5 without stability estimates in the same norms, however. In this section, we will prove continuity and stability estimates in Sobolev norms.

We define first Sobolev spaces on Σ and $\mathbf{R}\times S^2$. For any $s\in\mathbf{R}$, set

$$\begin{aligned} \|g\|_{\bar H^s(\Sigma)} &= \left\|(1-\Delta_z)^{s/2} g\right\|_{L^2(\Sigma)},\\ \|g\|_{\bar H^s(\mathbf{R}\times S^{n-1})} &= \left\|(1-\mathrm{d}_p^2)^{s/2} g\right\|_{L^2(\mathbf{R}\times S^{n-1})}, \end{aligned} \tag{3.1}$$

where Δ_z is the Laplacian on each fiber $\theta^\perp$ of Σ, while $d_p = \partial/\partial p$, as above. Fractional powers of the operators in the definition are defined through the partial Fourier transforms $\mathcal{F}_{\theta^\perp}$ in the z variable, and $\mathcal{F}_p$ in the p-variable, respectively. Note that those are not the standard Sobolev spaces on the manifolds Σ and $\mathbf{R}\times S^2$, respectively because the definition we gave includes derivatives w.r.t. some of the variables only. The next theorem shows that the norms above norms are sharp — we get estimates form above *and below* of Xf and Rf. This proves not only the continuity of R and X in those norms but proves as well stability of the inversion.

THEOREM 3.1. *For any bounded domain $\Omega\subset\mathbf{R}^n$ with smooth boundary, and any s, we have*

$$\|f\|_{H^s(\mathbf{R}^n)}/C \le \|Xf\|_{\bar H^{s+1/2}(\Sigma)} \le C\|f\|_{H^s(\mathbf{R}^n)}, \tag{3.2}$$

$$\|f\|_{H^s(\mathbf{R}^n)}/C \le \|Rf\|_{\bar H^{s+(n-1)/2}(\mathbf{R}\times S^{n-1})} \le C\|f\|_{H^s(\mathbf{R}^n)} \tag{3.3}$$

for all $f\in H^s(\mathbf{R}^n)$ supported in $\bar\Omega$.

replace d_p by D_p?

PROOF. Consider Rf. Let us start with the following observation. Set

$$R^\sharp f := C_n^{1/2}|D_p|^{(n-1)/2} Rf, \tag{3.4}$$

where C_n is as in Theorem 2.6. Then

$$\|R^\sharp f\|_{L^2(\mathbf{R}\times S^{n-1})} = \|f\|_{L^2(\mathbf{R}^n)}, \quad \forall f\in C_0^\infty(\mathbf{R}^n), \tag{3.5}$$

i.e., $R^\sharp$ is an isometry.[1] We show later in Theorem 6.1 that $R^\sharp$ is also surjective. Relation (3.5) follows directly from (2.11). This proves the first inequality in (3.3) for $s=0$. For the second one, we see that we have to deal with the fact that when we apply $\mathcal{F}_p$, $|d_p|^{(n-1)/2}$ transforms into multiplication by $|\rho|^{(n-1)/2}$, where ρ is dual variable ρ of p, and that factor vanishes for $\rho=0$.

[1] i.e., norm preserving. We do not include the requirement to be surjective to call it isometry.

The next step is to prove the first inequality in (3.3) for any s. Using the Fourier Slice Theorem, we get

$$
\begin{aligned}
\|f\|_{H^s(\mathbf{R}^n)} &= \|(1-\Delta)^{s/2} f\|_{L^2(\mathbf{R}^n)} \\
&= (2\pi)^{-n} \int_{\mathbf{R}^n} \left(1+|\xi|^2\right)^s |\hat f(\xi)|^2 \,\mathrm{d}\xi \\
&= (2\pi)^{-n} \int_{\mathbf{R}_+\times\Omega} \left(1+r^2\right)^s |\hat f(r\omega)|^2 r^{n-1} \,\mathrm{d}r\,\mathrm{d}\omega \\
&= \frac12 (2\pi)^{-n} \int_{\mathbf{R}\times\Omega} \left(1+r^2\right)^s |\hat f(r\omega)|^2 |r|^{n-1} \,\mathrm{d}r\,\mathrm{d}\omega \\
&\le \frac12 (2\pi)^{-n} \int_{\mathbf{R}\times\Omega} \left(1+r^2\right)^{s+\frac{n-1}{2}} |\hat f(r\omega)|^2 \,\mathrm{d}r\,\mathrm{d}\omega \\
&= \frac12 (2\pi)^{-n} \int_{\mathbf{R}\times\Omega} \left(1+r^2\right)^{s+\frac{n-1}{2}} |\mathcal{F}_p Rf(r\omega)|^2 \,\mathrm{d}r\,\mathrm{d}\omega \\
\\
&= C_n \left\|(1-d_p^2)^{s+\frac{n-1}{2}} Rf\right\|_{L^2(\mathbf{R}\times S^{n-1})} \\
&= C_n \|Rf\|_{\bar H^{s+\frac{n-1}{2}}(\mathbf{R}\times S^{n-1})}.
\end{aligned}
$$

This proves the first inequality in (3.3). To prove the second one, write

$$
\begin{aligned}
\|Rf\|^2_{\bar H^{s+\frac{n-1}{2}}(\mathbf{R}\times S^{n-1})} &= \int_{\mathbf{R}\times S^{n-1}} (1+\rho^2)^{s+\frac{n-1}{2}} |(\mathcal{F}_p Rf)(\rho,\omega)|^2 \,\mathrm{d}\rho\,\mathrm{d}\omega \\
&= 2\int_{\mathbf{R}^n} \frac{(1+|\xi|^2)^{s+\frac{n-1}{2}}}{|\xi|^{n-1}} |\hat f(\xi)|^2 \,\mathrm{d}\xi,
\end{aligned}
$$

where we applied the usual argument again — we spilt the ρ-integral in two parts, for $\pm\rho > 0$, and used the fact that the integrand is even in (ρ,ω).

Split the latter integral into two parts: The low frequency one, I_1, where we integrate for $|\xi|\le 1$; and the high frequency one, I_2, where we integrate on $|\xi|\ge 1$. Clearly,

$$I_2 \le C\|f\|^2_{H^s(\mathbf{R}^n)}, \tag{3.6}$$

with some $C = C(n)$. The same inequality for I_1 is trickier because of the negative power $|\xi|^{1-n}$ of $|\xi|$ appearing in the integral. Since this term is integrable in the unit ball, we have

$$I_1 \le C(n,s) \sup_{|\xi|\le 1} |\hat f(\xi)|^2. \tag{3.7}$$

We regards $\hat f(\xi)$ as the action of f on the test function $\phi_\xi := e^{-\mathrm{i}x\cdot\xi}\chi(x)$, where $\chi \in C_0^\infty(\mathbf{R}^n)$ equals 1 in a neighborhood of $\bar\Omega$. Then

$$|\hat f(\xi)| = |\langle f, \phi_\xi\rangle| \le \|f\|_{H^s} \|\phi_\xi\|_{H^{-s}}.$$

Since $\hat\phi_\xi(\eta) = \hat\chi(\xi+\eta)$, we easily get that $\|\phi_\xi\|_{H^{-s}} \le C(n,s)$ with some $C(n,s) > 0$ for each $|\xi|\le 1$. That proves (3.6) for I_1, as well; and this completes the proof of (3.3).

To prove (3.2), we proceed in the same way starting with (2.5), that implies

$$\|Xf\|_{\bar H^{1/2}(\Sigma)} \ge c_n^{-1/2} \|f\|_{L^2(\mathbf{R}^n)}.$$

This is the first inequality in (3.2) for $s=0$. Using the Fourier Slice Theorem, we extend it for any s as above. The proof of the second inequality in (3.2) is similar to the proof above. In particular, the low frequency integral in this case is a constant times

$$I_1' = \int_{|\xi|\le 1} \frac{(1+|\xi|^2)^{s+\frac{1}{2}}}{|\xi|} |\hat{f}(\xi)|^2 \, \mathrm{d}\xi. \tag{3.8}$$

Then I_1' still satisfies (3.7) because $|\xi|^{-1}$ is integrable in the unit ball, and we proceed as above. □

REMARK 3.1. It follows form the proof that the constant C on the right of each of the inequalities (3.2), (3.3) can be chosen dependent on n only. Then f does not need to be supported in $\bar{\Omega}$. The constant on the right depends on Ω, s and n.

Theorem 3.1 shows that we "gain 1/2 derivative" with the operator X, and $(n-1)/2$ derivatives with the operator R. Each one of those two operators involves an integration that has a smoothing effect. The gain is a half of the dimension of the linear submanifolds over which we integrate.

For future references, it is worth noticing that we established the following fact. If $k<n$, then for any s and any $f\in H^s$ supported in $\bar{\Omega}$, we have

$$\int_{|\xi|\le 1} \frac{|\hat{f}(\xi)|^2}{|\xi|^k} \, \mathrm{d}\xi \le C_{\Omega,k,s} \|f\|^2_{H^s(\Omega)} \qquad \int_{|\xi|\le 1} \frac{|\hat{f}(\xi)|}{|\xi|^k} \, \mathrm{d}\xi \le C_{\Omega,k,s} \|f\|_{H^s(\Omega)}. \tag{3.9}$$

In particular, the operator with kernel $\chi_{|\xi|\le 1}|\xi|^{-k}$, where $\chi_{|\xi|\le 1}$ is the characteristic function of the ball $B(0,1)$, is smoothing for $k<n$.

somewhat misleading title

3.1. Stability estimates for X in a bounded domain. We present here a different kind of stability estimates. We parameterize the lines through Ω by initial points on $\partial\Omega$ and initial incoming (we can use outgoing as well) directions, i.e., by elements in $\partial_- S\Omega$, see Section 1.3.1.

THEOREM 3.2. *Let Ω be a convex bounded domain with smooth boundary, and let $K\subset\Omega$ be compact. Then for any s, there exists $C=C_{K,s,n}>0$ so that for any $f\in H^s(\mathbf{R}^n)$ supported in K,*

$$\|f\|_{H^s(\mathbf{R}^n)}/C \le \|Xf\|_{H^{s+1/2}(\partial_- S\Omega)} \le C\|f\|_{H^s(\mathbf{R}^n)}$$

PROOF. The points on $\partial_- S\Omega$ corresponding to lines hitting K form a compact subset, call it $K_1\subset\partial_- S\Omega$. The same subset of lines, parameterized by points in Σ form another compact subset $K_2\subset\Sigma$. Then the map (1.13) is a diffeomorphism from K_1 to K_2. Then the theorem follows from Theorem 3.1. □

***** Range for s? *****

3.2. Stability estimates in terms of $X'X$ and $R'R$. We turn our attention to other types of stability estimates: estimating f in terms of $X'Xf$ and $R'Rf$ in case of compactly supported f. Assume now that $\operatorname{supp} f\subset\bar{\Omega}$, where Ω is a bounded open domain. The reconstruction formula (2.3) requires knowledge of $X'Xf$ in the whole $\mathbf{R}^n$ because $|D|$ is a non-local operator. Similarly, in (2.8), we need to know $R'Rf$ in the whole $\mathbf{R}^n$ for n even. On the other hand, it is easy to see that $X'Xf|_\Omega$ determines f uniquely — multiply $X'Xf|_\Omega=0$ by $\bar{f}$ and write the result as $\|Xf\|^2=0$. A similar remark applies to Rf. The next theorem says that knowing $X'Xf$, $R'Rf$ in some neighborhood of $\bar{\Omega}$ recovers f in a stable way.

THEOREM 3.3. *Let $\Omega \subset \mathbf{R}^n$ be open and bounded, and let $\Omega_1 \supset \bar{\Omega}$ be another such set. Then for any integer $s = 0, 1, \dots,$ there is a constant $C > 0$ so that for any $f \in H^s(\mathbf{R}^n)$ supported in $\bar{\Omega}$, we have*

$$\|f\|_{H^s(\mathbf{R}^n)}/C \le \|X'Xf\|_{H^{s+1}(\Omega_1)} \le C\|f\|_{H^s(\mathbf{R}^n)}, \tag{3.10}$$

$$\|f\|_{H^s(\mathbf{R}^n)}/C \le \|R'Rf\|_{H^{s+n-1}(\Omega_1)} \le C\|f\|_{H^s(\mathbf{R}^n)} \tag{3.11}$$

PROOF. The second inequality in (3.10) and in (3.11), respectively follows immediately from the fact that $X'X$ is the Fourier multiplier $c_n^{-1}|\xi|^{-1}$, see (2.3); and $R'R$ is the Fourier multiplier $C_n|\xi|^{n-1}$, see (2.8). The singularity at $\xi = 0$ can be dealt with using (3.9).

To prove the first inequality in (3.10), write

$$(1 + |\xi|^2)^{s/2}\hat{f}(\xi) = c_n(1 + |\xi|^2)^{s/2}|\xi|\mathcal{F}X'Xf.$$

This implies

$$\|f\|^2_{H^s(\mathbf{R}^n)} \le c_n\|X'Xf\|^2_{H^{s+1}(\mathbf{R}^n)} = c_n\|X'Xf\|^2_{H^{s+1}(\Omega_1)} + c_n\|X'Xf\|^2_{H^{s+1}(\mathbf{R}^n\setminus\Omega_1)}.$$

The operator

$$H_0^s(\Omega) \ni f \mapsto X'Xf|_{\mathbf{R}^n\setminus\Omega_1} \in H^{s+1}(\mathbf{R}^n \setminus \Omega),$$

with $H_0^s(\Omega)$ considered as a subspace of $H^s(\mathbf{R}^n)$, is compact because it has a C^∞ kernel, and $\bar{\Omega}$ is compact. Indeed, the kernel of $X'X$ is given by (2.1), and for $x \in \mathbf{R}^n \setminus \Omega$ and $y \in \bar{\Omega}$, it is smooth. On the other hand, $X'X : H_0^s(\Omega) \to H^{s+1}(\Omega_1)$ is injective. ***** it would be easier to assume $s \ge 0$; then the injectivity follows easily *****. By Lemma 3.4 below, the first estimate in (3.10) follows. The proof of the first estimate in (3.11) is similar. □

The following lemma was used in the proof above.

LEMMA 3.4. *Let X, Y, Z be Banach spaces, let $A : X \to Y$ be a bounded linear operator, and $K : X \to Z$ be a compact linear operator. Let*

$$\|f\|_X \le C\left(\|Af\|_Y + \|Kf\|_Z\right), \quad \forall f \in X. \tag{3.12}$$

Assume that A is injective. Then there exists $C' > 0$ so that

$$\|f\|_X \le C'\|Af\|_Y, \quad \forall f \in X.$$

PROOF. Assume the opposite. Then there exists a sequence f_n in X with $\|f_n\|_X = 1$ and $Af_n \to 0$ in Y. Since $K : X \to Z$ is compact, there exists a subsequence, that we will still denote by f_n, such that Kf_n converges in Z, therefore is a Cauchy sequence in Z. Applying (3.12) to $f_n - f_m$, we get that $\|f_n - f_m\|_X \to 0$, as $n \to \infty$, $m \to \infty$, i.e., f_n is a Cauchy sequence in X. Therefore, there exists $f \in X$ such that $f_n \to f$ and we must have $\|f\|_X = 1$. Then $Af_n \to Af = 0$. This contradicts the injectivity of A thus proving the lemma. □

4. The Radon transform in polar coordinates

Another view on the Radon transform is its representation in polar coordinates. Let us represent $f = f(r\omega)$ in the spherical harmonics basis $Y_l^m(\omega)$ with coefficients $f_{lm}(r)$ depending on the radius r, and let us write $Rf(p, \theta)$, in the spherical harmonic basis related to the ω variable, with coefficients $g_{lm}(p)$. It turns out that R is diagonal in that representation with diagonal entries integral operators of Abel type which can be inverted explicitly. This was first done by Allan Cormack in his 1963–64 papers [...]. In 1979, Allan Cormack and Godfrey N. Hounsfield were awarded the Nobel prize in Physiology or Medicine "for the development of computer assisted tomography".

We will study the $n = 2$ case only for simplicity. In fact, the analysis can be generalized to any n, see [**25**].

A basis for the spherical harmonics on S^1 is given by $e^{\mathrm{i}k\phi}$, where ϕ is the argument of the polar angle. Given $f \in C_0^\infty$, expand $f(r\theta)$ in Fourier series

$$f(r\theta) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} f_k(r)e^{\mathrm{i}k\arg(\theta)}. \tag{4.1}$$

We also write $g(p,\omega) = Rf$, and expand g as well

$$g(p,\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} g_k(p)e^{\mathrm{i}k\arg(\omega)}.$$

Note that g_k is an even function for k even, and an odd function for k odd. We also introduce the Chebyshev polynomials of the first kind given by

$$T_n(x) = \cos(n\arccos(x)) = \cosh(n\,\mathrm{arccosh}(x)), \quad n = 0, 1, \dots.$$

The first definition makes sense for $|x| \le 1$; the second one for $|x| \ge 1$ but in both cases we get same polynomials of degree n, which easy to see using elementary trigonometry. The first three of them are given by $T_0 = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$. The polynomials T_n are even when n is even and odd when n is odd.

THEOREM 4.1. *For every* $f \in \mathcal{S}(\mathbf{R}^2)$,

$$g_k(p) = 2\int_{|p|}^{\infty} T_{|k|}\left(\frac{p}{r}\right)\left(1 - \frac{p^2}{r^2}\right)^{-1/2} f_k(r)\,\mathrm{d}r$$
$$f_k(r) = -\frac{1}{\pi}\int_r^{\infty} T_{|k|}\left(\frac{p}{r}\right)\left(p^2 - r^2\right)^{-1/2} g_k'(p)\,\mathrm{d}p.$$

The theorem says that in the spherical harmonics representation, R is diagonal, indeed, since g_k depends on f_k only, with the same k. Moreover, the second equation provides yet another inversion formula.

PROOF. We will prove the first part of the theorem only. For a proof of the second formula, see [**25**]. First, for $p \ge 0$, we compute

$$h_k(p,\omega) := \int_{x\cdot\omega = p} e^{\mathrm{i}k\arg(x)} f_k(r)\,\mathrm{d}S_x,$$

see (4.1). The line over which we integrate can be parameterized as $s \mapsto x = p\omega + s\omega^\perp$ and $\mathrm{d}S_x = \mathrm{d}s$. Then $r^2 = p^2 + s^2$. Split the range $s \in \mathbf{R}$ into two parts: $s > 0$ and $s < 0$. Then $\theta := \arccos(p/r)$ is the angle between the vectors $x(s)$ and $p\omega$ in both cases. Therefore, for $\pm s > 0$, $\arg(x) = \arg(\omega) \pm \theta$. Then

$$\begin{aligned} h_k(p,\omega) &= \int_0^\infty e^{\mathrm{i}k(\arg(\omega)+\theta)} f(\sqrt{p^2+s^2})\,\mathrm{d}s + \int_0^\infty e^{\mathrm{i}k(\arg(\omega)-\theta)} f(\sqrt{p^2+s^2})\,\mathrm{d}s \\ &= 2e^{\mathrm{i}k\arg(\omega)}\int_0^\infty \cos(k\theta) f(\sqrt{p^2+s^2})\,\mathrm{d}s. \end{aligned}$$

By our definition of θ, we get $\cos(k\theta) = T_{|k|}(p/r)$. Make the substitution $r = \sqrt{p^2+s^2}$ in the integral to get

$$h_k(p,\omega) = 2e^{\mathrm{i}k\arg(\omega)}\int_0^\infty T_{|k|}(p/r)f(r)\frac{r}{\sqrt{r^2-p^2}}\,\mathrm{d}r.$$

The formula holds for $p<0$ as well as it can be easily seen by using the fact that h_k is even with respect to (p,ω) and $T_{|k|}$ is even/odd when k is even/odd.

We can now take the Radon transform of (4.1) to get the first identity in the theorem. □

5. Support Theorems

5.1. Support theorems for R.

THEOREM 5.1. *Let $f \in C(\mathbf{R}^n)$ be such that*

(i) *$|x|^k f(x)$ is bounded for any integer k,*
(ii) *there exists a constant $A>0$ so that $Rf(p,\omega)=0$ for $|p|>A$.*

Then $f(x)=0$ for $|x|>A$.

PROOF. If $f \in \mathcal{S}(\mathbf{R}^n)$ at least, the theorem follows directly from Theorem 4.1. We will reproduce the original Helgason's proof here, which uses only $K=1$ part of Theorem 4.1, which proof is simpler and can easily be justified for rapidly decaying $C(\mathbf{R}^n)$ functions as in the theorem.

***** do it *****

Let f be non necessarily radial. Note first that it is enough to prove the theorem for $f \in C^\infty(\mathbf{R}^n)$. Indeed, let $\phi \in C_0^\infty(\mathbf{R}^n)$ be such that $\int \phi\,\mathrm{d}x = 1$. Set $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$. Then $f*\phi_\varepsilon \in C^\infty$ satisfies the assumptions with A replaced by $A+\varepsilon$, see Theorem 1.13. After we prove the theorem for $f*\phi_\varepsilon$, we can take the limit $\varepsilon \to 0$ to prove it for f as well. we could also refer to the distributions chapter

Write in polar coordinates $f = f(r\omega)$ and set $F(r) = \int_{S^{n-1}} f(r\omega)\,\mathrm{d}\omega$. Then $F(|x|)$ is a radial function of x for $|x|>A$ and integrals over any plane at distance greater than A to the origin, vanish. Therefore, $F(r)=0$ for $r>A$. In other words, integrals of f over any sphere $|x|=R>A$ vanish. Move the origin of the coordinate system, and apply the same arguments. We then get that integrals of f over any sphere encompassing the ball $|x|\le A$ (in the original coordinates) vanish. Also, f satisfies (i). explain

Therefore, if $R>A$, and $|y|<R-A$, $f \in C^1$?

$$\int_{S^{n-1}} f(y+R\omega)\,\mathrm{d}\omega = 0. \tag{5.1}$$

Multiply by R^{n-1} and integrate in R from a fixed $R_0 \ge A$ to ∞ to get

$$\int_{|x|\le R} f(y+x)\,\mathrm{d}x = \int_{\mathbf{R}^n} f(x)\,\mathrm{d}x = \text{const}.$$

Differentiate w.r.t. y^i to get

$$\int_{|x|\le R}\frac{\partial f}{\partial x^i}(y+x)\,\mathrm{d}x = 0.$$

By the divergence theorem applied to the function $F(x) = f(x+y)$,

$$\int_{|x|=R} x^i f(y+x)\,\mathrm{d}S_x = 0.$$

By (5.1),

$$\int_{|x|=R} y^i f(y+x)\,\mathrm{d}S_x = 0.$$

Those two inequalities imply that

$$\int_{|x|=R} (y^i + x^i) f(y+x)\,\mathrm{d}S_x = 0.$$

Therefore, (5.1) also holds for $x^i f(x)$. Iterating this argument, we get that for any polynomial $P(x)$, the function $P(x)f(x)$ integrates to zero over any sphere encompassing $B(0,A)$. In particular, on any such sphere, f is orthogonal to all spherical harmonics on it. Therefore, $f = 0$ on any such sphere, and therefore outside $B_R(0)$. Since $R > A$ is arbitrary, this completes the proof. □

In case we a priori know that f is of compact support, Strichartz [**39**] gave the following simple proof.

PROPOSITION 5.2. *Let $f \in C_0(\mathbf{R}^n)$ satisfy condition (ii) of Theorem 5.1. Then $f(x) = 0$ for $|x| > R$.*

PROOF. Again, we can assume that f is smooth. For any $A' > A$, and for $|\phi|$ small enough,

$$\int_{\mathbf{R}^{n-1}} f(A' + x^2 \sin\phi, x^2\cos\phi, x^3, \ldots, x^n)\,\mathrm{d}x' = 0,$$

where $x = (x^1, x')$. Differentiate w.r.t. ϕ, at $\phi = 0$ to get

$$\int_{\mathbf{R}^{n-1}} x^2 \frac{\partial f}{\partial x^1}(A', x')\,\mathrm{d}x' = 0.$$

The convergence of the integrals and a justification for the differentiation is guaranteed by (i). It also allows us to write

$$\frac{\partial}{\partial x^1}\int_{\mathbf{R}^{n-1}} x^2 f(x^1, x')\,\mathrm{d}x' = 0, \quad x^1 > A. \tag{5.2}$$

The integral above is a C^1 function of x^1 that tends to 0, as $x^1 \to \infty$. Integrate (5.2) w.r.t. x^1 from $x_0^1 \geq A$ to infinity to get

$$\int_{\mathbf{R}^{n-1}} x^2 f(x^1, x')\,\mathrm{d}x' = 0, \quad x^1 \geq A.$$

Rotating the coordinate system, we see that for any polynomial P_1 of order 1, the assumptions of the theorem hold for $P_1 f$, as well. Repeat this argument to get that for any polynomial P (of finite degree), integrals of Pf over any plane as in (ii) vanish. Since f has compact support, this implies that $f = 0$ on any such plane. □

The support theorem generalizes easily in several directions. First, f can be a distribution, and the ball $B(0,A)$ can be replaced by a convex compact set.

COROLLARY 5.3. *Let $K \subset \mathbf{R}^n$ be a convex compact set. Let $f \in C(\mathbf{R}^n)$ satisfy the assumption (i) of Theorem 5.1. Assume also that $Rf(\pi) = 0$ for any hyperplane π not intersecting K. Then $f = 0$ outside K.*

PROOF. Choose $x_0 \notin K$. Since K is compact and convex, there exists a hyperplane π_0 so that x_0 and K are on different sides (open half-spaces) of it. Then one can easily construct a closed ball B with a large enough radius, tangent to π_0 so that $K \subset B$, $x_0 \notin B$. Choose a Cartesian coordinate system centered at x_0 and apply Theorem 5.1 to conclude that $f(x_0) = 0$. $\square$

COROLLARY 5.4. *Let $K \subset \mathbf{R}^n$ be a convex compact set. Let $f \in \mathcal{E}'(\mathbf{R}^n)$, and assume that $Rf(\pi) = 0$ in the open set of hyperplanes π not intersecting K. Then $f = 0$ outside K.*

The proof follows by smoothing out f by a convolution, as in the proof of Theorem 5.1. We will leave the details to the reader.

5.2. Support theorems for X. Support theorems of the kind above for X can be derived directly from those for R by working in various 2-dimensional planes, where R and X are the same transforms, or by expressing each hyperplane as a union of lines. This reflects the fact that for $n \geq 3$, the problem of inverting X is overdetermined, i.e., we have "too much information." On the other hand, one can formulate stronger results for X due to the fact that lines in $\mathbf{R}^n$ are "thinner" and can fit into smaller "holes." One possible case is to require K to have the property that any cross section with a plane of the kind $x^n =$ const. to be convex without the need of K to be convex. In ..., we will formulate even stronger results.

6. Range conditions

6.1. Range conditions for the Radon transform R. We start with analyzing the range of $R^\sharp = C_n^{1/2}|D_p|^{(n-1)/2}Rf$, see (3.4). Let $L^2_e(\mathbf{R} \times S^{n-1})$ denote the subspace of $L^2(\mathbf{R} \times S^{n-1})$ of the even functions in that space. That subspace is closed, therefore $L^2_e(\mathbf{R} \times S^{n-1})$ is a Hilbert space itself.

THEOREM 6.1. *$R^\sharp : L^2(\mathbf{R}^n) \to L^2_e(\mathbf{R} \times S^{n-1})$ is unitary.*

PROOF. We show first that for any $f \in L^2(\mathbf{R}^n)$, $R^\sharp f$ is even.

$$R^\sharp f(-p, -\omega) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-\mathrm{i}pr} \mathcal{F}_p R^\sharp f(r, -\omega)\, \mathrm{d}p = \frac{C_n^{1/2}}{2\pi} \int_{\mathbf{R}} e^{-\mathrm{i}pr} |r|^{\frac{n-1}{2}} \hat{f}(-r\omega)\, \mathrm{d}r.$$

Change r to $-r$ to get

$$R^\sharp f(-p, -\omega) = \frac{C_n^{1/2}}{2\pi} \int_{\mathbf{R}} e^{\mathrm{i}pr} |r|^{\frac{n-1}{2}} \hat{f}(r\omega)\, \mathrm{d}r = R^\sharp f(p, \omega).$$

We showed already that $R^\sharp$ is an isometry. Since it is a map between complex Hilbert spaces, it preserves the inner product, as well. What remains to be shown is that it is surjective.

Let $g \in L^2_e(\mathbf{R} \times S^{n-1})$. We want to solve $R^\sharp f = g$. Taking Fourier transform in the p variable, we see that this equation is equivalent to the following

$$\mathcal{F}_p R^\sharp f(r, \omega) = \mathcal{F}_p g(r, \omega) = C_n^{1/2} |r|^{\frac{n-1}{2}} \hat{f}(r\omega) \tag{6.1}$$

for (almost) all $(r, \omega) \in \mathbf{R} \times S^{n-1}$. Based on this, we set

$$\hat{f}(\xi) = C_n^{-1/2} |\xi|^{\frac{1-n}{2}} \mathcal{F}_p g(|\xi|, \xi/|\xi|).$$

We show first that $\hat f$ is well defined and belongs to $L^2(\mathbf{R}^n)$. Indeed,

$$\begin{aligned}\|\hat f\|^2_{L^2(\mathbf{R}^n)} &= C_n^{-1}\int_{\mathbf{R}_+\times S^{n-1}} r^{1-n}|\mathcal{F}_p g(r,\omega)|^2 r^{n-1}\mathrm{d}r\,\mathrm{d}\omega\\ &= 2(2\pi)^n\|g\|_{L^2(\mathbf{R}\times S^{n-1})}<\infty.\end{aligned}$$

Finally, we need to show that (6.1) is indeed satisfied. For $r>0$, we let (r,ω) to be polar coordinates for ξ, and then (6.1) holds in a trivial way. Notice next that since g is an even function, then so is $\mathcal{F}_p g$. Indeed,

$$\begin{aligned}\mathcal{F}_p g(-r,-\omega) &= \int_{\mathbf{R}} e^{\mathrm{i}pr} g(p,-\omega)\,\mathrm{d}p = \int_{\mathbf{R}} e^{\mathrm{i}pr} g(-p,\omega)\,\mathrm{d}p\\ &= \int_{\mathbf{R}} e^{-\mathrm{i}qr} g(q,\omega)\,\mathrm{d}q = \mathcal{F}_p g(r,\omega),\end{aligned}$$

where we used the fact that g is even, and applied the change of variables $q=-p$. For $r<0$, we write $\xi=r\omega$ again, where $\omega=-\xi/|\xi|$. Then by the definition of $\hat f$ we have

$$C_n^{1/2}|r|^{\frac{n-1}{2}}\hat f(r\omega)=\mathcal{F}_p g(-r,-\omega)=\mathcal{F}_p g(r,\omega).$$

Therefore, (6.1) holds for $r<0$, as well. The remaining set $r=0$ is of measure zero in $\mathbf{R}\times S^{n-1}$. □

We consider now the range of R acting on the Schwartz class $\mathcal{S}(\mathbf{R}^n)$. It turns out that the range of R is quite restricted unlike what we got above for $R^\sharp$ on L^2.

Given $k=0,1,\dots$, denote by

$$\mu_k Rf(\omega)=\int_{\mathbf{R}} p^k Rf(p,\omega)\,\mathrm{d}p \tag{6.2}$$

the k-th moment of Rf in the p variable. It follows immediately that

$$\mu_k Rf(\omega)=\int_{\mathbf{R}} s^k Rf(p,\omega)\,\mathrm{d}p=\int_R p^k\int_{x\cdot\omega=p} f(x)\,\mathrm{d}S_x\,\mathrm{d}p=\int_{\mathbf{R}^n}(x\cdot\omega)^k f(x)\,\mathrm{d}x.$$

For a fixed x, the integrand is a homogeneous polynomial of ω of degree k restricted to the sphere S^{n-1}. Integrating in x, we get the same conclusion for $\mu_k Rf(\omega)$. Therefore, if g is in the range of R, acting on the Schwartz class, then $\mu_k g(\omega)$ must be a homogeneous polynomial restricted to the sphere S^{n-1}. In other words, the homogeneous extension of $\mu_k g(\omega)$ from the sphere to $\mathbf{R}^n$ as a homogeneous function of order k must be a polynomial. This is a rather restrictive property.

Let $\mathcal{S}(\mathbf{R}\times S^{n-1})$ be the linear space of all $g\in C^\infty(\mathbf{R}\times S^{n-1})$ with the property that for any $k\ge 0$, $\ell\ge 0$, and any differential operator P on S^{n-1},

$$\sup_{(p,\omega)\in\mathbf{R}\times S^{n-1}}\left|(1+|p|^k)\frac{\mathrm{d}^\ell}{\mathrm{d}p^\ell}Pg(p,\omega)\right|<\infty. \tag{6.3}$$

The expression above defines a countable set of seminorms on $\mathcal{S}(\mathbf{R}\times S^{n-1})$. We know that for f in the Schwartz class, Rf is even and must satisfy the moment condition discussed above. Let $\mathcal{S}_H(\mathbf{R}\times S^{n-1})$ be the subspace of $\mathcal{S}(\mathbf{R}\times S^{n-1})$ subject to those conditions.

Definition 6.2. *Denote by $\mathcal{S}_H(\mathbf{R}\times S^{n-1})$ the subspace of $\mathcal{S}_H(\mathbf{R}\times S^{n-1})$ consisting of all functions $g\in\mathcal{S}_H(\mathbf{R}\times S^{n-1})$ satisfying*

(i) $g(p,\omega)=g(-p,-\omega)$

(ii) *For any* $k = 0, 1, \dots,$ $\mu_k g(\omega)$ *is a homogeneous polynomial of* ω *of degree* k*.*

THEOREM 6.3. $R : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}_H(\mathbf{R} \times S^{n-1})$ *is a linear bijection.*

PROOF. We start with an informal discussion emphasizing the role of the moment conditions. To prove the theorem, given $g \in \mathcal{S}_H(\mathbf{R} \times S^{n-1})$, we want to solve the equation $Rf = g$, and get a solution f in the Schwartz class. By the Fourier Slice Theorem,

$$\hat{f}(r\omega) = \mathcal{F}_p g(r, \omega). \tag{6.4}$$

We can use this as a definition of f, at least when $r > 0$. Since g is even, nothing will change if we take $r < 0$. Clearly then $\hat{f} \in \mathcal{S}(\mathbf{R}^n \setminus 0)$ but the regularity of $\hat{f}$ at 0 is under question because the polar coordinates are singular at the origin. To check the smoothness of $\hat{f}$ at 0 let us first see whether $\hat{f}(0)$ is well defined. We get

$$\hat{f}(0) = \mathcal{F}_p(0, \omega) = \int_{\mathbf{R}^n} g(p, \omega)\, \mathrm{d}p = \mu_0 g(\omega).$$

Now, in order that $\hat{f}(0)$ be well defined, it must be independent of ω, at least. The 0-th order moment condition however says exactly that, since homogeneous polynomials of degree 0 are the constant ones.

Next, let us see whether the first derivatives of $\hat{f}$ at 0 are well defined. Let us first try to understand what they should be. Since $\hat{f}$ has to be smooth at 0, it has a finite Taylor expansion of the form $\hat{f}(\xi) = \hat{f}(0) + a \cdot \xi + O(|\xi|^2)$ near $\xi = 0$, where $a = \nabla_\xi \hat{f}(0)$. Writing this in polar coordinates, we get $\hat{f}(r\omega) = \hat{f}(0) + ra \cdot \omega + O(r^2)$, near $r = 0$. Differentiate with respect to r at $r = 0$ to get

$$\partial_r \hat{f}(r\omega)|_{r=0} = a \cdot \omega.$$

In other words, the left-hand side above must be a homogeneous polynomial of ω of degree 1. We need to check now that our definition (6.4) of $\hat{f}$ satisfies this condition, at least. Differentiate (6.4) to get

$$\partial_r \hat{f}(r\omega)|_{r=0} = \partial_r\big|_{r=0} \int_{\mathbf{R}} e^{-\mathrm{i}pr} g(p, \omega)\, \mathrm{d}p = \int_{\mathbf{R}} (-\mathrm{i}p) g(p, \omega)\, \mathrm{d}p = -\mathrm{i}\mu_1 g(\omega).$$

The latter is a linear form of ω, by the 1-st moment condition, and this is exactly what we wanted to check.

Continuing in this manner, we see that

$$\partial_r^k \hat{f}(r\omega)|_{r=0} = \sum_{|\alpha|=k} a_\alpha \omega^\alpha,$$

where a_α is proportional to the partial derivative $\partial^\alpha \hat{f}(0)$ of order k at 0. Our definition (6.4) of $\hat{f}$ should be consistent with the requirement that the left-hand side above is a homogeneous polynomial of degree k. Differentiate (6.4) at $r = 0$ k times to get

$$\partial_r^k \hat{f}(r\omega)|_{r=0} = \partial_r^k\big|_{r=0} \int_{\mathbf{R}} e^{-\mathrm{i}pr} g(p, \omega)\, \mathrm{d}p = \int_{\mathbf{R}} (-\mathrm{i}p)^k g(p, \omega)\, \mathrm{d}p = (-\mathrm{i})^k \mu_k g(\omega).$$

That is a homogeneous polynomial of degree k, indeed, by the moment conditions.

We proceed with the formal proof now. We will show first that R maps $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}_H(\mathbf{R} \times S^{n-1})$. It is enough to work in a neighborhood $U \subset S^{n-1}$, of a point where $\omega' = (\omega^1, \dots, \omega^{n-1})$ can be chosen to be local coordinates. We are going to use the Fourier Slice Theorem next, so for this reason we verify (6.3) for $g = \hat{f}(s\omega)$ on $\mathbf{R} \times S^{n-1}$ first. Since f, and therefore, $\hat{f}$ are in the Schwartz class, it follows immediately that

$$\operatorname{supp}_{(p,\omega)\in\mathbf{R}\times U} \left|(1+|s|^k)\frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell}(P\hat{f})(p,\omega)\right| < \infty. \tag{6.5}$$

By the Fourier Slice Theorem,

$$Rf(p,\omega) = (2\pi)^{-1}\int_{\mathbf{R}} e^{\mathrm{i}ps}\hat{f}(s\omega)\,\mathrm{d}s.$$

Then for any differential operator P on S^{n-1},

$$(1+|p|^2)^k\frac{\mathrm{d}^\ell}{\mathrm{d}p^\ell}PRf(p,\omega) = (2\pi)^{-1}\int e^{\mathrm{i}ps}\left(1-\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right)^k(\mathrm{i}s)^\ell P\hat{f}(s\omega)\,\mathrm{d}s.$$

Estimate (6.5) then implies that the r.h.s. above is bounded on $\mathbf{R}\times U$, which proves that $Rf \in \mathcal{S}_H(\mathbf{R}\times S^{n-1})$.

We prove now that the map R is surjective, i.e., that given $g \in \mathcal{S}_H(\mathbf{R}\times S^{n-1})$, we can solve the equation $Rf = g$ with $f \in \mathcal{S}(\mathbf{R}^n)$.

Set

$$\psi(s,\omega) = \int_{\mathbf{R}} e^{-\mathrm{i}ps}g(p,\omega)\,\mathrm{d}p.$$

If such an f exists, by the Fourier Slice Theorem, its Fourier transform $\hat{f}$ will be given by $\hat{f}(\xi) = \psi(|\xi|, \xi/|\xi|)$. Note that ψ is an even function of (s,ω) first. Next, the zero order moment condition implies that $\psi(0,\omega)$ is a constant. Therefore, the following function is well defined

$$F(\xi) = \psi(|\xi|, \xi/|\xi|). \tag{6.6}$$

We prove next that $F \in \mathcal{S}(\mathbf{R}^n)$. Assume for a moment that we have proved that. Then we define f by $\hat{f} = F$; then $f \in \mathcal{S}(\mathbf{R}^n)$ as well. By the Fourier Slice Theorem, for $s \geq 0$,

$$\int_{\mathbf{R}} e^{-\mathrm{i}ps}Rf(p,\omega)\,\mathrm{d}p = \hat{f}(s\omega) = F(s\omega) = \int_{\mathbf{R}} e^{-\mathrm{i}ps}g(p,\omega)\,\mathrm{d}s.$$

For $s \leq 0$, by (6.6) we have $F(s\omega) = \psi(-s,-\omega) = \psi(s,\omega)$, and the formula above still holds. Therefore, $Rf = g$ as claimed.

We prove now that $F \in \mathcal{S}(\mathbf{R}^n)$. Write

$$e^{-\mathrm{i}ps} = \sum_{j=0}^{k}\frac{(-\mathrm{i}ps)^j}{j!} + e_k(-\mathrm{i}ps),$$

$e_k(-\mathrm{i}ps)$ is the remainder in the Taylor expansion of the function $s \mapsto e^{-\mathrm{i}ps}$ about $s = 0$. Its integral representation is

$$e_k(-\mathrm{i}ps) = \int_0^s \frac{(-\mathrm{i}p)^{k+1}}{(k+1)!}e^{-\mathrm{i}pt}(s-t)^k\,\mathrm{d}t = \frac{(-\mathrm{i}p)^{k+1}}{(k+1)!}\rho_k(s,p),$$

where

$$\rho_k(s,p) = \int_0^s e^{-\mathrm{i}pt}(s-t)^k\,\mathrm{d}t. \tag{6.7}$$

Clearly, for $s \geq 0$,

$$\left|\frac{\partial^j \rho_k(s,p)}{\partial s^j}\right| \leq \frac{k!}{(k-j)!} s^{k+1-j}, \quad 0 \leq j \leq k. \tag{6.8}$$

Then

$$\begin{aligned} F(s\omega) &= \sum_{j=0}^{k} \int_{\mathbf{R}} \frac{(-\mathrm{i}ps)^j}{j!} g(p,\omega)\,\mathrm{d}p + \int_{\mathbf{R}} e_{k+1}(-\mathrm{i}ps) g(p,\omega)\,\mathrm{d}p \\ &= \sum_{j=0}^{k} \frac{(-\mathrm{i})^j}{j!} s^j \mu_j g(\omega) + R_k(s,\omega), \end{aligned}$$

where

$$R_k(s,\omega) = \int_{\mathbf{R}} \frac{(-\mathrm{i}p)^{k+1}}{(k+1)!} \rho_k(s,p) g(p,\omega)\,\mathrm{d}p. \tag{6.9}$$

Now, by assumption,

$$s^j \mu_j g(\omega) = \sum_{|\alpha|=j} a_\alpha (s\omega)^\alpha$$

for some constants a_α, where α denotes a multiindex. We therefore get

$$F(\xi) = \sum_{|\alpha|\leq k} a_\alpha \xi^\alpha + R_k(|\xi|, \xi/|\xi|). \tag{6.10}$$

It is enough to show that for any k, the remainder term above is in C^k (near $\xi = 0$). By (6.7) and (6.8) for $k = j = 0$, $|\rho_0| \leq s$; therefore by (6.9), $|R_0(s,\omega)| \leq Cs$. Then (6.10) implies that F is continuous at $\xi = 0$, and is therefore a continuous function.

The differentiability of F at the origin follows in a similar way from (6.10) for $k = 1$, by definition. Indeed,

$$F(\xi) = F(0) + \sum_{|\alpha|=1} a_\alpha \xi^\alpha + R_1(|\xi|, \xi/|\xi|), \tag{6.11}$$

and (6.8) for $k = 1$, $j = 0$ and (6.9) imply $R_1(|\xi|, \xi/|\xi|) = O(|\xi|^2)$. Then F is differentiable at the origin by definition, and of course, it is differentiable everywhere else.

We show next that the first derivatives of F are continuous. Compute first

$$\frac{\partial}{\partial \xi_j} R_k(|\xi|, \xi/|\xi|) = \frac{\partial R_k}{\partial s} + \frac{1}{s}\frac{\partial R_k}{\partial \omega_i}(\delta_{ij} - \omega_i\omega_j) \quad \text{evaluated at } s = |\xi|,\ \omega = \xi/|\xi|.$$

We regard $R_k(s,\omega)$ as a (smooth) function defined in $\mathbf{R} \times (\mathbf{R}^n \setminus 0)$ obtained form the original one by a homogeneous extension of order 0 with respect to ω.

***** to be continued ***** □

6.2. Range conditions for X; John's Equations. For $f \in \mathcal{S}(\mathbf{R}^n)$, we extend Xf to $\mathbf{R}^n \times (\mathbf{R}^n \setminus 0)$ by keeping (1.2) unchanged but allowing θ, that we call ξ now, to be non-unit:

$$Xf(x,\xi) = \int_{\mathbf{R}} f(x + s\xi)\,\mathrm{d}s, \quad (x,\xi) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus 0). \tag{6.12}$$

It is straightforward to check that $Xf(x,\xi)$ is a positively homogeneous of order -1 in ξ. The parameterization by points in Σ, see section 1.3, and the present extension are related by

$$Xf(x,\xi) = \frac{1}{|\xi|}Xf(z,\theta), \quad z = x-(x\cdot\theta)\theta, \quad \theta = \xi/|\xi|. \tag{6.13}$$

PROPOSITION 6.4 (John's equations). *Let $f \in \mathcal{S}(\mathbf{R}^n)$. Then the following partial differential equations hold for all $(x,\xi) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus 0)$:*

$$\left(\frac{\partial^2}{\partial x^i\partial\xi^j} - \frac{\partial^2}{\partial x^j\partial\xi^i}\right)Xf(x,\xi) = 0, \quad \forall i,\, j. \tag{6.14}$$

PROOF. In fact, the integrand satisfies those equations:

$$\left(\frac{\partial^2}{\partial x^i\partial\xi^j} - \frac{\partial^2}{\partial x^j\partial\xi^i}\right)f(x+s\xi) = 0$$

because

$$\frac{\partial^2}{\partial x^i\partial\xi^j}f(x+s\xi) = s\frac{\partial^2 f}{\partial x^i\partial x^j}(x+s\xi)$$

is symmetric with respect to (i,j). Integrate in s to get John's equations. □

It turns out that in dimension $n \geq 3$, John's equations uniquely characterize the range of X.

THEOREM 6.5. *Let $n \geq 3$, $g \in C_0^\infty(\Sigma)$, and define its extension to $\mathbf{R}^n \times (\mathbf{R}^n \setminus 0)$ as in (6.13), i.e.,*

$$g(x,\xi) = \frac{1}{|\xi|}g\left(z - \left(\frac{\xi}{|\xi|}\cdot x\right)\frac{\xi}{|\xi|}, \frac{\xi}{|\xi|}\right). \tag{6.15}$$

Then there exists $f \in C_0^\infty(\mathbf{R}^n)$ so that $g = Xf$ if an only if $g(x,\xi) = g(x,-\xi)$ for all x, ξ, and

$$\left(\frac{\partial^2}{\partial x^i\partial\xi^j} - \frac{\partial^2}{\partial x^j\partial\xi^i}\right)g(x,\xi) = 0, \quad \forall i,\, j. \tag{6.16}$$

PROOF. ++++++++++++++ □

PROBLEM 6.1. Show that Theorem 6.5 does not hold for $n = 2$. More precisely, let g be as in (6.15); then show that (6.16) holds for $i = 1$, $j = 2$ (the only interesting case), regardless of whether g is the Radon transform of some function.

7. The Euclidean Doppler Transform

Let $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$ be a complex-valued vector field in $\mathbf{R}^n$. We define the X-ray transform of f as the following map

$$Xf(\ell) = \int_\ell f_j(x)\, \mathrm{d}x^j, \tag{7.1}$$

if the integral exists, where ℓ runs over the set of all *directed* lines in $\mathbf{R}^n$. If the lines are parameterized by (x, θ) as in section 1.1, then we can write

$$Xf(x,\theta) = \int_{\mathbf{R}} f_j(x + s\theta)\theta^j\, \mathrm{d}s, \tag{7.2}$$

compare with (1.2). As before, the parameterization is not unique but now Xf is an odd function of θ:

$$Xf(x,\theta) = Xf(x + t\theta, \theta), \quad Xf(x,\theta) = -Xf(x, -\theta). \tag{7.3}$$

The integral makes sense for any $f \in L^1(\mathbf{R}^n, \mathbf{C}^n)$.

Unique parameterization of Xf on the manifold of the directed line can be obtained by restricting (x, θ) to Σ as in section 1.1. When f is supported in a bounded strictly convex domain $\Omega \subset \mathbf{R}^n$, one can also think of Xf as a map from $\partial_- S\Omega$ to $\mathbf{C}^n$.

7.1. Motivation. Doppler tomography, also called vector (field) tomography appeared first in a work by Norton [**26**]. The motivating example there was acoustic imaging of a vector field, for example imaging blood flow by ultrasound. If the speed of sound is $c = \text{const.}$, and if that the velocity field f of the fluid is much smaller than c, using approximation, one of the assumptions is that the trajectories through the fluid can be approximated by straight lines but the speed along the line would be $c + f(x)\cdot\theta$. Then the travel time between two points a and b on the boundary of a domain, connected by a line is

$$\int_a^b \frac{\mathrm{d}l}{c + f(x)\cdot\theta} \approx \int_a^b \left(\frac{1}{c} - \frac{1}{c^2} f(x)\cdot\theta\right) \mathrm{d}l = \frac{b-a}{c} - \frac{1}{c^2}\int_a^b f(x)\cdot\theta\, \mathrm{d}l.$$

If we measure the travel time in the opposite direction, we get the same expression with a sum instead of a difference. We can therefore recover both c and the Doppler transform of f.

7.1.1. *The transpose and solenoidal injectivity.* We will see in section ??? that the transformation law of f under coordinate changes is as a covector field. We therefore can identify f with the form $f = f_j(x)\mathrm{d}x^j$, see also (7.1). This point of view is adopted in ..., while here, we fix the coordinate system. We still use lower indices for the components of f to conform with the notation in the next chapters but we identify $\{f_j\}$ and $\{f^j\}$.

The natural pairing of vector fields is given by

$$(f, v) = \int_{\mathbf{R}^n} f_j(x) v^j(x)\, \mathrm{d}x, \tag{7.4}$$

where $v = (v^1(x), \dots, v^n(x))$ is another vector field. In particular, distribution valued vector fields $f \in \mathcal{D}'(\mathbf{R}^n, \mathbf{C}^n)$ are defined as continuous linear functionals on the space $C_0^\infty(\mathbf{R}^n, \mathbf{C}^n)$. When $f \in L^1_{\text{loc}}(\mathbf{R}^n, \mathbf{C}^n)$, we identify f and the linear functional given by (7.4).

PROBLEM 7.1. Show that the transpose X' is given by

$$(X'\psi(x))^j = \int_{S^{n-1}} \theta^j \psi(x-(x\cdot\theta)\theta,\theta)\,\mathrm{d}\theta, \quad \psi\in C_0^\infty(\Sigma). \tag{7.5}$$

One can extend X to $\mathcal{E}'(\mathbf{R}^n,\mathbf{C}^n)$ in the same way as in (1.3), see Definition 1.4. Similarly, in a bounded open set Ω, one can define $Xf\in\mathcal{E}'(\partial_- S\Omega)$, and

$$X : L^2(\Omega,\mathbf{C}^n)\to L^2(\partial_- S\Omega,\mathrm{d}\mu)$$

is bounded, where

$$\|f\|^2_{L^2(\Omega,\mathbf{C}^n)} = \int_\Omega \sum_j |f_j(x)|^2\,\mathrm{d}x. \tag{7.6}$$

Then X^* is well defined.

In contrast to the X-ray transform of functions, now X has an obvious infinite dimensional kernel. By the Fundamental Theorem of Calculus, for any $\phi\in\mathcal{S}(\mathbf{R}^n)$, the vector field $f=\mathrm{d}\phi$ (i.e., $f=\{f_j\}$ with $f_j=\partial\phi/\partial x^j$) belongs to the kernel of X:

$$X(\mathrm{d}\phi)=0, \quad \text{for any } \phi\in\mathcal{S}(\mathbf{R}^n). \tag{7.7}$$

It turns out that this is the only obstruction to uniqueness, as we will see below.

The transforms X of vector fields is closely related to the Fourier transform, as well. The next theorem can be considered as the Fourier Slice Theorem for X acting on vector fields.

THEOREM 7.1. *For any $f\in L^1(\mathbf{R}^n)$,*

$$\hat f(\zeta)\cdot\theta = \int_{\theta^\perp} e^{-\mathrm{i}z\cdot\zeta} Xf(z,\theta)\,\mathrm{d}S_z, \quad \forall\theta\perp\zeta,\ \theta\in S^{n-1}. \tag{7.8}$$

The proof follows directly from Theorem 1.7 by applying the latter to each component f_j or by repeating the proof.

should it be introduced elsewhere?

We recall that the exterior derivative of the 1-form f is given by

$$\mathrm{d}f = \frac{\partial f_j}{\partial x^i}\mathrm{d}x^i\wedge\mathrm{d}x^j = \sum_{i<j}\left(\frac{\partial f_j}{\partial x^i}-\frac{\partial f_i}{\partial x^j}\right)\mathrm{d}x^i\wedge\mathrm{d}x^j. \tag{7.9}$$

The exterior differential $\mathrm{d}f$ can also be considered as an anti-symmetric tensor field with components $(df)_{ij}=(1/2)(\partial f_j/\partial x^i-\partial f_i/\partial x^j)$. We also introduce the notation

$$\langle df, v\otimes w\rangle = (df)_{ij}v^iw^j = \frac12\left(\frac{\partial f_j}{\partial x^i}-\frac{\partial f_i}{\partial x^j}\right)v^iw^j. \tag{7.10}$$

Clearly, that is an anti-symmetric form.

THEOREM 7.2. *Let $f\in C(\mathbf{R}^n)$, and let $|f(x)|\le C(1+|x|)^{-n-\varepsilon}$ with some $C>0$, $\varepsilon>0$. The following statements are equivalent:*

(i) $Xf=0$,
(ii) $f=\mathrm{d}\phi$ *for some* $\phi\in C^1(\mathbf{R}^n)$,
(iii) $\mathrm{d}f=0$.

***** One could try $|f(x)|\le C(1+|x|)^{-1-\varepsilon}$. It works, but one has to extend the Fourier Slice Theorem for that class ($\hat f$ is a distribution then). Difficulties: one has to explain why $\hat f$ has a trace on $\theta^\perp$. *****

PROOF. Note first that the decay condition on f implies that Xf is well defined. The implication (ii) $\Rightarrow$ (i), as explained above, is trivial when f is compactly supported. Under the assumptions of the theorem, write in polar coordinates, using (ii),

$$\phi(r\theta) - \phi(0) = \int_0^r \frac{\mathrm{d}}{\mathrm{d}t}\phi(t\theta)\,\mathrm{d}t = \int_0^r f_j(t\theta)\theta^j\,\mathrm{d}t, \tag{7.11}$$

therefore, $|\phi(x)| \le C(1+|x|)^{-n+1-\varepsilon}$. This yields easily $X(\mathrm{d}\phi) = 0$.

Next, (ii) and (iii) are equivalent by the Poncaré Lemma in $\mathbf{R}^n$. It remains to prove the implication (i) $\Rightarrow$ (iii).

Assume (i). Since $f \in L^1(\mathbf{R}^n, \mathbf{C}^n)$, $\hat f$ is continuous by the Riemann-Lebesgue Lemma. By the Fourier Slice Theorem, $\hat f(\xi)\cdot\eta = 0$ for any η with $\eta\cdot\xi = 0$. On the other hand, for any v, w,

$$\begin{aligned}\frac{1}{\mathrm{i}}\langle \widehat{\mathrm{d}f}(\xi), v\otimes w\rangle &= \frac12\left(\hat f_j(\xi)\xi_i v^i w^j - \hat f_i(\xi)\xi_j v^i w^j\right)\\ &= \frac12 \hat f(\xi)\cdot(w(\xi\cdot v) - v(\xi\cdot w)) = 0\end{aligned}$$

because the vector $w\xi\cdot v - v\xi\cdot w$ is orthogonal to ξ. Therefore, $\mathrm{d}f = 0$. □

It may seem unexpected that we did not require $\phi(x)\to 0$, as $|x|\to\infty$ in (ii). In fact, up to adding a constant to ϕ, the latter is always true, by (7.11). On the other hand, that constant would be annihilated by the differential. Therefore, each of the conditions in the theorem is also equivalent to the following

(ii′) $f = \mathrm{d}\phi$ for some $\phi\in C^1(\mathbf{R}^n)$, $\phi(x)\to 0$, as $|x|\to\infty$.

We refer to the implication (i) $\Rightarrow$ (ii') as *solenoidal injectivity* of X.

COROLLARY 7.3. *Assume that $f \in C_0(\mathbf{R}^n)$. If $Xf = 0$, than $f = \mathrm{d}\phi$ with some $\phi \in C_0^1(\mathbf{R}^n)$, and $\phi = 0$ in the largest unbounded connected open set where $f = 0$. In particular,* $\operatorname{supp} f$ *is contained in the convex hull of* $\operatorname{supp}\phi$.

PROOF. For any x_1 and x_2 belonging to the same connected component of $\mathbf{R}^n\setminus\operatorname{supp} f$, integrate $f = \mathrm{d}\phi$ from x_1 to x_2 along a smooth curve that connects them to get $\phi(x_1) = \phi(x_2)$. Therefore, ϕ is constant in any such component. In the unbounded one it has to vanish because $\phi(x)\to 0$, as $|x|\to\infty$ by the proof of Theorem 7.2. □

7.2. Support Theorems. The following proposition allows us to use the support theorems for functions to formulate support theorems for vector fields.

PROPOSITION 7.4. *Let $n = 2$, and let $f \in C^1(\mathbf{R}^2, \mathbf{C}^2)$ be such that $|f|$ and $|\partial f_i/\partial x^j|$ are bounded by $C(1+|x|)^{-1-\varepsilon}$ with some $C>0$, $\varepsilon>0$. Then for any x, θ,*

$$\frac{\partial}{\partial x^2}Xf(x,\theta) = \theta^1\int_{\ell_{x,\theta}}\left(\frac{\partial f_1}{\partial x^2} - \frac{\partial f_2}{\partial x^1}\right)\mathrm{d}s.$$

PROOF. Let $n\ge 2$ be arbitrary first. Fix $\theta\in S^{n-1}$, and let $v\in\mathbf{R}^n$. Then

$$(v\cdot\nabla_x)Xf(x,\theta) = \int_{\mathbf{R}}(v\cdot\nabla_x)f(x+t\theta)\cdot\theta\,\mathrm{d}t.$$

On the other hand,

$$\int_{\mathbf{R}} (\theta\cdot\nabla_x) f(x+t\theta)\cdot v\,\mathrm{d}t = 0$$

in a trivial way, by the Fundamental Theorem of Calculus. Subtracting those two identities, we get

$$(v\cdot\nabla_x)Xf(x,\theta) = \int_{\mathbf{R}} \big((v\cdot\nabla_x)f(x+t\theta)\cdot\theta - (\theta\cdot\nabla_x)f(x+t\theta)\cdot v\big)\mathrm{d}t. \tag{7.12}$$

Using the notation (7.10), we can write this as

$$(v\cdot\nabla_x)Xf(x,\theta) = \int_{\ell_{x,\theta}} \langle df, v\otimes\theta\rangle\,\mathrm{d}s \tag{7.13}$$

where $\ell_{x,\theta}$ is the line determined by (x,θ). Since the form (7.10) is anti-symmetric, we can replace θ by $\theta+Cv$ for any scalar C in the integrand above without changing it. Assume now that v and w are unit and orthogonal to each other, and θ belongs to their span. Then $(w\cdot\theta)w = \theta-(v\cdot\theta)v$. Therefore, we then get

$$(v\cdot\nabla_x)Xf(x,\theta) = (w\cdot\theta)\int_{\ell_{x,\theta}} \langle df, v\otimes w\rangle\,\mathrm{d}s. \tag{7.14}$$

The assumptions on f guarantee that f and the first derivatives of f are absolutely integrable over any line, and justify the differentiation under the integral sign.

Let now $n=2$, and take now $w=e_1$, $v=e_2$ to finish the proof. □

REMARK 7.1. An alternative way to prove the proposition is to apply the divergence theorem to half-discs and allow their radii to converge to infinity. Note also that applying the same formula for the x^1 derivative, and combining them both, we get the more symmetric expression

$$-\theta^\perp\cdot\partial_x Xf(x,\theta) = \int_{\ell_{x,\theta}} \left(\frac{\partial f_1}{\partial x^2}-\frac{\partial f_2}{\partial x^1}\right)\mathrm{d}s, \quad \theta^\perp := (\theta^2,-\theta^1). \tag{7.15}$$

Let now $Xf(\ell)=0$ for all lines ℓ not intersecting some set $K\subset\mathbf{R}^n$. Then we can work in any plane π parallel to the coordinate plane x^1x^2 first, and consider $Xf(\ell)$ restricted to lines ℓ in π. Then $Xf(\ell)=0$ for ℓ not interesting K implies that the X-ray transform of the function $\partial f_1/\partial x^2-\partial f_2/\partial x^1$ vanishes over all lines in π not intersecting $\pi\cap K$. Then one can apply the support theorem for functions, provided that the regularity and the decay assumptions are met, and that $\pi\cap K$ is compact and convex. Now, one can repeat this argument for all planes parallel to any other 2-dimensional coordinate plane of the type x^ix^j to get $\mathrm{d}f=0$ outside K.

Those arguments allow us to formulate the following analogue to Corollary 5.3 for the the X-ray transform of vector fields.

THEOREM 7.5. *Let $K\subset\mathbf{R}^n$ be a convex compact set. Let the vector field $f\in C^1(\mathbf{R}^n)$ be such that f and its first partial derivatives satisfy the assumption (i) of Theorem 5.1. Assume also that $Xf(\ell)=0$ for any line ℓ not intersecting K. Then $\mathrm{d}f=0$ outside K, and there exists a function $\phi\in C^2(\mathbf{R}^n\setminus K)$ so that $f=\mathrm{d}\phi$ in $\mathbf{R}^n\setminus K$ with ϕ satisfying the rapid decay condition (i) of Theorem 5.1.*

PROOF. We already showed how Theorem 5.1 implies $\mathrm{d}f=0$ outside K. The conclusion that the form f must be exact then seems unjustified at first glance because when $n = 2$, $\mathbf{R}^2 \setminus K$ is not simply connected. The reason that we can still claim that f is exact are the decay conditions at infinity.

We can always assume that the origin is in K. Using polar coordinates, set

$$\phi(x) = -\int_r^\infty f_j(t\theta)\theta^j\,\mathrm{d}t, \tag{7.16}$$

see also (7.11). The integral converges absolutely and can be differentiated once under the integral sign by the assumption of the theorem. We claim that $f = \mathrm{d}\phi$ outside K. One way to check this is to differentiate (7.16) directly. Another way is to do the following. Fix $x_0 \in \mathbf{R}^n \setminus K$. For any y in some ball $B(x_0, \varepsilon)$, with $\varepsilon > 0$ so that the later ball is outside K, we can deform the contour of integration in (7.16) from the radial line segment $[y, \infty y)$ to the path $p_{y,R} := [y, Ry] \cup [Ry, Rx_0] \cup [Rx_0, \infty x_0)$. When R is large enough, that path is outside K. The integral

$$\phi_R(y) := -\int_{p_{y,R}} f\,\mathrm{d}x$$

of $f\mathrm{d}x$ over that path converges to the integral (7.16), equal to $\phi(y)$, as $R \to \infty$ as a consequence of the rapid decay of f. We also have $\mathrm{d}\phi_R \to \mathrm{d}\phi$ because the first derivatives of f decay rapidly as well.

On the other hand, in a small enough simply connected neighborhood of the ray $[y, \infty y)$, $f = \mathrm{d}\psi$ for some C^2 function ψ. Then $\phi_R(y) = \psi(y) - \psi(Rx_0)$. Differentiate to get $\mathrm{d}\phi_R(y) = f(y)$. Take the limit $R \to \infty$ to conclude that $f = \mathrm{d}\phi$. □

7.3. Decomposition into solenoidal and potential parts. As we showed above, we can only recover f up to a differential $\mathrm{d}\phi$, with ϕ decaying in a certain way at infinity. Theorem 7.2 says that within the class of f's studied there, $Xf_1 = Xf_2$ if and only if there exists ϕ decaying at ∞ (and this property determines ϕ uniquely), so that $f_1 = f_2 + \mathrm{d}\phi$. The latter condition is an equivalence relation, and another way to formulate the theorem is to say that Xf recovers uniquely the equivalence class.

The non-uniqueness of recovery of f raises the following question. Is there some "natural" representative of f in each class, that we can recover easily and explicitly? The answer depends on what we think is "natural" and what recovery looks "easy", and here we present one way of doing that.

Since X vanishes on all $\mathrm{d}\phi$ with certain decay at infinity, we can try to find the orthogonal complement of that space with respect to some Hilbert structure. One such choice is $L^2(\mathbf{R}^n)$ defined in (7.6). We do not have a definition of Xf on that space, however. We can still use that orthogonality structure but restricted to a subspace. When we work with fields supported in a fixed compact, then L^2 functions are L^1, as well, therefore X is well defined on such functions.

Let $f \in \mathcal{S}(\mathbf{R}^n)$ be orthogonal to $\mathrm{d}\phi$ for all $\phi \in \mathcal{S}(\mathbf{R}^n)$. Integrating by parts, we get

$$0 = \int_{\mathbf{R}^n} f \cdot \mathrm{d}\phi\,\mathrm{d}x = \int_{\mathbf{R}^n} f_i \frac{\partial \bar{\phi}}{\partial x^i}\,\mathrm{d}x = -\int_{\mathbf{R}^n} \sum_i \frac{\partial f_i}{\partial x^i}\bar{\phi}\,\mathrm{d}x$$

for any such ϕ. Therefore, the divergence of f must vanish:

$$\delta f = 0, \quad \text{where } \delta f = \sum_i \frac{\partial f_i}{\partial x^i}.$$

This is the basis for the following.

DEFINITION 7.6. *The vector field f is called solenoidal in $\mathbf{R}^n$, if $\delta f = 0$. It is called potential in $\mathbf{R}^n$, if $f = \mathrm{d}\phi$ with some $\phi(x) \to 0$, as $|x| \to \infty$.*

We intentionally did not specify the space of where f and ϕ belong because that can vary with the applications. Of course, we must requite at least Xf and $X\mathrm{d}\phi$ to be well defined. That can include spaces of distributions as well.

THEOREM 7.7. *For any $f \in L^2(\mathbf{R}^n)$ of compact support, there exists a uniquely determined vector field $f^s_{\mathbf{R}^n} \in L^2(\mathbf{R}^n)$ and a function $\phi_{\mathbf{R}^n} \in H^1_{\text{loc}}(\mathbf{R}^n)$, smooth away from* $\operatorname{supp} f$*, so that*

$$f = f^s_{\mathbf{R}^n} + \mathrm{d}\phi_{\mathbf{R}^n}, \quad \delta f^s_{\mathbf{R}^n} = 0, \quad \phi_{\mathbf{R}^n}(x) \to 0, \ \text{as } |x| \to \infty.$$

Moreover, the following estimates are satisfied for $|x|$ large enough

$$|f^s_{\mathbf{R}^n}(x)| + |\mathrm{d}\phi_{\mathbf{R}^n}(x)| \le C(1+|x|)^{-n}, \quad |\phi_{\mathbf{R}^n}(x)| \le C(1+|x|)^{1-n}. \tag{7.17}$$

PROOF. Assume first that the decomposition can be done. Then $\delta f = \delta \mathrm{d}\phi_{\mathbf{R}^n} = \Delta_{\mathbf{R}^n}\phi$. Therefore, $\phi_{\mathbf{R}^n}$ has to solve the equation

$$\Delta\phi_{\mathbf{R}^n} = \delta f. \tag{7.18}$$

To prove the uniqueness it is enough to show that if $f = 0$, then $\phi_{\mathbf{R}^n} = 0$. That follows from the fact that the only harmonic function decaying at infinity is the zero one.

To prove the existence, we set $\phi_{\mathbf{R}^n} = \delta G_n * f$, where G_n is the standard fundamental solution of the Laplacian in $\mathbf{R}^n$

$$G_n f = \frac{1}{(2-n)|S^{n-1}|}|x|^{2-n}, \quad n \ge 3, \quad G_2(x) = \frac{1}{2\pi}\log|x|.$$

In both cases, we get

$$\phi_{\mathbf{R}^n}(x) = \frac{1}{|S^{n-1}|} f_i * \frac{x^i}{|x|^n}. \tag{7.19}$$

To see that this is well defined, choose $\chi \in C_0^\infty(\mathbf{R})$ with $\chi = 1$ near the origin. Then write

$$|S^{n-1}|\phi_{\mathbf{R}^n}(x) = f_i * \left(\chi(|x|)\frac{x^i}{|x|^n}\right) + f_i * \left((1-\chi(|x|))\frac{x^i}{|x|^n}\right). \tag{7.20}$$

The first term on the right is in $L^2(\mathbf{R}^n)$ by Young's inequality. To prove that it belongs to H^1, one can use the theory of operators with singular kernels but we will use elliptic regularity below instead. This also follows from the Fourier transform arguments follwoing the proof. The second term is a convolution of the compactly supported $f \in L^2$ with a smooth function, and this convolution is a smooth function.

We have $\Delta\phi_{\mathbf{R}^n} = \Delta\delta G_n * f = \Delta G_n * (\delta f) = \delta f$.

Since $\Delta\phi_{\mathbf{R}^n} = \delta f \in H^{-1}(\mathbf{R}^n)$, then $\phi_{\mathbf{R}^n} \in H^1_{\text{loc}}$ by elliptic regularity. Formula (7.19) implies immediately the estimates on $\phi_{\mathbf{R}^n}$ and $\mathrm{d}\phi_{\mathbf{R}^n}$. Indeed, let $\operatorname{supp} f \subset B(0,R)$ with some $R > 0$. Then for $x \notin B(0,2R)$,

$$\begin{aligned} |x|^{n-1}\,|\phi_{\mathbf{R}^n}(x)| &\le C\int_{B(0,R)} \frac{|x|^{n-1}|f(y)|}{|x-y|^{n-1}}\mathrm{d}y \le C'\int_{B(0,R)} |f(y)|\,\mathrm{d}y \\ &\le C''(R)\|f\|_{L^2(\mathbf{R}^n)}. \end{aligned} \tag{7.21}$$

Similarly, we prove estimate (7.17) for $\mathrm{d}\phi_{\mathbf{R}^n}$ by differentiating (7.19) for $x \notin \operatorname{supp} f$, and using the same arguments as above. Set $f^s_{\mathbf{R}^n} = f - \mathrm{d}\phi_{\mathbf{R}^n}$. Then $\delta f^s_{\mathbf{R}^n} = \delta f - \Delta\phi_{\mathbf{R}^n} = 0$, and $f^s_{\mathbf{R}^n}$ satisfies the decay estimate because so does $\mathrm{d}\phi_{\mathbf{R}^n}$. $\square$

An equivalent way to define $\phi_{\mathbf{R}^n}$ is to Fourier transform (7.18) and to write

$$\hat{\phi}_{\mathbf{R}^n}(\xi) = -\mathrm{i}\frac{\xi^i}{|\xi|^2}\hat{f}_i(\xi). \tag{7.22}$$

By the assumptions of the theorem, $\hat{f} \in L^2$ and f is of compact support; therefore the right-hand side above is locally in L^1, and it is also in $\mathcal{S}'$. The inverse Fourier transform then exists and defines $\phi_{\mathbf{R}^n}$ correctly. We would like to note that by dividing by $|\xi|^2$ to obtain (7.22), we actually made a choice among the many solutions of (7.18).

The equivalence of (7.19) and (7.22) will be established, if we can show that

$$\mathcal{F}^{-1}\frac{\xi^i}{|\xi|^2} = \frac{\mathrm{i}}{|S^{n-1}|}\frac{x^i}{|x|^n}. \tag{7.23}$$

When $n \neq 2$, we can use Lemma A.2.1 with $\mu = 2 - n$ to compute $\mathcal{F}^{-1}|\xi|^{-2}$; and then $\mathcal{F}^{-1}\xi^i|\xi|^{-2}$ by applying the operator $-\mathrm{i}\partial/\partial x^i$ to the result. A direct calculation then proves (7.23). When $n = 2$ we can not do this because $\mu = 2$ is then a pole in the formula in Lemma A.2.1. On the other hand, we can first differentiate the left-hand side of (A.2.1) with respect to x^i for those μ that are not poles to get

$$-\mathrm{i}\mu\Gamma\left(-\frac{\mu}{2}\right)\mathcal{F}x^i|x|^{\mu-2} = 2^{n+\mu}\pi^{n/2}\Gamma\left(\frac{n+\mu}{2}\right)\xi^i|\xi|^{-\mu-n} \tag{7.24}$$

for $-\mu,\ -n-\mu \notin 2\mathbf{Z}_+$. Let $n = 2$ now. Then the Gamma function on the left has a simple pole at $\mu = 0$ that is cancelled by the factor μ. On the other hand, $\mathcal{F}x^i|x|^{\mu-2}$ is an analytic function of μ with values in $\mathcal{S}'$ in a neighborhood of $\mu = 0$; and so is the right-hand side. Taking analytic extension at $\mu = 0$, we prove (7.23) in the case $n = 2$, as well.

To show that $\phi_{\mathbf{R}^n}$, defined by (7.22) belongs to H^1, notice first that $(1+|\xi|)\hat{\phi}_{\mathbf{R}^n}$, restricted to $|\xi| > 1$, clearly belongs to L^2. In the unit ball $|\xi| \leq 1$, we apply (3.9) with $k = 1$ to conclude the same.

The problem of inverting X can now be formulated in the following way. Given Xf, find $f^s_{\mathbf{R}^n}$. Alternatively, given g in the range of X, find f with $\delta f = 0$ so that $Xf = g$.

Note that we actually got

$$(\hat{f}^s_{\mathbf{R}^n})_i = (\delta^j_i - \xi_i\xi^j/|\xi|^2)\hat{f}_j. \tag{7.25}$$

We therefore get that $\phi_{\mathbf{R}^m}$, $f^s_{\mathbf{R}^n}$ are in $L^2(\mathbf{R}^n)$, but the compactness of the support is not preserved, in general. Indeed, for $f \in C_0^\infty$, $f^s_{\mathbf{R}^n}$ is smooth but not in $\mathcal{S}(\mathbf{R}^n)$ in general. Indeed, for many f, $\hat{f}^s_{\mathbf{R}^n}$ cannot be smooth because of the singularity in (7.25) at $\xi = 0$. Also, when f is compactly supported, $f^s_{\mathbf{R}^n}$ and $\mathrm{d}\phi_{\mathbf{R}^n}$ in general are not because if $f^s_{\mathbf{R}^n} \in C_0^\infty$ we would then get $f^s_{\mathbf{R}^n} \in \mathcal{S}$, that is not always true.

It is worth noting that the equivalence of (i) and (ii) in Theorem 7.2, that can now be formulated as $Xf = 0 \Leftrightarrow f = \mathrm{d}\phi$ admits the following simple proof using the solenoidal/potential decomposition. If $Xf = 0$, then we also have $Xf^s = 0$.

Then $\hat{f}^s_{\mathbf{R}^n}(\xi)\cdot\theta = 0$ for $\theta\perp\xi$, by the Fourier Slice Theorem, and since $\delta f^s = 0$, we also have $\xi^i(f^s_{\mathbf{R}^n})_i(\xi) = 0$. Now, clearly

$$\hat{f}^s_{\mathbf{R}^n}(\xi)\cdot\xi = 0, \quad \text{and} \quad \hat{f}^s_{\mathbf{R}^n}(\xi)\cdot\theta = 0,\ \forall\theta\perp\xi \quad \Longrightarrow \quad f^s_{\mathbf{R}^n} = 0. \tag{7.26}$$

In particular, we get solenoidal injectivity of X for f with compactly supported L^2 components, as in Theorem 7.7.

When f is supported in a bounded set $\bar{\Omega}$, the decomposition above is inconvenient because it forces us to work in the whole $\mathbf{R}^n$. On the other hand, one can use the Fourier transform. We present now another solenoidal-potential decomposition, this time in the domain Ω.

THEOREM 7.8. *Any $f\in L^2(\Omega)$ admits an unique orthogonal decomposition*

$$f = f^s + \mathrm{d}\phi, \quad \textit{where } f^s\in L^2(\Omega),\ \delta f^s = 0 \textit{ in } \Omega, \quad \textit{and } \phi\in H^1_0(\Omega).$$

Moreover, $f^s = \mathcal{S}f$, $\mathrm{d}\phi = \mathcal{P}f$, where $\mathcal{P}$ and $\mathcal{S}$ are orthogonal projections in $L^2(\Omega)$ with $\mathcal{P}+\mathcal{S} = \mathrm{Id}$, and the map $L^2(\Omega)\ni f\mapsto\phi\in H^1_0(\Omega)$ is bounded.

PROOF. Assuming that such a decomposition exists, we get $\Delta\phi = \delta f$ in Ω. Based on that and on the condition $\phi = 0$ on $\partial\Omega$, we define ϕ as the solution of the elliptic boundary value problem

$$\Delta\phi = \delta f \quad \text{in } \Omega, \quad \phi|_{\partial\Omega} = 0. \tag{7.27}$$

Since $\delta f\in H^{-1}(\Omega)$, this problem has a unique solution $\phi\in H^1_0(\Omega)$, that we denote by $\phi = \Delta_D^{-1}\delta f$, where Δ_D stands for the Dirichlet Laplacian in Ω. Moreover, the map $L^2(\Omega)\ni f\mapsto\phi\in H^1_0(\Omega)$ is bounded. With that definition of ϕ, set $f^s = f - \mathrm{d}\phi$, i.e.,

$$\mathcal{P} = \mathrm{d}\Delta_D^{-1}\delta, \quad \mathcal{S} = \mathrm{Id} - \mathrm{d}\Delta_D^{-1}\delta.$$

It is easy to check that $\mathcal{P}$, $\mathcal{S}$ are orthogonal projections and that $\delta\mathcal{S} = 0$. □

As pointed out above, this decomposition is different from the decomposition of f, extended in the whole $\mathbf{R}^n$, even when $f\in C_0(\Omega)$. For vector fields supported in $\bar{\Omega}$, then we can pose the question of invertibility of X as follows. Given Xf, find f^s. The latter is defined in Ω only, by definition, but its extension as zero outside $\bar{\Omega}$ will have the same X-ray transform.

Next theorem is a version of Theorem 7.2 for $f\in L^2(\Omega)$ that we consider as a subspace of $L^2(\mathbf{R})$.

THEOREM 7.9. *Let Ω be a bounded domain with a connected exterior. Then $Xf = 0$ for $f\in L^2(\Omega)$ if and only if $f^s = 0$.*

PROOF. To prove the "if" part, we need to show first that $X\mathrm{d}\phi = 0$ for any $\phi\in H^1_0(\Omega)$. Since f is not necessarily in C^1, instead of applying the Fundamental Theorem of Calculus, we will apply the Fourier Slice Theorem. Let $f = \mathrm{d}\phi$ and denote by f_{e}, ϕ_{e} the extensions of f, ϕ as zero outside $\bar{\Omega}$. Then $f_{\mathrm{e}} = \mathrm{d}\phi_{\mathrm{e}}$ in the whole $\mathbf{R}^n$ because $\phi = 0$ on $\partial\Omega$. We drop the subscript e in the rest of the proof. Take Fourier transform to get $\hat{f}_i(\xi) = \mathrm{i}\xi_i\hat{\phi}(\xi)$. By the Fourier Slice Theorem (Theorem 7.1), $\mathcal{F}_{\theta^\perp}Xf(\cdot,\theta) = \mathrm{i}\theta^i\xi_i\hat{\phi}(\xi)$ for any $\xi\perp\theta$. Therefore, $\mathcal{F}_{\theta^\perp}Xf(\cdot,\theta) = 0$, that implies $Xf = 0$.

Now, let $Xf = 0$. Note that we cannot apply the argument (7.26) because f^s, extended as zero outside $\bar{\Omega}$, may fail to be solenoidal in $\mathbf{R}^n$ due to possible jumps at the boundary. Instead, we write $f = f^s_{\mathbf{R}^n} + \mathrm{d}\phi_{\mathbf{R}^n}$, by Theorem 7.7, and we have

$f = \mathrm{d}\phi_{\mathbf{R}^n}$ by (7.26). In $\mathbf{R}^n \setminus \bar\Omega$, $\phi_{\mathbf{R}^n}$ is smooth. As in the proof of Corollary 7.3, we then get $\phi_{\mathbf{R}^n} = 0$ in $\mathbf{R}^n \setminus \bar\Omega$. Therefore, $\operatorname{supp}\phi_{\mathbf{R}^n} \subset \bar\Omega$; and since $\phi_{\mathbf{R}^n}$ is locally in H^1, we then get $\phi_{\mathbf{R}^n} \in H^1_0(\Omega)$. Therefore, $\phi_{\mathbf{R}^n}$ equals ϕ in Theorem 7.8, therefore, $f^s = 0$. In particular, when $Xf = 0$, we do get $\phi_{\mathbf{R}^n} = \phi$ and $f^s = f^s_{\mathbf{R}^n}$ in Ω (the latter is actually 0) but that is not in general true for f not in the kernel of X. □

7.4. Inversion formulas. We start with analyzing the Schwartz kernel of $X'X$. The analog of Proposition 2.2 in this case is the following.

PROPOSITION 7.10. *For any* $f \in C_0^\infty(\mathbf{R}^n)$,

$$(X'Xf)^i = 2\frac{x^i x^j}{|x|^{n+1}} * f_j.$$

PROOF. The proof is similar to that of Proposition 2.2. By Problem 7.1,

$$\begin{aligned}(X'Xf)^i(x) &= \int_{S^{n-1}} \theta^i Xf(x-(x\cdot\theta)\theta,\theta)\,\mathrm{d}\theta\\ &= \int_{S^{n-1}}\int_{\mathbf{R}} \theta^i f_j(x+s\theta-(x\cdot\theta)\theta,\theta)\theta^j\,\mathrm{d}s\,\mathrm{d}\theta \\ &= \int_{S^{n-1}}\int_{\mathbf{R}} \theta^i f_j(x+s\theta)\theta^j\,\mathrm{d}s\,\mathrm{d}\theta.\end{aligned} \tag{7.28}$$

We split again the s-integral in two parts: over $s>0$ and $s<0$. Then we make the change of variables $(s,\theta)\mapsto(-s,-\theta)$ in the second one. Thus we get

$$\begin{aligned}(X'Xf)^i(x) &= 2\int_{S^{n-1}}\int_0^\infty \theta^i f_j(x+s\theta)\theta^j\,\mathrm{d}s\,\mathrm{d}\theta\\ &= 2\int_{\mathbf{R}^n}\frac{z^iz^jf_j(x+z)}{|z|^{n+1}}\mathrm{d}z = 2\int_{\mathbf{R}^n}\frac{(x^i-y^i)(x^j-y^j)}{|x-y|^{n+1}}f_j(y)\,\mathrm{d}y.\end{aligned}$$

□

COROLLARY 7.11. *For any* $f \in C_0^\infty(\mathbf{R}^n)$,

$$\mathcal{F}(X'Xf)_i(\xi) = C_n\frac{\delta_{ij}-\xi_i\xi_j/|\xi|^2}{|\xi|}\hat f_j(\xi),\quad C_n = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} = |S^n|.$$

PROOF. By Proposition 7.10, $X'X$ is a Fourier multiplier with the Fourier transform of $2x^ix^j|x|^{-n-1}$. By Lemma A. 2.1,

$$\mathcal{F}|x|^{-n-1} = \tilde C_n|\xi|,\quad \tilde C_n := 2^{-1}\pi^{n/2}\frac{\Gamma\left(-\frac12\right)}{\Gamma\left(\frac{n+1}{2}\right)} = -\frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} = -\frac12|S^n|.$$

Therefore,

$$\mathcal{F}\left(2x^ix^j|x|^{-n-1}\right) = -2\tilde C_n\partial_{\xi_i}\partial_{\xi_j}|\xi| = -2\tilde C_n\frac{\delta_{ij}-\xi_i\xi_j/|\xi|^2}{|\xi|}. \tag{7.29}$$

This completes the proof. □

THEOREM 7.12. *For any* $f \in C_0^\infty(\mathbf{R}^n)$, *we have*

$$f^s_{\mathbf{R}^n} = C_n^{-1}|D|X'Xf, \tag{7.30}$$

where C_n *is the constant in Corollary 7.11.*

PROOF. Follows immediately from (7.25) and Corollary 7.11. □

Comparing Theorem 2.6 and Theorem 7.12, we see that the inversion operator is the same, up to a constant factor.

The analog of Theorem 2.4 holds as well. Before that, we need the following generalization of Lemma 2.1.

LEMMA 7.13. *For any matrix valued $F \in \mathcal{S}(\mathbf{R}^n)$,*

$$\int_{S^{n-1}} \int_{\omega^\perp} F_{ij}(x)\omega^i\omega^j \, \mathrm{d}S_x \, \mathrm{d}\omega = \frac{|S^n|}{2\pi} \int_{\mathbf{R}^n} \frac{F_{ij}(x)}{|x|} \left(\delta_{ij} - \frac{x^i x^j}{|x|^2} \right) \mathrm{d}x.$$

PROOF. By the proof of Lemma 2.1,

$$\int_{\omega^\perp} F_{ij}(x) \, \mathrm{d}S_x = \frac{1}{\pi} \int_{\mathbf{R}_+} \hat{F}_{ij}(r\omega) \, \mathrm{d}r.$$

Therefore,

$$\begin{aligned}\int_{S^{n-1}} \int_{\omega^\perp} F_{ij}(x)\omega^i\omega^j \, \mathrm{d}S_x \, \mathrm{d}\omega &= \frac{1}{\pi} \int_{S^{n-1}} \int_{\mathbf{R}} \hat{F}_{ij}(r\omega)\omega^i\omega^j \, \mathrm{d}r \, \mathrm{d}\omega \\ &= \frac{1}{\pi} \int_{\mathbf{R}^n} \hat{F}_{ij}(\xi)\xi^i\xi^j |\xi|^{-n-1} \, \mathrm{d}\xi \\ &= \frac{1}{\pi}(2\pi)^n \int_{\mathbf{R}^n} F_{ij}(x) \frac{|S^n|}{2(2\pi)^n} \left(\delta_{ij} - \frac{x^i x^j}{|x|^2} \right) |x|^{-1} \, \mathrm{d}x,\end{aligned}$$

and we used the Plancherel equality together with (7.29). □

THEOREM 7.14. *For any $f \in \mathcal{S}(\mathbf{R}^n)$,*

$$f^s_{\mathbf{R}^n} = C_n^{-1} X'|D_z|Xf, \tag{7.31}$$

where C_n is the constant in Corollary 7.11.

PROOF. Let f, g be in $\mathcal{S}(\mathbf{R}^n)$, and let $f^s_{\mathbf{R}^n}$ be as in Theorem 7.7. Then

$$\begin{aligned}(X'|D_z|Xf^s_{\mathbf{R}^n}, g)_{L^2(\mathbf{R}^n)} &= (|D_z|Xf^s_{\mathbf{R}^n}, Xg)_{L^2(\Sigma)} \\ &= (2\pi)^{1-n}(\mathcal{F}_z|D_z|Xf^s_{\mathbf{R}^n}, \mathcal{F}_z Xg)_{L^2(\Sigma)}.\end{aligned}$$

As before, for any fixed θ, let ρ be the dual variable to z. Then $\mathcal{F}_z|D_z| = |\rho|\mathcal{F}_z$. Combining this with the Fourier Slice Theorem for vector fields (Theorem 7.1), we get

$$\begin{aligned}(X'|D_z|Xf^s_{\mathbf{R}^n}, g)_{L^2(\mathbf{R}^n)} &= (2\pi)^{1-n}(|\rho|\mathcal{F}_z Xf^s_{\mathbf{R}^n}, \mathcal{F}_z Xg)_{L^2(\Sigma)} \\ &= (2\pi)^{1-n} \int_{S^{n-1}} \int_{\theta^\perp} (\hat{f}^s_{\mathbf{R}^n}(\rho)\cdot\theta)(\hat{\bar{g}}(\rho)\cdot\theta)|\rho| \mathrm{d}S_\rho \, \mathrm{d}\theta.\end{aligned}$$

We apply Lemma 7.13 with $F_{ij} = (\hat{f}^s_{\mathbf{R}^n})_i \hat{\bar{g}}_j$ and x in the lemma replaced by ρ. Note that $\rho \cdot \hat{f}^s_{\mathbf{R}^n}(\rho) = 0$ because $f^s_{\mathbf{R}^n}$ is divergence free. Therefore,

$$\begin{aligned}(X'|D_z|Xf^s_{\mathbf{R}^n}, g)_{L^2(\mathbf{R}^n)} &= (2\pi)^{1-n} \frac{|S^n|}{2\pi} \int_{R^n} \hat{f}^s_{\mathbf{R}^n}(\rho) \cdot \hat{g}(\rho) \, \mathrm{d}\rho \\ &= |S^n|(f^s_{\mathbf{R}^n}, g)_{L^2(\mathbf{R}^n)}.\end{aligned} \tag{7.32}$$

This completes the proof. □

Formulas (7.30) and (7.31) reconstruct a vector field $f^s_{\mathbf{R}^n}$ with the same X-ray transform as f but they are inconvenient when we know a priori that f is supported in a bounded domain Ω. In that case, we would like to recover the solenoidal projection f^s of Theorem 7.8. This can be done as follows. We have

$$f^s = f - \mathrm{d}\phi \quad \text{in } \Omega, \quad \text{where } \Delta\phi = \delta f,\ \phi \in H_0^1(\Omega).$$

On the other hand, we have a reconstruction formula for

$$f^s_{\mathbf{R}^n} = f - \mathrm{d}\phi_{\mathbf{R}^n} \quad \text{in } \mathbf{R}^n, \quad \text{where } \Delta\phi_{\mathbf{R}^n} = \delta f \quad \text{in } \mathbf{R}^n, \quad \lim_{|x|\to\infty} \phi_{\mathbf{R}^n}(x) = 0. \tag{7.33}$$

We can therefore obtain f^s in Ω from $f^s_{\mathbf{R}^n}$ by adding a suitable correction term:

$$f^s = f^s_{\mathbf{R}^n} + \mathrm{d}\psi \quad \text{in } \Omega, \quad \text{with } \psi := \phi_{\mathbf{R}^n} - \phi,\ \Delta\psi = 0. \tag{7.34}$$

Since ψ is harmonic in Ω, it is uniquely determined by its boundary value. The latter is well defined because $\psi \in H^1(\Omega)$. Since $\phi = 0$ on $\partial\Omega$, we get $\psi = \phi_{\mathbf{R}^n}$ on $\partial\Omega$. We will recover the boundary value of $\phi_{\mathbf{R}^n}$ from (7.33). We have

$$\mathrm{d}\phi_{\mathbf{R}^n}|_{\mathbf{R}^n\setminus\bar\Omega} = f^s_{\mathbf{R}^n}|_{\mathbf{R}^n\setminus\bar\Omega}. \tag{7.35}$$

To recover $\phi_{\mathbf{R}^n}$ on $\partial\Omega$, and therefore f^s, is to integrate (7.35) from any $x \in \partial\Omega$ along any curve outside Ω starting from x and going to infinity. That however requires computations in an unbounded domain. On the other hand, one can fix $y \notin \bar\Omega$ and integrate from $x \in \partial\Omega$ to y, staying outside Ω. This will recover $\phi_{\mathbf{R}^n}$ on $\partial\Omega$ up to the constant $\phi_{\mathbf{R}^n}(y)$. Then ψ in (7.34) will be known up to a constant. That constant however will be annihilated by the differential in (7.34).

7.5. Stability estimates. We will study stability estimates for f supported in $\bar\Omega$, where Ω is a smooth bounded domain in $\mathbf{R}^n$. We are looking for sharp stability estimates of the type (III.1.3). Since f^s does not necessarily have a smooth extension as 0 outside Ω, even when $f \in C_0^\infty(\Omega)$, we cannot expect the analog of the estimate (3.2) in Theorem 3.1 to hold for all s without some modifications, at least. The same remarks applies to estimate (3.10) in Theorem 3.3.

THEOREM 7.15. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary and connected exterior. Then*

$$\|f^s\|_{L^2(\Omega)}/C \le \|Xf\|_{\bar H^{1/2}(\Sigma)} \le C\|f\|_{L^2(\Omega)} \tag{7.36}$$

for any $f \in L^2(\Omega)$.

PROOF. It is enough to prove the estimates for $f \in C_0^\infty(\Omega)$. Let $f^s_{\mathbf{R}^n}$ be the solenoidal projection in $\mathbf{R}^n$ of f, extended as 0 to $\mathbf{R}^n\setminus\bar\Omega$. By (7.25), $f^s_{\mathbf{R}^n} \in L^2(\mathbf{R}^n)$, and then by (7.32), with $g = f^s_{\mathbf{R}^n}$, we get

$$\|f^s_{\mathbf{R}^n}\|^2_{L^2(\mathbf{R}^n)} = C_N^{-1}\left\||D_z|^{1/2} X f^s_{\mathbf{R}^n}\right\|^2_{L^2(\Sigma)}, \tag{7.37}$$

that in particular proves the first inequality in (7.36) with f^s there replaced by $f^s_{\mathbf{R}^n}$. The second one follows from the fact that $Xf(z,\theta) = \theta^i X_0 f_i(z,\theta)$, where X_0 is the X-ray transform of functions, and from Theorem 3.1. Therefore

$$\|f^s_{\mathbf{R}^n}\|_{L^2(\mathbf{R}^n)}/C \le \|Xf\|_{\bar H^{1/2}(\Sigma)} \le C\|f^s_{\mathbf{R}^n}\|_{L^2(\mathbf{R}^n)}.$$

By (7.25), $f^s_{\mathbf{R}^n}$ is obtained by applying a bounded operator in $L^2(\mathbf{R}^n)$, an orthogonal projection actually, to f. Therefore,

$$\|f^s_{\mathbf{R}^n}\|_{L^2(\mathbf{R}^n)}/C \le \|Xf\|_{\bar H^{1/2}(\Sigma)} \le C\|f\|_{L^2(\Omega)}. \tag{7.38}$$

In particular, this is an analog of estimate (3.2) of Theorem 3.1 for $s = 0$. The proof is then completed by the lemma below. □

LEMMA 7.16. *Under the assumptions of Theorem 7.15, there exists a constant* $C = C(\Omega)$, *so that for any* $f \in L^2(\mathbf{R}^n)$,

$$\|f^s\|_{L^2(\Omega,\mathbf{R}^n)} \le C\|f^s_{\mathbf{R}^n}\|_{L^2(\mathbf{R}^n)} \tag{7.39}$$

PROOF. We follow the arguments of the reconstruction of f^s once we have reconstructed $f^s_{\mathbf{R}^n}$, described above. By (7.34), we need control of $\mathrm{d}\psi$ in Ω. To determine the harmonic function ψ in Ω, we need its boundary values on $\partial\Omega$ that coincide with $\mathrm{d}\phi_{\mathbf{R}^n}$ on $\partial\Omega$. This is done by (7.35).

Let $\Omega_1 \subset \mathbf{R}^n$ be a bounded open set so that $\bar\Omega \subset \Omega_1$. It is convenient but not essential to assume that $\partial\Omega_1$ is given by $\operatorname{dist}(x,\partial\Omega) = \varepsilon$ with $\varepsilon > 0$ small enough so that $\partial\Omega_1$ is smooth. Let $f^s_{\Omega_1}$ be the solenoidal projection of f, extended as 0 outside Ω (that we always assume), to Ω_1; and let $\mathrm{d}\phi_{\Omega_1}$ be its potential one. Comparing $f^s_{\Omega_1}$ and $f^s_{\mathbf{R}^n}$, we get

$$f^s_{\Omega_1} = f^s_{\mathbf{R}^n} + \mathrm{d}\varphi_1 \quad \text{in } \Omega_1, \quad \varphi_1 := \phi_{\mathbf{R}^n} - \phi_{\Omega_1}, \quad \Delta\varphi_1 = 0 \quad \text{in } \Omega_1. \tag{7.40}$$

We claim that

$$C_0^\infty(\Omega) \ni f \mapsto \mathrm{d}\varphi_1 \in C^\infty(\bar\Omega_1) \tag{7.41}$$

extends as a compact operator from $L^2(\Omega)$ to $\in L^2(\Omega_1)$. Indeed, as a harmonic function, φ_1 is determined by $\varphi_1|_{\partial\Omega_1}$. The latter equals $\phi_{\mathbf{R}^n}|_{\partial\Omega_1}$, see (7.19). Since $\operatorname{supp} f \subset \bar\Omega$, and the kernel of the convolution operator in (7.19) is smooth away from the diagonal, we get that $f \mapsto \phi_{\mathbf{R}^n}|_{\partial\Omega_1}$ extends to a bounded map from $L^2(\Omega)$ to $H^s(\partial\Omega_1)$ for any s. Next, $\phi_{\mathbf{R}^n}|_{\partial\Omega_1} = \varphi_1|_{\partial\Omega_1} \mapsto \mathrm{d}\varphi_1|_\Omega$ is a bounded map from $H^{3/2}(\partial\Omega)$ to $H^1(\Omega_1)$ by standard elliptic estimates. Therefore, the map (7.41) is bounded from $L^2(\Omega)$ to $H^1(\Omega_1)$, thus compact as a map from $L^2(\Omega)$ to $L^2(\Omega_1)$.

To summarize, we showed so far in (7.40) that $f^s_{\Omega_1}$ and $f^s_{\mathbf{R}^n}$ coincide in Ω_1 up to a compact operator applied to f, i.e,

$$f^s_{\Omega_1} = f^s_{\mathbf{R}^n} + K_1 f \quad \text{in } \Omega_1, \tag{7.42}$$

where $K_1 : L^2(\Omega) \to L^2(\Omega_1)$ is compact. Next, we will compare f^s (the solenoidal projection of f in Ω, that could also be denoted by f^s_Ω), extended as 0 outside Ω, and $f^s_{\Omega_1}$.

In Ω, we have $f^s = f - \mathrm{d}\phi$, $\phi \in H^1_0(\Omega)$. Extend ϕ and f^s as 0 outside Ω. Then we still have $f^s = f - \mathrm{d}\phi$ in the whole $\mathbf{R}^n$ because there is no jump of ϕ at $\partial\Omega$. Comparing f^s and $f^s_{\Omega_1}$, we get

$$f^s = f^s_{\Omega_1} + \mathrm{d}\varphi_2 \quad \text{in } \Omega_1, \quad \varphi_2 := \phi_{\Omega_1} - \phi. \tag{7.43}$$

Since $f^s = \phi = 0$ outside Ω, we have

$$\mathrm{d}\phi_{\Omega_1} = \mathrm{d}\varphi_2 = -f^s_{\Omega_1}, \quad \text{in } \Omega_1 \setminus \Omega, \tag{7.44}$$

therefore,

$$\|\mathrm{d}\varphi_2\|_{L^2(\Omega_1\setminus\Omega)} = \|f^s_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}. \tag{7.45}$$

We claim that we actually have

$$\|\varphi_2\|_{H^1(\Omega_1\setminus\Omega)} \le C\|f^s_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}. \tag{7.46}$$

Estimate (7.46), requires a Poincaré type of inequality that would estimate $\|\varphi_2\|_{L^2(\Omega_1\setminus\Omega)}$ as well, through $\|\mathrm{d}\varphi_2\|_{H^1(\Omega_1\setminus\Omega)}$. Those types of inequalities require knowledge of φ_2 on a part of the boundary, at least, or some average value of φ_2 in an open set because adding a constant to φ_2 does not change φ_2. We know however that $\varphi_2 = 0$ on $\partial\Omega_1$; and that was the reason actually to do the step $f^s_{\mathbf{R}^n} \mapsto f^s_{\Omega_1}$.

With this in mind, let for $x \in \Omega_1 \setminus \Omega$, $y^n(x)$ be the distance from x to $\partial\Omega$. Then $\bar\Omega_1 \setminus \Omega$ is given by $0 \le y^n \le \varepsilon$. For $\varepsilon > 0$ small enough, the map $[0,\varepsilon] \times \partial\Omega \ni (y^n, y') \mapsto x \in \Omega_1$, where $y'(x)$ is the closest point to x on $\partial\Omega$, is a diffeomorphism.

Notice next that if we regard f as the components of the form $f = f_j \mathrm{d}x^j$, then under coordinate changes, f behaves as a covector field. In other words, if $y = y(x)$ is a local diffeomorphism, then $f_i \mathrm{d}x^i = \tilde f_i \mathrm{d}y^i$ with

$$\tilde f_j(y) = f_i(x(y)) \frac{\mathrm{d}x^i}{\mathrm{d}y^j}.$$

This is the point of view that we adopt in the next chapters. Then (7.44) is preserved under the coordinate change, as it can be easily seen. More precisely, let $y' = (y^1, \dots, y^{n-1}) \in U$ be local coordinates on $\partial\Omega_1$ near a fixed $x_0 \in \partial\Omega$. Set $\tilde\phi_{\Omega_1}(y) = \phi_{\Omega_1}(x(y))$; then we have

$$\mathrm{d}_y \tilde\phi_{\Omega_1} = \tilde f^s_{\Omega_1} \tag{7.47}$$

in the subset of $[0,\varepsilon] \times U$, where (y', y^n) are defined.

Integrating in normal directions, and using the fact that $\tilde\phi_{\Omega_1} = 0$ for $y^n = \varepsilon$, we get

$$\tilde\phi_{\Omega_1}(y) = -\int_{y^n}^{\varepsilon} \partial_{y^n} \tilde\phi_{\Omega_1}(y', t)\,\mathrm{d}t$$

for $0 \le y^n \le \varepsilon$. By the Cauchy inequality,

$$|\tilde\phi_{\Omega_1}(y)|^2 \le (\varepsilon - y^n) \int_{y^n}^{\varepsilon} |\partial_{y^n} \tilde\phi_{\Omega_1}(y', t)|^2\,\mathrm{d}t \le \varepsilon \int_0^{\varepsilon} |\partial_{y^n} \tilde\phi_{\Omega_1}(y', t)|^2\,\mathrm{d}t \tag{7.48}$$

for $0 \le y^n \le \varepsilon$. Integrate in $[0,\varepsilon] \times U$ to get

$$\|\tilde\phi_{\Omega_1}\|^2_{L^2([0,\varepsilon]\times U)} \le \varepsilon^2 \int_0^{\varepsilon} |\partial_{y^n} \tilde\phi_{\Omega_1}(y)|^2\,\mathrm{d}y \le \|\mathrm{d}\tilde\phi_{\Omega_1}\|_{L^2([0,\varepsilon]\times U)}.$$

This immediately implies the same estimate for ϕ_{Ω_1} through $\mathrm{d}\varphi_{\Omega_1} = f^s_{\Omega_1}$ in the domain of validity of the coordinates y. We can do this near any point $x_0 \in \partial\Omega$. Using the compactness of $\partial\Omega$, we can choose a finite subset to get

$$\|\phi_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)} \le C \|f^s_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}.$$

Combining this with (7.45), we thus complete the proof of the claim (7.46).

By the trace theorem,

$$\|\varphi_2\|_{H^{1/2}(\partial\Omega)} \le C \|f^s_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}.$$

In Ω, $\Delta\varphi_2 = \Delta(\phi_{\Omega_1} - \phi) = \delta f - \delta f = 0$. Then by standard elliptic estimates,

$$\|\varphi_2\|_{H^1(\Omega)} \le C \|f^s_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}. \tag{7.49}$$

This and (7.43) imply

$$\|f^s\|_{L^2(\Omega)} \le C \|f^s_{\Omega_1}\|_{L^2(\Omega_1)}. \tag{7.50}$$

We are ready to complete the proof now, using (7.42) and (7.50).

$$\begin{aligned} \|f^s\|_{L^2(\Omega)} &\le C \|f^s_{\mathbf{R}^n}\|_{L^2(\Omega_1)} + \|K_1 f\|_{L^2(\Omega_1)} \\ &\le C \|f^s_{\mathbf{R}^n}\|_{L^2(\mathbf{R}^n)} + \|K_1 f\|_{L^2(\Omega_1)}. \end{aligned}$$

Let us first assume that f is solenoidal, i.e., $f = f^s$ or, equivalently, $\delta f = 0$ in Ω. We now apply Lemma 3.4 with f belonging to the closed subspace $\mathcal{S}L^2(\Omega)$ of solenoidal vector fields in $L^2(\Omega)$. As it follows from the proof of Theorem 7.9, the bounded map $\mathcal{S}L^2(\Omega) \ni f \to f^s_{\mathbf{R}^n} \in L^2(\mathbf{R}^n)$ is injective. The estimate above then implies that (7.39) holds with a different constant. □

***** One can prove the lemma with a certain version of the Poincaré inequality, including an average value in an open set.**
****

We now turn our attention to estimating f^s through $X'Xf$.

THEOREM 7.17. *Let $\Omega \subset \mathbf{R}^n$ be open and bounded, and let $\Omega_1 \supset \bar{\Omega}$ be another such set. Then there is a constant $C > 0$ so that for any $f \in H^s(\mathbf{R}^n)$ supported in $\bar{\Omega}$, we have*

$$\|f^s\|_{L^2(\mathbf{R}^n)}/C \leq \|X'Xf\|_{H^1(\Omega_1)} \leq C\|f\|_{L^2(\mathbf{R}^n)} \tag{7.51}$$

PROOF. The second inequality in (7.51) follows as in the proof of Theorem 3.3. To prove the first inequality, use Corollary 7.11 and (7.25) to get

$$\hat{f}^s_{\mathbf{R}^n}(\xi) = C_n^{-1}|\xi|\mathcal{F}X'Xf.$$

Therefore,

$$C_n\|f^s_{\mathbf{R}^n}\|^2_{L^2(\mathbf{R}^n)} \leq \|X'Xf\|^2_{H^1(\mathbf{R}^n)} = \|X'Xf\|^2_{H^1(\Omega_1)} + \|X'Xf\|^2_{H^1(\mathbf{R}^n\setminus\Omega_1)}.$$

By Lemma 7.16,

$$C_n\|f^s\|^2_{L^2(\mathbf{R}^n)} \leq \|X'Xf\|^2_{H^1(\Omega_1)} + \|X'Xf\|^2_{H^1(\mathbf{R}^n\setminus\Omega_1)}.$$

Similarly to the proof of Theorem 3.3, the operator

$$L^2(\Omega) \ni f \mapsto X'Xf|_{\mathbf{R}^n\setminus\Omega_1} \in H^1(\mathbf{R}^n \setminus \Omega),$$

is compact because it has a C^∞ kernel, and $\bar{\Omega}$ is compact. Indeed, the kernel of $X'X$ is given by (2.1), and for $x \in \mathbf{R}^n \setminus \Omega$ and $y \in \bar{\Omega}$, it is smooth. On the other hand, $X'X : \mathcal{S}L^2(\mathbf{R}^n) \to H^1(\Omega_1)$ is injective. Indeed, if $X'Xf = 0$ with some $f \in \mathcal{S}L^2$, then $0 = (X'Xf, f)_{L^2(\Omega)} = \|Xf\|^2_{L^2(\Omega)}$, hence $f = 0$ by Theorem 7.9. Then an application of Lemma 3.4 completes the proof. □

8. The Euclidean X-ray transform of tensor fields of order two

Let $f(x) = \{f_{ij}(x)\}^n_{i,j=1}$ be a complex matrix valued function on $\mathbf{R}^n$. We define the X-ray transform of f as

$$Xf(x,\theta) = \int_{\mathbf{R}} f_{ij}(x + s\theta)\theta^i\theta^j \,\mathrm{d}s, \tag{8.1}$$

where, as before, we use $(x,\theta) \in \mathbf{R}^n \times S^{n-1}$ to parameterize all directed lines in $\mathbf{R}^n$. As before, the parameterization is not unique and Xf is an even function of θ:

$$Xf(x,\theta) = Xf(x + t\theta, \theta), \quad Xf(x,\theta) = Xf(x, -\theta), \tag{8.2}$$

as in the case of functions. The integral makes sense for any f with $L^1(\mathbf{R}^n)$ entries.

Unique parameterization of Xf on the manifold of the directed lines can be obtained by restricting (x,θ) to Σ as in section 1.1. When f is supported in a bounded strictly convex domain $\Omega \subset \mathbf{R}^n$, one can also think of Xf as a map from $\partial_-S\Omega$ to $\mathbf{C}^n$, as before.

The integrand of (8.1) is a symmetric quadratic form of θ, and will not change if we replace f by its symmetric part $\frac{1}{2}(f_{ij}+f_{ji})$. For that reason, we will assume that f is symmetric itself, i.e., $f_{ij}=f_{ji}$ for any i, j.

8.0.1. *Motivation.* The basic motivation to study this transform is that it is a linearized version of the non-linear boundary rigidity problem, see section ???. In the Euclidean case that we study here, it is a linearization near the Euclidean metric.

We will see in section ??? that we f behaves as a covariant tensor field of order 2, i.e., as a tensor of type $(0,2)$, under a change of variables. The integrand in (8.1) can be viewed as such a tensor f applied to $\theta\otimes\theta$. We can therefore identify f with the quadratic form $f=f_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j$, see also (7.1). This explains our choice to use lower indices for f_{ij}.

We will use the notation $ST_2^0(\mathbf{C}^n)$ to denote the space of symmetric tensors of type $(0,2)$ over the vector space $\mathbf{C}^n$, $T_2^0(\mathbf{C}^n)$ being all tensors of that type. When it is clear from the context that we mean tensor spaces, we will often omit the notation $ST_2^0(\mathbf{C}^n)$. For example, $L^2(\mathbf{R}^n)$ below is actually : $L^2(\mathbf{R}^n;ST_2^0(\mathbf{C}^n))$, etc.

The natural way to define pairing of symmetric tensor fields in $\mathbf{R}^n$ is given by

$$(f,h)=\int_{\mathbf{R}^n} f_{ij}(x)h^{ij}(x)\,\mathrm{d}x, \tag{8.3}$$

where $h=\{h^{ij}\}$ is another symmetric tensor field; and in this chapter, we identify h_{ij} and h^{ij} for any such filed. The distribution space $\mathcal{D}'(\mathbf{R}^n)$ of symmetric 2-tensor fields (i.e., $\mathcal{D}'(\mathbf{R}^n,ST_2^0(\mathbf{C}^n))$) is defined as the space of all continuous linear functionals on $C_0^\infty(\mathbf{R}^n)$. The elements of $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ are naturally identified with distributions by (8.3).

PROBLEM 8.1. Show that the transpose X' is given by

$$(X'\psi(x))^{ij}=\int_{S^{n-1}}\theta^i\theta^j\psi(x-(x\cdot\theta)\theta,\theta)\,\mathrm{d}\theta, \quad \psi\in C_0^\infty(\Sigma). \tag{8.4}$$

One can extend X to $\mathcal{E}'(\mathbf{R}^n)$ in the same way as in (1.3), see Definition 1.4. The $L^2(\mathbf{R}^n)$ space is defined by the norm

$$\|f\|^2_{L^2(\mathbf{R}^n)}=\int_\Omega\sum_{ij}|f_{ij}(x)|^2\,\mathrm{d}x, \tag{8.5}$$

and the inner product in it is given by (8.3), with h replaced by $\bar{h}$.

As in the previous sections, if Ω is a strictly convex bounded domain with smooth boundary, one can define $Xf\in\mathcal{E}'(\partial_-S\Omega)$, and

$$X: L^2(\Omega)\to L^2(\partial_-S\Omega,\mathrm{d}\mu)$$

is bounded. Then X^* is well defined.

The X-ray transform on symmetric 2-tensor fields has an obvious infinitely dimensional kernel as well. Given a vector field $v=\{v_i\}$, let d^sv be the *symmetric differential* of v:

$$(\mathrm{d}^sv)_{ij}=\frac{1}{2}\left(\partial_iv_j+\partial_jv_i\right). \tag{8.6}$$

The latter is a symmetric 2-tensor field. Then is is easy to see that

$$X(\mathrm{d}^sv)=0, \quad \forall v\in\mathcal{S}(\mathbf{R}^n). \tag{8.7}$$

This follows from the identity

$$\frac{\mathrm{d}}{\mathrm{d}s} v_i(x+s\theta)\theta^i = (\mathrm{d}^s v)_{ij}(x+s\theta)\theta^i\theta^j, \tag{8.8}$$

and the Fundamental Theorem of Calculus. We show below that this is the only obstruction to uniqueness.

The Fourier Slice Theorem takes the following form in this case.

THEOREM 8.1. *For any* $f \in L^1(\mathbf{R}^n)$,

$$\hat{f}_{ij}(\zeta)\theta^i\theta^j = \int_{\theta^\perp} e^{-\mathrm{i}z\cdot\zeta} Xf(z,\theta)\,\mathrm{d}S_z, \quad \forall \theta \perp \zeta,\ \theta \in S^{n-1}. \tag{8.9}$$

We omit the proof.

8.1. Solenoidal – potential decomposition. We will construct a solenoidal – potential decomposition in $\mathbf{R}^n$ first, and then in a bounded domain, similarly to section 7.3. In $\mathbf{R}^n$, we describe first the space of symmetric 2-tensor fields f in the Schwartz class, orthogonal in $L^2(\mathbf{R}^n)$ sense to all $\mathrm{d}^s v$, with v in the Schwartz class, again. For any such f,

$$0 = \int_{\mathbf{R}^n} f^{ij}(x)\frac{1}{2}\left(\partial_i \bar{v}_j + \partial_j \bar{v}_i\right)\,\mathrm{d}x = -\int_{\mathbf{R}^n} (\partial_i f^{ij})(x)\bar{v}_j(x)\,\mathrm{d}x. \tag{8.10}$$

Therefore, f must be divergence free, i.e.,

$$\delta f = 0, \quad \text{where } (\delta f)^i = \partial_i f^{ij}.$$

We recall that we freely raise and lower indices in this chapter. Then the divergence is a differential operator that sends symmetric 2-tensor fields into vector fields. As the calculation above shows, $-\delta$ is the formal adjoint to d^s, the latter acting on vector fields. This motivates the following.

DEFINITION 8.2. *The symmetric 2-tensor field f is called solenoidal, if $\delta f = 0$. It is called potential, if $f = \mathrm{d}^s v$ with $v(x) \to 0$, as $|x| \to \infty$.*

THEOREM 8.3. *For any $f \in L^2(\mathbf{R}^n)$ of compact support, there exist a uniquely determined $f^s_{\mathbf{R}^n} \in L^2(\mathbf{R}^n)$ and a vector field $v_{\mathbf{R}^n} \in H^1_{\mathrm{loc}}(\mathbf{R}^n)$, smooth outside* supp f, *so that*

$$f = f^s_{\mathbf{R}^n} + \mathrm{d}^s v_{\mathbf{R}^n}, \quad \delta f^s_{\mathbf{R}^n} = 0, \quad v_{\mathbf{R}^n}(x) \to 0, \ \text{as } |x| \to \infty.$$

Moreover, the following estimates are satisfied for $|x|$ large enough

$$|f^s_{\mathbf{R}^n}(x)| + |\mathrm{d}^s v_{\mathbf{R}^n}(x)| \le C(1+|x|)^{-n}, \quad |v_{\mathbf{R}^n}(x)| \le C(1+|x|)^{1-n}. \tag{8.11}$$

PROOF. As in the proof of Theorem 7.7, assume first that the decomposition can be done. Then $\delta f = \delta \mathrm{d}^s v_{\mathbf{R}^n}$. The conditions on v guarantee that the latter is in $L^2(\mathbf{R}^n)$. Therefore, $\hat{v}_{\mathbf{R}^n}$ exists.

The operator $\delta\mathrm{d}^s$ requires more attention. It is a second order differential operator that maps vector fields into vector fields. We have

$$(\delta \mathrm{d}^s v)_i = \frac{1}{2}\left(\Delta v_i + \partial_i\partial_j v^j\right).$$

Take Fourier transform to get

$$\mathcal{F}(\delta\mathrm{d}^s v)^i(\xi) = -\frac{1}{2}\left(|\xi|^2\delta^{ij} + \xi^i\xi^j\right)\hat{v}_j(\xi) =: A^{ij}(\xi)\hat{v}_j(\xi).$$

Therefore, $\delta \mathrm{d}^s$ is Fourier multiplier with the matrix $A(\xi)$. The latter is homogeneous of order 2, and negative definite for $\xi\neq 0$ because for any $a\in\mathbf{R}^n$,

$$-A^{ij}(\xi)a_ia_j = \frac{1}{2}\left(|\xi|^2|a|^2+(\xi\cdot a)^2\right)\geq \frac{1}{2}|\xi|^2|a|^2.$$

The matrix A can be easily inverted. Indeed, if $\hat w = A\hat v$, take a dot product with ξ to get $\hat w\cdot\xi = -|\xi|^2\hat v\cdot \xi$. Then

$$-2\hat w = |\xi|^2\hat v+\xi\hat v\cdot\xi = |\xi|^2\hat v - |\xi|^{-2}\xi\hat w\cdot\xi,$$

and we can solve for $\hat v$ now. This implies that the equation $\mathcal{F}\delta\mathrm{d}^s v_{\mathbf{R}^n} = \mathcal{F}\delta f$ has a possible solution

$$(\hat v_{\mathbf{R}^n})_i(\xi) = \mathrm{i}|\xi|^{-2}\left(-2\delta_i^j+|\xi|^{-2}\xi_i\xi^j\right)\xi^k\hat f_{jk}(\xi). \tag{8.12}$$

The singularity at $\xi=0$ is locally integrable in all dimensions. Therefore $\hat v_{\mathbf{R}^n}$ belongs to $\mathcal{S}'(\mathbf{R}^n)$, and so does $v_{\mathbf{R}^n}$. Note that the equation $\mathcal{F}\delta\mathrm{d}^s v_{\mathbf{R}^n} = \mathcal{F}\delta f$ has many other solutions, and they all differ by distributions supported at 0. The one that we chose will give us the desired behavior of $v_{\mathbf{R}^n}$ at infinity however. Now, we take (8.12) as a definition of $v_{\mathbf{R}^n}$; and then we set $f^s_{\mathbf{R}^n} = f-\mathrm{d}^s v_{\mathbf{R}^n}$. Clearly, $\delta f^s_{\mathbf{R}^n}=0$.

To prove the estimates on $v_{\mathbf{R}^n}$, notice first that $\hat v_{\mathbf{R}^n}$ is a linear combination of the terms $\xi^k|\xi|^{-2}\hat f_{ik}$ and $\xi_i\xi_j\xi_k|\xi|^{-4}\hat f_{ik}$. We already found the inverse Fourier transform of $\xi^k|\xi|^{-2}$ in (7.23) and showed that it decays in the required way. To find the inverse Fourier transform of $\xi_i\xi_j\xi_k|\xi|^{-4}$, we apply Lemma A. 2.2 to find $\mathcal{F}^{-1}|\xi|^{-4}$, and then differentiate three times. That would not work in dimensions $n=2$ and $n=4$ due to restriction on μ in Lemma A. 2.2. Assume first that $n\geq 5$. Then we get $v_{\mathbf{R}^n} = P*f$, where $P(x)$ is a matrix with entries that are homogeneous functions of x of order $-n+1$; and they in fact equal $|x|^{-n+1}$ multiplied by polynomials of $x/|x|$. When $n=2$ or $n=4$, we argue as in (7.24). Then estimates (8.11) follow as in the proof of Theorem 7.7, where $\phi_{\mathbf{R}^n}$ plays the role of $v_{\mathbf{R}^n}$ here, see (7.19).

To prove uniqueness of the decomposition, let $f=0$. Then we need to show that the corresponding $v_{\mathbf{R}^n}$ vanishes. The equation $\delta\mathrm{d}^s v_{\mathbf{R}^n}=0$ can be written as $A(\xi)\hat v_{\mathbf{R}^n}(\xi)=0$, therefore, $\operatorname{supp}\hat v_{\mathbf{R}^n}\subset\{0\}$. The only function that satisfies the conditions of the theorem with that property is the zero one. $\square$

Using (8.12), we get

$$(\hat f^s_{\mathbf{R}^n})_{kl}(\xi) = \lambda^{kl}_{ij}(\xi)\hat f^{ij}(\xi),\quad \lambda^{kl}_{ij}(\xi):=\left(\delta^k_i-\frac{\xi_i\xi^k}{|\xi|^2}\right)\left(\delta^l_j-\frac{\xi_j\xi^l}{|\xi|^2}\right). \tag{8.13}$$

We are ready now to prove the following analog of Theorem 7.2.

Theorem 8.4. *Let $f\in L^2(\mathbf{R}^n)$ be of compact support. Then $Xf=0$ if and only if $f=\mathrm{d}^s v$ with some $v(x)\in H^1(\mathbf{R}^n)$ that vanishes in the unbounded connencted component of $\mathbf{R}^n\setminus\operatorname{supp} f$.*

Proof. We already know that the "if" part holds for smooth f. For f as in the theorem, we can use the Fourier Slice Theorem to prove it. We prove the "only if" part next. Let $Xf=0$. Let $f=f^s_{\mathbf{R}^n}+\mathrm{d}^s v_{\mathbf{R}^n}$ be the decomposition of f. Then

$$\xi^i(\hat f^s_{\mathbf{R}^n})_{ij}(\xi)=0$$

because $\delta f^s_{\mathbf{R}^n} = 0$. By the Fourier Slice Theorem (Theorem 8.1),

$$\theta^i\theta^j(\hat{f}^s_{\mathbf{R}^n})_{ij}(\xi) = 0$$

for any $\theta \perp \xi$. Then for any constants $a_{1,2}$ and $b_{1,2}$, and any $\theta \perp \xi$,

$$(a_1\theta + b_1\xi)^i(a_2\theta + b_2\xi)^j(\hat{f}^s_{\mathbf{R}^n})_{ij}(\xi) = 0.$$

Fix $\xi \neq 0$. Then any vector can be expressed in the form $a\theta + b\xi$ with some constants a, b, and some $\theta \perp \xi$. We therefore get $\eta^i\zeta^j(\hat{f}^s_{\mathbf{R}^n})_{ij}(\xi)$ for any $\eta \in \mathbf{R}^n$, $\zeta \in \mathbf{R}^n$. Therefore, $\hat{f}^s_{\mathbf{R}^n}(\xi) = 0$ for $\xi \neq 0$, and since $\hat{f}^s_{\mathbf{R}^n}$ cannot be supported at 0, we get $f^s_{\mathbf{R}^n} = 0$. Then $f = \mathrm{d}^s v_{\mathbf{R}^n}$.

To prove the statement about $\operatorname{supp} v$, we start with

$$\mathrm{d}^s v_{\mathbf{R}^s} = 0 \quad \text{in } \mathbf{R}^n \setminus \bar{\Omega}, \quad \bar{\Omega} := \operatorname{supp} f.$$

We also know that $v_{\mathbf{R}^s}$ is smooth in the indicated domain. Moreover, estimates (8.11) hold for large $|x|$. We will prove that this implies $v_{\mathbf{R}^s} = 0$ in $\mathbf{R}^n \setminus \bar{\Omega}$.

Integrate (8.8) along any line segment $[x_0, x_0 + t\theta_0]$ in $\mathbf{R}^n \setminus \bar{\Omega}$ to get

$$v_{\mathbf{R}^n}(x_0) \cdot \theta_0 = v_{\mathbf{R}^n}(x_0 + t\theta_0) \cdot \theta_0. \tag{8.14}$$

Assume that the ray $x_0 + t\theta_0$, $t \geq 0$ is contained in $\mathbf{R}^n \setminus \bar{\Omega}$. Taking the limit $t \to \infty$ and using (8.11), we get

$$v_{\mathbf{R}^n}(x) \cdot \theta_0 = 0.$$

Since $\mathbf{R}^n \setminus \bar{\Omega}$ is open, this is true for θ in a small enough neighborhood in S^{n-1} of θ_0. Any fixed vector is uniquely determined by its dot product with n linearly independent unit vectors; and in any open set on the sphere, such vectors exist. Therefore, $v_{R^n}(x_0) = 0$. Since $\mathbf{R}^n \setminus \bar{\Omega}$ is open, we can perturb x_0 a bit, and we still get $v_{R^n}(x) = 0$.

To summarize, under the assumption that the ray $x_0 + t\theta_0$, $t \geq 0$ is contained in $\mathbf{R}^n \setminus \bar{\Omega}$, we showed that $v = 0$ in some open set $U_0 \ni x_0$. Now, let $x_1 \in \mathbf{R}^n \setminus \bar{\Omega}$, be such that the line segment $[x_1, x_0]$ is in $\mathbf{R}^n \setminus \bar{\Omega}$. Then we get as before

$$v_{\mathbf{R}^n}(x_1) \cdot \theta_1 = v_{\mathbf{R}^n}(x_1 + t\theta_1) \cdot \theta_1 = 0, \quad x_0 - x_1 = t\theta_1,\ t = |x_0 - x_1|.$$

For θ unit, set $\tilde{x}_0 = x_0 + t\theta$. Then $\tilde{x}_0 \in U_0$ if θ is close enough to θ_1, and we get again $v_{\mathbf{R}^n}(x_1) \cdot \theta = 0$. Therefore, $v_{\mathbf{R}^n}(x_1) = 0$. The assumption on x_1 is an open one, therefore we get $v_{\mathbf{R}^n} = 0$ in an open set $U_1 \ni x_1$.

Iterate this argument to get the following. Let $y \in \mathbf{R}^n \setminus \bar{\Omega}$, so that there exists a polygon $[y, x_N] \cup [x_N, x_{N-1}] \cup [x_1, x_0] \cup \cdots \cup [x_0, \infty\theta_0]$ (with an obvious definition of the latter) in $\mathbf{R}^n \setminus \bar{\Omega}$ that connects y to "infinity", we get $v_{\mathbf{R}^n}(y) = 0$. This is what we had to prove. $\square$

The uniqueness result of the theorem, up to potential fields, can be formulated as before in any of the following ways. For f as in the Theorem,

- If $Xf = 0$, then $f^s_{\mathbf{R}^n} = 0$,
- If $Xf = 0$, then $f = \mathrm{d}^s v$ with $v = 0$ for large $|x|$,
- If $Xf = 0$ and $\delta f = 0$, then $f = 0$.

We call this property *solenoidal injectivity* of X.

In Theorem 7.2, the condition $\mathrm{d}f = 0$ was shown to be equivalent there to $Xf = 0$ and $f = \mathrm{d}\phi$. Similar condition exists in the case of symmetric 2-tensor fields. The role of the differential d is played by the *Saint-Venant operator*

$$(Wf)_{ijkl} = \partial_i\partial_j f_{kl} + \partial_k\partial_l f_{ij} - \partial_k\partial_j f_{il} - \partial_i\partial_l f_{kj}.$$

PROBLEM 8.2. Under the assumptions of Theorem 8.4, prove that each of the conditions $Xf = 0$ and $f = \mathrm{d}^s v$ is equivalent to $Wf = 0$.

We turn our attention now to tensor fields supported in a fixed bounded domain Ω with smooth boundary. If v is smooth and $v = 0$ on $\partial\Omega$, then its zero extension outside Ω is in the kernel of X again. In fact, the symmetric differential d^s commutes with the extension as zero outside the domain, let us call it E, for v vanishing on $\partial\Omega$. In other words, $E\mathrm{d}^s v = \mathrm{d}^s E v$ for such v. Then $XE\mathrm{d}^s v = X\mathrm{d}^s E v = 0$, by (8.7). Of course, $XE\mathrm{d}^s v = 0$ can be checked directly for v smooth enough by using (8.8). We will not use the notation E below.

By Theorem 8.4, $X\mathrm{d}^s v = 0$ for any $v \in H_0^1(\Omega)$. The symmetric 2-tensor fields orthogonal to all $\mathrm{d}^s v$ with $v \in H_0^1(\Omega)$ must satisfy the first equality on (8.10) with the integral taken in Ω. Since $v = 0$, the integration by parts is justified, at least for f smooth, to give us $\delta f = 0$ in Ω. When f is only in L^2, the this is still true but δf should be considered as an tensor field in H^{-1}.

DEFINITION 8.5. *Let $\Omega \subset \mathbf{R}^n$ be an open set. Then $f \in L^2(\Omega)$ is called solenoidal in Ω, if $\delta f = 0$ in Ω. It is called potential, if $f = \mathrm{d}^s v$ with some $v \in H_0^1(\Omega)$.*

THEOREM 8.6. *Any $f \in L^2(\mathbf{R}^n)$ admits an unique orthogonal decomposition*

$$f = f^s + \mathrm{d}^s v, \quad \text{where } f^s \in L^2(\Omega), \quad \delta f^s = 0 \text{ in } \Omega, \quad \text{and } v \in H_0^1(\Omega).$$

Moreover, $f^s = \mathcal{S}f$, $\mathrm{d}^s v = \mathcal{P}f$, where $\mathcal{P}$ and $\mathcal{S}$ are orthogonal projections in the space $L^2(\Omega)$ with $\mathcal{P} + \mathcal{S} = \mathrm{Id}$, and the map $L^2(\Omega) \ni f \mapsto v \in H_0^1(\Omega)$ is bounded.

PROOF. Arguing as in the proof of Theorem 7.8, we define v as the unique solution of the elliptic boundary value problem with Dirichlet boundary conditions

$$\delta \mathrm{d}^s v = \delta f \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0. \tag{8.15}$$

The matrix-valued operator $\delta\mathrm{d}^s$ is an elliptic one, and the Dirichlet boundary conditions are regular ones for it. Since $\delta f \in H^{-1}(\Omega)$, we get that there exists unique solution $v \in H_0^1(\Omega)$. We will use the notation reference?

$$v = (\delta \mathrm{d}^s)_D^{-1} \delta f,$$

where $(\delta\mathrm{d}^s)_D^{-1}$ is the solution of the non-homogeneous boundary value problem for $\delta\mathrm{d}^s$ with Dirichlet boundary conditions. Then we set $f^s = f - \mathrm{d}^s v$. Then

$$\mathcal{P} = \mathrm{d}^s(\delta\mathrm{d}^s)_D^{-1}\delta, \quad \mathcal{S} = \mathrm{Id} - \mathcal{P}.$$

The properties of $\mathcal{P}$ and $\mathcal{S}$ are mow easy to check. Indeed, $\mathcal{P}$ is bounded by standard elliptic estimates. Next, $\mathcal{P}$ is symmetric operator on smooth vector fields, and since it is a bounded operator, it is also self-adjoint. It is straightforward to check that $\mathcal{P}^2 = \mathcal{P}$. □

The decomposition of the theorem is different than the decomposition of f, extended as 0 outside Ω provided by Theorem 8.3. Similarly to the analysis in section 7, we see from (8.12) and (8.13) that for $f \in C_0^\infty(\mathbf{R}^n)$, $\mathcal{F}\mathrm{d}^s v_{\mathbf{R}^n}$ and $\hat{f}^s$ have singularities at the origin for generic f; and therefore they are not in the Schwartz class in those cases, thus $\mathrm{d}^s v$ and f even though smooth, do not belong to C_0^∞.

In next theorem, $L^2(\Omega)$ is regarded as a subspace of $L^2(\mathbf{R}^n)$.

COROLLARY 8.7. *Let Ω be as in Theorem 8.6 and let $f \in L^2(\Omega)$.*
(a) If $Xf = 0$, then $f = \mathrm{d}^s v$ with some $v \in H_0^1(\Omega)$,
(b) If $Xf = 0$, then $f^s = 0$,
(c) If $Xf = 0$ and $\delta f = 0$, then $f = 0$.

PROOF. Follows directly from Theorem 8.4. □

We call each of the properties (i), (ii), (ii) above solenoidal injectivity of X in Ω. They show that X is injective on the closed subspace $\mathcal{S}L^2(\Omega)$ of $L^2(\Omega)$ consisting of solenoidal tensors.

8.2. Inversion Formulas. We start with analyzing the Schwartz kernel of $X'X$.

PROPOSITION 8.8. *For any $f \in C_0^\infty(\mathbf{R}^n)$,*

$$(X'Xf)^{ij} = 2\frac{x^i x^j x^k x^l}{|x|^{n+3}} * f_{kl}. \tag{8.16}$$

The proof is similar to that of Proposition 7.10 and is left to the reader as an exercise.

COROLLARY 8.9.

$$\mathcal{F}(X'Xf)_{ij}(\xi) = \sigma(X'X)(\xi)_{ijkl}\hat{f}^{kl}(\xi) \tag{8.17}$$

where

$$\sigma(X'X)(\xi)_{ijkl} = C_n'\partial_{\xi_i}\partial_{\xi_j}\partial_{\xi_k}\partial_{\xi_l}|\xi|^3, \quad C_n' = \frac{\pi^{\frac{n+1}{2}}}{3\Gamma\left(\frac{n+3}{2}\right)}. \tag{8.18}$$

We also have

$$\sigma(X'X)(\xi)_{ijkl} = 9C_n'|\xi|^{-1}\operatorname{symm}\left[\left(\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}\right)\left(\delta_{kl} - \frac{\xi_k\xi_l}{|\xi|^2}\right)\right], \tag{8.19}$$

where $\operatorname{symm}(h_{ijkl})$ *is the symmetrization of h, i.e., the average of h_{ijkl} over all 24 permutations of the indices.*

PROOF. By Proposition 8.8, $X'X$ is a convolution type of operator. Therefore, it is a Fourier multiplier with the Fourier transform of $2x^ix^jx^kx^l|x|^{-n-3}$, and the latter is in $\mathcal{S}'$. Then

$$\sigma(X'X)(\xi)_{ijkl} = \mathcal{F}\left(2x^ix^jx^kx^l|x|^{-n-3}\right).$$

Let $|x|^{-n-3}$ be the distribution defined by analytic extension in Lemma II.2.1. Formally,

$$\mathcal{F}\left(2x^ix^jx^kx^l|x|^{-n-3}\right) = 2\partial_{\xi_i}\partial_{\xi_j}\partial_{\xi_k}\partial_{\xi_l}\mathcal{F}|x|^{-n-3} = C_n'\partial_{\xi_i}\partial_{\xi_j}\partial_{\xi_k}\partial_{\xi_l}|\xi|^3, \tag{8.20}$$

by Lemma II.2.2, as claimed. To justify this computation, we use the fact that the distribution $x^ix^jx^kx^l|x|^{-\mu}$ depends analytically on μ in the half-plane $\Re\mu > -n-4$, while $|x|^{-\mu}$ is analytic for $\Re\mu > -n$, and has a meromorphic extension with poles at $\mu = -n, -n-2, -n-4, \dots$. For $\Re\mu > -n$, the former distribution is obtained from the latter by multiplication by the smooth function $x^ix^jx^kx^l$. That relation is preserved under analytic continuation from $\Re\mu > -n$ to some neighborhood of $\mu = -n-3$. This justifies (8.20).

Equality (8.19) can be verified by direct differentiation in (8.18). □

The symmetrization in (8.19) has only 3 distinct terms, therefore

$$
\begin{aligned}
[\sigma(X'X)(\xi)]_{ijkl} = 3C_n'|\xi|^{-1}\bigg[&\left(\delta_{ij}-\frac{\xi_i\xi_j}{|\xi|^2}\right)\left(\delta_{kl}-\frac{\xi_k\xi_l}{|\xi|^2}\right)\\
&+\left(\delta_{ik}-\frac{\xi_i\xi_k}{|\xi|^2}\right)\left(\delta_{jl}-\frac{\xi_j\xi_l}{|\xi|^2}\right)\\
&+\left(\delta_{il}-\frac{\xi_i\xi_l}{|\xi|^2}\right)\left(\delta_{kj}-\frac{\xi_k\xi_j}{|\xi|^2}\right)\bigg].
\end{aligned}
\tag{8.21}
$$

The operators $f\mapsto f^s_{\mathbf{R}^n}$ and $f\mapsto \mathrm{d}^s v_{\mathbf{R}^n}$ in Theorem 8.3 are Fourier multipliers on the space $ST_2^0(\mathbf{C}^n)$ with $S(\xi)$ and $P(\xi)$ defined as

$$[S(\xi)f]_{ij}=\lambda_{ij}^{kl}(\xi)f_{kl},\quad P(\xi)=\mathrm{Id}-S(\xi),$$

see (8.13). The operators δ and d^s are Fourier multipliers, too, with $\sigma(\delta)(\xi)$ and $\sigma(\mathrm{d}^s)(\xi)$ defined as

$$\frac{1}{\mathrm{i}}[\sigma(\delta)f]_i=\xi^i f_{ij},\quad \frac{1}{\mathrm{i}}[\sigma(\mathrm{d}^s)v]_{ij}=\frac{1}{2}(\xi_i v_j+\xi_j v_i)$$

It is straightforward to see that $\sigma(\xi)$, viewed as a linear operator in $ST_2^0(\mathbf{C}^n)$ given by $f_{ij}\mapsto[\sigma(X'X)(\xi)]_{ijkl}f^{kl}$, satisfies

$$\sigma(X'X)\sigma(\mathrm{d}^s)=\sigma(\delta)\sigma(X'X)=0,\quad S\sigma(\mathrm{d}^s)=\sigma(\delta)S=0.$$

The first two equalities also follow from (8.7), while the last two — from the definition of the solenoidal/potential decomposition in $\mathbf{R}^n$. Therefore,

$$S\sigma(X'X)=\sigma(X'X)S=\sigma(X'X),\quad P\sigma(X'X)=\sigma(X'X)P=0.$$

The first two equalities follow also from $Xf=Xf^s_{\mathbf{R}^n}$. Therefore, if we write

$$\mathbf{S}(\xi)=S(\xi)(ST_2^0(\mathbf{C}^n)),\quad \mathbf{P}(\xi)=P(\xi)(ST_2^0(\mathbf{C}^n)),$$

then $ST_2^0(\mathbf{C}^n)=\mathbf{S}(\xi)\oplus\mathbf{P}(\xi)$ is an orthogonal decomposition for any $\xi\neq 0$; $S\sigma(X'X)(\xi)$ leaves those two subspaces invariant, and vanishes on $\mathbf{P}(\xi)$. We will show now that it is an isomorphism on $\mathbf{S}(\xi)$ and will invert it there.

Set $\omega=\xi/|\xi|$. Let $f\in\mathbf{S}(\xi)$, then $\xi^i f_{ij}=0$. We solve the equation

$$\sigma(X'X)(\xi)f=h \tag{8.22}$$

below. Let f be a solution. By (8.21),

$$h_{kl}=3C_n'|\xi|^{-1}\left((\operatorname{tr}f)(\delta_{kl}-\omega_k\omega_l)+2f_{kl}\right), \tag{8.23}$$

where $\operatorname{tr}f=f_i^i=\sum_i f_{ii}$. Take trace of both sides to get

$$\operatorname{tr}h=3C_n'|\xi|^{-1}\left((n-1)\operatorname{tr}f+2\operatorname{tr}f\right)=3C_n'|\xi|^{-1}(n+1)\operatorname{tr}f.$$

Solve this for $\operatorname{tr}f$ and substitute in (8.23) to get

$$f_{ij}={C_n'}^{-1}|\xi|\left(\frac{1}{6}h_{ij}-\frac{1}{n+1}(\delta_{ij}-\omega_i\omega_j)\operatorname{tr}h\right). \tag{8.24}$$

It follows immediately that $f\in\mathbf{S}$ (i.e., $\xi^j f_{ij}=0$, if $h\in\mathbf{S}$, as well. What we showed so far is that if (8.22) is solvable, the solution must be given by that formula. Assume $f\in\mathbf{S}$. Then by (8.23),

$$\langle\sigma(X'X)f,f\rangle=3C_n'|\xi|^{-1}\left(|\operatorname{tr}f|^2+2|f|^2\right),$$

where $\langle f,h\rangle=f_{ij}\bar h^{ij}$ is the inner product in $ST_2^0(\mathbf{C}^n)$, and $|f|^2=\langle f,f\rangle$. Therefore, $\sigma(X'X)(\xi)\geq C|\xi|^{-1}$ in operator sense on the finitely dimensional space $\mathbf{S}$, and is therefore an isomorphism there. Therefore, (8.22) is always solvable on $\mathbf{S}$ with a solution given by (8.24).

We therefore proved the following.

THEOREM 8.10. *For any $f \in C_0^\infty(\mathbf{R}^n)$,*

$$[f^s_{\mathbf{R}^n}]_{ij} = {C'_n}^{-1}|D|\left(\frac{1}{6}\delta_{ik}\delta_{jl} - \frac{1}{n+1}\left(\delta_{ij} - \partial_{x^i}\partial_{x^j}\Delta^{-1}\right)\delta_{kl}\right)(X'Xf)_{kl},$$

where Δ^{-1} denotes the Fourier multiplier by $-|\xi|^{-2}$, [2] *and C'_n is as in (8.18).*

8.3. Stability Estimates. We will prove here analogs of the results in Section 7.5. The following lemma plays the role of Lemma 7.16 in the present case.

LEMMA 8.11. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary and connected exterior. Then there exists a constant $C = C(\Omega) > 0$, so that for any $f \in L^2(\mathbf{R}^n)$,*

$$\|f^s\|_{L^2(\Omega)} \le C\|f^s_{\mathbf{R}^n}\|_{L^2(\mathbf{R}^n)} \tag{8.25}$$

PROOF. We follow the proof of Lemma 7.16 but we will see that the 2-tensor case involves additional difficulties.

Let as before, $\Omega_1 \subset \mathbf{R}^n$ be a bounded open set so that $\bar\Omega \subset \Omega_1$. Assume that $\partial\Omega_1$ is given by $\operatorname{dist}(x, \partial\Omega) = \varepsilon$ with $\varepsilon > 0$ small enough so that $\partial\Omega_1$ is smooth. Let $f^s_{\Omega_1}$ be the solenoidal projection of f, extended as 0 outside Ω, that we always assume, to Ω_1; and let $\mathrm{d}^s v_{\Omega_1}$ be its solenoidal one. Comparing $f^s_{\Omega_1}$ and $f^s_{\mathbf{R}^n}$, we get

$$f^s_{\Omega_1} = f^s_{\mathbf{R}^n} + \mathrm{d}^s w_1 \quad \text{in } \Omega_1, \quad w_1 := v_{\mathbf{R}^n} - v_{\Omega_1}, \quad \delta\mathrm{d}^s w_1 = 0 \quad \text{in } \Omega_1. \tag{8.26}$$

We claim that

$$C_0^\infty(\Omega) \ni f \mapsto \mathrm{d}^s w_1 \in C^\infty(\bar\Omega_1) \tag{8.27}$$

extends as a compact operator from $L^2(\Omega)$ to $\in L^2(\Omega_1)$. Indeed, $\mathrm{d}^s w_1$ solves the elliptic system $\delta\mathrm{d}^s w_1 = 0$ in Ω_1 and hence is determined by $w_1|_{\partial\Omega_1}$. The latter equals $v_{\mathbf{R}^n}|_{\partial\Omega_1}$. The field $v_{\mathbf{R}^n}$ is obtained from f by a convolution with a tensor field $P(x)$homogeneous of order $-n+1$ as we saw in the proof of Theorem 8.3. Then we conclude as in the proof of Lemma 7.16 that (8.27) is a compact map, indeed. Therefore,

$$f^s_{\Omega_1} = f^s_{\mathbf{R}^n} + K_1 f \quad \text{in } \Omega_1, \tag{8.28}$$

where $K_1 : L^2(\Omega) \to L^2(\Omega_1)$ is compact. Next, we will compare f^s extended as 0 outside Ω, and $f^s_{\Omega_1}$.

We have $f^s = f - \mathrm{d}^s v$ in in Ω, with $v \in H^1_0(\Omega)$. Extend v and f^s as 0 outside Ω. Then we still have $f^s = f - \mathrm{d}^s v$ in the whole $\mathbf{R}^n$ because there is no jump of v at $\partial\Omega$. Comparing f^s and $f^s_{\Omega_1}$, we get

$$f^s = f^s_{\Omega_1} + \mathrm{d}^s w_2 \quad \text{in } \Omega_1, \quad w_2 := v_{\Omega_1} - v. \tag{8.29}$$

Since $f^s = v = 0$ outside Ω, we have

$$\mathrm{d}^s v_{\Omega_1} = \mathrm{d}^s w_2 = f^s_{\Omega_1}, \quad \text{in } \Omega_1 \setminus \Omega, \tag{8.30}$$

therefore,

$$\|\mathrm{d}^s w_2\|_{L^2(\Omega_1\setminus\Omega)} = \|f^s_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}. \tag{8.31}$$

[2] more precisely, $\partial_{x^i}\partial_{x^j}\Delta^{-1}$ is the Fourier multiplier by $\xi_i\xi_j/|\xi|^2$

As before, we claim that we actually have

$$\|w_2\|_{H^1(\Omega_1\setminus\Omega)} \le C\|f^s_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}. \tag{8.32}$$

So far, we followed closely the proof of Lemma 7.16. At this point however we see an essential difference. To prove (8.32), we still need to prove some version of the Poincaré inequality to estimate the L^2 norm of w_2 through the L^2 norm of its first order derivatives. We first need however to estimate those derivatives through the L^2 norm of $\mathrm{d}^s w_2$!

In the case of vector fields, $\mathrm{d}\phi$ determines trivially all first order derivatives of ϕ. However, if v is a vector field, $\partial v_i/\partial x_j$ cannot be obtained from $(\mathrm{d}^s v)_{ij}$ by algebraic operations. Indeed, at any fixed point, there are n^2 independent possible values for $\partial v_i/\partial x_j$, and only $n(n+1)/2$ ones for $(\mathrm{d}^s v)_{ij}$. Therefore, if we view (8.6) as an algebraic system for $\partial v_i/\partial x_j$, we only have $n(n+1)/2$ equations for n^2 unknowns, so this system is under-determined. Some derivatives however can be obtained easily: $\partial_i v_i = v_{ii}$.

On the other hand, the operator $\mathrm{d}^s : C_0^\infty(\mathbf{R}^n) \to C_0^\infty(\mathbf{R}^n)$ is elliptic in the sense that it has a left inverse of order -1 (mapping H^s to H^{s+1} locally) given by $(\delta\mathrm{d}^s)^{-1}\delta$, where, somewhat incorrectly, the later denotes the map (8.12). Then $\partial v_i/\partial x_j$ can be recovered by taking derivatives of $(\delta\mathrm{d}^s)^{-1}\delta(\mathrm{d}^s v)$. That operator however is a Fourier multiplier by a non-polynomial, and therefore is a non-local operator. One could try to use the pseudo-differential calculus, see Chapter..., but the problem here is that we need to estimate the H^1 norm of w_2 in $\Omega_1\setminus\Omega$, all the way to $\partial\Omega$. The standard pseudo-differential calculus would only give us estimates in open sets in $\Omega_1\setminus\Omega$.

To resolve this problem we use Korn's inequality, see ,... Applied to $\Omega_1\setminus\Omega$, it says that

$$\|v\|_{H^1(\Omega_1\setminus\Omega)} \le C\|\mathrm{d}^s v\|_{L^2(\Omega_1\setminus\Omega)} + C\|v\|_{L^2(\Omega_1\setminus\Omega)}$$

for all smooth vector fields on the closure of $\Omega_1\setminus\Omega$. We therefore get from (8.31),

$$\|w_2\|_{H^1(\Omega_1\setminus\Omega)} \le C\|f^s_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)} + C\|w_2\|_{L^2(\Omega_1\setminus\Omega)}.$$

It remains now to estimate the L^2 norm of w_2 in $\Omega_1\setminus\Omega$:

$$\|w_2\|_{L^2(\Omega_1\setminus\Omega)} \le C\|f^s_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}. \tag{8.33}$$

The starting point for this is the second inequality in (8.30), that we will integrate along various lines, compare (8.14) that follows from (8.8) by integration. We see another important difference between the X-ray transform of 2-tensors and 1-tensors (1-forms). While an integral of $\mathrm{d}\phi$ over some path, where ϕ is a function, depends only on the end-points but not on the path itself; an integral of $\mathrm{d}^s v$ may not equal (8.14) if not taken over a straight line segment.

It is enough to prove (8.33) with the L^2 norm on the left restricted to a neighborhood of any point in the closure of $\Omega_1\setminus\Omega$, and then use a partition of unity. Choose $x_0\in\partial\Omega$ and let $U\supset x_0$ be open. At least one of the coordinate vectors e_j at x_0 is transversal to $\partial\Omega$, we can always assume that it is e_n. We can also assume that $x_0=0$ and that e_n points away from $\partial\Omega$. Then near x_0, $\partial\Omega$ is locally given by $x^n = F(x')$, while $\partial\Omega_1$ is defined by $x^n = G(x')$, $G>F$, if ε is small enough.

Let $\theta\in S^{n-1}$ be close enough to e_n so that $\theta_n>0$ and θ is still transversal to both $\partial\theta$ at $x_0=0$, and to $\partial\theta_1$ at the point of intersection of it with the ray $s\theta$, $s>0$. Let $\delta>0$ be such that those properties are preserved with x_0 replaced by $(x',0)$, $|x'|\le\delta$, more precisely, we require that the ray $(x',0)+s\theta$ hits both $\partial\Omega$

and $\partial\Omega_1$ transversely. Choose $U \ni 0$ so that all those rays sweep U; and that this is also true for θ replaced by $\tilde\theta$ with $|\tilde\theta^n| \le \theta_n$ (because we would need to perturb Ω later). Then as in (7.48), we get that $|w_2\cdot\theta|$ restricted to the intersection of the ray $(x',0)+s\theta$ with $\Omega_1\setminus\Omega$, is pointwise bounded by

$$\int_{\tau_-(x,\theta)}^{\tau^+(x,\theta)} |f^s_{\Omega_1}((x',0)+t\theta)|^2\,\mathrm{d}t,$$

where $\tau_\mp(x,\theta)$ correspond to the values of t, where the ray over which we integrate hits $\partial\Omega_1$ nd $\partial\Omega$, respectively. Integrate with respect to s first, and then with respect to x' in $|x'|\le\delta$ to get

$$\|w_2\cdot\theta\|_{L^1(U)} \le C(\theta)\|f^s_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}.$$

We repeat that with n linear independent vectors ω without increasing the absolute value of the n-th component. In other words, all those vectors are close enough to e_n, and form a basis in $\mathbf{R}^n$. This proves (8.33) with the norm on the left taken in $U\cap(\Omega^1\setminus\Omega)$. Take a finite cover of $\partial\Omega$ of such U_s, and moving $\partial\Omega_1$ closer to $\partial\Omega$, if needed, we complete the proof of (8.33).

Therefore, (8.32) holds. By the trace theorem,

$$\|w_2\|_{H^{1/2}(\partial\Omega)} \le C\|f^s_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}.$$

In Ω, $\delta\mathrm{d}^s w_2 = 0$. Then by standard elliptic estimates,

$$\|w_2\|_{H^1(\Omega)} \le C\|f^s_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}. \tag{8.34}$$

This and (8.29) imply

$$\|f^s\|_{L^2(\Omega)} \le C\|f^s_{\Omega_1}\|_{L^2(\Omega_1)}. \tag{8.35}$$

Therefore, by (8.28) and (8.35),

$$\begin{aligned}\|f^s\|_{L^2(\Omega)} &\le C\|f^s_{\mathbf{R}^n}\|_{L^2(\Omega_1)} + \|K_1 f\|_{L^2(\Omega_1)}\\ &\le C\|f^s_{\mathbf{R}^n}\|_{L^2(\mathbf{R}^n)} + \|K_1 f\|_{L^2(\Omega_1)}.\end{aligned}$$

Assume first that f is solenoidal, i.e., $f=f^s$ or, equivalently, $\delta f = 0$ in Ω. Apply Lemma 3.4 with f belonging to the closed subspace $\mathcal{S}L^2(\Omega)$ of solenoidal vector fields in $L^2(\Omega)$. We proved in Theorem 8.4 that the bounded map

$$\mathcal{S}L^2(\Omega)\ni f \to f^s_{\mathbf{R}^n}\in L^2(\mathbf{R}^n)$$

is injective. The estimate above then implies that (8.25) holds with a different constant. □

THEOREM 8.12. *Let $\Omega\subset\mathbf{R}^n$ be a bounded domain with smooth boundary and connected exterior. Then*

$$\|f^s\|_{L^2(\Omega)}/C \le \|Xf\|_{\bar H^{1/2}(\Sigma)} \le C\|f\|_{L^2(\Omega)} \tag{8.36}$$

for any $f\in L^2(\Omega)$.

***** to be continued *****

9. The attenuated X-ray transform

10. The Light Ray transform

In the theory of time-dependent wave-like type PDEs (like the wave equation with a time dependent potential $q(t,x)$), one arrives naturally at the following transform

$$Lf(t,x,\theta) = \int f(t+s, x+s\theta)\,\mathrm{d}s, \quad t\in\mathbf{R},\ x\in\mathbf{R}^n,\ \theta\in S^{n-1}. \tag{10.1}$$

We assume $n\ge 2$ in what follows. Here, Lf is just an integral of f over the lines in $R^{n+1}\ni(t,x)$ parallel to $(1,\theta)$, $|\theta|=1$. Those are actually the characteristic lines of the wave operator with speed one. The transform L is an example of a restricted X-ray transform (in R^{n+1}), where the lines over which we integrate belong to a submanifold instead of belonging to an open subset. In relativity, those are called light rays and represent (trajectories) of photons. They coincide with the zero geodesics in the Minkowski metric $-\mathrm{d}t^2+d(x^1)^2+\cdots+d(x^1)^n$ (parameterized by time); i.e., lines with tangent vectors having zero length in that metric.

The parameterization of the light rays by (t,x,θ) is overdetermined, of course. The freedom we have to change parameterization is given by

$$Lf(t,x,\theta) = Lf(t+T, x+T\theta, \theta). \tag{10.2}$$

Notice that L is the forward light ray transform because the time t increases along the light rays involved in it. One can define also the backward transform by changing $t+s$ by $t-s$ in (10.1) but the latter is directly related to L and does not provide extra information.

One way to fix a parameterization on the (forward pointing) light rays is to fix the initial condition on a fixed space-like surface, see the definitions below. The simplest one is perhaps $t=0$. Then each forward pointing light ray has unique parameterization by $(x,\theta)\in\mathbf{R}^n\times S^{n-1}$ (the unit sphere bundle $S\mathbf{R}^n$ in invariant terms) given by

$$s\longrightarrow (s, x+s\theta).$$

We chose the natural measure $\mathrm{d}x\,\mathrm{d}\theta$ on $\mathbf{R}^n\times S^{n-1}$. In particular, we see that the complex of light rays has the natural structure of a manifold of dimension $2n-1$; which is 1 less (because of the characteristic restriction on the directions) than the dimension $2(n+1)-2=2n$ of all lines in $\mathbf{R}^{n+1}$. From now on, we will write

$$Lf(x,\theta) = \int f(s, x+s\theta)\,\mathrm{d}s, \quad (x,\theta)\in\mathbf{R}^n\times S^{n-1}. \tag{10.3}$$

10.1. Motivation. Consider the wave equation with a time-dependent potential $q(t,x)$ such that $q=0$ for $|x|>R$ for some $R>0$:

$$u_{tt}-\Delta u+q(t,x)u=0 \quad (t,x)\in\mathbf{R}^{n+1}. \tag{10.4}$$

The inverse scattering problem for this equation consists of determining q given measurements of the solution u outside the ball $B(0,R)$ where the potential $q(\cdot,t)$ is supported, or at infinity. One simple choice of waves we can send are plane waves of the type $\delta(t-s-x\cdot\theta)$, $|\theta|=1$ which solve the wave equation with $q=0$. They are supported on the plane $x\cdot\theta=t+s$ and propagate in the direction θ. The parameter s measures the time-delay. If $t\ll 0$, the plane wave does not intersect the ball $B(0, R)$ and is moving towards it. Let $u(t, x; s, \theta)$ be the solution of (10.4) with initial data

$$u|_{t<-s-R} = \delta(t - s - x \cdot \theta).$$

We can differentiate with respect to t there, therefore we have Cauchy data for any fixed $t < -s-R$ which leads to a well posed Cauchy problem. The fact that we have distributions is not a problem. Indeed, set $h_j(s) = s^j/j!$ for $s \geq 0$ and $h_j(s) = 0$ for $s < 0$. Then we can replace δ by h_1, and then differentiate the solution twice with respect to s.

To find an ansatz for the solution u, we seek u as a formal expansion

$$u(t, x; s, \theta) = \delta(t - s - x \cdot \theta) + A_0(t, x, \theta)h_0(t - s - x \cdot \theta) + A_1 h_1(t - s - x \cdot \theta) + \dots.$$

The initial condition for u implies $A_j = 0$ for $t < -s - R$. Apply $\partial_t^2 - \Delta + q$ to u and rearrange the terms in order of their singularity. To kill the most singular term involving δ after that rearrangement, we need A_0 to solve

$$(\partial_t + \theta \cdot \nabla_x)A_0(t, x, \theta) = -\frac{1}{2}q(t, x), \quad A_0|_{t<-s-R} = 0.$$

Therefore,

$$A_0(t, x, \theta) = -\frac{1}{2}\int_{-\infty}^{0} q(t + \sigma, x + \sigma\theta)\, \mathrm{d}\sigma.$$

We make observation after the most singular part of the wave (and therefore, all singularities) has left the ball $B(0, R)$, i.e., for $t = s + x \cdot \theta$ (where the h_0 term is singular) and $x \cdot \theta > R$. Therefore, we measure

$$\begin{aligned} A_0(s + x \cdot \theta, x, \theta) &= -\frac{1}{2}\int_{-\infty}^{\infty} q(s + x \cdot \theta + \sigma, x + \sigma\theta)\, \mathrm{d}\sigma \\ &= -\frac{1}{2}\int_{-\infty}^{\infty} q(\tau, (x - (x \cdot \theta)\theta - s\theta) + \tau\theta)\, \mathrm{d}\sigma \end{aligned}$$

for $x \cdot \theta > R$. This is just $L(-q/2)(z, \theta)$ with $z = x - (x \cdot \theta)\theta - s\theta$. It is easy to see that this way, we get $L(-q/2)(z, \theta)$ for *all* (z, θ). Therefore, to find q, we need to invert L.

The ansatz above does not prove yet that there is a solution u with that properties — this can be done using standard PDE techniques. We refer to [**35**] for details.

We actually got $L(-q/2)$ by measuring the jump (a singularity) of the second term in the singular expansion of u. This is a strong suggestion that there is no loss if stability in this step, see also Chapter III. The proof of this statement however is beyond the scope of this work.

10.2. Fourier Slice Theorem and corollaries. The Fourier Slice Theorem in this case is a direct consequence of its version for X. Notice that even for $f(t, x) = f_0(x) \in C_0^\infty(\mathbf{R}^n)$, we have $\hat{f}(\tau, \xi) = 2\pi\delta(\tau)\hat{f}_0(\xi)$, therefore $\hat{f}$ is in general a distribution and restricting (τ, ξ) to the plane $\tau + \theta \cdot \xi = 0$, needs to be justified. We study f in the Schwartz class first.

THEOREM 10.1 (Fourier Slice Theorem). *For any $f \in \mathcal{S}(\mathbf{R}^{n+1})$,*

$$\hat{f}(\zeta) = \int_{\mathbf{R}^n} e^{-\mathrm{i}x\cdot\xi} Lf(x, \theta)\, \mathrm{d}x, \quad \text{when } (1, \theta) \perp \zeta,\ \theta \in S^{n-1},$$

where $\zeta = (\tau, \xi)$.

PROOF. The integral on the right equals

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}} e^{-\mathrm{i}x\cdot\xi} f((0,x)+s(1,\theta))\,\mathrm{d}s\,\mathrm{d}x.$$

Set $w=(0,x)+s(1,\theta)$ and note that $w\cdot\zeta = x\cdot\xi$ when $\zeta\perp(1,\theta)$. Performing the change of variables $(s,x)\mapsto w$, we see that the integral above equals $\hat f(\zeta)$. □

REMARK 10.1. Let

$$\pi_\theta = \{(\tau,\xi);\ \tau+\theta\cdot\xi = 0\} \tag{10.5}$$

be the light-like hyperplane normal to $(1,\theta)$. One can also write

$$\hat f|_{\pi_\theta} = \hat f(-\theta\cdot\xi,\xi) = \int_{\mathbf{R}^n} e^{-\mathrm{i}x\cdot\xi} Lf(x,\theta)\,\mathrm{d}x, \quad \forall\theta\in S^{n-1}. \tag{10.6}$$

In the terminology of relativity theory, vectors $v=(v^0,v')$ satisfying $|v_0|<|v'|$ are called *spacelike*. The simplest example are vectors $(0,v')$, $v'\neq 0$. Vectors with $|v_0|>|v'|$ are *timelike*; an example is $(1,0)$ which points along the time axis. For covectors, it is the opposite: $\zeta=(\tau,\xi)$ is spacelike if $|\tau|>|\xi|$; timelike if $|\tau|<|\xi|$ and lighlike if we have equality. Surfaces with spacelike normals (which are covectors) are spacelike, etc.

As a consequence of the theorem, $Lf(\cdot,\theta)$ known for some unit θ determines $\hat f$ for all $\zeta=(\tau,\xi)$ on the lightlike plane $\tau+\xi\cdot\theta=0$. In particular, those ζ must satisfy $|\tau|\le|\xi|$, i.e., they are timelike or lightlike. It easy to see however that the former condition describes all such vectors.

COROLLARY 10.2. *If $f\in\mathcal{S}(\mathbf{R}^{n+1})$ and $Lf=0$, then $\hat f(\zeta)=0$ in the cone $|\tau|\le|\xi|$.*

Note that we get more than formulated in the corollary — by (10.6), given Lf, we can compute $\hat f$ directly by the Fourier Slice Theorem by choosing, for any timelike ζ, a unit θ so that $(1,\theta)\cdot\zeta=0$. Clearly, this can be done, in infinitely many ways. We can also formulate a stability estimate in any compact subset of the time like cone. do it

COROLLARY 10.3. *L is injective on $C_0^\infty(\mathbf{R}^{n+1})$.*

PROOF. For any $f\in C_0^\infty(\mathbf{R}^{n+1})$, $\hat f$ is real analytic. By Corollary 10.2, $\hat f=0$ in an opens set; therefore, $f=0$. □

The same proof works for $f\in L^1_{\mathrm{comp}}(\mathbf{R}^{n+1})$, for example.

Notice that we used analytic continuation in the proof. This, and the fact that we could not constructively (and therefore, stably) reconstruct $\hat f$ in the spacelike cone (of covectors $|\tau|>|\xi|$) is a strong indication, but not a proof yet, that there is no stability in the sense of Chapter III. We will see later that this is true, in fact.

COROLLARY 10.4. *L is not injective on $\mathcal{S}(\mathbf{R}^{n+1})$.*

PROOF. Let $\psi\in C_0^\infty(\mathbf{R}^{n+1})$ supported in the cone $|\tau|>|\xi|$ and set $f=\check\psi\in\mathcal{S}$. Then $\hat f(-\theta\cdot\xi,\xi)=0$ for any unit θ and any ξ because $|\tau|=|\theta\cdot\xi|\le|\xi|$. By (10.6), $Lf=0$. □

This uniqueness result was easy to get but the compact support assumption with respect to the t variable is too restrictive for applications, since it excludes even time-independent functions $f(x)$. A stronger result can be obtained by assuming a uniformly compact support in the x variable only and exploit the partial analyticity of $\hat{f}$ in the ξ variable.

THEOREM 10.5. *Let $f \in \mathcal{S}(\mathbf{R}^n)$ and let $f(t,x) = 0$ for $|x| > R$ for some $R > 0$. Then $Lf(\cdot,\theta) = 0$ for θ in some open set implies $f = 0$.*

PROOF. The idea of the proof is to use the analyticity of the partial Fourier transform of f with respect to x. For any fixed τ (and we can fix τ because $\hat{f} \in \mathcal{S}(\mathbf{R}^{n+1})$ as well),

$$\hat{f}(\tau,\xi) = \int e^{-\mathrm{i}(t\tau + x\cdot\xi)} f(t,x)\,\mathrm{d}t\,\mathrm{d}x = \int e^{-\mathrm{i}x\cdot\xi}\tilde{f}(\tau,x)\,\mathrm{d}x, \tag{10.7}$$

where

$$\tilde{f}(\tau,x) = \int e^{-\mathrm{i}t\tau} f(t,x)\,\mathrm{d}t$$

is the partial Fourier transform of f with respect to the t variable. It is clear from (10.7) that $\hat{f}$ extends to an analytic function of ξ for any fixed τ. Assume first that $Lf = 0$ for all (x,θ); then $\hat{f}(\tau,\xi) = 0$ for $|\xi| > |\tau|$ by Corollary 10.2. By the analyticity, $\hat{f} = 0$ for that fixed τ and all ξ. We have the same conclusion for any τ, therefore $f = 0$.

Now, let $Lf = 0$ for $\theta \in U$, where U is an open subset of S^{n-1}. Then

$$\hat{f}(\tau,\xi) = 0 \quad \text{in } V := \bigcup_{\theta\in U} \pi_\theta, \tag{10.8}$$

see (10.5). It follows from the lemma below (and it is intuitively clear) that for any τ^0, the set $W := V \cap \{\tau = \tau^0\} \cap \{|\tau| < |\xi|\}$ has a non-empty interior in $\{\tau = \tau^0\}$. Then by Corollary 10.2 and analytic continuation, $\hat{f} = 0$ for $\tau = \tau^0$, and since τ^0 is arbitrary, $f = 0$. □

LEMMA 10.6. *For every open subset $U \subset S^{n-1}$, the set*

$$V \setminus \{|\tau| = |\xi|\}$$

is open in $\mathbf{R}^{n+1}$.

In particular, for every $\theta_0 \in U$, the set

$$\pi_{\theta_0} \setminus \{(-\lambda, \lambda\theta_0) |\ \lambda \in \mathbf{R}\}$$

belongs to the interior of V.

PROOF. Note first that V does not intersect the space-like cone $|\tau| > |\xi|$. Take (τ^0,ξ^0) with $|\tau^0| < |\xi^0|$ so that $(\tau^0,\xi^0) \in \pi_{\theta_0}$ with some $\theta_0 \in U$. We want to solve $F := \tau + \theta\cdot\xi = 0$ for θ close to θ_0, and we want the solution to be close to (τ^0,ξ^0). Take local coordinates $\theta = \theta(p^1,\dots,p^{n-1})$ on the unit sphere and compute the differential of F with respect to p. If that differential at the point p_0 corresponding to θ_0 is not zero, a solution (not necessarily unique) exists because we can apply the implicit function theorem when only one of the p_k varies (the one that contributes a non-zero derivative), and the rest fixed. Since $\mathrm{d}_p\theta(p)$ spans all vectors tangent to θ, we get that $\mathrm{d}_pF(p_0) \neq 0$ if and only if ξ^0 is not collinear with θ_0, i.e., $\xi^0 = \lambda\theta_0$ for some $\lambda \in \mathbf{R}$ does not hold. If it does, then $\tau^0 = -\lambda$. We therefore get that as along as $\tau^0\theta_0 + \xi^0 \neq 0$, a solution exists. Since $|\tau^0| < |\xi^0|$ by assumption, that condition is satisfied. □

10.3. Extending L to a larger class of functions or distributions. To compute the dual L' of L in this parameterization, write

$$\begin{aligned}\langle Lf, \phi\rangle &= \int_{S^{n-1}}\int_{\mathbf{R}^n}\int_{\mathbf{R}} f(s, x+s\theta)\phi(x,\theta)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\theta\\ &= \int_{S^{n-1}}\int_{\mathbf{R}^n}\int_{\mathbf{R}} f(s, x)\phi(x-s\theta,\theta)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\theta.\end{aligned}$$

Therefore,

$$L'\phi(t,x) = \int_{S^{n-1}} \phi(x-t\theta,\theta)\,\mathrm{d}\theta, \quad \phi\in C_0^\infty(\mathbf{R}^n\times S^{n-1}). \tag{10.9}$$

The operator L' does not preserve compactness of the support, therefore we cannot define L on $\mathcal{D}'(\mathbf{R}^n)$ by duality, but this is to be expected. On the other hand, L can be defined by duality on compactly supported distributions $\mathcal{E}'(\mathbf{R}^{n+1})$ but even that is too restrictive for some applications because it excludes time-independent distributions or even time-independent C_0^∞ functions of x.

If f is continuous, Lf is well defined at least when any light ray has a compact intersection with $\operatorname{supp} f$. We will extend this requirement a bit. We call $\chi \in C^\infty(\mathbf{R}^{n+1})$ *properly supported*, if for each compact set $K_1 \subset \mathbf{R}^n\times S^{n-1}$ (i.e., for any compact set K_1 of light rays), there exists a compact set $K_2\subset \mathbf{R}$ so that $s\mapsto \chi(s, x+s\theta)$ is supported in K_2 when $(x,\theta)\in K_1$. One such example is an s-independent C_0^∞ function of x. Viewing χ as an operator of multiplication, then $(L\chi)' = \chi L'$ does preserve compactness of the supports, and we can therefore define $L\chi$ on $\mathcal{D}'(\mathbf{R}^n)$, see Section A.4. Also, $\chi L' : \mathcal{S}\to\mathcal{S}$ continuously, see (10.9); therefore, $L\chi$ extends to a continuous operator $L\chi : \mathcal{S}'(\mathbf{R}^{n+1}) \to \mathcal{S}'(\mathbf{R}^n\times S^{n-1})$ as well. This allows us to define L on $\mathcal{S}'(\mathbf{R}^{n+1})$ distributions vanishing for $|x|>R$ for some $R>0$.

We will show next that for any f as above, $\mathrm{WF}(\hat f)\subset\{(t,x,\tau,0)\}$. Indeed, let $\phi\in C_0^\infty$ be such that $\phi(\tau^0,\xi^0)\neq 0$ for some (τ^0,ξ^0). Then $\mathcal{F}^{-1}(\phi\hat f) = \check\phi * f$ (we can use $\mathcal{F}^{-1}$ instead of $\mathcal{F}$ to test for wave front, as well) decays rapidly in any direction different from $(\pm 1, 0)$. This proves the claim. By ...., the trace of $\hat f$ on any hyperplane $\tau+\theta\cdot\xi = 0$ is well defined then. Moreover, the trace depends smoothly on θ. ref?

If we parameterize L as in (10.1), we can check directly that L commutes with the convolution with respect to the (t,x) variable, i.e., $L\phi * f = \phi * Lf$ for any $\phi\in C_0^\infty$.

Next theorem presents a more general uniqueness result; it does not require compact support with respect to t.

THEOREM 10.7. *Let $Lf(\cdot,\theta) = 0$ for $f\in\mathcal{S}'(\mathbf{R}^{n+1})$ vanishing for some $|x|>R$ and all unit θ in some open set. Then $f=0$.*

PROOF. Take $\phi\in C_0^\infty$ supported in $|x|+|t|\le 1$. Then $L\phi * f = 0$ as well with $\phi * f$ smooth and supported in $B(0,R+1)$. Therefore, without loss of generality, we can assume that f is smooth as well. Note that $\hat f$ might still be singular.

The idea of the proof is to generalize the argument we used for the proof of Theorem 10.5. Since $\hat f(\tau,\xi)$ is a distribution now, we cannot fix τ now and exploit the analyticity with respect to ξ.

For any $f\in\mathcal{S}$ and any $\phi\in\mathcal{S}$, by the Fourier Slice Theorem,

$$\int_{S^{n-1}}\int \hat f(-\theta\cdot\xi,\xi)\phi(\xi,\theta)\,\mathrm{d}\xi\,\mathrm{d}\theta=\int_{S^{n-1}}\int L\chi f(x,\theta)\hat\phi(x,\theta)\,\mathrm{d}x\,\mathrm{d}\theta, \tag{10.10}$$

where $\chi\in C^\infty$ is as above and $\chi f=f$. Above, $\hat\phi$ is the partial Fourier transform with respect to the x variable.

Let $(\tau,p)\in\mathbf{R}\times\mathbf{R}^{n-2}$ be a local parameterization of S^{n-1} near some θ_0 so that $-\theta\cdot\xi=\tau$ and p is independent of ξ. Then the left-hand side of (10.10) becomes

$$\iiint \hat f(\tau,\xi)\tilde\phi(\tau,\xi,p)J(\tau,\xi,p)\,\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}p \tag{10.11}$$

where $J\neq 0$ is the corresponding Jacobian, where $\tilde\phi(\tau,\xi,p)$ is equal to $\phi(\xi,\theta)$ in the new variables.

Let U be a neighborhood of some θ_0. By Lemma 10.6, for any τ_0, we can find ξ^0 so that (τ^0,ξ^0) belongs to the interior of V defined in (10.8), i.e., there are open set $W_1\ni\tau^0$, $W_2\ni\xi^0$ so that $\overline{W_1\times W_2}\subset V$. Let $\operatorname{supp}\phi\subset U\times W_2$. By shrinking U and W_2 if necessary, we can guarantee that $\tau\in W_1$ on $\operatorname{supp}\tilde\phi$. Since (10.11) equals the right-hand side of (10.10) for every $f\in\mathcal{S}$, we can extend this equality by continuity to every $f\in\mathcal{S}'$. Let now f be as in the theorem: so that $Lf(\theta,\cdot)=0$ for $\theta\in U$. Then the integral in (10.11) vanishes for every ϕ with support close enough to (τ^0,ξ^0,p_0), where p_0 corresponds to θ_0. This means that $\hat f=0$ near (τ_0,ξ_0).

On the other hand, the function $\xi\to\langle\hat f(\cdot,\xi),\varphi\rangle$ is analytic for every $\varphi\in C_0^\infty(\mathbf{R})$. Choose φ supported in W_2. Then $\langle\hat f(\cdot,\xi),\varphi\rangle=0$, which proves that $\hat f=0$ on $W_2\times\mathbf{R}^n$. Since τ_0 can ne arbitrary, and the only requirement for $W_2\ni\tau_0$ is to be small enough, we get $f=0$. $\square$

10.4. The Schwartz kernel of $L'L$. By (10.3) and (**??**),

$$\begin{aligned}L'Lf(t,x)&=\int_{S^{n-1}}\int_{\mathbf{R}} f(s,x-t\theta+s\theta)\,\mathrm{d}s\,\mathrm{d}\theta\\&=\int_{S^{n-1}}\left(\int_{s<t}+\int_{s>t}\right)f(s,x-t\theta+s\theta)\,\mathrm{d}s\,\mathrm{d}\theta.\end{aligned}$$

For the first integral, we get

$$\begin{aligned}\int_{S^{n-1}}\int_{s<t} f(s,x-t\theta+s\theta)\,\mathrm{d}s\,\mathrm{d}\theta&=\int_{S^{n-1}}\int_{-\infty}^0 f(t+\sigma,x+\sigma\theta)\,\mathrm{d}\sigma\,\mathrm{d}\theta\\&=\int_{S^{n-1}}\int_0^\infty f(t-\sigma,x+\sigma\theta)\,\mathrm{d}\sigma\,\mathrm{d}\theta\\&=\int_{\mathbf{R}^n} f(t-|z|,x+z)|z|^{1-n}\,\mathrm{d}z\\&=\int_{\mathbf{R}^n}\frac{f(t-|x-x'|,x')}{|x-x'|^{n-1}}\,\mathrm{d}x'.\end{aligned} \tag{10.12}$$

For the second one, we have

$$\begin{aligned}\int_{S^{n-1}}\int_{s>t} f(s,x-t\theta+s\theta)\,\mathrm{d}s\,\mathrm{d}\theta&=\int_{S^{n-1}}\int_0^\infty f(t+\sigma,x+\sigma\theta)\,\mathrm{d}\sigma\,\mathrm{d}\theta\\&=\int_{\mathbf{R}^n}\frac{f(t+|x-x'|,x')}{|x-x'|^{n-1}}\,\mathrm{d}x'.\end{aligned}$$

We therefore get the following.

THEOREM 10.8.
(a) We have

$$L'Lf(t,x) = \int_{\mathbf{R}^n} \frac{f(t-|x-x'|,x') + f(t+|x-x'|,x')}{|x-x'|^{n-1}}\,\mathrm{d}x'.$$

(b) For the Schwartz kernel $N(t,x;t',x')$ we have

$$N(t,x;t',x') = \frac{\delta(t-t'-|x-x'|,x') + \delta(t-t'+|x-x'|,x')}{|x-x'|^{n-1}}$$

(c) $L'L$ is a convolution:

$$L'Lf = \mathcal{N} * f, \quad \mathcal{N}(t,x) = \frac{\delta(t-|x|) + \delta(t+|x|)}{|x|^{n-1}}.$$

REMARK 10.2. Since $t\pm|x|$ is not a smooth function of (t,x), the expression for $\mathcal{N}$ requires some clarification. We define $\delta(t\mp|x|)/|x|^{n-1}$ as the linear functionals

$$\phi(t,x) \longmapsto \int \frac{\phi(\pm|x|,x)}{|x|^{n-1}}\,\mathrm{d}x$$

which are clearly distributions in $\mathcal{D}'(\mathbf{R}^{1+n})$. Then for $\psi \in C_0^\infty((\mathbf{R}\times\mathbf{R}^n)^2)$, by definition, ref Friedlander-Joshi

$$\langle \mathcal{N} * f, \psi\rangle = \langle \mathcal{N}\otimes f, \rho(t',x')\psi(t'+t,x'+x)\rangle,$$

where $\rho \in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ equals 1 near $\operatorname{supp} f$. The last two identities prove (b) and (c).

Another way to justify $\delta(t\mp|x|)$ is to notice that it is a well defined distribution away from the origin because the functions $t\mp|x|$ are smooth near $t\mp|x|=0$. Then we define $\delta(t\mp|x|)$ on the whole $\mathbf{R}^{1+n}$ as the unique homogeneous extension of the same distribution from $\mathbf{R}^{1+n}\setminus 0$ to $\mathbf{R}^{1+n}$, see section A.2.2. We define $\mathcal{N}$ in a similar way. We used essentially the fact that their orders of homogeneity, -1 and $-n$ respectively, was greater than $-n-1$.

This reveals an interesting relationship with the wave operator: for $n=3$, if we replace $|x|^{n-1}$ by $|x|$ (i.e., if you apply the ΨDO $|D|$ to $\mathcal{N}$), we get a sum of the classical incoming and outgoing fundamental solutions of the wave operator.

Since $L'L$ is a convolution, it must be a Fourier multiplier with the Fourier transform of $\mathcal{N}$ that we compute below. This leads to the following.

THEOREM 10.9.

$$L'Lf = 2\pi|S^{n-2}|\mathcal{F}^{-1}\frac{(|\xi|^2-\tau^2)_+^{\frac{n-3}{2}}}{|\xi|^{n-2}}\mathcal{F}f, \quad \forall f\in\mathcal{S}(\mathbf{R}^{n+1}).$$

This confirms what we noticed earlier: the Fourier transform $\hat{f}$ can be constructed stably in the timelike cone; and the estimate deteriorates at its boundary: the lightlike cone. No stable inversion can be done in the space-like cone.

PROOF. Notice first that $\mathcal{N}$ is a (tempered) distribution homogeneous of order $-n$. Therefore, it has a Fourier transform $\hat{\mathcal{N}}$, homogeneous of order -1, see section A.2.2.

To compute the Fourier transform of $\mathcal{N}$, we write formally

$$\begin{aligned}
\hat{\mathcal{N}}(\tau,\xi) &= \int e^{-\mathrm{i}(t\tau+x\cdot\xi)}\frac{\delta(t-|x|)+\delta(t+|x|)}{|x|^{n-1}}\,\mathrm{d}t\,\mathrm{d}x\\
&= \int e^{-\mathrm{i}x\cdot\xi}\frac{e^{-\mathrm{i}\tau|x|}+e^{\mathrm{i}\tau|x|}}{|x|^{n-1}}\,\mathrm{d}x\\
&= \int_{S^{n-1}}\int_0^\infty e^{-\mathrm{i}r\theta\cdot\xi}\left(e^{-\mathrm{i}\tau r}+e^{\mathrm{i}\tau r}\right)\mathrm{d}r\,\mathrm{d}\theta\\
&= \frac{1}{2}\int_{S^{n-1}}\int_{-\infty}^\infty e^{-\mathrm{i}r\theta\cdot\xi}\left(e^{-\mathrm{i}\tau r}+e^{\mathrm{i}\tau r}\right)\mathrm{d}r\,\mathrm{d}\theta\\
&= \pi\int_{S^{n-1}}\left(\delta(\tau+\theta\cdot\xi)+\delta(\tau-\theta\cdot\xi)\right)\mathrm{d}\theta\\
&= 2\pi\int_{S^{n-1}}\delta(\tau+\theta\cdot\xi)\,\mathrm{d}\theta.
\end{aligned}$$

The integrals above are in distribution sense and can be justified by computing the action of $\mathcal{N}$ on a test function as we do below. We used also the fact that the Fourier transform of 1 is $2\pi\delta$. To complete the computation, we can apply formally Lemma 10.10 to $\psi=\delta$.

The formal computation above can be justified in the following way. For $\phi\in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$, we write $\phi_{\mathrm{e}}(\tau) := (\phi(\tau,\xi)+\phi(-\tau,\xi))/2$. The kernel $\mathcal{N}$ is an even function of t (and x); therefore, $\widehat{\mathcal{N}}$ is an even distribution as well. To compute the latter, it is enough to compute first $\widehat{\chi\mathcal{N}}$ for every even function $\chi(t)\in C_0^\infty(\mathbf{R})$. The reason for choosing such a function is to make sure that all integrals below are absolutely convergent. Fix such a function and write

$$\begin{aligned}
\langle\widehat{\chi\mathcal{N}},\phi\rangle &= \langle\chi\mathcal{N},\hat{\phi}\rangle\\
&= \int_{\mathbf{R}^n}\chi(|x|)\frac{\hat{\phi}(|x|,x)+\hat{\phi}(-|x|,x)}{|x|^{n-1}}\,\mathrm{d}x\\
&= \int_{\mathbf{R}_+\times S^{n-1}}\chi(r)\left(\hat{\phi}(r,r\theta)+\hat{\phi}(-r,r\theta)\right)\,\mathrm{d}r\,\mathrm{d}\theta\\
&= \int_{\mathbf{R}\times S^{n-1}}\chi(r)\hat{\phi}(r,r\theta)\,\mathrm{d}r\,\mathrm{d}\theta\\
&= \int_{\mathbf{R}\times S^{n-1}}\int_{\mathbf{R}^{1+n}}\chi(r)e^{-\mathrm{i}r(\tau+\xi\cdot\theta)}\phi(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}r\,\mathrm{d}\theta\\
&= \int_{S^{n-1}}\int_{\mathbf{R}^{1+n}}\hat{\chi}(\tau+\xi\cdot\theta)\phi(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}\theta\\
&= \int_{\mathbf{R}^{1+n}}\left[\int_{S^{n-1}}\hat{\chi}(\tau+\xi\cdot\theta)\,\mathrm{d}\theta\right]\phi(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi.
\end{aligned}\tag{10.13}$$

To obtain the fourth line, we performed the change $(r,\theta)\mapsto(-r,-\theta)$. In particular,

$$\widehat{\chi\mathcal{N}}(\tau,\xi) = \int_{S^{n-1}}\hat{\chi}(\tau+\xi\cdot\theta)\,\mathrm{d}\theta,\quad \xi\neq 0.$$

This is an even function of τ because χ is even and we can change θ to $-\theta$ in the integral. By Lemma 10.10, for every even $\chi\in C_0^\infty(\mathbf{R})$,

$$\widehat{\chi\mathcal{N}}(\cdot,\xi) = \hat{\chi}*F(\cdot,\xi),\quad \xi\neq 0,\tag{10.14}$$

where

$$F(\tau,\xi) := |S^{n-2}|(|\xi|^2-\tau^2)_+^{\frac{n-3}{2}}|\xi|^{2-n}, \quad \xi\neq 0. \tag{10.15}$$

When $n=2$, we have to exclude $|\xi|=|\tau|$ above, as well. Note that the left-hand side of (10.14) is smooth, in fact real analytic functions of (τ,ξ) but F is singular.

Since F is homogeneous of order -1 and locally integrable, it defines a distribution in $\mathbf{R}^{1+n}$. It has the unique homogeneous of order -1 extension of F as a distribution, see section A.2.2. That extension is given by the same formula, with the exclusion of the singular set $\xi=0$, $|\xi|=|\tau|$.

We need to show that $\hat{\mathcal{N}}=2\pi F$. Take $\chi=\psi_\varepsilon(x)=\psi(\varepsilon x)$, $0<\varepsilon\to 0$, where $\psi\in C_0^\infty(\mathbf{R})$ is even with $2\pi\psi(0)=1$. Then $\hat\psi_\varepsilon(\tau)=\varepsilon^{-n}\hat\psi(\tau/\varepsilon)$ is a Friedrichs mollifier and

$$\langle\widehat{\psi_\varepsilon\mathcal{N}},\phi\rangle=\int_{\mathbf{R}^{1+n}}\int_{\mathbf{R}}\hat\psi_\varepsilon(\tau-\tau')F(\tau',\xi)\phi(\tau,\xi)\,\mathrm{d}\tau'\,\mathrm{d}\tau\,\mathrm{d}\xi$$

As $\varepsilon\to 0$, the right-hand side converges to $\langle F,\phi\rangle$ because the convolution of $\phi(\cdot,\xi)$ with the molllifier ψ_ε converges to $\phi(\cdot,\xi)$ uniformly in ξ. The left-hand side equals $\langle\mathcal{N},\psi_\varepsilon\hat\phi\rangle$ which is easily seen to converge to $(2\pi)^{-1}\langle\mathcal{N},\hat\phi\rangle=(2\pi)^{-1}\langle\hat{\mathcal{N}},\phi\rangle$. □

We used the following lemma in the proof.

LEMMA 10.10. *For every* $\psi\in\mathcal{S}(\mathbf{R}^{1+n})$,

$$\int_{S^{n-1}}\psi(\theta\cdot\xi)\,\mathrm{d}\theta=|S^{n-2}||\xi|^{2-n}\int_{\mathbf{R}}\psi(s)(|\xi|^2-s^2)_+^{\frac{n-3}{2}}\,\mathrm{d}s, \quad \xi\neq 0, \tag{10.16}$$

where $x_+=x$ *if* $x\geq 0$ *and* $x_+=0$ *if* $x<0$, *and* $|S^{n-2}|$ *is the area of* S^{n-2} *if* $n\geq 3$*; equal to* 2 *when* $n=2$.

PROOF. The result depends on $|\xi|$ only, so we can take $\xi=|\xi|(0,\dots,0,1)$; and then $\theta\cdot\xi=|\xi|\theta_n$. Note that the result is a smooth function of ξ, so it is enough to compute it for $\xi\neq 0$ and then we can take the limit $\xi\to 0$.

Let $n\geq 2$ first. We have, with ψ_{e} being the even part of ψ,

$$\begin{aligned}\int_{S^{n-1}}\psi(\theta\cdot\xi)\,\mathrm{d}\theta&=\int_{S^{n-1}}\psi(|\xi|\theta_n)\,\mathrm{d}\theta\\&=2\int_{S^{n-1},\,\theta^n>0}\psi_{\mathrm{e}}(|\xi|\theta^n)\,\mathrm{d}\theta\\&=2\int_{B^{n-1}}\psi_{\mathrm{e}}\left(|\xi|\sqrt{1-|\theta'|^2}\right)\frac{\mathrm{d}\theta'}{\sqrt{1-|\theta'|^2}}\\&=2\int_{\mathbf{R}_+\times S^{n-2}}\psi_{\mathrm{e}}(|\xi|\sqrt{1-r^2})\frac{r^{n-2}\,\mathrm{d}r\,\mathrm{d}\theta}{\sqrt{1-r^2}},\\&=2|S^{n-2}|\int_0^\infty\psi_{\mathrm{e}}(|\xi|\sqrt{1-r^2})\frac{r^{n-2}\,\mathrm{d}r}{\sqrt{1-r^2}}.\end{aligned} \tag{10.17}$$

where B^{n-1} is the unit ball in $\mathbf{R}^{n-1}$. To pass from the θ to the θ'-integral above, we parameterized the upper hemisphere, and the lower one, by their projections θ' on the ball B^{n-1} which is given by the intersection if $\theta^n=0$ with S^{n-1}. Then $\theta^n=\pm\sqrt{1-|\theta'|^2}$ and we pass to polar coordinates in B^{n-1}.

We now make the change of coordinates $|\xi|\sqrt{1-r^2} = s$ in (10.17) to get

$$
\begin{aligned}
\int_{S^{n-1}} \psi(\theta\cdot\xi)\,\mathrm{d}\theta &= 2|S^{n-2}| \int_0^{|\xi|} \psi_{\mathrm{e}}(s)|\xi|^{2-n}(|\xi|^2 - s^2)^{\frac{n-3}{2}}\,\mathrm{d}s \\
&= |S^{n-2}| \int_{-|\xi|}^{|\xi|} \psi(s)|\xi|^{2-n}(|\xi|^2 - s^2)^{\frac{n-3}{2}}\,\mathrm{d}s \qquad (10.18)\\
&= |S^{n-2}||\xi|^{2-n} \int_{\mathbf{R}} \psi(s)(|\xi|^2 - s^2)_+^{\frac{n-3}{2}}\,\mathrm{d}s,
\end{aligned}
$$

This completes the proof of the lemma.

For $n = 2$, we write $\theta = (\cos\alpha, \sin\alpha)$. Then

$$
\int_{S^1} \psi(\theta\cdot\xi)\,\mathrm{d}\theta = 2\int_0^{\pi} \psi(|\xi|\cos\alpha)\,\mathrm{d}\alpha = 2\int_{-|\xi|}^{|\xi|} \frac{1}{|\xi|\sqrt{1 - s^2/|\xi|^2}}\psi(s)\,\mathrm{d}s, \qquad (10.19)
$$

which proves the lemma for $n = 2$, as well. □

11. Remarks

Besides the classical Helgason book [**17**], the Euclidean X-ray and Radon transforms are analyzed also in [**25**] and [**32**]. Our goal in this chapter was not so much to present a full account of the Euclidean theory but to present it in a way that would make the microlocal analysis of non-Euclidean geometries more clear. We follow mainly [**17**] but we included stability estimates (some of them also found in [**25**]) and the analysis of the X-ray transform on tensor fields, some of which can be found in [**32**]. The book [**32**] has further details on the tensor transform but we include some theorems not present there.

We restricted ourselves to proving many properties of X and R on C_0^∞ functions. Since this space is dense in our spaces of interest, this is not a restriction.

Writing $f = C_n|D|^\alpha R'|D_p|^{n-1-\alpha}Rf$, one can "interpolate" between Theorem 2.6 and Theorem 2.7, and a similar remark applies to the X-ray transform, see Theorem 2.3 and Theorem 2.4. We refer to [**25**] for details.

Doppler and tensor transform: [**13, 28, 30, 31, 45**]

The X-ray transform of functions, differential fields or forms, and tensor fields of order 2 can be combined into one case as X-ray transform of tensor fields of any order. Some of the explicit formulas generalize in an obvious way, like (8.16), (8.18), (8.19). For more details, we refer to [**32**]

Support Theorem: [**39**]

Light Ray transform: [**35**], also Ramm&Sjöstrand.

Stability for incomplete data (maybe those remarks should be in next chapter): [**11, 2**].

CHAPTER III

Stability of Linear Inverse Problems

1. Sharp stability

Consider the following abstract linear "inverse problem". Let

$$A : \mathcal{B}_1 \longrightarrow \mathcal{B}_2 \tag{1.1}$$

be a bounded linear map between two Banach spaces, $\mathcal{B}_1$ and $\mathcal{B}_2$. The inverse problem consists of finding f given the "measurement" Af. Examples are find f given Xf or Rf with the spaces as in Theorem II.3.1; or $A = X_w^* X_w$, where X_w is the weighted X-ray transform, see Chapter IV, and $\mathcal{B}_1 = L^2(\Omega)$ and $\mathcal{B}_2 = H^1(\Omega_1)$ with $\Omega_1 \Supset \Omega$.

The first question is about *uniqueness*. Does Af determine f uniquely? Since A is a linear operator, uniqueness is equivalent to injectivity.

In practical terms, if f is the object we want to recover, then Af is the "data" or the "measurement(s)". Assume that we have already answered the uniqueness question. In practice, we can never measure Af perfectly, or at all infinitely many points (typically, Af is a function of several continuous variables). So we always have some errors (noise) in the data. A fundamental question to ask is the *stability* question: do "small" errors in the data lead to "small "errors in f? To make this question more precise we need to clarify how to measure the errors; what we mean by "small", and even what we mean by the recovered f with perturbed data. The latter is related to knowing the range of A; since the equation $Af_\varepsilon = d + \varepsilon$, where d is the data, and ε is the perturbation may not be solvable even when d is an actual measurement, i.e., when $d = Af$ with some f. In practice, even when $d+\varepsilon$ is not in the range of A, often a certain "solution" is still possible to compute by minimizing the error $Af_\varepsilon - (d+\varepsilon)$ in one way or another. We are not discussing reconstructions or approximate reconstructions now; we want to understand if the problem itself is stable or not. Assume that we are lucky enough to have perturbed data in the range of A; then we know that $Af_\varepsilon = d + \varepsilon$ is uniquely solvable. Is f_ε close to f? This is equivalent to asking if $f - f_\varepsilon$ is small if $A(f - f_\varepsilon)$ is small; i.e., if h is small if Ah is small for an arbitrary h.

If it is (in a certain sense), we call the problem *stable*. If not, we call it *unstable*. If it is unstable, no stable recovery is possible regardless of how clever we are since even if the perturbed data is in the range of A, there is no stability. If it is stable, we can hope for stable recovery. One abstract way is to project the data onto the range of A (thus introducing a small error) and then solve a well posed problem. How do we actually do the latter, is another question.

There are several ways we can measure "smallness". Since $A : \mathcal{B}_1 \to \mathcal{B}_2$, we can use the $\mathcal{B}_1$ norm to measure the errors in the reconstruction of the object we want to recover; and the the $\mathcal{B}_2$ norm to measure the errors in the data. A natural way to define sharp stability then is to require

$$\|f\|_{\mathcal{B}_1} \le C\|Af\|_{\mathcal{B}_2}, \quad \forall f \in \mathcal{B}_1 \tag{1.2}$$

for some constant $C > 0$. A priori, we could have asked for an expression of the kind $\phi(\|Af\|_{\mathcal{B}_2})$ on the right with some $\phi(t) \to 0$ as $t \searrow 0$ but since A is linear, the natural choice is a linear function. We call the estimate (1.2) a *sharp* stability estimate. Together with the assumed continuity of (1.1), it implies trivially

$$\|f\|_{\mathcal{B}_1}/C \le \|Af\|_{\mathcal{B}_2} \le C\|f\|_{\mathcal{B}_1} \tag{1.3}$$

with a possibly different constant $C > 0$.

The "natural" Banach spaces in (1.1) are not always obvious. For any two other Banach spaces $\mathcal{B}_1' \subset \mathcal{B}_1$, $\mathcal{B}_2' \supset \mathcal{B}_2$,

$$\|Af\|_{\mathcal{B}_2'} \le C\|f\|_{\mathcal{B}_1'} \tag{1.4}$$

but then (1.3) will fail in general. Therefore, the notion of stability depends on the choice of the spaces.

In the examples above, "stable" means that there is sharp stability.

Example 1.1.

(a) The Fourier transform $\mathcal{F} : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is trivially stable because it is unitary after rescaling.

(b) Let $P = (-\Delta + 1)^{-1}$, which can be defined through the Fourier transform on the Schwartz class distributions. Clearly, $P : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is bounded and injective. It is not stable however in those spaces! Note that the range $\operatorname{Ran} P$ of P is dense but not closed. On the other hand, we can redefine P as $P : L^2(\mathbf{R}^n) \to H^2(\mathbf{R}^n)$. Then it is bounded and stable.

(c) The X-ray transform

$$X : H_0^s(\Omega) \longrightarrow H^{s+1/2}(\Sigma)$$

is stable for any $s = 0, 1, \dots$, see Theorem II.3.1, where Ω is a fixed bounded domain. If we view X as the operator $X : L^2(\Omega) \to L^2(\Sigma)$, for example, then X is bounded but there is no (sharp) stability. The only reason for that would be the unnatural choice of the norms. On the other hand, in applications, measuring Xf in $H^{1/2}$ may mean applying a half derivative to the actual measurements Xf, that may naturally belong to $L^2(\Sigma)$ of even $L^\infty(\Sigma)$. Then the problem of inverting X is unstable with a loss of $1/2$ derivative in the former case.

(d) For any bounded domain $\Omega_1 \Supset \Omega$, the normal operator

$$X'X : H_0^s(\Omega) \longrightarrow H^{s+1/2}(\Omega_1)$$

is stable for any $s = 0, 1, \dots$, see Theorem II.3.3.

(e) A classical example of a unstable transform is the convolution with a smooth function. Let $\phi \in \mathcal{S}(\mathbf{R}^n)$. Then the operator

$$\Phi f = \phi * f$$

is unstable in $L^2(\mathbf{R}^n)$ (and in any "reasonable" space). Assume for simplicity that $\hat\phi$ has no zeros. Then Φ has a left inverse (a deconvolution operator)

$$\Phi^{-1} g := \mathcal{F}^{-1}\hat\phi^{-1}\mathcal{F}g$$

which is unbounded, with domain $\{g|\ \hat\phi^{-1}\mathcal{F}g \in L^2\}$. Since $\hat\phi$ decreases rapidly at infinity, the domain consists of g with rapidly decaying Fourier transform, as well. Clearly, Φ^{-1} is unbounded, and it will remain so in any pair of Sobolev spaces. Then

Φ is injective but the estimate $\|f\|_{H^{s_1}} \le C\|\phi * f\|_{H^{s_2}}$ fails (for all f) regardless of the choice of s_1, s_2.

In the literature, it is the deconvolution operator Φ^{-1} which is called unstable, not the convolution one.

If A is injective, as we assumed, then there is a well-defined left inverse B : $\operatorname{Ran} A \mapsto \mathcal{B}_1$ so that $BA = \mathrm{Id}$. Note that $\operatorname{Ran} A$ is a Banach space itself if and only if it is closed, which is not a priori guaranteed, see Example 1.1(b). Also, B does not need to be a right inverse since $\operatorname{Ran} A$ might be smaller than $\mathcal{B}_2$ even if we close the former space.

THEOREM 1.1. *The following statements are equivalent:*
(a) (1.2) holds;
(b) A is injective and $\operatorname{Ran} A$ *is closed in $\mathcal{B}_2$;*
(c) A has a bounded left inverse $B : \operatorname{Ran} A \to \mathcal{B}_1$.

PROOF. Clearly, (c) implies (a) by writing $f = BAf$ for any f.

Assume (b). Then there is a well-defined left inverse $B : \operatorname{Ran} A \to \mathcal{B}_1$. Since $\operatorname{Ran} A$ is a Banach space itself, by the closedness of $\operatorname{Ran} A$; the operator B is actually a bijection between two Banach spaces. By the open mapping theorem, B is bounded, which proves (c).

It remains to show that (a) implies (b). Injectivity follows trivially. For the closedness of $\operatorname{Ran} A$, choose f_n so that $Af_n \to g$. Then Af_n is a Cauchy sequence, and so is f_n, by (1.2). Therefore, $f_n \to f$ for some f; and then $Af_n \to Af$; so $g = Af \in \operatorname{Ran} A$. $\square$

We may think of B above as the reconstruction operator. Then sharp stability means existence of a bounded reconstruction operator.

A classical example of operators which do not have closed ranges is given below.

EXAMPLE 1.2 (Smoothing operators are unstable). Let Ω be a bounded domain in $\mathbf{R}^n$ with a smooth boundary (or a compact manifold). Let $A : H^{s_1}(\Omega) \to H^{s_2}(\Omega)$ be bounded and injective. Let $\operatorname{Ran} A \subset H^{s_2'}(\Omega)$ with $s_2' > s_2$, i.e., A is smoothing (of degree $s_s' - s_2$). For the purpose of this example, it is enough to assume that all Sobolev exponents are non-negative integers. Then A is unstable. To prove the latter, notice first that A is compact by the Rellich compactness criterion. Assume that A is stable. Then $BA = \mathrm{Id}$ on $H^{s_1}(\Omega)$ with some bounded $B : H^{s_2}(\Omega) \to H^{s_1}(\Omega)$. So we get that identity is a compact operator which is not true. The convolution example above belongs to this class.

In particular, operators with range included in C^∞ are unstable regardless of the choice of the Sobolev exponents s_1 and s_2. In Section 5 we formulate a microlocal analog of this.

2. Fredholm properties

Injectivity can be tricky to prove and fails sometimes. On the other hand, microlocal methods provide powerful tools for proving that certain class of operators have left "pseudo-inverses", i.e., bounded operators B so that $BA = \mathrm{Id} + K$, where K is compact. We may think of this as stability even when injectivity may fail. In that case, $\operatorname{Ker} A$ is finitely dimensional only; and on its orthogonal complement, we have actual (sharp) stability.

We start with the following lemma, which we will use frequently, see also [**41**, Proposition V.3.1].

LEMMA 2.1. *Let $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ be Banach spaces, let $A : \mathcal{B}_1 \to \mathcal{B}_2$ be a closed linear operator with domain $\mathcal{D}(A)$, and $K : \mathcal{B}_1 \to \mathcal{B}_3$ be a compact linear operator. Let*

$$\|f\|_{\mathcal{B}_1} \le C\left(\|Af\|_{\mathcal{B}_2} + \|Kf\|_{\mathcal{B}_3}\right), \quad \forall f \in \mathcal{D}(A). \tag{2.1}$$

Assume that A is injective. Then

$$\|f\|_{\mathcal{B}_1} \le C'\|Af\|_{\mathcal{B}_2}, \quad \forall f \in \mathcal{D}(A). \tag{2.2}$$

with a possibly different constant C'.

PROOF. We show first that one can assume that A is bounded. Indeed, let $\|\cdot\|_{\mathcal{D}(A)}$ denote the graph norm. Then (3.12) implies

$$\|f\|_{\mathcal{D}(A)} \le C\left(\|Af\|_{\mathcal{B}_2} + \|Kf\|_{\mathcal{B}_3}\right), \quad \forall f \in \mathcal{D}(A).$$

Assuming the lemma for bounded operators, we get $\|f\|_{\mathcal{D}(A)} \le C\|Af\|_{\mathcal{B}_2}$ and this implies the estimate we want to prove.

For bounded A, assume the opposite. Then there exists a sequence f_n in $\mathcal{B}_1$ with $\|f_n\|_{\mathcal{B}_1} = 1$ and $Af_n \to 0$ in $\mathcal{B}_2$. Since $K : \mathcal{B}_1 \to \mathcal{B}_3$ is compact, there exists a subsequence, that we will still denote by f_n, such that Kf_n converges in $\mathcal{B}_3$, therefore is a Cauchy sequence in $\mathcal{B}_3$. Applying (3.12) to $f_n - f_m$, we get that $\|f_n - f_m\|_{\mathcal{B}_1} \to 0$, as $n \to \infty$, $m \to \infty$, i.e., f_n is a Cauchy sequence in $\mathcal{B}_1$. Therefore, there exists $f \in \mathcal{B}_1$ such that $f_n \to f$ and we must have $\|f\|_{\mathcal{B}_1} = 1$. Then $Af_n \to Af = 0$. This contradicts the injectivity of A thus proving the lemma. □

This is a very helpful result for proving stability, but we want to point out a weakness. While the constant in estimates of the kind (2.1) can be controlled in principle because a typical proof is a microlocal parametrix construction; we have no control over the constant C in (2.2) — how it depends on the constant in (2.1).

DEFINITION 2.2. *The bounded operator $A : \mathcal{B}_1 \to \mathcal{B}_2$ is called* upper semi-Fredholm, *if* $\operatorname{Ker} A$ *is finite dimensional and* $\operatorname{Ran} A$ *is closed. It is called* lower semi-Fredholm, *if* $\operatorname{Coker} A$ *is finite dimensional and* $\operatorname{Ran} A$ *is closed. It is called* Fredholm *if it is both upper and lower semi-Fredholm.*

THEOREM 2.3. *Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator between two Hilbert spaces. Then the following statements are equivalent:*

(a) A is upper semi-Fredholm;

(b) $\operatorname{Ker} A$ *is finitely dimensional and there exists a bounded operator $B : \operatorname{Ran} A \to \mathcal{H}_1$ so that $BA = \operatorname{Id}$ on $(\operatorname{Ker} A)^\perp$.*

(c) There exists a bounded operator $B : \operatorname{Ran} A \to \mathcal{H}_1$ so that $BA = \operatorname{Id} + K$ on $\mathcal{H}_1$ with K compact.

(d) There is $C > 0$ so that

$$\|f\|_{\mathcal{H}_1} \le C\|Af\|_{\mathcal{H}_2} + C\|Kf\|_{\mathcal{H}_1}, \quad \forall f \in \mathcal{H}_1 \tag{2.3}$$

with K compact.

PROOF. (a) ⇔(b): Apply Theorem 1.1 to the operator $A : (\operatorname{Ker} A)^\perp \to \operatorname{Ran} A$.
(a) ⇒(d): By the same argument,

$$\|f\|_{\mathcal{H}_1} \le C\|Af\|_{\mathcal{H}_2}, \quad \forall f \in (\operatorname{Ker} A)^\perp.$$

Let Π be the orthogonal projection on the finitely dimensional $\operatorname{Ker} A$. Then $A\Pi = 0$ and Π is compact. Write $f = \Pi f + (\mathrm{Id} - \Pi)f$ and apply the estimate above to $(\mathrm{Id} - \Pi)f$.

(b)⇒(c): With B as in (b), we have $BA(\mathrm{Id} - \Pi) = \mathrm{Id} - \Pi$, therefore, $BA = \mathrm{Id} - \Pi$.

(c)⇒(d): obvious.

(d)⇒(a): Any $f \in \operatorname{Ker} A$ satisfies

$$\|f\|_{\mathcal{H}_1} \le C\|Kf\|_{\mathcal{H}_1}.$$

Choose an infinite orthonormal sequence f_n in $\operatorname{Ker} A$. Then $f_n \to 0$ weakly; therefore $Kf_n \to 0$. On the other hand, $1 \le C\|Kf_n\|_{\mathcal{H}_2}$, which is a contradiction. Therefore, $\operatorname{Ker} A$ is finitely dimensional.

We prove now that $\operatorname{Ran} A$ is closed. Let f_n be such that $Af_n \to g$. Let f_n' be the projection of f_n onto $(\operatorname{Ker} A)^\perp$. Then $Af_n = Af_n' \to g$. Since $A : (\operatorname{Ker} A)^\perp \to \mathcal{H}_1$ is injective, by Lemma 2.1, $\|f_n' - f_k'\| \le C\|Af_n' - Af_k'\|$; therefore, f_n' is a Cauchy sequence. Then $f_n' \to f'$ for some f'; therefore, $Af_n' \to Af'$; hence $g = Af_n' \in \operatorname{Ran} A$. □

3. Non-sharp stability estimates

Let $A : \mathcal{B}_1 \to \mathcal{B}_2$, be as above and let $\mathcal{B}_1' \supset \mathcal{B}_1$, $\mathcal{B}_2' \subset \mathcal{B}_2$ with at least one of those inclusions strict. If (1.2) fails (or if we cannot prove it) but we can prove the weaker estimate

$$\|f\|_{\mathcal{B}_1'} \le C\|Af\|_{\mathcal{B}_2'},$$

we say that we have a *non-sharp* stability estimate.

EXAMPLE 3.1. Let $A = \mathrm{d}/\mathrm{d}x^n$, $\mathcal{B}_1 = C_0^1(\Omega)$, $\mathcal{B}_2 = C(\mathbf{R}^n)$, $n \ge 2$, where Ω is a bounded domain. Then A is bounded and the spaces above are "natural", i.e., we cannot improve one of them by keeping the other one if we want to stay in the C^k class of spaces. The the equation $Af = g$ can be solved by integration along lines parallel to the x^n-axis with initial condition 0 either for $x^n \ll 0$ or for $x^n \gg 0$. We see here that the left inverse is not unique — we can even change the direction of the integration from line to line. Clearly, A is injective but the sharp stability estimate

$$\|f\|_{C^1} \le C\|f_{x^n}\|_C$$

fails because the right-hand side cannot control the other partial derivatives. On the other hand, the weaker estimate

$$\|f\|_C \le C\|f_{x^n}\|_C$$

holds (we used C above to denote both a constant and the space of continuous functions), which is actually the Poincaré inequality. There is a "loss of one derivative" in that estimate.

Notice that this non-sharp estimate cannot be made sharp only if we insist on working in C^k spaces. If we replace the $\mathcal{B}_1$ norm by the graph one, we get sharp stability. On the other hand, the graph norm depends on the operator A which in many applications is not very explicit.

EXAMPLE 3.2. Let $\Phi f(x) = \phi * f$, where $\phi(x) = e^{-|x|^2/2}$. Then Φ is unstable in any pair of Sobolev spaces, as we saw before. Since $\hat\phi(\xi) = C\phi(\xi)$, we may define the space $H^\sharp$ through the norm $\|f\|_{H^\sharp} = \|e^{|\xi|^2/2}\hat f(\xi)\|$. Then $\Phi : H^\sharp \to L^2$ is stable;

in fact, it is unitary, up to rescaling. On the other hand, functions in $H^\sharp$ must be analytic, which makes that space "too small" and unsuitable for applications.

Other examples of non-sharp stability estimates can be constructed by choosing A to be a hypoelliptic but not elliptic ΨDO.

4. Conditional Stability

In some cases, we can only show that f is small if Af is small and f is a priori bounded in a smaller space (and has to belong there, of course). Let $A : \mathcal{B}_1 \to \mathcal{B}_2$ as before, and let $\mathcal{B}_1' \subsetneq \mathcal{B}_1$. Then conditional stability would mean that $\|f\|_{\mathcal{B}_1} \to 0$ as $\|Af\|_{\mathcal{B}_2} \to 0$ and $\|f\|_{\mathcal{B}_1'} < M$ with some fixed constant M. The rate of convergence of $\|Af\|_{\mathcal{B}_2}$ is an important element of the stability. The extra boundedness condition restricts f to a non-linear subset (a ball), and we cannot have a linear rate of convergence anymore. A fractional law is considered still "stable"; i.e., one way to formulate conditional stability is to require the inequality

$$\|f\|_{\mathcal{B}_1} \le C\|Af\|^p_{\mathcal{B}_2}, \quad \text{when } \|f\|_{\mathcal{B}_1'} \le M \tag{4.1}$$

for some $M > 0$ and $p \in (0, 1)$. A quick look at the estimates above shows that they do not scale well, which is unnatural for a linear problem. To write the estimates in a different way, given $f \in \mathcal{B}_1'$, set $g = Mf/\|f\|_{\mathcal{B}_1'}$. Then the boundedness condition of (4.1) is satisfied for g and after some trivial simplification we get

$$\|f\|_{\mathcal{B}_1} \le C'\|Af\|^p_{\mathcal{B}_2}\|f\|^{1-p}_{\mathcal{B}_1'}, \quad \forall f \in \mathcal{B}_1'. \tag{4.2}$$

Conversely, from (4.2) we can easily get (4.1) with some C and M (actually, for every M there is $C = C(M)$).

We call estimates of the kind (4.2) *Hölder conditional stability* estimates. They can also be non-sharp conditional stability estimates if $\|\cdot\|_{\mathcal{B}_1}$ is replaced there by a weaker norm; and/or $\|\cdot\|_{\mathcal{B}_2}$ is replaced by a stronger one. An example of such estimate is the estimate obtained by Sharafutdinov in tensor tomography using the energy method [**32**].

5. Microlocal Stability. Visible and Invisible singularities

In many cases, stability is lost. A typical example is the X-ray transform with a limited angle. On the other hand, using regularization techniques, one can still construct an "aproximate" image, showing some of the "features" of the original — like some of the jumps across somooth boundaries, for example. To understand this, we need to start thinking in microlocal terms.

Consider the following example. Let P be a ΨDO of order m elliptic in some open conic set Γ_1 and smoothing (i.e., of order $-\infty$) in another open conic set Γ_2, with $\Gamma_1 \cap \Gamma_2 = \emptyset$. What can we construct by knowing $Pf = g$? For any $\chi \in \Psi^0$ with essential support in Γ_1, we can apply a parametrix Q in Γ_1 to get

$$\chi(x, D)f = Qg + Rf, \tag{5.1}$$

where Q is a ΨDO of order $-m$ and R is smoothing. This recovers $\mathrm{WF}(f)$ in any compact subset there in a direct way. On the other hand, $\mathrm{WF}(f) \cap \Gamma_2$ does not affect $\mathrm{WF}(Pf)$ at all. We can call the essential support of P the set of visible singularities; and the set where P is of order $-\infty$ invisible ones. This definition is open to some interpretation — we may want to include in the set of the visible singularities sets, where P drops its order, for example belongs to Ψ^{m-k}, $k > 0$, and is elliptic as such operator there. We call this stability with a loss of k derivatives. Or, we may declare those singularities as the invisible ones.

Relation (5.1) implies the following estimate

$$\|\chi(x,D)f\|_{H^s} \le C_s\|Pf\|_{H^{s-m}} + C_{t,s}\|f\|_{H^l} \tag{5.2}$$

for any s and l. Assume now that P is not necessarily a ΨDO but we know that

$$P : H^s \to H^{s-m} \quad \text{is bounded} \tag{5.3}$$

for some s and m. One possible definition of visible singularities then is to declare them to be the largest open conic set Γ_1 with the property that (5.1) holds with $Q : H^s \to H^{s+m}$ for any $\chi \in \Psi^0$ with essential support. In that case, we also say that the singularities in Γ_1 are stably recoverable. Note that we do not really mean only that if f had singularities there, we would be able to recover them stably — we mean the estimate (5.2) which makes sense for smooth f as well, for example for sequencies of f with oscillations of increasing frequencies. The set of the invisible singularities then is the largest open Γ_2 with the property that $\mathrm{WF}(f) \subset \Gamma_2$ implies $Pf \in C^\infty$. Then (5.2) is not possible with the essential support of χ in Γ_2 regardless of m, s and l. Notice that $\Gamma_1 \cup \Gamma_2$ does not need to cover the whole cotangent bundle (minus the zero section), even after closure. For example, we may have an open set with a loss of finitely many derivatives.

In particular, if $P \in \Psi^m$ is an elliptic, then we can choose $\chi = 1$ in (5.2).

As a consequence of Theorem 2.3 and the ellipticity of P, we get the following.

THEOREM 5.1.
(a) Let $P \in \Psi^m(\mathbf{R}^n)$, with P elliptic in $\bar\Omega$. Then for any s, $l < s$ we have

$$\|f\|_{H^s} \le C_s\|Pf\|_{H^{s-m}} + C_{l,s}\|f\|_{H^l} \tag{5.4}$$

for any $f \in H^s_0(\Omega)$.

(b) The kernel of P on the space of the distributions with support in $\bar\Omega$ is finitely dimensional and consists of C_0^∞ functions.

(c) Assume in addition that A is injective on some closed subspace $\mathcal{L}$ of H^s. Then

$$\|f\|_{H^s} \le C_s\|Pf\|_{H^{s-m}}, \quad \forall f \in \mathcal{L}. \tag{5.5}$$

with a possibly different constant $C_s > 0$.

PROOF. Estimate (5.4) follows from (5.1). T prove (b), notice first that any f with $Pf = 0$ and support in $\bar\Omega$ satisfies $f = QPf + Rf = Rf$ by (5.1) since wa can choose χ to be dependent on x only and to be equal to one near $\bar\Omega$, and zero outside a larger domain. The finiteness of teh kernel and part (c) follow from Lemma 2.1. □

The X-ray and the Radon transforms we study are actually FIOs, not ΨDOs. Fortunately, the normal operators $X'X$ and $R'R$ happen to be ΨDOs (but $L'L$ is not!). We can apply this theory to the normal operators. The normal operators are of interest as well because $A'A$ is injective if and only if A is, see Proposition IV.5.5, and $X'Xf$, and $R'Rf$ map f back to the x-space.

EXAMPLE 5.1. Let $0 \le p \in C^\infty$, $p(\xi) = 1$ for $|\xi| \ge 1$. Then $p(D) \in \Psi^0$ is elliptic. It may not have a left inverse because p may have zeros in the unit ball — it actually can be zero in an open set. Let $E : L^2(\Omega) \to L^2(\mathbf{R}^n)$ be the extension as zero, and let $R = E' : L^2(\mathbf{R}^n) \to L^2(\Omega)$ be the restriction to Ω. Then

$Af := Rp(D)Ef$ is injective. Indeed, $Af = 0$ implies $(p(D)Ef, Ef)_{L^2(\mathbf{R}^n)} = 0$; and passing to the Fourier transform we get $\widehat{Ef}(\xi) = 0$ for $|\xi| > 1$. Since $\widehat{Ef}$ is analytic, this implies $f = 0$. By Theorem 5.1, A is stable, i.e.,

$$\|f\| \leq C\|p(D)Ef\|, \quad \forall f \in L^2(\Omega).$$

This is actually true if $p = 0$ in the unit ball, and $p = 1$ outside of it but then p is not a symbol anymore. Then $A = \mathrm{Id} + K$, where K is compact, and we need to show that -1 is not an eigenvalue of K. We leave the details to the reader. In particular, this shows that recovery of $\hat{f}$ in a compact set, if known outside of it, is a stable operation; for f a priori supported in a fixed bounded set.

6. Concluding Remarks

As pointed out above, those methods can be very powerful but have a few shortcomings.

First, we do not have control over the constant in the estimates obtained with the use of Lemma 2.1.

Second, if we rely on microlocal arguments and ellipticity, we formally need smooth coefficients. For example, to apply the theory to $X_w^* X_w$, where X_w is the weighted X-ray transform, we need $w \in C^\infty$. In fact, symbols of finite smoothness are sufficient because in the ellipticity argument, we only need $QP = \mathrm{Id} + K$, where K is of order -1, hence compact when restricted to compact sets. This can be done with symbols of finite smoothness, hence w in X_w need to be C^k only with $k \gg 1$. The required k would be too large for what we would expect it to be however; for example, just to prove that $p(x, D)$ is bounded in L^2, we need $n+1$ derivatives, see [**19**, Theorem 18.1.11′]; for $a(x, y, D)$, we need $2n+1$ [**36**]. On the the hand, sometimes a reduction of the smoothness requirements is possible, see Corollary IV.5.8.

Third (related to our first point), the constants in the estimates can make a big difference. Even though convolution is unstable, if we convolute with a very well concentrated $\phi \in \mathcal{S}$, the deconvolution can be done and it is practically done all the time with a small error because the instability manifests itself for functions f with $\hat{f}$ decaying extremely slow at infinity (looking almost as Dirac δ, or highly oscillating, for example), and in practical applications, the functions of interest have some limit of how high frequency content they can have. Similarly, a stable operator with a very large sharp constant C in (1.2) may behave as unstable for practical purposes. Indeed, (1.2) is a bound on the asymptotic behavior of $\|f\|$ as $\|Af\| \to 0$ and if Af is "small", for a large C, $\|f\|$ would not be "so small". In Example 5.1, the constant C would grow with R if we want to recover $\hat{f}$ for $|\xi| < R$; and for large R, the problem will be unstable for practical purposes because $C \gg 1$.

CHAPTER IV

The Weighted Euclidean X-ray transform

1. Introduction

The object of study in this chapter is the weighted X-ray transform

$$X_w f(x,\theta) = \int_{\mathbf{R}} w(x+t\theta,\theta) f(x+t\theta)\,\mathrm{d}t, \quad (x,\theta)\in \mathbf{R}^n\times S^{n-1}, \tag{1.1}$$

compare with (II.1.1). Here $w = w(x,\theta)$ is a smooth weight depending in general not only on the point x but also on the direction θ. We can parametrize X_w by $(z,\theta)\in\Sigma$ as in Section II.1.3. In general, $X_w f$ is not an even function of θ any more. One can think of w as a function of points x and lines ℓ restricted to the sumbanifold $x\in\ell$.

There are several reasons to study this transform. First, the attenuated X-ray transform, see section II.9, is a transform of this kind with a weight function

$$w(x,\theta) = \exp\left(-\int_0^\infty a(x+t\theta)\,\mathrm{d}t\right), \tag{1.2}$$

where $a(x)$ is the attenuation at any given x. This is a weight of a special type with the property that $\theta\cdot\nabla_x \log w = -a(x)$ is independent of θ. In particular,

$$\mathrm{d}_\theta\, \theta\cdot\nabla_x \log w = 0,$$

where d_θ is the differential on the unit sphere; and at least locally, this characterizes such weights uniquely. Even though there are explicit inversion formulas [...] for X_w, certain properties like stability and recovery of singularities are independent of the special nature of the weight and are better understood if we think about the attenuated X-ray transform as a weighted transform with a positive weight. In media with non-isotropic attenuation, i.e., when a depends on x and θ, formula (1.2) still holds with $a = a(x+t\theta,\theta)$ in the integral but w does not have any special structure, at least locally.

Another important reason motivating the study of X_w is the microlocal inversion of the localized X-ray transform even when $w=1$. One way to model the fact that we know Xf only for an open subset of lines is to multiply Xf by a smooth cutoff function on the set of lines, i.e., a function $\psi\in C_0^\infty(\Sigma)$. Then ψXf is a weighted X-ray transform of f with a weight ψ constant along each line.

A third reason is to explain the main ideas used in the more general case of geodesics or geodesic-like curves in this relatively simple case. This allows for a simplified exposition, avoiding the Riemannian geometry terminology and notation, which still demonstrates the power of the microlocal analysis to derive non-obvious results.

Finally, we have more complete results for X_w than we have for the geodesic X-ray transform.

An essential difference between X (where $w = 1$) and X_w is not just the introduction of a non-trivial weight but that the transform loses its analyticity. Indeed, X is related to the Euclidean metric (and so is X_w) which is analytic; while X_w has a non-analytic weight, in general. We will see in Theorem 5.6 that analyticity makes possible to prove support theorems while C^∞ regularity only does not in general, see Theorem 5.1.

The structure of this chapter is the following. In Section 2, we derive formulas for the Schwartz kernel of $X_w^*X_w$ and show that the later is a ΨDO. While we do not have inversion formulas (and no uniqueness in general), we show that the problem is Fredholm for non-vanishing weights with the standard consequences of that. We also show how one can reduce the regularity requirements on the weight using the Calderón-Zygmund theory. In Section 3, we microlocalize the problem, i.e., we study which singularities can be recovered with partial data observations. The answer is pretty straightforward — we just need to find out where the localized normal operator is elliptic, and it then turns out that we can recover singularities conormal to the lines in our (open) set, if w does not vanish there. This allows for a microlocal treatment of the limited angle problem (for which there is uniqueness but no stability), and of the Region of Interest (ROI) Problem in Section 4, for which there is no uniqueness but (microlocally) there is stability.

Section 5 goes beyond the standard consequences of the microlocal ellipticity and it is perhaps the most interesting part of this chapter. First, we recall Boman's examples of lack of support theorems or injectivity for $n = 2$. Next, we show how one can use the analytic microlocal calculus in Theorem 5.10 to prove support theorems and injectivity for analytic non-vanishing weights. This presents another point of view about the support theorem for constant weights. To the author's knowledge, this point of view appeared for the first time in [**8**]. Based on Boman's 2D examples, it is a bit surprising to see that in dimensions $n \geq 3$, support theorems and injectivity hold (for non-vanishing weights). This was proven for the more delicate geodesic case by Uhlmann and Vasy in [**44**] and later generalized to more general transforms and weights by ... Both results appeared after we started working on this book project. The proof we present here however is much more elementary, the stability is stronger; and we are able to do this because of the simpler Euclidean geometry, of course. Note that there are no analyticity assumptions here. Uhlmann and Vasy's results and the related Theorem 5.11 can be considered as unique continuation (for smooth coefficients), while support theorems for analytic coefficients are a kind of (microlocal analytic) continuation, in some sense similar to Holmgren's type of theorems.

2. Basic Properties

2.1. The transpose X_w'. We follow II.1.3. In the same way, we define X_w' with respect to the measure $\mathrm{d}\sigma$.

Proposition 2.1. *For any $\psi \in C^\infty(\Sigma)$,*

$$X_w'\psi(x) = \int_{S^{n-1}} w(x,\theta)\psi(x-(x\cdot\theta)\theta,\theta)\,\mathrm{d}\theta.$$

Proof. Let $\phi \in C_0^\infty(\mathbf{R}^n)$, $\psi \in C^\infty(\Sigma)$. We have

$$\int_\Sigma (X_w\phi)\psi\,\mathrm{d}\sigma = \int_\Sigma\int_{\mathbf{R}} w(z+s\theta,\theta)\phi(z+s\theta)\,\psi(z,\theta)\,\mathrm{d}s\,\mathrm{d}S_z\,\mathrm{d}\theta. \tag{2.1}$$

Set $x = z + s\theta$, where $z \in \theta^\perp$. For a fixed $\theta \in S^{n-1}$, $(z,s) \mapsto x$ is an isomorphism with a Jacobian equal to 1. The inverse is given by

$$z = x - (x\cdot\theta)\theta, \quad s = x\cdot\theta.$$

We therefore have

$$\int_\Sigma (X_w\phi)\psi\,\mathrm{d}\sigma = \int_{S^{n-1}}\int_{\mathbf{R}^n} w(x,\theta)\phi(x)\,\psi(x-(x\cdot\theta)\theta,\theta)\,\mathrm{d}x\,\mathrm{d}\theta.$$

This completes the proof. □

2.2. Properties of the normal operator $X_w^*X_w$. We study $X_w^*X_w$ instead of $X_w'X_w$ to allow for complex valued weights. Then $X_w^*X_w$ is injective if and only X_w is, see Proposition 5.5 below.

2.2.1. *The Schwartz kernel of the normal operator.* We compute next $X_w^*X_w$. Clearly, $X_w^* = X'_{\bar w}$.

Proposition 2.2. *For any two L^∞ weights a, b,*

$$X_b'X_af(x) = \int \frac{W\big(x,y,\frac{x-y}{|x-y|}\big)}{|x-y|^{n-1}}f(y)\,\mathrm{d}y,$$

where

$$W(x,y,\theta) = b(x,\theta)a(y,\theta) + b(x,-\theta)a(y,-\theta) \tag{2.2}$$

Proof. By Proposition 2.1,

$$\begin{aligned} X_b'X_af(x) &= \int_{S^{n-1}} b(x,\theta)\int a(x-(x\cdot\theta)\theta+t\theta,\theta)f(x-(x\cdot\theta)\theta+t\theta)\,\mathrm{d}t\,\mathrm{d}\theta\\ &= \int_{S^{n-1}} b(x,\theta)\int a(x+t\theta,\theta)f(x+t\theta)\,\mathrm{d}t\,\mathrm{d}\theta. \end{aligned}$$

Split the t-integral in two parts: for $t>0$ and for $t<0$, and replace t by $-t$ in the second one to get

$$\begin{aligned} X_b'X_af(x) = &\int_{S^{n-1}} b(x,\theta)\int_0^\infty a(x+t\theta,\theta)f(x+t\theta)\,\mathrm{d}t\,\mathrm{d}\theta\\ &+\int_{S^{n-1}} b(x,\theta)\int_0^\infty a(x-t\theta,\theta)f(x-t\theta)\,\mathrm{d}t\,\mathrm{d}\theta. \end{aligned} \tag{2.3}$$

Replace $-\theta$ by θ in the second integral to get

$$\begin{aligned} X_b'X_af(x) = \int_{S^{n-1}}\int_0^\infty &[b(x,\theta)a(x+t\theta,\theta)\\ &+b(x,-\theta)a(x+t\theta,-\theta)]f(x+t\theta)\,\mathrm{d}t\,\mathrm{d}\theta. \end{aligned} \tag{2.4}$$

Pass to polar coordinates $y = x+t\theta$, centered at x to finish the proof. □

2.2.2. *The normal operator is a ΨDO of order −1.* To write $X_b'X_a$ as a ΨDO, recall that if the Schwartz kernel of a linear operator is given by $K(x,y,x-y)$, then it is a formal ΨDO with an amplitude given by the Fourier transform of K w.r.t. the third variable, see Theorem 5.3. We will repeat the proof of that theorem in this context. Therefore, $X_b'X_a$ is a formal ΨDO with amplitude that can be computed formally as

$$\begin{aligned}\int_{\mathbf{R}^n} e^{-\mathrm{i}z\cdot\xi}\frac{W(x,y,z/|z|)}{|z|^{n-1}}\,\mathrm{d}z &= \int_{\mathbf{R}_+\times S^{n-1}} e^{-\mathrm{i}r\theta\cdot\xi}W(x,y,\theta)\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \pi\int_{S^{n-1}} W(x,y,\theta)\delta(\theta\cdot\xi)\,\mathrm{d}\theta.\end{aligned} \tag{2.5}$$

We used here the fact that W is an even function of θ and that the inverse Fourier transform of 1 is δ. We refer to Theorem B.5.3 for details. If $n=2$, we get

$$\begin{aligned}\int_{\mathbf{R}^2} e^{\mathrm{i}z\cdot\xi}\frac{W(x,y,z/|z|)}{|z|}\,\mathrm{d}z &= \frac{\pi}{|\xi|}\left(W(x,y,\xi^\perp/|\xi|)+W(x,y,-\xi^\perp/|\xi|)\right) \\ &= \frac{2\pi}{|\xi|}W(x,y,\xi^\perp/|\xi|),\end{aligned} \tag{2.6}$$

where $\xi^\perp := (-\xi_2,\xi_1)$. Since this is a homogeneous function of ξ, with an integrable singularity that can be cut-off resulting in a smoothing operator. Therefore, we proved the following.

THEOREM 2.3. *Let a, b be smooth. Then $X_b'X_a$ is a classical ΨDO of order -1 with amplitude given by (2.5), (2.6) and principal symbol*

$$\sigma_p(X_b'X_a) = 2\pi\int_{S^{n-1}} b(x,\theta)a(x,\theta)\delta(\theta\cdot\xi)\,\mathrm{d}\theta,$$

PROOF. We use (2.6) to get

$$\sigma_p(X_b'X_a) = \pi\int_{S^{n-1}} W(x,y,\theta)\delta(\theta\cdot\xi)\,\mathrm{d}\theta.$$

Use (2.2) to split the integral into two parts and make make the change of variable $\theta\mapsto-\theta$ in the second one. □

The theorem implies, in particular, that

$$X_w^*X_w : H^s_{\mathrm{comp}}(\mathbf{R}^n) \longrightarrow H^{s+1}_{\mathrm{loc}}(\mathbf{R}^n)$$

is continuous for w smooth. In Corollary 2.6 below we estimate its norm in terms of w for $s=0$.

If $n=2$, the integral is understood in the sense (2.6).

Theorem 2.3 implies a necessary and sufficient condition for ellipticity: $X_b'X_a$ is an elliptic ΨDO of order -1 at (x,ξ) if and only if the average of $(ab)(x,\theta)$ over the $(n-2)$-dimensional sphere $|\theta|=1$, $\theta\perp\xi$ is not zero. If $n=2$, there are only two such $\theta's$, namely $\pm\xi^\perp/|\xi|$.

COROLLARY 2.4. *Let $w\in C^\infty(\mathbf{R}^n\times S^{n-1})$. Then $X_w^*X_w$ is an elliptic ΨDO of order -1 at (x,ξ) if and only if there exists a unit $\theta\perp\xi$ so that $w(x,\theta)\neq0$.*

In particular, let $\Omega\subset\mathbf{R}^n$ be open and bounded. Then $X_w^*X_w$ is an elliptic ΨDO of order -1 in a neighborhood of $\bar\Omega$ if and only if

(2.7) *for every $(x,\xi)\in\bar\Omega\times\mathbf{R}^n\setminus 0$ there exists a unit $\theta\perp\xi$ so that $w(x,\theta)\neq0$.*

In invariant terms, (x,ξ) is a covector, while (x,θ) is a vector.

EXAMPLE 2.1. Consider the X-ray transform in $\mathbf{R}^2$ with weight $w = 1$. Assume that we know $Xf(z,\theta)$ for θ restricted to an open set $U \subset S^1$ given by $0 \le \alpha < \arg\theta < \beta \le 2\pi$, and all corresponding $z \in \theta^\perp$, knowing a priori that f is continuous and of compact support. Is that enough to recover f? By the Fourier Slice Theorem, we can uniquely determine $\hat f(\xi)$ for all ξ so that $\xi\cdot\theta = 0$ for some θ as above. In particular, if $\beta - \alpha > \pi$ (we have more than "half" of the angles), we can recover $\hat f(\xi)$ for all ξ. Of course, then we have all the lines as well, and we have stability. should this example be here?

What if we knew $Xf(z,\theta)$ for $\alpha < \arg\theta < \beta$ with $\beta-\alpha < \pi$? Then we can still recover easily $\hat f(\xi)$ for all $\xi \in U^\perp$ but the latter does not cover the whole $\mathbf{R}^n$. If $\alpha = 0$, $\beta = \pi/2$, for example, then we only get $\hat f(\xi)$ for $\arg\xi$ in $[\pi/2,\pi]\cup[3\pi/2,2\pi]$. On the other hand, $\hat f(\xi)$ is a real analytic function, and then by analytic continuation, we can recover $\hat f(\xi)$ even for ξ in the missing sector. Therefore, the so restricted Xf recovers f uniquely. Note that we could have used the support theorem to get the same conclusion.

The use of analytic continuation is a strong suggestion (but not a proof!) of possible instability. As we see in the next section, in the second case, $\beta - \alpha < \pi$, stability is lost, indeed.

2.2.3. *Mapping properties of the normal operator.* Next proposition is part of the the Calderón-Zygmund theory of singular operators.

PROPOSITION 2.5. *Let A be the operator*

$$[Af](x) = \int_{\mathbf{R}^n} \frac{\alpha\Big(x,y,|x-y|,\frac{x-y}{|x-y|}\Big)}{|x-y|^{n-1}} f(y)\,\mathrm{d}y \tag{2.8}$$

with $\alpha(x,y,r,\theta)$ compactly supported in x,y. Then

(a) If $\alpha\in C^2$, then $A : L^2 \to H^1$ is continuous with a norm not exceeding $C\|\alpha\|_{C^2}$.

(b) Let $\alpha(x,y,r,\theta) = \alpha'(x,y,r,\theta)\psi(\theta)$. Then We may not need (b). Proof not included

$$\|A\|_{L^2\to H^1} \le C\|\alpha'\|_{C^2}\|\psi\|_{H^1(S^{n-1})}.$$

PROOF. We recall some facts about the Calderón-Zygmund theory of singular operators, see [**22**]. First, if K is an integral operator with singular kernel $k(x,y) = \phi(x,\theta)r^{-n}$, where $\theta = (x-y)/|x-y|$, $r = |x-y|$, and if the "characteristic" ϕ has a mean value 0 as a function of θ, for any x, i.e.,

$$\int_{S^{n-1}} \phi(x,\theta)\,\mathrm{d}\theta = 0,$$

then K is a well defined operator on test functions, where the integral has to be understood in the principle value sense. Moreover, K extends to a bounded operator to L^2 with a norm not exceeding $C\sup_x \|\phi(x,\cdot)\|_{L^2(S^{n-1})}$, see [**22**, Theorem XI.3.1].

Also, see [**22**, Theorem XI.11.1], if B is an operator with a weakly singular kernel $\beta(x,\theta)r^{-n+1}$, then $\partial_x B$ is an integral operator with singular kernel $\partial_x[\beta(x,\theta)r^{-n+1}]$ plus the operator of multiplication by $-\int_{S^{n-1}}\theta\beta(x,\theta)\,\mathrm{d}\theta$. The former, up to a weakly singular operator, has a singular kernel of the type ϕr^{-n}, and the integration is again understood in the principle value sense, see the next paragraph. In particular, the zero mean value condition is automatically satisfied.

In our case, $\beta = \alpha$ depends on y and r as well. Assume first that it does not, i.e., B is as above. The multiplication operator is easy to analyze, so we focus our attention on the kernel $\partial_x[\beta(x,\theta)r^{-n+1}]$. Extend β as a homogeneous function of θ of order 0 near S^{n-1}. Then

$$\begin{aligned}\partial_{x_i}\frac{\beta(x,\theta)}{r^{n-1}} &= (1-n)\frac{\theta_i}{r^n}\beta+\sum_j\frac{\partial\beta/\partial\theta_j}{r^{n-1}}\frac{\partial\theta_j}{\partial x_i}+\frac{\beta_{x_i}(x,\theta)}{r^{n-1}}\\ &= (1-n)\frac{\theta_i}{r^n}\beta+\sum_j\frac{\partial\beta/\partial\theta_j}{r^n}\left(\delta_{ij}-\theta_i\theta_j\right)+\frac{\beta_{x_i}(x,\theta)}{r^{n-1}}\\ &= \frac{(1-n)\theta_i\beta+\partial\beta/\partial\theta_i}{r^n}+\frac{\beta_{x_i}(x,\theta)}{r^{n-1}} \qquad (2.9)\end{aligned}$$

We used the fact that $\sum_j \theta_j\partial\beta/\partial\theta_j = 0$ because β is homogeneous of order 0 in θ. It is not hard to show that the "characteristic"

$$\phi(x,\theta) = (1-n)\theta_i\beta + \partial\beta/\partial\theta_i$$

has zero mean over S_θ^{n-1}, see [**22**, p. 243]. In this particular case ($\alpha(x,y,\theta) = \beta(x,\theta)$, independent of y, r), statement (a) can be proven as follows. Choose a finite atlas of charts for S^{n-1} so that for each chart, $n-1$ of the θ coordinates (that we keep fixed in $\mathbf{R}^n$) can be chosen as local coordinates. By rearranging the x, and respectively, the θ coordinates, in each fixed chart, we can assume that they are $\theta' = (\theta_1,\dots,\theta_{n-1})$. Then $\partial\beta/\partial\theta_n = -\sum_{i=1}^{n-1}\partial\beta/\partial\theta_i$. Then in (2.9), we have derivatives of β w.r.t. θ' (and x) with smooth coefficients. The contribution of the first term then can be estimated by the Calderón-Zygmund theorem. The second term is a kernel of a weakly singular operator. The following criterion can be applied to it: If K has an integral kernel $k(x,y)$ with the property

$$\sup_x\int|k(x,y)|\mathrm{d}x\le M,\quad \sup_y\int|k(x,y)|\mathrm{d}y\le M, \qquad (2.10)$$

then K is bounded in L^2 with a norm not exceeding M [**41**, Prop. A.5.1].

This proves (a) for $\alpha=\alpha(x,\theta)$. For general $\alpha(x,y,r,\theta)$, write

$$\alpha(x,y,r,\theta) = \alpha(x,x,0,\theta) + r\gamma(x,y,r,\theta).$$

The second term on the right can expressed in terms of the first derivatives of α and contributes a weakly singular integral operator G with kernel $\gamma(x,y,r,\theta)/r^{n-2}$. Then $\partial_x G$ is still a weakly singular operator with a kernel $\partial_x(\gamma(x,y,r,\theta)/r^{n-2})$, recall that r and θ depend on x as well. It is bounded in L^2 under the assumptions of the theorem. □

In the remainder of the chapter, Ω denotes a bounded domain, and Ω_1 denotes another one with $\Omega_1 \Supset \Omega$. To apply Proposition 2.5 to $X_w^*X_w$, we can cut off w smoothly to make it zero near $\partial\Omega_1$.

Corollary 2.6. *If $w\in C^2$, then for every $f\in L^2(\Omega)$ we have*

$$\|X_w^*X_w f\|_{H^1(\Omega_1)} \le C\|w\|_{C^2}^2\|f\|_{L^2(\Omega)}.$$

The constant C depends on Ω and Ω_1 only, and the C^2 norm of w above is taken in some fixed neighborhood of the unit sphere bundle $S\bar\Omega$. Actually, one can take the C^2 norm of w there on $S\bar\Omega$ since one can always take an extension from $S\bar\Omega$ to $S\mathbf{R}^n$ which could only change the C^2 norm at most by a multiplication by a fixed constant.

2.2.4. *Mapping properties of X_w.*

THEOREM 2.7. *Let $w \in C^\infty$. Then for every $s \geq 0$,*

$$\begin{aligned} X_w : & H^{s-1/2}_{\text{comp}}(\mathbf{R}^n) \longrightarrow H^s(\Sigma) \\ X_w^* : & H^{s-1/2}(\Sigma) \longrightarrow H^s_{\text{loc}}(\mathbf{R}^n) \end{aligned}$$

are continuous.

PROOF. It is enough to prove the theorem for w supported in a small neighborhood of a fixed line ℓ_0; then we can use a partition of unity. Fix a plane S transversal to ℓ_0 and parametrize Xf by initial points on S, and a corresponding direction. Let $y = (y^1, \dots y^{n-1})$ be the coordinates on S and let $(\zeta^1, \dots \zeta^{n-1})$ be the projection of θ onto S; in other words, (y, ζ) are local of coordinates on $\Sigma \times S^{n-1}$. Write $X_w f(y, \zeta)$ in the new coordinates. Then for any $k = 0, 1, \dots,$

$$\|X_w f\|^2_{H^k} = \sum_{|\alpha| \leq k} \|\partial^\alpha_{y,\zeta} X_w f\|^2 = \sum_{|\alpha| \leq k} |(X_w^* \partial^{2\alpha}_{y,\zeta} X_w f, f)_{L^2(\Sigma)}|. \tag{2.11}$$

Our goal next is to analyze the operator $X_w^* \partial^{2\alpha}_{y,\zeta} X_w$. We will show that it is a ΨDO of order $2|\alpha| - 1$.

Assume below that $\operatorname{supp} f \in \bar{\Omega}$. We have

$$\begin{aligned} \partial_{y^j} X_w f = & \int_{\mathbf{R}} w_{y^j}(x + t\theta, \theta) f(y + t\theta)\, \mathrm{d}t \\ & + \int_{\mathbf{R}} w(x + t\theta, \theta)) f_{y^j}(y + t\theta)\, \mathrm{d}t. \end{aligned}$$

This is a sum of the weighted X-ray transforms with weight w_{y^j} acting on f and X_w acting on ∂f, i.e.,

$$\partial_{y^j} X_w f = X_{w^j} f + X_w \partial_{y^j} f.$$

Therefore,

$$X_w^* \partial_{y^j} X_w = X_w^* X_{w^j} + X_w^* X_w \partial_{y^j}$$

is a ΨDO of order 0 by Theorem 2.6.

Similarly,

$$\begin{aligned} \partial_{\zeta^j} X_w f = & \int_{\mathbf{R}} t(\partial\theta^k / \partial\zeta^j) w_{y^k}(x + t\theta, \theta) f(y + t\theta)\, \mathrm{d}t \\ & + \int_{\mathbf{R}} (\partial\theta^k / \partial\zeta^j) w_{\theta^k}(x + t\theta, \theta) f(y + t\theta)\, \mathrm{d}t \\ & + \int_{\mathbf{R}} w(x + t\theta, \theta)) t(\partial\theta^k / \partial\xi^j) f_{\theta^k}(y + t\theta)\, \mathrm{d}t. \end{aligned}$$

The terms t there is a weight factor, i.e., it can be written as a smooth function of $z := x + t\theta$ and θ. Indeed, t can be uniquely determined by the condition $z - t\theta \in S$ with z and θ given, because we assumed that S is transversal to ℓ_0, and therefore this property would be preserved if $\operatorname{supp} w$ is concentrated close enough to the direction θ_0 corresponding to θ_0. Therefore, $\partial_{\zeta^j} X_w$ has the same structure as $\partial_{y^j} X_w$.

We can take higher order derivatives to conclude that $X_w^* \partial^{2\alpha}_{y,\zeta} X_w$ is a ΨDO of order $2|\alpha| - 1$, as claimed.

Let $\Lambda = (\mathrm{Id} - \Delta)^{1/2}$. Write $f = \Lambda^{-|\alpha|+1/2}\Lambda^{|\alpha|-1/2} f$ in (2.11). Then

$$\left(X_w^* \partial_{y,\zeta}^{2\alpha} X_w f, f\right)_{L^2(\Sigma)} = \left(\Lambda^{-|\alpha|+1/2} X_w^* \partial_{y,\zeta}^{2\alpha} X_w f, \Lambda^{|\alpha|-1/2} f\right)_{L^2(\Sigma)} \leq C\|f\|^2_{H^{|\alpha|-1/2}}.$$

Then (2.11) yields

$$\|X_w f\|_{H^k} \leq C\|f\|_{H^{k-1/2}}, \quad \forall f \in H_0^{k-1/2}(\Omega).$$

This proves the first statement for $s \geq -1/2$ half-integer. We can prove it for every $s \geq -1/2$ by interpolation.

To prove the second part, notice that $\partial_x^\alpha X_w^*$ is a sum of operators of the kind X_w with weights obtained from w by differentiation, composed with derivatives $\partial_{y,\zeta}^\beta$ of order $|\beta| \leq |\alpha|$. Then

$$\begin{aligned} |(f, \partial_x^\alpha X_w^* \psi)| &\leq C \sum_{|\beta|\leq|\alpha} |(f, X_{w_\beta}^* \partial_{y,\zeta}^\beta \psi)| = C \sum_{|\beta|\leq|\alpha} |(X_{w_\beta} f, \partial_{y,\zeta}^\beta \psi)| \\ &\leq C\|X_{w_\beta} f\|_{H^{1/2}} \|\partial_{y,\zeta}^\beta \psi\|_{H^{-1/2}} \leq C\|f\|_{L^2}\|\psi\|_{H^{|\alpha|-1/2}}. \end{aligned}$$

This proves the second statement for $s \geq 0$ integer. The general statement follows by interpolation. □

2.2.5. *Finiteness of the kernel and stability under the ellipticity condition.*

THEOREM 2.8. *Assume that $w \in C^\infty$ satisfies the ellipticity condition (2.7). Then*

(a) $\operatorname{Ker} X_w \cap L^2(\Omega)$ *is a finite dimensional space consisting of $C_0^\infty(\mathbf{R}^n)$ functions.*

(b) For any $s \geq 0$, there exists constants $C > 0$ (independent of s) and C_s so that

$$\|f\|_{L^2(\Omega)} \leq C\|X_w^* X_w f\|_{H^1(\Omega_1)} + C_s\|f\|_{H^{-s}(\mathbf{R}^n)}, \quad \forall f \in L^2(\Omega). \tag{2.12}$$

(c) If X_w is injective on $L^2(\Omega)$, then estimate (2.12) holds without the last term (and a possibly different C), i.e.,

$$\|f\|_{L^2(\Omega)} \leq C\|X_w^* X_w f\|_{H^1(\Omega_1)}, \quad \forall f \in L^2(\Omega). \tag{2.13}$$

PROOF. Follows directly from Theorem III.1.1. □

3. Visible and Invisible singularities. The limited angle problem

3.1. Visible and invisible singularities. Let us consider X_w restricted to a small neighborhood of the single directed line $\ell_0 = x_0 + t\theta_0$. We would like to find out what singularities (elements of $\mathrm{WF}(f)$) can be recovered from knowing $X_w f$ near ℓ_0. To be more precise, we want to know what part of $\mathrm{WF}(f)$ can be recovered *in a stable way* in the sense of Section III.5.

As an example, consider a function f a priori supported in the unit ball in $\mathbf{R}^2$. If $Xf(\ell) = 0$ for all lines ℓ away from the ball $B_{1-\varepsilon}(0)$, $0 < \varepsilon < 1$, by the support theorem, $f = 0$ outside $B_{1-\varepsilon}(0)$. In particular, if $Xf = Xg$ for such lines, then $f = g$ outside $B_{1-\varepsilon}(0)$. Therefore, we have unique "recovery" (only a uniqueness statement, actually) there. Then any element $(x, \xi) \in \mathrm{WF}(f)$ with $|x| > 1 - \varepsilon$ is uniquely determined by the data simply because the whole f is determined outside $B_{1-\varepsilon}(0)$. That determination is however done by the use of the support theorem which is a kind of unique/analytic continuation. We will see below that if $n = 2$,

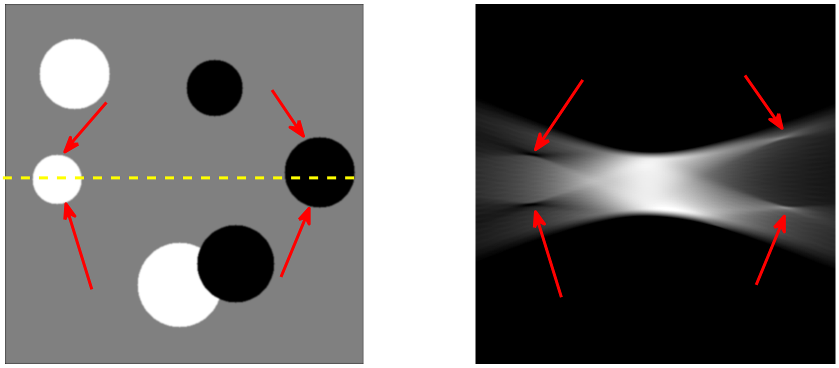

FIGURE IV.1. Singularities that can be recovered by $X_w f$, $w > 0$, restricted to a small neighborhood of the dotted line

only a small part of $\mathrm{WF}(f)$ over $1-\varepsilon < |x| < 1$ can be stably recovered; which consists of (x,ξ) with ξ (co)normal to some line of our set. Roughly speaking, those are (x,ξ) with $1-\varepsilon < |x| < 1$ and ξ close to a radial (co)direction λx, $\lambda \neq 0$. If $n \geq 3$ however, the $f|_{|x|>1}$ can be stably recovered; more precisely, f can be stably recovered in any compact set in $|x| > 1$.

To get an idea of what we can expect, notice first that X_w has a Schwartz kernel $\mathcal{X}_w(x,\ell) = w(x,l)\delta_\ell(x)$, where δ_ℓ is the delta function in the x variable on the line l. The kernel $\mathcal{X}_w$ is a smooth function of ℓ with values in $\mathcal{S}'(\mathbf{R}^n)$ having wave front set $N^*\ell$. One would intuitively expect that $\mathrm{WF}(X_w f)$ could be only affected by $\mathrm{WF}(f)$ lying on the union of $N^*\ell$ for all line ℓ over which we integrate. Indeed, by Theorem ..., $\mathrm{WF}(X_w f) \subset \mathrm{WF}'(\mathcal{X}_w) \circ \mathrm{WF}(f)$. Here $\mathrm{WF}'(\mathcal{X}_w)$ is the twisted wave front set of $\mathcal{X}_w$ as a function of (x,ℓ). One can see that the projection of $\mathrm{WF}'(\mathcal{X}_w)$ onto $T^*\mathbf{R}^n_x \setminus 0$, for any fixed ℓ, is indeed $N^*\ell$ which confirms our expectation.

Therefore, we can only hope to recover (in a stable way) $\mathrm{WF}(f)$ on $N^*\ell$ for all lines ℓ over which we integrate. Below we show that we can actually do that, if the weight does not vanish.

By Theorem 2.3, the operator $X_w^* X_w$ is a ΨDO of order -1 with principal symbol

$$\sigma_p(X_w^* X_w)(x,\xi) = 2\pi \int_{S^{n-1}} |w(x,\theta)|^2 \delta(\theta\cdot\xi)\,\mathrm{d}\theta, \tag{3.1}$$

and a full symbol

$$a(X_w^* X_w)(x,y,\xi) = 2\pi \int_{S^{n-1}} \bar{w}(x,\theta) w(y,\theta)\delta(\theta\cdot\xi)\,\mathrm{d}\theta. \tag{3.2}$$

If $n = 2$, the first formula above takes the form

$$\sigma_p(X_w^* X_w) = \frac{2\pi}{|\xi|}\left(\left|w\Big(x,\frac{\xi^\perp}{|\xi^\perp|}\Big)\right|^2 + \left|w\Big(x,-\frac{\xi^\perp}{|\xi^\perp|}\Big)\right|^2\right), \tag{3.3}$$

with $\xi^\perp := (-\xi^2,\xi^1)$; and similarly for the second one.

This shows that the visible singularities, in the sense of Section III.5, are given by

$$\mathcal{V} = \{(x,\xi) \in T^*\mathbf{R}^n \setminus 0 |\ \exists \theta \perp \xi \text{ so that } w(x,\theta) \neq 0\}. \tag{3.4}$$

Here, θ is a vector, while ξ is a covector, so $\theta \perp \xi$ actually means that $\xi(\theta) = 0$, i.e., $\xi_j\theta^j = 0$ in coordinate notation. Since this observation is based on the principal symbol only, we need to analyze the full symbol for possible non-sharp microlocal stability. One can compute the latter by (3.2) and (B.3.9) to see that away from $\bar{\mathcal{V}}$, $X_w^*X_w$ is smoothing. In particular, the invisible singularities are

$$\mathcal{U} = T^*\mathbf{R}^n \setminus \bar{\mathcal{V}}, \tag{3.5}$$

where $\mathcal{V}$ is considered as a conic set, and the closure is taken in that sense.

The geometric interpretation of this is simple: to be able to recover a singularity (x, ξ), we need to have a line trough x in a direction θ normal to ξ (there are two such directed lines if $n = 2$ and infinitely many if $n \geq 3$), so that $w(x, \theta) \neq 0$.

We then have the following corollary, see Section III.5.

THEOREM 3.1. *Fix a bounded domain $\Omega \subset \mathbf{R}^n$. Let $\chi \in S^0$ have essential support in $\mathcal{V}$.*

(a) Then for every s and $l < s$, we have

$$\|\chi(x, D)f\|_{H^s} \leq C_s\|X_w^*X_w\|_{H^{s+1}} + C_{s,l}\|f\|_{H^l}$$

for every $f \in H_0^s(\Omega)$.

(b) The estimate on (a) does not hold when $\mathrm{ES}(\chi) \cap \mathcal{V} = \emptyset$, regardless of the choice of s and $l < s$.

Since we restricted $\operatorname{supp} f$ above to $\bar{\Omega}$, we can replace $\mathbf{R}^n$ in (3.4) by $\bar{\Omega}$.

This theorem does not directly answer the question what singularities we can recover from knowledge of $X_w f$, not $X_w^*X_w f$. In terms of uniqueness, it is a trivial observation that X_w is injective if and only if X_w is, see Proposition 5.5. We show below that there is a microlocal equivalent to this.

***** stability thm in terms of $X_w^* f$ here *****

***** We can use X^*X_w *****.

3.2. The limited angle and the partial data X-ray transform. Assume that $Xf(\ell)$ (weight $w = 1$ for simplicity) is known for lines ℓ in an open set $\mathcal{L}$. What can we recover from that information? The requirement that $\mathcal{L}$ open is essential. If we restrict to a submanifold of positive codimension, the microlocal nature of the problem changes dramatically, see [**15, 16**].

We can model such a case by choosing weights w constant along all lines, i.e., $\theta \cdot \nabla_x w = 0$, compactly supported in $\mathcal{L}$. In the Σ parameterization for the lines we adopted in Section II.1.3, this means $w(x, \theta) = w_0(\theta, x - (\theta \cdot x)\theta)$ with some $w_0 \in C_0^\infty$. Then we get that the singularities that can be stably recovered are the ones in

$$N^*\mathcal{L} := \{(x, \xi) \in N^*\ell |\, \ell \in \mathcal{L}\},$$

where N^* stands for the conormal bundle. In other words, those are all (x, ξ) with the property that (x, ξ) is conormal to some line in $\mathcal{L}$. All singularities outside the closure of $N^*\mathcal{L}$ are not stably recoverable.

FIGURE IV.2. Limited angle tomography: Left: original; Center: reconstruction with the angles in $[-30°, 30°]$ missing; Right: the "un-filtered $X^*X_w f$

4. Recovery in a Region of Interest (ROI)

In practical applications, the following problem is of great interest. Assume that we only want to recover f, given Xf or Rf, in a subdomain called a Region of Interest (ROI). For example, we want to create the image of the hart only. Is it enough to illuminate that region only? In other words, if we know $Xf(\ell)$ for all lines through the ROI; or Rf for all planes through the ROI, does this determine f restricted to the ROI uniquely?

In odd dimensions, for the Radon transform R, this follows immediately from the reconstruction formula in Theorem II.2.6. In even dimensions for R. and in all dimensions for Xf, the reconstruction contains a fractional power of $-\Delta$, see also Theorem II.2.3; and the same argument does not work.

4.1. Non-uniqueness. We will show that there is no unique solution to that problem for the Radon transform R (and therefore, for X) in two dimensions, based on the representation of R in Theorem II.4.1.

Given $a > 0$, take any even function $g \in C_0^\infty(\mathbf{R})$ so that $g(p) = 0$ for $|p| \le a$. Then $g(p)$, considered as a function of (p, ω) independent of ω, satisfies the range conditions in Definition II.6.2 (i), (ii) in a trivial way. Indeed, $\int g(p)p^k \, \mathrm{d}p$ vanishes for k odd, and is constant for k even. In the later case, it is the homogeneous polynomial $C|\omega|^k$ restricted to the unit sphere. Therefore, there exists $f \in \mathcal{S}$ with $g = Rf$. By the support theorem, $f \in C_0^\infty$. Then $Rf = 0$ for $|p| \le a$ by construction. Theorem II.4.1 allows us to express f explicitly:

$$f(r) = -\frac{1}{\pi}\int_r^\infty (p^2 - r^2)^{-\frac{1}{2}} g'(p) \, \mathrm{d}p.$$

It remains to see that we can choose g so that $f \not\equiv 0$ in $B(0, a)$. This can be done in several ways. For $0 < r < a$, integrate by parts above to get

$$f(r) = \frac{1}{\pi}\int_r^\infty p(p^2 - r^2)^{-\frac{3}{2}} g(p) \, \mathrm{d}p, \quad r < a.$$

When $g \ge 0$ but $g \not\equiv 0$ (recall that $g = 0$ in $[-a, a]$), we get $f(r) > 0$ for $0 \le r \le a$.

We can extend this construction to the higher order harmonics. Given an integer $k \ge 0$, let $0 \not\equiv g_k \in C_0^\infty$ be even when k is even and odd otherwise. Let $g_k = 0$ in $[-a, a]$. Set $g(p, \omega) := g_k(p)e^{\mathrm{i}k\omega}$. We want to solve $Rf = g$ for f. Let us check the range conditions of Definition II.6.2. The first one, requiring g to be an

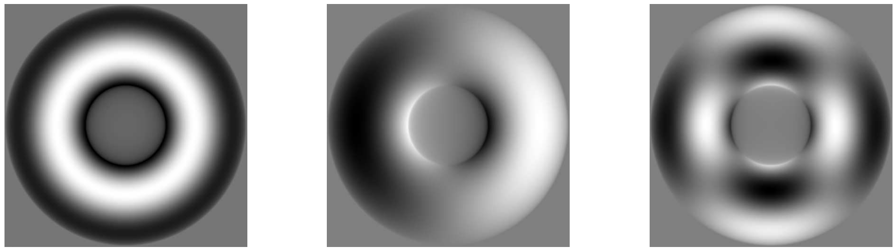

FIGURE IV.3. Examples of non-uniqueness for the ROI problem: harmonics of order 0, 1 and 2. The functions are non-zero (but "small") in the inner disks. Medium gray corresponds to 0.

FIGURE IV.4. Computed Radon transforms of the functions in Figure IV.3. The horizontal axis represents the angle ranging in $[0, 180°]$; the vertical one corresponds to p. Medium gray corresponds to 0. The functions are even/odd/even, from left to right, as functions of p. The third function has zero integrals in the p variable.

even function, is guaranteed by construction. For the second one, consider

$$\int p^m g_k(p) e^{\mathrm{i}k \arg\omega}\,\mathrm{d}p = e^{\mathrm{i}k\arg\omega} \int p^m g_k(p)\,\mathrm{d}p. \tag{4.1}$$

Condition (ii) is satisfied if the expression above is a homogeneous polynomial of ω of degree m, restricted to the unit sphere, for every m. Notice first that every such polynomial $P_m(\omega)$, is also the restriction of a homogeneous of order $m+2j$, $j = 0, 1, \dots$ polynomial on the unit sphere because $P_m(\omega) = |\omega|^{2j} P_m(\omega)$ there. Let k be even first. Then g_k is even and (4.1) vanishes for m odd. For m even, it is enough to have condition (ii) for $m \le k$, However, if $m < k$, (4.1) shows that we get a homogeneous polynomial of order k which cannot be reduced. Therefore, we need

$$\int p^m g_k(p)\,\mathrm{d}p = 0, \quad m = 0, 1, \dots, k-1.$$

Let now k be odd. Then by the same arguments, we get the same condition.

For $k = 1$, there is no restriction, since $\int g_1(p)\,\mathrm{d}p = 0$ because g_1 is odd. Choose any odd function $g_1 \in C_0^\infty$ vanishing near the origin. Then

$$g(p,\omega) = e^{\mathrm{i}\arg\omega} g_1(p)$$

is in the range of R, and there exists $f \in \mathcal{S}$ so that $g = Rf$. By the support theorem, $f \in C_0^\infty$. By Theorem II.2.6, $f_1 \not\equiv 0$ and all other f_k vanish. Then

$$f = e^{\mathrm{i}\arg x} f_1(|x|) = (x^1 + \mathrm{i}x^2) f_1(|x|)$$

with f_1 related to g_1 as in Theorem II.2.6. If we need real functions, we can take the real parts. It remains to show that for generic g_1, f cannot be identically zero in a neighborhood of 0, which can be done as above. The middle plot of Figure IV.3 represents such a function and the middle plot in Figure IV.4 represents the corresponding g.

For $k = 2$, we pick an even function $g_2 \in C_0^\infty$ vanishing near 0 so that $\int g_2(p)\,\mathrm{d}p = 0$. Then

$$g(p,\omega) = e^{2\mathrm{i}\arg\omega} g_2(p)$$

satisfies the range conditions. Then, in the same way as before, we get $g = Rf$ with some $f \in C_0^\infty$, with f_2 related to g_2 as in Theorem II.4.1, and all other f_k zero. One can also take the real parts of f and g. One such function g is plotted in Figure IV.4 on the right, where the vertical axis is the p one; and the corresponding f is plotted in Figure IV.4 on the right.

The functions plotted in Figure IV.4 are non-zero in the ROI (the inner disk) but they are still "small". A discussion of that phenomenon and how it depends on the radii of the inner and on the outer circles can be found in [**25**]. We will see below that they must be smooth in the ROI.

4.2. Parametrix reconstruction. Even though we cannot reconstruct f in the ROI uniquely, we can reconstruct it up to a smooth error. This allows us to recover the singularities. Indeed, let $\chi(\ell)$ be a smooth function on the line manifold equal to one on all lines through the ROI, and zero outside a larger neighborhood. Then $X^*\chi Xf$ (or $R^*\chi Rf$) is a ΨDO of order -1, elliptic in the ROI. Indeed, by Theorem 2.3, with $a = 1$ and $b = \chi$, the principal symbol of $X^*\chi X$, and actually the whole symbol or amplitude, equals 1 in a neighborhood of the ROI. Therefore, for $\psi \in C_0^\infty$ with $\chi = 1$ near the ROI and $\operatorname{supp}\psi$ close enough to the ROI, we have $\psi f = QX^*\chi Xf + Rf$, where R is smoothing. Therefore, we can recover f in the ROI up to a smooth term; more precisely, up to a smoothing operator applied to f.

In Figure IV.5, we show one such reconstruction. The error, plotted on the right, is smooth in the ROI, consisting of the small disk in the center. The figure in the middle is just a parametrix and the low frequency part is not optimized. In fact, we subtracted a small constant to make the zero values black; and this clipped some negative values outside the ROI which appear black again. Since constant functions are smooth, the reconstruction shown has the same singularities in the ROI, of course.

5. Support Theorems and Injectivity

5.1. Lack of injectivity in the 2D case. Generic Injectivity.

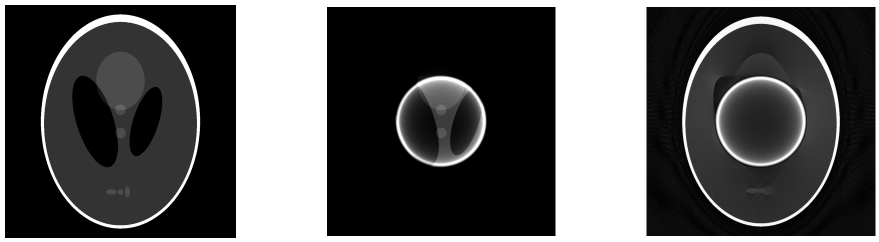

FIGURE IV.5. ROI: Left: the original; Center: ROI recovered; Right: Absolute value of the difference

5.1.1. *Boman's example.* Jan Boman constructed in [**6**] a weight w for which the support theorem for X_w fails in the plane; and he showed in [**7**] that the set of weights for which it fails is dense. We will present those results here without the proofs.

THEOREM 5.1 (the support theorem fails). *There exists a smooth $w(x,\theta) > 0$ defined on $\mathbf{R}^2 \times S^1$ and a smooth function $f(x)$ on $\mathbf{R}^2$ supported in $|x^1| \le x^2$ with $0 \in \operatorname{supp} f$ so that $Xf(\ell) = 0$ for all lines ℓ of the kind $x^2 = ax^1 + b$, $|a| < 1$, $b < 1$.*

THEOREM 5.2 (uniqueness fails). *There exists a smooth $w(x,\theta) > 0$ defined on $\mathbf{R}^2 \times S^1$ and a smooth function $f(x)$ on $\mathbf{R}^2$ not identically zero so that $Xf(\ell) = 0$ for all lines ℓ in the plane.*

THEOREM 5.3 (density of the weights). *The set of smooth positive weights w for which Theorem 5.1 holds (with $f = f_w$) is dense in $C^k(\mathbf{R}^2 \times S^1)$ for every $k = 0, 1, \dots$.*

In dimensions $n \ge 3$ however, the support theorem holds as we show below. Note that in this case, X_w is formally overdetermined.

5.1.2. *Injectivity on small domains.* Here and below, if $U \subset \mathbf{R}^n$, we consider $L^2(U)$ as a subspace of $L^2(\mathbf{R}^n)$. Let $\mathcal{E}'(K)$ the space all all distributions supported in the compact set K. We formulate many injectivity theorems below in terms of $L^2(\Omega)$ for simplicity. When w does not vanish in $\bar\Omega$, $X_w^* X_w$ is elliptic, and its kernel consist of smooth functions. Therefore, injectivity on $L^2(\Omega)$ is equivalent to injectivity on $\mathcal{E}'(K)$.

On small domains, X_w is injective and stable. Next theorem is valid for all dimensions $n \ge 2$ but for $n \ge 3$, we prove a stronger theorem below.

THEOREM 5.4 (injectivity on small domains). *Let $w \in C^\infty(\bar\Omega \times S^{n-1})$ and assume that $w(x,\theta) \neq 0$, $\forall (x,\theta) \in \bar\Omega \times S^{n-1}$. Then there exists $\varepsilon > 0$ so that for any compact set $K \subset \bar\Omega$ with measure $|K| \le \varepsilon$, the weighted transform X_w is injective on $L^2(K)$ and*

$$\|f\|_{L^2} \le C \|X_w^* X_w f\|_{H^1(\Omega_1)}, \quad \forall f \in L^2(K). \tag{5.1}$$

PROOF. Any smooth extension of w outside $\bar\Omega$ will not vanish for x close enough to $\partial\Omega$. Take one such extension and choose $\chi \in C_0^\infty(\Omega_1)$ equal to one in a neighborhood of the closure of the set where $w \neq 0$. The operator $\chi X_w^* X_w \chi$ then is a properly supported ΨDO in Ω_1, elliptic of order -1. Applying the "small domain" Theorem B.3.1, we get (5.1). ∎

REMARK 5.1. Another way to think about small domains is through a scaling argument. If we scale a small ball $B(0,\varepsilon) \ni x$ to the ball $B(0,1) \ni y$ by the transformation $x = \varepsilon y$, then the weight becomes $w(\varepsilon y, \theta)$ (in fact, vectors scale by ε as well but if in the new coordinates we keep the direction unit, $\varepsilon\theta$ becomes θ again, at the expense of rescaling the parameter t in (1.1)). Now, in any C^k, $w(\varepsilon y, \theta)$ is close to $w(0, \varepsilon)$ for $0 < \varepsilon \ll 1$, and by the perturbation argument below, it is enough to have injectivity for weights independent of x. Such weights however are trivial because they are constant along each line, and injectivity then reduces to that for $w = 1$, which is known.

REMARK 5.2. We could have formulated the theorem without the need of Ω and Ω_1 by assuming that w has uniformly bounded derivatives in $\mathbf{R}^n$ of some fixed order $k \gg 1$, see the remarks following Theorem B.3.1.

We show next that injectivity of X_w is equivalent to injectivity of $X_w^* X_w$, either in Ω or Ω_1.

PROPOSITION 5.5. *Let $\Omega_1 \Supset \Omega$. The following statements are equivalent.*
(a) $X_w^* X_w : L^2(\Omega) \to L^2(\Omega_1)$ is injective.
(b) $X_w^* X_w : L^2(\Omega) \to L^2(\Omega)$ is injective.
(c) X_w is injective on $L^2(\Omega)$.

PROOF. Clearly,

$$\|X_w f\|^2_{L^2(\Sigma)} = (X_w^* X_w f, f)_{L^2(\Omega)} = (X_w^* X_w f, f)_{L^2(\Omega_1)}, \quad \forall f \in L^2(\Omega).$$

This shows that both (a) and (b) are equivalent to (c). □

Therefore, for small domains, X_w is injective as well.

REMARK 5.3. The domain Ω_1 in (5.1) can be replaced with any $U \Supset K$, once we fix K for which the theorem holds. For that, we use Proposition 5.5 and Section III.5. Then we lose control over the constant C however.

5.1.3. *Injectivity for analytic and for generic weights.*

THEOREM 5.6. *Let w be analytic and non-vanishing in a neighborhood of $\bar{\Omega} \times S^{n-1}$. Then X_w is injective on $L^2(\Omega)$.*

PROOF. We will use the analytic ΨDO calculus here, see Section B.7. It is enough to prove the theorem for $n = 2$ only. Indeed, any function in the kernel of X_w is smooth. If we can prove that f, restricted to any 2-plane vanishes, then $f = 0$. On any 2-plane, on the other hand, we have to invert just the two-dimensional X_w.

By (2.6), the operator $X_w^* X_w$ is a ΨDO with amplitude

$$a(x,y,\xi) = \frac{2\pi}{|\xi|} W(x,y,\xi^\perp/|\xi|)$$

where

$$W(x,y,\theta) = \bar{w}(x,\theta)w(y,\theta) + \bar{w}(x,-\theta)w(y,-\theta).$$

This is the "full amplitude", i.e., $X_w^* X_w$ is given by the oscillatory integral (B.3.7) with that amplitude and there is no smoothing operator error. Take $\phi \in C_0^\infty(\mathbf{R}^n)$ with $\phi = 1$ near 0 and $\phi(\xi) = 0$ for $|\xi| \geq 1$. Then $\phi(\xi)a(x,y,\xi)$ is an analytically regularizing amplitude, see (7.5) (note that the lack of analyticity of ϕ is not a problem for this argument). On the other hand, $(1-\phi(\xi)a(x,y,\xi)$ is pseudoanalytic. Indeed, since a is a homogeneous function of ξ for $|\xi| \gg 1$, it is enough to verify

(7.5) for ξ in any compact outside the support of ϕ, i.e, for $1 \le |\xi| \le 2$. Then the estimate follows from the assumed analyticity of $w(x,\theta)$. □

THEOREM 5.7 (perturbing the weight). *Let X_w be injective on $L^2(\Omega)$ with some nowhere vanishing $w \in C^\infty(\bar\Omega_1 \times S^{n-1})$. Then there exists $\varepsilon > 0$ so that for any $v \in C^2(\bar\Omega_1 \times S^{n-1})$ with*

$$\|w - v\|_{C^2} \le \varepsilon, \tag{5.2}$$

X_v is injective as well. Moreover, there is a constant $C > 0$ (depending on w but not on v) so that

$$\|f\|_{L^2(\Omega)} \le C\|X_v^* X_v f\|_{H^1(\Omega_1)} \tag{5.3}$$

for any such v.

PROOF. By Proposition 2.2, the operator $X_w^*X_w - X_v^*X_v$ is of the form (2.8) with

$$\alpha(x,y,\theta) = \bar w(x,\theta)w(y,\theta) + \bar w(x,-\theta)w(y,-\theta) - \bar v(x,\theta)v(y,\theta) + \bar v(x,-\theta)v(y,-\theta)$$

(independent of r). Then

$$\|\alpha\|_{C^2(\bar\Omega_1\times\bar\Omega_1\times S^{n-1})} \le C\|w - v\|_{C^2(\bar\Omega_1\times S^{n-1})}$$

with $C = C(w)$. Therefore,

$$\|(X_w^*X_w - X_v^*X_v)\, f\|_{H^1(\Omega_1)} \le C\varepsilon\|f\|_{L^2(\Omega)}$$

when (5.2) holds. Then, by Theorem 2.8,

$$\|f\|_{L^2(\Omega)} \le C\|X_w^*X_w f\|_{H^1(\Omega_1)} \le \|X_v^*X_v f\|_{H^1(\Omega_1)} + C\varepsilon\|f\|_{L^2(\Omega)}.$$

We can therefore absorb the last term by the left-hand side one for $\varepsilon \ll 1$. □

COROLLARY 5.8 (generic uniqueness). *The transform X_w is injective on $L^2(\Omega)$ for an open dense set of nowhere vanishing weights $w \in C^2(\bar\Omega_1 \times S^{n-1})$.*

Of course, for such weights, we have the locally uniform estimate (5.3).

REMARK 5.4. We did not prove that the set of non-vanishing C^2 weights for which X_w is injective, is open (but we did prove that it is dense). If we replace C^2 by C^k, $k \gg 1$ (regularity sufficient to apply the ΨDO arguments) then this statement follows from our proofs because then injectivity implies stability, that we can perturb. If $w \in C^2$ only, and X_w is injective, we do not have an argument showing that it is stable at the same time (unless w is smoother).

5.2. Support theorem and uniqueness for $n \ge 3$. The "exterior problem". The *exterior problem* asks whether we can reconstruct f outside a compact convex set K if $X_w f$ is known for all lines not intersecting K. Support theorems allow us to claim uniqueness of the exterior problem if the function satisfies some conditions at infinity. We want to analyze the stability, as well.

5.2.1. *First things first, microlocal considerations.* We start with microlocal considerations. Let us say that we want to reconstruct WF(f) outside a K not the whole f. Can we do this? Why this would not solve the support theorem question, it would at least tell us if the reconstruction could be stable.

The analysis so far tells us that $X_w f$ restricted to an open set of lines, recovers conormal singularities to those line, assuming w nowhere vanishing along the lines. Therefore, the visible singularities are the ones described below, for $n = 2$ and $n \geq 3$, respectively:

$$\begin{aligned}\mathcal{U}_2 := \Big\{ &(x,\xi) \in T^*\mathbf{R}^n \setminus 0 |\; |x| > R, \;\text{ the line trough } x \text{ normal to } \xi \\ &\text{does not intersect } K \Big\},\\ \mathcal{U}_n := \big\{ &(x,\xi) \in T^*\mathbf{R}^2 \setminus 0 |\; x \notin K \big\}.\end{aligned}$$

Moreover, the invisible ones are those in the complement of the closures of those sets. Note that if $n \geq 3$, the visible singularities include all covectors with a base point outside K, while this is not true when $n = 2$. Since the visible set outside K for $n \geq 3$ includes all of the exterior of K, and it is not limited to certain codirections, the exterior problem then would be well a posed one for such n, and an ill posed one for $n = 2$. We make this more precise below.

5.2.2. *Support theorem for analytic weights, $n = 2$.* Here, we present a microlocal proof of the support theorems for X and R, and also for X_w for analytic weights w. We restrict ourselves to the case where f is a priori compactly supported. In fact, one can use analytic microlocal techniques even when f is only rapidly decreasing, as in Theorem II.5.1, as shown by Boman [**5**].

To author's knowledge, the fist application of the analytic micorolocal calculus to proving support theorems is due to by Boman and Quinto [**8**] for the weighted Radon transform in n dimensions

$$R_\mu f(p,\omega) = \int_{x\cdot\omega = p} \mu(x,\omega) f(x)\, \mathrm{d}S_x$$

with an analytic positive weight. The theorem says, that if μ is analytic, $f \in \mathcal{E}'(\mathbf{R}^n)$, and $Rf = 0$ for all planes $x\cdot\omega = p$ with ω close to a fixed ω_0, and $p > p_0$, then $f = 0$ on the half-plane $x \cdot \omega_0 > p_0$. Of course, this theorem applies to X_w in two dimensions with $w(x,\theta) = \mu(x,\theta^\perp)$; and support theorems for X_w in two dimensions imply similar ones on all dimensions $n \geq 2$.

We will formulate and prove a bit more general theorem, in terms of the transfrom X_w which assumes local analyticity only and proves a local result.

The proof is based on the following theorem, which say that we can resolve the conormal analytic singularities to some line if we know $X_w f$ near that line. We already established this in the C^∞ category but in the analytic one, the proof is more delicate. We formulate the theorem for the line $x' = 0$ but by a linear change of coordinates, we can apply it to every line, of course.

THEOREM 5.9. *Fix $(x_0,\theta_0) \in \Omega\times S^{n-1}$ and let ℓ_0 be the directed line determined by it. Let $w(x,\theta)$ a weight function defined and analytic for x in some neighborhood L of $\ell_0 \cap \bar\Omega$ and for θ in some neighborhood V of θ_0. For $f \in \mathcal{E}'(\Omega)$, let $X_w f(\ell)$ be an analytic function of ℓ in a neighborhood of ℓ_0. If $w(x_0,\theta_0) \neq 0$, then*

$$\mathrm{WF_A}(f) \cap N^*_{x^0}\ell_0 = \emptyset.$$

PROOF. By assumption, $X_w f(\ell_{x,\theta})$ is analytic for $x \in U$ and $\theta \in V$, where $U \ni x_0$ is open, where we shrink V if needed to make sure that all lines issued from points in $U \times V$ stay in some compact subset of L. If we can extend w analytically for all x and θ, then $X_w^* X_w$ would be an analytic ΨDO elliptic near (x_0, ξ) for all $\xi \perp \theta_0$, with an amplitude $a(z, y, \xi)$ given by (2.5) and (2.6), respectively, depending on the dimension, with W as in (2.2). Such an extension may not exist however. Instead, we consider

$$N^R f(x) = (2\pi)^{-n} \int e^{\mathrm{i}(x-y)\cdot\xi} a(x, y, \xi) g^R(\xi)\, u(y)\, \mathrm{d}y\, \mathrm{d}\xi,$$

see (B.7.11) with a suitable g^R equal to 1 in a small conic neighborhood of all codirections conormal to θ_0. This is an analytic ΨDO in a conic neighborhood of $N^*\ell_0$; elliptic at (x_0, ξ) for all $\xi \perp \theta_0$ under the assumptions of the theorem. If we can show that $N^R f$ is analytic at conormal directions to ℓ_0 at x_0, we are done.

Let $V_0 \Subset V$ be another open set containing θ_0. Let $\psi(\theta)$ be a smooth function so that $\operatorname{supp}\psi \subset V$ and $\psi = 1$ near $\bar{V}_0$. Then $X_w^* \psi X_w$ is a (non-analytic) ΨDO with "full amplitude"

$$a_\psi = \frac{\pi}{|\xi|} \int_{S^{n-1}} W(x, y, \theta)\psi(\theta)\delta(\theta \cdot \xi/|\xi|)\, \mathrm{d}\theta,$$

with the integral reducing to a sum of two terms when $n = 2$, see (2.6) and (2.3) again. By Proposition 2.1, and by the assumptions of the theorem,

$$X_w^* \psi X_w f|_U = \mathrm{Op}(a_\psi) f|_U \in \mathcal{A}(U).$$

Note that the non-analyticity of ψ is not a problem for this argument.

Let $\Gamma \Subset \Gamma^*$ be two open cones in $\mathbf{R}^n \setminus 0$ so that $\bar{\Gamma}^*$ is included in the dual cone

$$V_0^\perp := \{\xi |\ \xi \perp \theta \text{ for some } \theta \in V_0\}.$$

Let $g^R = 1$ in Γ and be supported in Γ^*. Then $a(x, y, \xi)$ is analytic for $x \in U$, $y \in L$ and $\xi \in \operatorname{supp} g^R \setminus 0$.

Now, the operator $N^R - \mathrm{Op}(a_\psi)$ has an amplitude

$$a(x, y, \xi) g^R(\xi) - a_\psi(x, y, \xi)$$

vanishing for $\xi \in \Gamma$, $x \in \Omega$ and $y \in L$. That amplitude is analytic in the x variable, and moreover satisfies the assumptions of Lemma B.7.3. Therefore, $\mathrm{WF_A}(N^R f)$ and $\mathrm{WF_A}(\mathrm{Op}(a_\psi) f)$ coincide on $U \times \Gamma$. On the other hand, restricted to $U \times \Gamma$, the latter is empty, therefore, $\mathrm{WF_A}(N^R f)$ is microlocally analytic there. □

THEOREM 5.10 (support theorem for X_w with w analytic). *Let $f \in \mathcal{E}'(\mathbf{R}^2)$. Let $X_w f(\ell)$ be analytic for all lines ℓ in some neighborhood of ℓ_{x_0,θ_0} with some $(x_0, \theta_0) \in \mathbf{R}^2 \times S^1$. Let $w(x, \theta)$ be analytic for x near $\ell_{x_0,\theta_0} \cap \operatorname{supp} f$ and for θ near θ_0. If there is $\varepsilon > 0$ so that f is supported on one side of the line segment $\ell_0 \cap B(x_0, \varepsilon)$, then $f = 0$ near x_0.*

PROOF. By Theorem 5.9, f is microlocally analytic at conormal directions at x_0. By the Sato-Kawai-Kashiwara Theorem B.7.1, $f = 0$ near x_0. □

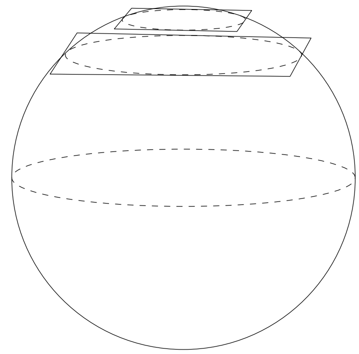

FIGURE IV.6. Illustration to the proof of Theorem 5.11. We apply the "small domain" theorem on planes close to ones tangent to the ball $B(0,R)$.

5.2.3. *Support theorem for smooth weights, $n \geq 3$.*

THEOREM 5.11. *Let $n \geq 3$ and let $w \in C^\infty(\mathbf{R}^n \times S^{n-1})$ be a nowhere vanishing weight. Let $f \in \mathcal{E}'(\mathbf{R}^n)$ and assume that $X_w f(\ell) = 0$ for all lines ℓ not intersecting the ball $B(0,R)$. Then $f(x) = 0$ for $|x| > R$.*

In particular, if $X_w f = 0$, then $f = 0$.

PROOF. Since all singularities outside $\overline{B}(0,R)$ are stably recoverable, f is smooth there.

The idea is to apply the "small domain" result in Theorem 5.4 on a family of hyperplanes parallel to each other, cutting small disks from the ball $B(0,\rho)$, $\rho > R$, starting from a tangent hyperplane, see Figure IV.6.

Let $\rho > 0$ be the least number ρ so that $\operatorname{supp} f \subset B(0,\rho)$, i.e., $\rho = \max_{\operatorname{supp} f} |x|$. If $\rho = R$, then the assertion of the theorem holds. Assume $\rho > R$. For a fixed $\theta \in S^{n-1}$ and $s \leq \rho$, restrict X_w to the hyperplanes $\pi_{s,\theta} = \{x|\ x\cdot\theta = s\}$. Then for $f_s := f|_{\pi_s}$, we have $X_{w_s} f_s = 0$, where w_s is w restricted to $T\pi_s$. We can always assume that $\theta = (0,\dots,0,1)$. Then $w_s(x',\theta') = w((x',s),(\theta',0))$ for $\theta' \in S^{n-2}$, $x' \in \mathbf{R}^{n-1}$ and we can think of all of X_{w_s} as acting on the same hyperplane, $x^n = 0$, that we identify with $\mathbf{R}^{n-1}$. Then $s \mapsto w_s \in C^2$ is a continuous family of weights; and therefore, the family $X^*_{w_s} X_{w_s}$, restricted to the same domain on $\mathbf{R}^{n-1}$, is continuous as well. Set $\Omega_1 = B(0,1)$, $\Omega_2 = B(0,\delta)$, with some $0 < \delta < 1$ where the balls are in $\mathbf{R}^{n-1}$. Then $X^*_{w_s} X_{w_s} : L^2(B(0,\delta)) \to L^2(B(0,2))$ is injective for $s = \rho$ and $\delta \ll 1$, by Theorem 5.4. By Theorem 5.7, this property is preserved for s such that $0 \leq \rho - s \ll 1$. Therefore, X_{w_s} is injective on $L^2(B(0,\delta))$ for such s as well.

Therefore, $f = 0$ on each disk

$$\pi_{s,\theta} \cap B(0,\rho), \quad \sqrt{\rho^2 - \delta^2} < s \leq \rho, \tag{5.4}$$

with $\delta = \delta(\theta)$. In particular, for any $\theta \in S^{n-1}$, we get $f = 0$ in a neighborhood of $x = \rho\theta$. Take a finite cover of the unit sphere with those neighborhoods to get that $f = 0$ in a neighborhood of $|x| = \rho$. This contradicts the choice of ρ however. □

COROLLARY 5.12 (injectivity for $n \geq 3$). *Let $w \in C^\infty(\bar{\Omega} \times S^{n-1})$ be nowhere vanishing and let $n \geq 3$. Then X_w is injective (and therefore, stable) on $L^2(\Omega)$.*

5.2.4. *Stability and instability in the support theorems.* ...

...

***** An inaccurately formulated theorem, fix it *****

THEOREM 5.13.

(a) Let $n = 2$ and assume that w is nowhere vanishing on points and directions determining lines not crossing $B(0,R)$. Then for any symbol $\chi \in S^1(\mathbf{R}^n)$ supported in $\mathcal{U}_2$ there exists $\psi \in C^\infty(\Sigma)$ supported away from the lines intersecting $\overline{B(0,R)}$ so that for any $l < s$, we have

$$\|\chi(x,D)f\|_{H^s} \le C_s\|X_w^*\psi X_w f\|_{H^{s+1}} + C_{l,s}\|f\|_{H^l}, \quad \forall f \in H_0^s(\Omega).$$

Moreover, this estimate fails for any symbol supported outside of $\bar{\mathcal{U}}_2$.

(b) Let $n \ge 3$. Then for every $\chi \in C_0^\infty(\mathbf{R}^n)$ supported away form the ball $B(0,R)$, there exists ψ as above so that

$$\|\chi f\|_{H^s} \le C_s\|X_w^*\psi X_w f\|_{H^{s+1}}, \quad \forall f \in H_0^s(\Omega).$$

***** Stability estimate in terms of X_w? Range for s? *****

PROOF. The proof of (a) follows from the micorlocal ellipticity of $X_w^* X_w$ in $\mathcal{U}_2$.

To prove (b), take χ as in (a) independent of ξ. Then ... □

It is worth noticing that the final step of the proof of Theorem 5.11 implies the following.

PROPOSITION 5.14. *Let $n \ge 3$. Let f be a smooth function supported in $\overline{B(0,R)}$, and let $|w| > 0$ for $|x| = R$. Let $R_1 > R$. Then there exists $\delta > 0$ and $C > 0$ so that for $R - \delta < s \le R$ and every unit ω,*

$$\|f\|_{L^2(\pi_{s,\omega})} \le C\Big\| \int_{|\theta|=1,\ \theta\perp\omega} \bar{w}(\cdot,\theta)X_w(\cdot,\theta)\,\mathrm{d}\theta \Big\|_{H^1(\pi_{s,\omega}\cap B(0,R_1))}$$

with $\pi_{s,\omega} := \{x|\ x\cdot\omega = s\}$. Moreover, C can be chosen the same under small variations of R.

This a stability estimate for the exterior problem locally, near the boundary of $B(0,R)$. Note that the smoothness of f is a stronger assumption that needed and the regularity of f can be deduced form that of $X_w f$.

6. Concluding Remarks

small domains: [**21**]
Weighted Radon [**14**].
Section 3.2: smooth cutoff.

CHAPTER V

The Geodesic X-ray transform

1. Introduction

1.1. Definition. Let (M,g) be a compact manifold with boundary ∂M. The geodesic X-ray transform of functions on M is given by

$$Xf(\gamma) = \int f(\gamma(t))\,\mathrm{d}t \tag{1.1}$$

defined for all maximal finite geodesics in the interior of M. If M is not strictly convex, the structure of those geodesics can be quite complex with geodesic hitting ∂M tangentially, possibly having contact of infinite order or a sequence of tangent points having a point of accumulation. In that case, we extend g to some neighborhood M_1 of M and it is more convenient to study Xf on geodesics with endpoints in $M_1\backslash$M, see [**37**].

If M is strictly convex, we can parameterize all such geodesics by initial points on ∂M and initial unit directions pointing into M, i.e., by elements in ∂_+SM, where

$$\partial_\pm SM = \{(x,v)|\; x\in\partial M,\; v\in T_xM,\; \pm\langle\nu,v\rangle\ge 0\},$$

compare with (II.1.11), where ν is the unit exterior normal at x. Then we write $Xf(x,v)$ for $(x,v)\in\partial_-SM$. The domain of X consists of those (x,v) for which $\gamma_{x,v}$ is of finite length. If all of then are finite, then (M,g) is called *non-trapping*. So, if ∂M is strictly convex and if M is non-trapping,

$$X: C(M)\ni f\to Xf\in C(\partial_-SM)$$

is a well defined continuous map.

The question we study is if Xf determines f uniquely; and stably, if uniqueness holds.

One can naturally define the X-ray transform of tensor fields of any order. For covector fields f (identified with vector ones by the metric), we have

$$Xf(\gamma) := \int\langle f\circ\gamma,\dot\gamma\rangle\mathrm{d}t = \int f_j(\gamma(t))\dot\gamma^j(t)\,\mathrm{d}t.$$

The second integral is a written in a bit incorrect form — it uses a coordinate representation assuming that a global chart exists (which is not always true but near a fixed geodesic, we can always choose a single chart). For contravariant tensors of any order m, we integrate $\langle f,\dot\gamma^m\rangle$. The most important examples are $m=0$ (functions), $m=1$ (1-forms) and $m=2$ (2-tensors). The case $m=4$ appears in linearization of problems arising in elasticity. We will see later that if $m\ge 1$, there is a natural obstruction to uniqueness of recovery of f that can be explained by the Fundamental Theorem of Calculus. We want to recover f then modulo that obstruction.

1.2. Motivation. The main motivation for studying the $m=0$ and the $m=2$ cases is that X appears as a linearization of the boundary rigidity and the lens rigidity problems. Actually, this is the reason the authors started working on the geodesic X-ray transform.

Let $\mathrm{d}_g(x,y)$ be the distance function on M. The *boundary rigidity* problem is the following. Can we recover (M,g) from the boundary distance function known on $\partial M\times\partial M$? Every diffeomorphism ψ on M acting as the identity on the boundary would not change the data; and there is no other obvious obstruction for all metrics (but there is for metrics not so close to flat ones, for example, see below). Therefore, we should expect to recover the manifold up to action of such a diffeomorphism. Let us fix the manifold M (its topology) but vary the metric. Then we have the following.

DEFINITION 1.1. *The metric g is called boundary rigid, if for any other metric $\tilde g$ with $\mathrm{d}_g=\mathrm{d}_{\tilde g}$ on $\partial M\times\partial M$, there exists a diffeomorphism ψ on M with $\psi|_{\partial M}=\mathrm{Id}$ and $\psi^*\tilde g=g$.*

There are obvious counter examples to boundary rigidity. If we have a metric in a bounded domain, which is very "slow" somewhere in the interior (g_{ij} small), then all minimizing curves will avoid that region and a small change of g there will not affect the boundary distance function. To exclude such examples, one usually requires that the metric be simple: every two points can be connected by a unique minimizing geodesics, smoothly depending on those points, and that the boundary is strictly convex.

Lens rigidity is defined similarly but the data now is the *lens relation* L and the *travel time* ℓ defined below. For every (x,v) in the interior $\partial_-SM^{\mathrm{int}}$ of ∂_-SM (i.e., v not tangent to ∂M), let $L(x,v)=(y,w)$ be point where the geodesic $\gamma_{x,v}$ hits ∂M for the first (positive) time, if it is non-trapping, and let w be the (unit) direction at that point. Let $\ell(x,v)$ be the length of that geodesic (the travel time). Then

$$L:\partial_-SM^{\mathrm{int}}\to\partial_+SM,\quad \ell:\partial_-SM^{\mathrm{int}}\to[0,\infty)$$

is called the lens data, possibly defined on a subset of $\partial_-SM^{\mathrm{int}}$ if (M,g) is trapping. One may include the boundary of ∂_-SM consisting of tangent directions in the domain of definition of L and ℓ, see also [**38**]. Then we define lens rigid metrics as above but using L and ℓ as the data.

Assume that (M,g) is simple. Fix x, Y in M and let $v=\exp_x^{-1}y$. We will linearize $d_g(x,y)$ with respect to the metric. Let $g^\varepsilon=g+\varepsilon f$ be family of metrics (we work in a fixed coordinate system near the geodesic $\gamma(t)=\exp_x(tv)$, $0\le t\le 1$). Here, f is a symmetric 2-tensor contravariant field. Then g^ε is simple for $|\varepsilon|\ll 1$ and x and y are connected by a unique minimizing geodesic $\gamma_\varepsilon(t)$, $0\le t\le 1$. Then

$$\mathrm{d}_{g^\varepsilon}(x,y)=\int_0^1\sqrt{g^\varepsilon_{ij}\dot\gamma^i_\varepsilon(t)\dot\gamma^j_\varepsilon(t)}\,\mathrm{d}t$$

because the integrand is independent on t and equal to the left-hand side. Compute $\mathrm{d}/\mathrm{d}\varepsilon$ at $\varepsilon=0$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mathrm{d}_{g^\varepsilon}(x,y)&=\frac12\int_0^1 f_{ij}\dot\gamma^i(t)\dot\gamma^j(t)\,\mathrm{d}t+\frac12\int_0^1 g_{ij}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\left[\dot\gamma^i_\varepsilon(t)\dot\gamma^j_\varepsilon(t)\right]\,\mathrm{d}t\\&=\frac12\int_0^1 f_{ij}\dot\gamma^i(t)\dot\gamma^j(t)\,\mathrm{d}t=\frac12 Xf(\gamma).\end{aligned}$$

The reason for the second integral to vanish is the fact that the geodesics for g are locally minimizing. In $Xf(\gamma)$ above the parameter t along γ is not unit speed. To reparameterize, we can set $s = t\,\mathrm{dist}(x,y)$. Therefore, the linearization of d_g is $\frac{1}{2}Xf(x,v)/\,\mathrm{dist}(x,y)$, where $v = \exp_x^{-1} y/|\exp_x^{-1} y|$. This motivates the $m = 2$ transform.

If we want to recover a "sound speed", i.e., then the metric is $c^{-2}\mathrm{d}x^2$. More generally, let us assume that there is a know background metric g^0 and we consider a class of conformal metric $g = c^{-2}g^0$. Let us linearize as above about a fixed c_0, by writing $c^{-2} = c_0^{-2}(1+\varepsilon h)$. Then the integrand becomes $hc_0^{-2}g^0_{ij}\dot\gamma^i\dot\gamma^j = h$ because $c_0^{-2}g^0_{ij}\dot\gamma^i\dot\gamma^j = 1$ for unit speed geodesics. Then we get X acting on functions ($m = 0$).

The $m = 1$ can be motivated by ultrasonic imaging of moving fluids of variable speed c, as we did in Section II.7.1. Then the metric would be $c^{-2}\mathrm{d}x^2$. Another application is recovery of a magnetic potential α (an one form) from boundary data related to propagation of charged particles in an inhomogeneous medium described by a metric g. This is the equivalent of the boundary and the lens rigidity problems for charged particles, and we refer to [**12**] for more details. If the metric is unknown as well, in linearization, we get a difference $X_2f - X_1\alpha$ of X acting on a two tensor f_{ij} and X acting on an the form α_j. We can separate the two terms if we let the particles travel in the opposite direction; then we get the sum, compare with Norton's arguments in Section II.7.1. The $m = 1$ transform also appears in hyperbolic inverse problems of recovery all coefficients of a general self-adjoint second order differential operator

$$P = \frac{1}{\sqrt{\det g}}\left(\frac{1}{\mathrm{i}}\frac{\partial}{\partial x^i} + a_i\right) g^{ij}\sqrt{\det g}\left(\frac{1}{\mathrm{i}}\frac{\partial}{\partial x^j} + a_j\right) + q.$$

From the hyperbolic Dirichlet-to-Neumann map we can extract in a stable way the lens relation for g, the X-ray transform of $a_i\mathrm{d}x^i$ and that of q. Linearizing the lens relation, we get the X-ray transform of a two tensor $f_{ij}\mathrm{d}x^i\mathrm{d}x^j$, which is the perturbation of the metric. So we have the $m = 2$, $m = 1$ and the $m = 0$ cases together, and inverting these transforms (up their natural kernels when $m \geq 1$) allowed Montalto in [**23**] to show that one can recover in a stable way g, a_j and q up to a gauge transformation, under simplicity conditions for g and some generic assumptions for the recovery of g.

1.3. Microlocal considerations. Let us see first what we could expect not by actually proving anything at this point but by trying to develop a reasonable intuition. We have the integral geometry analog of variable coefficients PDEs. We can hardly expect easy uniqueness proofs and explicit inversion as in the Euclidean case. Fourier transform will not be so useful anymore but microlocal analysis should be since the latter is the natural extension of Fourier transform methods to variable coefficients problems. We already saw the power of microlocal methods for the weighted Euclidean X-ray transform. How will the new geometry change the analysis? Let us consider the $m = 0$ case only.

Can we expect the normal operator $N = X^*X$ to be a ΨDO (elliptic) again? For "small" time t, every geodesic $\gamma_{x,v}(t)$ looks like a line because of the Taylor expansion $\gamma_{x,v} = x + tv + O(t^2)$. This makes it plausible that the Schwartz kernel of N has a singularity of the kind $\sim |x-y|^{1-n}$ which is not enough of course, but a strong suggestion that N might be a ΨDO (of order -1). If the metric and the weight, assuming that there is a non-trivial weight, are analytic, we can expect the analytic microlocal arguments to work as well. We can expect to recover singularities conormal to the geodesics in our open family of geodesics over which we integrate. On the other hand, the arguments we used to prove support theorems for the weighted X_w for non-analytic weights, see Theorem IV.5.11, may not work since we do not, in general, have totally geodesic surfaces.

Those arguments can lead to wrong expectations if we fail to realize that *they are local.* They are fine near a fixed point, i.e., to get an idea how the Schwartz kernel $\mathcal{N}(x, y)$ would look like when y is close to x. But in order to claim that the latter is a kernel of a ΨDO we need to know, see Section B.5, aside from a detailed analysis near the diagonal $x = y$, that there are no other singularities on $M \times M$! Recall that in the derivation of the formula for the kernel of $X'X$ in the Euclidean case, see Proposition II.2.2, we made the change of variables $(s, \theta) \mapsto z = x + s\theta$, for x fixed, i.e., we passed from polar to Cartesian coordinates. In the geodesic case, the equivalent to that would be $z = \exp_x(t\theta)$. On T_xM, (t, θ) are polar coordinates for $v := t\theta$. The map $v \mapsto \exp_x v$ however is a local diffeomorphism near some v if and only if x and $y := \exp_x v$ are conjugate along the geodesic $t \mapsto \exp_x(\theta)$. This shows that $\mathcal{N}(x, y)$ would have singularities at pairs of conjugate points (along some geodesic). Therefore, it cannot be a ΨDO anymore.

This underlines an essential difference between the geodesic case and the Euclidean one. Geometry matters and conjugate points create a new phenomenon. That is why most but not all works so far assume that there are no conjugate points.

The next logical question is the following. We know that if there are conjugate points, $N = X^*X$ is not a ΨDO anymore. Does it mean that we cannot resolve the singularities? The answer to this question is not straightforward: sometimes we can, sometimes we cannot but we do not have a full understanding yet.

One case where we can give an immediate answer is the following. Let $n = 2$ and let γ be a geodesic with a pair of conjugate points like on the sphere — all rays issued from a single point x and some open cone focus at some other point y, see Figure B.D.1. Assume that this is a flattened piece of the sphere (wit the sphere metric, of course). Assume we want to recover the singularity at x normal to the "vertical geodesic" γ_0. The only geodesic that can possibly be used for that
kernel thm
is the same vertical one and a small neighborhood of it, by ... . On the other hand, every such geodesic, by the symmetry, carries the same information about the point y as well. If we place a distribution with a small support near x, and its antipodal image with the opposite sign at y, all integrals along geodesics close to γ_0 will be zero! In particular, those distributions can have the singularities we want to recover; conormal to γ_0 at x and y. So those would be invisible singularities. We will see later that in higher dimensions, the picture changes.

2. The Energy Method

The energy method started with a work by Mukhometov [**24**]. He proved that for a general family of geodesic-like curves, with a unit Euclidean speed and no conjugate points, the corresponding X-ray transform is invertible. After that, the method was extended and generalized to other cases which we review below.

Let us consider a well known example. Assume that we want to prove uniqueness for the elliptic boundary value problem

$$\Delta u = 0 \text{ in } \Omega; \quad u|_{\partial\Omega} = f, \tag{2.1}$$

where Ω is a bouded open domain, and f is regular enough. Multiply the PDE by $\bar{u}$ and integrate by parts to get

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \bar{f} \mathrm{d}S, \tag{2.2}$$

where ν is the exterior unit normal. This yields, for every s, the following estimate

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}x \le \|u_\nu\|_{H^{-s}(\partial\Omega)} \|f\|_{H^s(\partial\Omega)}. \tag{2.3}$$

We get uniqueness (assuming existence of a solution, regular enough up to the boundary) right away: $f = 0$ implies $u = \text{const}$. By the boundary condition, that constant must be zero. Estimate (2.3) is a conditional one, see ... because u appears on the right as well. The sharp well known estimate

$$\|u\|_{H^1(\Omega)} \le C\|f\|_{H^{1/2}(\partial\Omega)}$$

does not follow from (2.3) without much more detailed analysis, at least.

2.1. The energy method for the Euclidean X-ray transform X. To demonstrate the energy method, we will prove again that the Euclidean X-ray transform X is injective in a smooth bounded strictly convex domain Ω, with a non-sharp stability estimate.

Set

$$v(\theta) = (\cos\theta, \sin\theta).$$

As we saw in Chapter II, X can be defined through the solution u of the transport equation (II.1.14) and (II.1.15). If u solves

$$v(\theta) \cdot \nabla_x u = f(x), \quad u|_{\partial_- S\Omega} = 0, \tag{2.4}$$

then

$$Xf = u|_{\partial_+ S\Omega}. \tag{2.5}$$

In analogy to the Dirichlet problem for the Laplacian, we would like to have a homogeneous PDE, and then to prove that $Xf = 0$ implies $u = 0$; therefore, $f = 0$. So far, we have two unknowns, u and f. Differentiate with respect to θ to annihilate $f(x)$:

$$\partial_\theta v(\theta) \cdot \nabla_x u = 0. \tag{2.6}$$

Unfortunately, the operator on the left is not non-negative or non-positive. The non-negativity of the quadratic form (2.2) was crucial for the uniqueness argument above. So the analogy with the Laplace equation ends here. Multiplying by u and integrating by parts would not help much.

Set $G := v(\theta) \cdot \nabla_x$. This is the generator of the line flow in the phase space $\Omega \times S^1$. The PDE we have so far is $\partial_\theta G u = 0$. The main idea of the energy method is to multiply (2.6) not by u but by Pu with some cleverly chosen operator P so that an integration by parts (the divergence theorem) would produce a quadratic form with a fixed sign. One such successful choice turns out to be the operator $G_\perp := v^\perp(\theta) \cdot \nabla_x$. Note that $v_\perp(\theta) = \partial_\theta v(\theta)$; therefore, we can think of $G_\perp$ as being $\partial_\theta G$.

Multiply (2.6) by $G_\perp u$ to get

$$(G_\perp u)(\partial_\theta G u) = 0. \tag{2.7}$$

To integrate by parts, we need to put this PDE into a form close to a divergent one.

LEMMA 2.1.

$$2(G_\perp u)(\partial_\theta G u) = |\nabla_x u|^2 + \partial_{x^1}(u_{x^2}u_\theta) - \partial_{x^2}(u_{x^1}u_\theta) + \partial_\theta[(G_\perp u)(Gu)]. \tag{2.8}$$

PROOF. Notice first that for any two smooth functions u and v,

$$u_{x^2}v_{x^1} - u_{x^1}v_{x^2} = u_{y^2}v_{y^1} - u_{y^1}v_{y^2}$$

for any other coordinate system (y^1, y^2) obtained from the original one by a rotation. Therefore,

$$\partial_{x^1}(u_{x^2}u_\theta) - \partial_{x^2}(u_{x^1}u_\theta) = u_{x^2}u_{\theta x^1} - u_{x^1}u_{\theta x^2} = (Gu)G_\perp u_\theta - (G_\perp u)Gu_\theta. \tag{2.9}$$

For the last term in (2.8), we get

$$\partial_\theta[(G_\perp u)(Gu)] = -|Gu|^2 + (G_\perp u_\theta)Gu + (G_\perp u)\partial_\theta Gu. \tag{2.10}$$

Therefore, for the difference of the most left term in (2.8) and the most right one, we get

$$\begin{aligned} 2(G_\perp u)(\partial_\theta G u) - &\partial_\theta[(G_\perp u)(Gu)] \\ &= |Gu|^2 - (G_\perp u_\theta)Gu + (G_\perp u)\partial_\theta Gu \\ &= |Gu|^2 - (G_\perp u_\theta)Gu + |G_\perp u|^2 + (G_\perp u)Gu_\theta. \end{aligned} \tag{2.11}$$

Notice now that $|Gu|^2 + |G_\perp u|^2 = |\nabla_x u|^2$ because the length of the gradient at any point does not change if computed in a rotated coordinate system.

Therefore,

$$2(G_\perp u)(\partial_\theta G u) - \partial_\theta[(G_\perp u)(Gu)] = |\nabla_x u|^2 - (G_\perp u_\theta)Gu + (G_\perp u)Gu_\theta.$$

This, combined with (2.9) completes the proof. □

Let u solve (2.4), and therefore, (2.7). Use the lemma and apply the divergence theorem in $S^1 \times \Omega$. The boundary of the latter consists of $S^1 \times \partial\Omega$, and we get

$$\int_0^{2\pi}\int_\Omega |\nabla_x u|^2 \,\mathrm{d}x\,\mathrm{d}\theta = \int_0^{2\pi}\int_{\partial\Omega} (\nu_2 u_{x^1} - \nu_1 u_{x^2})\, u_\theta \,\mathrm{d}s\,\mathrm{d}\theta,$$

where ν is the unit exterior normal to Ω, and $\mathrm{d}s$ is the arc-length measure on $\partial\Omega$. The expression in the parentheses on the right is actually the derivative of u (as a function of x) in the direction of $-\nu^\perp$, i.e., (-1) times the derivative with respect to arclength parameter s, positively oriented (clockwise).

Therefore,

$$\int_0^{2\pi}\int_\Omega |\nabla_x u|^2 \,\mathrm{d}x\,\mathrm{d}\theta = -\int_0^{2\pi}\int_0^S u_s u_\theta \,\mathrm{d}s\,\mathrm{d}\theta, \tag{2.12}$$

with S being the length of $\partial\Omega$. We can declare cucess now because the form on the left is non-negative.

This equality already proves injectivity, as it is easy to check. Indeed, $Xf = 0$ implies $u_s = u_\theta = 0$ in the integral on the right (taken on the boundary!), and then u must be independent of x. Then $f = 0$ by (2.4).

To obtain a stability estimate, let us extend the domain of Xf from $\partial_+ S\Omega$ to the whole $\partial S\Omega$ by taking a zero extension to $\partial_- S\Omega$. This makes sense — lines exiting at points in $\partial_- S\Omega$ (which have incoming directions only) are just points, and integrals over them are zero. Recall that here, we parameterize lines by their exit points. We could also parameterize them by their entrance points, see (II.1.14) and (II.1.15) on one hand, vs. (II.1.16) and (II.1.17) on the other. In the later case, we choose a zero extension to $\partial_+ S\Omega$. By (2.4), $|f(x)| \le |\nabla_x u|$ pointwise. We assume at this point that $f \in C_0^\infty(\Omega)$ to avoid technical difficulties. Indeed, the boundaries of $\partial_\pm S\Omega$ consist of points on $\partial\Omega$ and unit tangent vectors at them, i.e., it coincides with $S\partial\Omega$. For $f \in C_0^\infty(\Omega)$, Xf vanishes near $S\partial\Omega$, therefore, the extension is smooth.

Then (2.12) implies the following.

PROPOSITION 2.2. *Let the Euclidean X-ray transform X be parameterized by $\partial\Omega \times S^1$ as explained above. Then we have the following stability estimate*

$$\|f\|_{L^2(\Omega)} \le \frac{1}{\sqrt{2}} \|Xf\|_{H^1(\partial\Omega\times S^1)}, \quad \forall f \in C_0^\infty(\Omega). \tag{2.13}$$

The estimate can be extended to functions f for which the norm on the right is finte but we will not pursue this. A quick comparison with Theorem II.3.2 shows that the estimate above is not sharp; there is a loss of an $1/2$ derivative.

2.2. Mukhometov's result for a general family in two dimensions. To describe Mukhometov's result, we want to define a general family of geodesic-like curves first. We work in a bounded smooth domain Ω in the Euclidean plane $\mathbf{R}^2$. We want from every point x and unit direction v to have a unique curve $\gamma_{x,v}(t)$ with $\gamma_{x,v}(0) = x$ and $\dot\gamma_{x,v}(0) = v$. A major assumption is that $\dot\gamma_{x,v}$ remains unit along the curve. We also want the curve issued from $(\gamma_{x,v}(t), \dot\gamma_{x,v}(t))$, which must exist by assumption, to be the same curve (the group property) Finally, we want a smooth dependence of (x, v) and t. Then $\ddot\gamma_{x,v}(t)$ will be a smooth function as well. Together with the group property, this actually implies that $\gamma_{x,v}(t)$ can be defined as the solution of the Newton's type of ODE

$$\ddot\gamma = \beta(\gamma, \dot\gamma), \quad (\gamma, \dot\gamma)|_{t=0} = (x, v) \in \bar\Omega \times S^1,$$

with $\beta : \bar\Omega \times S^1 \to \mathbf{R}^2$ is given. The requirement that $|\dot\gamma(t)|$ remains unit implies

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} |\dot\gamma|^2/2 = \ddot\gamma \cdot \dot\gamma = \beta(\gamma, \dot\gamma) \cdot \dot\gamma.$$

Introduce the notation $v_\perp = (-v^2, v^1)$ for the rotation by $+\pi/2$. We get that $\beta(x, v)$ must be collinear with $v^\perp$; therefore,

$$\beta(x, v) = \alpha(x, v) v_\perp.$$

On the other hand, any such ODE preserves the length along the curves. Therefore, our family is actually described by the ODE

$$\ddot\gamma = \alpha(\gamma, \dot\gamma)\dot\gamma_\perp.$$

One can interpret this as a Newton type of equation with a force always perpendicular to the trajectory. If α is independent of $\dot\gamma$, an example of such a dynamical system is a charged particle in the plane propagating in a magnetic field perpendicular to the plane. Then the right hand side of the ODE is the Lorentz force.

Writing this as a system $\dot{x} = v$, $\dot{v} = \alpha(x,v)v_\perp$ in $T\bar{\Omega}$, we get that this dynamical system consists of the integral curves of the following vector field

$$H := v^j \frac{\partial}{\partial x^j} + \alpha(x,v)\left(v^1 \frac{\partial}{\partial v^2} - v^2 \frac{\partial}{\partial v^1}\right).$$

As we established that already, those curves stay on the surface $|v|^2 = 1$ (the unit sphere tangent bundle). Indeed, H is tangent to it because $H|v|^2 = 0$. We denote such curves below (projections of integral curves of H on the base $\bar{\Omega}$) by γ. We impose additional assumptions below.

Note that the unit speed requirement is not a restrictive assumption since we can always re-parameterize the curves with their arc-length. There is an important "hidden" detail in that argument however. A re-parameterization would introduce a non-trivial weight in the definition of X. For the proof of the result below, it is very important that the weight is constant, when the t parameter is the Euclidean arc-length. We saw already in Chapter IV, Section 5 that a constant weight is not a requirement for injectivity even if the curves are lines; but that there are weights for which injectivity fails (Boman's counter examples).

We parameterize the boundary $\partial\Omega$ by its arc-length as follow

$$\partial\Omega = \left\{ z(s) = (z^1(s), z^2(s)) |\ 0 \le s \le S \right\},$$

where S is the length of $\partial\Omega$. We can think of $z(s)$ as a function with period S.

We now complete the definition of the family Γ of curves γ, imposing simplicity conditions as follows.

(i) Every two points in $\bar{\Omega}$ are joined by a unique curve γ.
(ii) The endpoints of each γ are on $\partial\Omega$ and all other points are in the interior. The lengths of the curves in Γ are uniformly bounded.

Those two conditions can be interpreted as a non-trapping condition plus a convexity one. Non-trapping means that each curve $\gamma_{x,v}(t)$ reaches $\partial\Omega$ for finite positive and negative t.

We then define the associated X-ray transform as in (1.1) for all $\gamma \in \Gamma$. The family Γ has the natural structure of a manifold given by $\partial\Omega \times \partial\Omega$ since every $\gamma \in \Gamma$ can be parameterized its endpoints $z(s_1)$ and $z(s_2)$, and therefore by (s_1, s_2). We use the notation $\gamma_{[s_1,s_2]}$ in this parameterization. We also assume sufficient regularity, and in fact, C^3 suffices. We refer to [**24, 33**] for more details. With that reparameterization, we write

$$Xf(s_1, s_2) = \int_{\gamma_{[s_1,s_2]}} f \,\mathrm{d}t$$

THEOREM 2.3. *Under the above conditions, the transform X is injective on $C^2(\bar{\Omega})$ and satisfies the stability estimate*

$$\|f\|_{L^2(\Omega)} \le C \left\|\partial_{s_1} Xf\right\|_{L^2([0,S]\times[0,S])}, \quad \forall f \in C^2(\bar{\Omega}).$$

SKETCH OF THE PROOF. We skip some of the regularity considerations near the boundary to simplify the exposition. The complete proof can be found in [**24, 33**]. The reader can just assume that supp f is contained in the (open) Ω.

Let $\tilde{\gamma}_{s,x}(t)$ be the oriented segment of the curve γ connecting $z(s) \in \partial\Omega$ and $x \in \bar{\Omega}$. Set

$$u(s,x) = \int_{\tilde{\gamma}_{s,x}} f \,\mathrm{d}t, \quad s \in [0,S],\ x \in \bar{\Omega}.$$

Let $\theta(s,x)$ be the argument (the polar angle) of $v(s,x) := \dot{\tilde{\gamma}}_{s,x}$ at its endpoint x, i.e.,

$$v(s,x) = (\cos\theta(s,x), \sin(s,x))$$

Then one verifies that u solves the following transport equation

$$v(s,x)\cdot\nabla_x u(s,x) = f(x), \tag{2.14}$$

with boundary condition

$$u(s_1, z(s_2)) = Xf(s_1,s_2). \tag{2.15}$$

The PDE (2.14) just reflects the fact that the derivative of u along the extension of $\tilde{\gamma}_{s,x}$ past its endpoint x is $f(x)$. It is very important for this argument that along that extension, s remains unchanged. This is a version of the transport equation (2.4) but the curves are parameterized differently now.

To make the PDE homogeneous, we use the fact that $f(x)$ is independent of s. Then we can differentiate with respect to s to get

$$Lu := \frac{\mathrm{d}}{\mathrm{d}s}\left[v(s,x)\cdot\nabla_x u(s,x)\right] = 0. \tag{2.16}$$

As above, set $G := v(s,x)\cdot\nabla_x$. Set also $G_\perp := v_\perp(s,x)\cdot\nabla_x$. Multiply (2.16) by the integrating factor $G_\perp u$ and put the resulting expression on a divergent form using the following identity, compare with Lemma 2.1:

$$2(G_\perp u)\partial_s Gu = \frac{\partial\theta}{\partial t}|\nabla_x u|^2 + \partial_{x^1}(u_{x^2}u_s) - \partial_{x^2}(u_{x^1}u_s) + \frac{\partial}{\partial s}[(Gu)(G_\perp u)].$$

Then we apply the divergence theorem as above. □

***** More references about the method. 2D vs. higher dimensions *****

3. Formulation of the problem and preliminaries

3.1. Non-trapping and simple manifolds.

Definition 3.1 (non-trapping manifold). *Let (M,g) be a compact Riemannian manifold with boundary ∂M. We call (M,g) non-trapping, if all maximal geodesics in M are of finite length.*

If (M,g) is non-trapping, then X given by (1.1) is well defined on $C(M)$ over the set of all maximal geodesics. One way to have a convenient parameterization of those geodesics is to require strict convexity of the boundary. This means that the second fundamental form on ∂M is strictly positive, see Section D.8.1. It also means that if we extend g outside M in a smooth way, then any geodesic $\gamma(t)$ tangent to ∂M at a boundary point $\gamma(0)$ has only that point in common with M (close to it), and $\mathrm{dist}(\gamma(t),\partial M)\sim t^2$. If (M,g) is non-trapping and has a strictly convex boundary, then we can use $\partial_- SM$ as a parameterization of all (non-zero length) maximal geodesics through M. This gives that set a structure of a manifold. Its boundary (not included in it by definition) consist of $S\partial M$.

In [**32**], non-trapping manifolds with a strictly convex boundary are called Compact Dissipative Riemannian Manifolds (CDRM). Notice that the no-conjugate points assumption below is not required for CDRM.

If (M,g) is non-trapping, then for every $v \in S_xM$, there exist "times" $\tau_-(x,v) \leq 0$ and $\tau_+(x,v) \geq 0$ so that

$$\gamma_{x,v}(\tau_-(x,v)) \in \partial M, \quad \gamma_{x,v}(\tau_+(x,v)) \in \partial M. \tag{3.1}$$

They are defined as the maximal interval $[\tau_-(x,v), \tau_+(x,v)]$ over which $\gamma_{x,v}$ is defined. Notice that $\gamma_{x,v}(t)$ may have other contacts with ∂M in this interval, including whole segments or infinitely many discrete points. We also allow that interval to be a point: $[0,0]$, which happens, for example, at points where ∂M is strictly convex. Clearly,

$$\tau_-(x,-v) = -\tau_+(x,v). \tag{3.2}$$

The functions $\tau_\pm$ may not be even continuous for non-convex boundaries. For non-trapping manifolds, one can define the geodesic X-ray transform X as a map parameterized by ∂_-SM:

$$Xf(x,v) = \int_0^{\tau_+(x,v)} \langle f(\gamma_{x,v}(t)), \dot\gamma^m \rangle \,\mathrm{d}t, \quad (x,v) \in \partial_-SM \tag{3.3}$$

where f is a symmetric tensor field of order m. If in addition, ∂M is strictly convex, then ∂_-SM has the structure of a compact manifold with boundary. In that case, we actually define $\tau_\pm$ uniquely by the conditions

$$\begin{aligned} &(\gamma_{x,v}(\tau_-(x,v)), \dot\gamma_{x,v}(\tau_-(x,v))) \in \partial_-SM, \\ &(\gamma_{x,v}(\tau_+(x,v)), \dot\gamma_{x,v}(\tau_+(x,v))) \in \partial_+SM. \end{aligned} \tag{3.4}$$

Obviously, if ∂M is strictly convex, $\tau_\pm$ are smooth in SM away from the boundary of $\partial_\pm SM = S\partial M$ because then the corresponding geodesic hits ∂M transversely and we can use the explicit function theorem. They are continuous on SM.

We will study *simple manifolds* first.

Definition 3.2 (simple manifold). *Let (M,g) be a compact Riemannian manifold with boundary ∂M. We call (M,g) simple, if*

(i) *∂M is strictly convex;*
(ii) *Every two points x and y in M can be connected by a unique geodesic smoothly depending on x and y. The later means that $\exp_x^{-1} y$ is a well defined map, smooth on $M \times M$.*

Proposition 3.3. *Let (M,g) be simple. Then*
(a) M is diffeomorphic to a closed ball.
(b) M is non-trapping.
(c) There are no conjugate points in M; i.e., every maximal geodesic in M is free of conjugate points.

Proof. Let x be an interior point. Then $\exp_x^{-1}(M) \subset T_XM$ contains some neighborhood of 0; in other words, $\pm\tau_\pm(x,v) > 0$ for every $v \in S_xM$. Next, $\exp_x^{-1}(M)$ is star-like, i.e., if v belongs to it, then so does sv for all $s \in [0,1]$. This means that

$$\exp_x^{-1}(M) = \{sv \in T_xM |\, v \in S_xM,\, \tau_-(x,v) \leq s \leq \tau_+(x,v)\}.$$

Clearly, this set is diffeomorphic to a ball. Since $\exp_x$ maps it to M diffeomorphically, then so is M. This proves (a) and (b).

Part (c) is a direct consequence of the assumption that $\exp_x$ is a diffeomorphism. □

The main questions we want to answer about X is inevrtibility, stability and recovery of singularities, and the same questions when X is restricted over an open subset of geodesics.

3.2. Functional spaces. We will work in Sobolev spaces of tensor fields on M. For functions, $L^2(M)$ is defined, in an invariant way, by the norm

$$\|f\|^2_{L^2(M)} = \int_M |f|^2 \,\mathrm{d}\,\mathrm{Vol}\,.$$

The associated inner product will be denoted by $(f,g)_{L^2(M)}$. If f is a symmetric tensor field of order m, we set similarly

$$(f,h)_{L^2(M)} = \int_M f_{i_1\dots i_m}\bar{h}^{i_1\dots i_m}\,\mathrm{d}\,\mathrm{Vol}\,. \tag{3.5}$$

Notice the raising of the indices of $\bar{h}$. If $h=f$, the the integrand is the natural magnitude $|f(x)|$ squared of $f(x)$ pointwise. Sobolev spaces of integer order $k = 0,1,\dots$ are defined by

$$\|f\|^2_{H^k(M)} = \int_M |\nabla\dots\nabla f|^2\,\mathrm{d}\,\mathrm{Vol},$$

where the covariant derivative is taken k times (making the resulting tensor field a field of order $m+k$). The Banach spaces C^k are defined similarly by using covariant derivatives. Since in local coordinates, covariant erivatives are just partial ones plus zero order terms; the standard, non-invariant H^k and C^k norms are equivalent to the invariant ones above. The constants in the estimates showing their equivalence however depend on the coordinate system. This is only a danger if we use infinitely many non-invariant norms. If in a proof, we use finitely many charts, and in each one we work with the invariant norms, at the end, we still get a norm equivalent to the invariant one.

One natural choice of a measure on $\partial_\pm SM$ is $\mathrm{d}S(x)\mathrm{d}\sigma_x(v)$, see Section D.4 and Section D.8.2. We saw however in the Euclidean case that such a measure is not invariant if you choose a different surface transversal to an open set of geodesics and parameterize the same way; for an invariant definition we need the factor $|\nu(x)\cdot\theta|$, see (II.1.12) and Corollary II.1.2.

3.3. Potential fields and potential-solenoidal decomposition. We derive here the Riemannian analog of the potential-solenoidal decomposition which we already encountered in Chapter II, Sections 7 and 8 in the Euclidean space.

An obvious counter-example to uniqueness when $m=1$ comes from the fundamental theorem of calculus. Let $v\in C^1_0(M)$ be a function (vanishing on ∂M). Then $X(dv)=0$ because

$$X(dv)(\gamma) = \int v_{x^j}(\gamma(t))\dot{\gamma}^j(t)\,\mathrm{d}t = \int_{\tau_-}^{\tau_+}\frac{\mathrm{d}}{\mathrm{d}t}v(\gamma(t))\,\mathrm{d}t = v(\tau_+)-v(\tau_-)=0,$$

where $\tau_\pm = \tau_\pm(\gamma)$ are the endpoints of γ. This observation extends to tensor fields of any positive order $m\ge 1$, as we show below. Notice that we did not use the fact that γ is a geodesic there, or even that there is a Riemannian structure in M; in fact dv is independent of the metric. Therefore, such dv would be in the kernel of X even if we integrate over other families of curves connecting boundary points.

Given a contravariant tensor field v of order m, written in local coordinates as $v=\{v_{i_1,\dots,i_m}\}$, we define the *symmetric differential* dv of v by $dv=\sigma\nabla v$, where σ

is the symmetrization operator, i.e., the average over all permutation of the indices of the $(m+1)$-tensor ∇v. For $m=1$, we have

$$(dv)_{ij} = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right).$$

Using the rules of covariant differentiation, it is straightforward to show the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v(\gamma(t)), \dot\gamma^m(t)\rangle = \langle dv(\gamma(t)), \dot\gamma^{m+1}(t)\rangle.$$

Therefore, if $v=0$ on ∂M, we have $X(dv)=0$. If $m=0$, v is a function and dv is just its differential. For $m\ge 1$, dv depends on the metric and the fact that γ was a geodesics (hence, $D_t^2\gamma = 0$) was used above.

Definition 3.4 (Potential fields). *Tensor fields of order $m\ge 1$ of the kind dv with $v\in H_0^1(M)$ are called potential fields.*

We will see below that the potential fields form a closed subspace of the tensorial $L^2(M)$ and that they belong to the kernel of X even when v has the regularity stated above (not necessarily in C_0^1).

Since potential filed belong to $\operatorname{Ker} X$ and there are no other obvious fields in $\operatorname{Ker} X$, it is natural to conjecture that $\operatorname{Ker} X$ coincides with the potential fields, when $m\ge 1$, and that it is trivial when $m=0$. We call this property *s-injectivity* of X. Naturally, for $m=0$, we just call it injectivity.

Definition 3.5. *X is called s-injective if $Xf=0$ for $f\in L^2(M)$ implies $f=0$ when $m=0$, and that f is potential when $m\ge 1$; i.e., that $f=dv$ for some $v\in H_0^1(M)$.*

The a priori assumption $f\in L^2(M)$ is not needed when (M,g) is simple, as we will see below even if f is a distribution supported in M, by ellipticity, we can prove this for $f+dv$ with some suitable v supported in M. For this, we extend (M,g) to a larger and still simple $(\tilde M, \tilde g)$.

Proving s-injectivity of X is open in general, and harder for $m\ge 2$. We prove and review below some known results.

One way to deal with the non-uniqueness due to potential fields when $m\ge 1$ is to try to prove that X is injective on the orthogonal complement of those fields. The "natural" Hilbert space is the $L^2(M)$ space of symmetric tensors introduced in Section 3.2 above. Then the symmetric m-field f is orthogonal to all potential ones if and only if

$$(f, dv) = 0 \quad \forall v\in H_0^1(M).$$

This is equivalent to $\delta f = 0$, where $\delta = -d^*$ is the *divergence* operator sending symmetric tensor fields of order $m\ge 1$ into symmetric tensor fields of order $m-1$ given in local coordinates by

$$(\delta f)_{i_1\dots i_{m-1}} = \nabla^{i_m} f_{i_1\dots i_{m-1} i_m}$$

where $\nabla^{i_m} f_{i_1\dots i_{m-1}i_m} := g^{i_m i_{m+1}}\nabla_{i_{m+1}} f_{i_1\dots i_m}$ If $f = f_j \mathrm{d}x^j$ is an one form, then

$$\delta f = \nabla^j f_j = g^{ij}\left(\partial_i f_j - \Gamma^k_{ij} f_k\right).$$

The divergence of symmetric two-tensor fields is

$$(\delta f)_j = \nabla^i f_{ij} = g^{ik}\nabla_k f_{ij}$$

and a coordinate representation of $\nabla_k f_{ij}$ can be found in equation (6.2) of Appendix D.

THEOREM 3.6. *In the $L^2(M)$ space of symmetric m-tensors, $m \geq 1$, there exist a unique choice of orthogonal projections $\mathcal{P}$ and $\mathcal{S}$, $\mathcal{P}+\mathcal{S}=\mathrm{Id}$, so that any $f \in L^2(M)$ admits the orthogonal decomposition*

$$f = f^s + dv, \quad f^s = \mathcal{S}f, \quad dv = \mathcal{P}f \tag{3.6}$$

with some symmetric $(m-1)$-tensor $v \in H_0^1(M)$, and $\delta f^s = 0$.

PROOF. Assume that the theorem holds and such projections exist. Then for any f, $f = f^s + dv$, with $\delta f^s = 0$. Take divergence of both sides to get $\delta f = \delta dv$, and $v \in H_0^1(M)$, i.e., $v \in H^1(M)$, $v = 0$ on ∂M. Therefore, v solves

$$\begin{cases} \delta dv = \delta f & \text{in } M,\\ v|_{\partial M} = 0. \end{cases} \tag{3.7}$$

It is not hard to see that $-\delta d$ is an elliptic non-negative differential operator of order 2. We can think of symmetric tensors as vector-valued functions (if $m = 2$, the dimension is $n(n+1)/2$). Then $-\delta d$ can be thought of as a matrix-valued differential operator (a system). Note first that $-\delta d$ is formally self-adjoint, and clearly non-negative because $(-\delta dv, v) = \|dv\|^2$ for any $v \in H_0^1(M)$. Denote by $\sigma_{\mathrm{p}}(\delta)$, $\sigma_{\mathrm{p}}(d)$ the principal symbols w.r.t. the scalar product defined by the integrand in (3.5) (which is the natural inner product in the space $ST_m^0(\mathbf{C}^n)$, see (II.8.3) and the paragraph preceding it).

One could actually write down $\sigma_{\mathrm{p}}(\delta)$, $\sigma_{\mathrm{p}}(d)$ explicitly. In the case $m = 2$, we get

$$\frac{1}{\mathrm{i}}\big(\sigma_{\mathrm{p}}(\delta)f\big)_i = \xi^j f_{ij}, \quad \frac{1}{\mathrm{i}}\big(\sigma_{\mathrm{p}}(d)v\big)_{ij} = \frac{1}{2}\big(\xi^j v_i + \xi^i v_j\big). \tag{3.8}$$

Recall that $\xi^i = g^{ij}(x)\xi_j$, so in particular, those symbols depend on x in a "hidden" way. The ellipticity is then easy to check directly. In fact, we get that $-\delta d$ is strongly elliptic, i.e., not only $\sigma_{\mathrm{p}}(x,\xi)$ vanishes for $\xi = 0$ only, but it in fact, is a strictly positive tensor (matrix) for $\xi \neq 0$. The Dirichlet boundary conditions for such a strongly elliptic system are automatically coercive [**40**, V.4]. Since the kernel and the cokernel of that system are trivial, we get that there is a unique solution satisfying the usual Sobolev estimates. We will denote the solution u to the system $\delta du = f$, $u = 0$ on ∂M by $u = (\delta d)_{\mathrm{D}}^{-1} u$. Then $(\delta d)_{\mathrm{D}}^{-1} : H^{-1} \to H_0^1$, see [**41**, p. 303]. Its norm depends continuously on $g \in C^1$, see [**37**, Lemma 1]. Also, $(\delta d)_{\mathrm{D}}^{-1} : H^s \to H^{s+2} \cap H_0^1$, $s = 0, 1, \dots$ with a norm bounded by a constant depending on an upper bound of $\|g\|_{C^k}$, $k = k(m) \gg 1$. By (3.7),

$$v = (\delta d)_{\mathrm{D}}^{-1}\delta f. \tag{3.9}$$

This motivates the following definition

$$\mathcal{P} = d(\delta d)_{\mathrm{D}}^{-1}\delta, \quad \mathcal{S} = \mathrm{Id} - \mathcal{P}. \tag{3.10}$$

It is not hard now to see that those two operators indeed have the properties required.

Notice that the $(m-1)$-form v so that $\mathcal{P}f = dv$, $v \in H_0^1(M)$, is uniquely determined. □

4. The geodesic X-ray transform of functions on simple manifolds

5. The geodesic X-ray transform of vector fields on simple manifolds

6. The geodesic X-ray transform of 2-tensor fields on simple manifolds

7. The geodesic X-ray transform of higher order tensor fields on simple manifolds

8. Manifolds with conjugate points

CHAPTER VI

The X-ray transform over general curves

1. Introduction

1.1. Definition. We study integral transforms over more general families of curves here. Examples include integrals over magnetic geodesics and the circular transform studied later in this chapter.

We work in $\mathbf{R}^n$ first. We want to have a "geodesic like" family of curves Γ having the following properties. For every point x and direction v (the length does not matter), we want to have a unique curve $\gamma_{x,v}(t)$ through x in the direction of v, i.e., so that $\gamma_{x,v}(0) = x$ and $\dot\gamma_{x,v}(0) = \mu v$ with some smooth $\mu(x, v) > 0$. We also want the dependence on (x, v) to be smooth and the curve to have a non-zero tangent everywhere. This naturally leads to the following ODE

$$\ddot\gamma = G(\gamma, \dot\gamma) \tag{1.1}$$

which just says that γ is determined by an initial position and direction and then the acceleration $\ddot\gamma$ depends smoothly on them. Then the weighted X-ray transform over Γ is then defined as

$$X_w f(\gamma) = \int w(\gamma(s), \dot\gamma(s)) f(\gamma(s), ds, \quad \gamma \in \Gamma, \tag{1.2}$$

with $w(x, v)$ a smooth function on $T\mathbf{R}^n$.

It is convenient but not really necessary to assume that we are given a Riemannian metric g. In many examples there is an underlying metric already and if there is none, we can choose some. Then we will take the second derivative in (1.1) in covariant sense writing

$$\nabla_t \dot\gamma = G(\gamma, \dot\gamma), \tag{1.3}$$

by modifying G. Here ∇_t is the covariant derivative along $\dot\gamma$. Explicitly, $\nabla_t\gamma = \ddot\gamma + \Gamma^k_{ij}\dot\gamma^i\gamma^j$. Then we add the second term to G to get the new generator. An example is the generator of the geodesic equation G, see (5.2) in Appendix D given by $G(x, v) = -\Gamma^k_{ij}(x)v^iv^j$ in (1.1) (in a non-invariant form). On the other hand, in covariant form, the geodesic equation is simply $\nabla_t\dot\gamma = 0$, therefore G in (1.3) is zero.

The formulation above has some inconveniences. The length $|\dot\gamma|$ of the curves is not necessarily conserved along each one of them. Indeed, we have $(\mathrm{d}/\mathrm{d}t)|\dot\gamma|^2 = 2\langle G(\gamma, \dot\gamma), \dot\gamma\rangle$ and the latter vanishes if and only if the acceleration G is normal to the curves at all points.

To fix a curve we need initial conditions for $(\gamma, \dot\gamma)$ at some point, say $t = 0$. Rescaling $\dot\gamma(0)$ by multiplying it by a positive constant may in principle lead to a different curve, not just a re-parameterization of the given one. This in fact happens in the circular transform example we study below. To guarantee that the curve is unique, we will do the following. Let H be a (small enough) hypersurface, let $x_0 \in H$, and let $0 \neq v_0$ be a vector at it transversal to H. For (x, v) in some neighborhood of (x_0, v_0), we define $\gamma_{x,v}(t)$ to be the solution of (1.1) with initial condition $\gamma_{x,v}(0) = x$, $\dot\gamma_{x,v}(0) = \mu v$ with some smooth $\mu(x, v) > 0$ defined for such (x, v). That solution is defined on some interval around zero which can be chosen uniform for all such (x, v) close enough to (x_0, v_0). We will re-parameterize those corves to force the speed to be unit (in the metric g). If $t = t(s)$ is such a re-parameterization for a fixed γ, then we have

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}\gamma}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}.$$

The make the $\mathrm{d}\gamma/\mathrm{d}s$ a unit vector, we need to solve the ODE

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = \Big|\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\Big|$$

with initial condition $s(0) = 0$ (for example), where $t = 0$, and therefore $s = 0$ correspond to $\gamma(0) \in H$. Since we assume that $\mathrm{d}\gamma/\mathrm{d}t$ never vanishes, the r.h.s. above is positive and smooth over each γ and we have a smooth solution $s(t)$ defined along the whole γ, which is also invertible with $t = t(s)$ smooth. We do that for every $\gamma \in G$ as above to get unit speed curves. Then $s(t)$ would depend on the initiel points and directions as well in a smooth way. The change of the variables $t = t(s)$ in (1.2) would change the weight by a multiplication by a non-vanishing Jacobian and by a change of variables in the weight. The property of the weight to be zero or not at a particular point and direction; or on some set, would not change. Also, if "everything is analytic", more preciely, H, w, G, then this would preserve analyticity. We will need this fact later. Finally, the new family of curves would solve (1.1) with a new generator G. ***** It is a bit more complicated because w and G depend on $\dot\gamma$ in the new variables also through the point where that geodesics hit H ***** Also,

$$\langle G(x, v), v\rangle = 0 \tag{1.4}$$

for all (x, v) which is equivalent to the unit speed property.

Having made this reduction, we get the following simplified problems.

The local problem of recovery of the singularities. Given $X_w f$ for γ close to a fixed γ_0, can we recover $\mathrm{WF}(f)$ on the conormals of the curves in that set? We assume that f is compactly supported and that γ_0 and the nearby curves have endpoints outside $\operatorname{supp} f$. As we saw in Chapter V, if the curves are geodesics and there are no conjugate point on them (restricted to $\operatorname{supp} f$), the answer is affirmative. If there are conjugate points, the nature of the problem changes. We will prove similar results in the general case.

Support theorems. We show that the classical Helgason support theorems generalize for analytic families of curves and weights (non-vanishing). In fact, even if the system is just smooth, there is still a support theorem when $n \geq 3$.

The global invertibility problem. Let Ω be a bounded domain with smooth boundary $\partial\Omega$, strictly convex with respect to the curves in Γ. Assume that the latter set consists of curves starting from all points on $\partial\Omega$ in all direction pointing into it. We want to find out if $X_w f$ is invertible, stably or not and prove stability estimates.

2. The local problem

Let H be a smooth oriented hypersface as above. Let Γ be some open sets of curves $\gamma_{x,v}$ solving (1.3) with a generator G satisfying (1.4) parameterized by initial points x and unit directions v consistent with the orientation. We fix one of them and call it $\gamma_0 := \gamma_{x_0,v_0}$. We assume that Γ is a "small enough" neoghborhhod of γ_0 in the following sense: during the proofs, we may have to shrink Γ finitely many times, and we reserve that freedom. Finally, all γ's in Γ are assumed to be defined over teh same interval $[0, T]$.

We are interested in inverting $X_w f$ microlocally, where f is a compactly supported distribution so that the endpoints of all $\gamma \in \Gamma$ are outside $\operatorname{supp} f$.

***** this is just the beginning *****

3. The magnetic geodesic X-ray transform

4. The circular transform

CHAPTER VII

The Generalized Radon transform

CHAPTER VIII

The X-ray and the Radon transforms as FIOs

APPENDIX A

Distributions and the Fourier Transform

1. Distributions

Stuff we need

[This is a collection of facts that we need, just a sketch]

1.1. Function Spaces. Let $\Omega \subset \mathbf{R}^n$ be an open set, which, in particular, can be the whole $\mathbf{R}^n$.

1.1.1. C^k *spaces.* $C_0^k(\Omega)$ consists of all functions f defined in Ω so that $\partial_\alpha f$ is continuous for $|\alpha| \le k$ and $\operatorname{supp} f \subset \Omega$. In particular, $\operatorname{supp} f$ is disconnected from $\partial\Omega$ (if there is a boundary, because $\Omega = \mathbf{R}^n$ has none). Just vanishing at $\partial\Omega$ is not enough.

$C_0^\infty(\Omega)$ is the space of all smooth functions f with supports in $\bar{\Omega}$. [define topology, call it $\mathcal{D}'$]

$C^k(\Omega)$ is the space of all functions f defined in Ω so that $\partial_\alpha f$ is continuous for $|\alpha| \le k$. There is no control of the behavior of such functions $f(x)$ as $x \to \partial\Omega$.

$C^\infty(\Omega)$ consists of functions smooth in Ω. [define topology]

If $K \subset \mathbf{R}^n$ is a compact set, then $C^k(K)$ consists of all f for which $\partial^\alpha f$ is continuous on K for $|\alpha| \le k$. This is a Banach space with norm $\|f\|_{C^k(K)} = \sum_{|\alpha|\le k} \max_K |\partial_\alpha f|$.

The space $C^\infty(K)$ consists of all f smooth in K, i.e., for which $\partial^\alpha f$ is continuous on K for each α. It is a Fréchet space with seminorms $\|\cdot\|_{C^k(K)}$, $k = 0, 1, \dots$.

$C_0^k(K)$ is the subspace of $C^k(K)$ consisting of all functions vanishing outside K with the same norm.

$C_0^\infty(K)$ is the Fréchet subspace of $C^\infty(K)$ consisting of all functions vanishing outside K.

1.1.2. *Sobolev Spaces.* If $k = 0, 1, \dots$ is an integer, then $H^k(\mathbf{R}^n)$ is the Hilbert space of functions f(equivalent classes, to be exact) so that

$$\|f\|^2_{H^k(\mathbf{R}^n)} := \sum_{|\alpha|\le k} \|\partial^\alpha f\|^2_{L^2(\mathbf{R}^n)}.$$

To be more precise, we view f a priori as a distribution; and if $\partial_\alpha f \in L^2$ for all $|\alpha| \le k$, where the derivatives are taken in a distribution sense, we declare it to be in H^k; otherwise — not.

Passing to the Fourier transform, we get the equivalent norm which for which we use the same notation (strictly speaking, we should replace ξ by a dot below):

$$\|f\|^2_{H^s(\mathbf{R}^n)} := \|(1 + |\xi|^2)^{s/2} \hat{f}(\xi)\|^2_{L^2(\mathbf{R}^n)}.$$

Note that this makes sense and defines a Hilbert space for any real s, which explains why we change the notation from k to s.

One can define Sobolev spaces $H^s(M)$ on a compact manifold M (without boundary) by requiring that any $f \in H^s(M)$ has to be in $H^s(\mathbf{R}^n)$ restricted to any local chart. A global definition is also possible: let P be a second order positive definite elliptic differential operator on M, for example the Laplacian associated to some Riemannian metric. Then $\|f\|_{H^s(M)} = \|(\mathrm{Id} + P)^{s/2} f\|_{L^2(M)}$, and one can use eigenfunction expansions to rewrite that norm.

Let $\Omega \subset \mathbf{R}^n$ be bounded, open, with smooth boundary. Then $H^k(\Omega)$ is defined by the norm

$$\|f\|^2_{H^k(\Omega)} := \sum_{|\alpha|\leq k} \|\partial^\alpha f\|^2_{L^2(\Omega)}.$$

We cannot use the Fourier transform anymore to define fractional or negative Sobolev spaces this way. For $s \geq 0$ non integer, we can define $H^s(\Omega)$ by complex interpolation; and one can also define this space for $s < 0$ [**41**]. The space $H^s_0(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$. Include now Ω in a larger compact manifold M of the same dimension. One can show (and for $s \geq 0$ integer this is straightforward) that $H^s_0(\Omega)$ consists of all $f \in H^s(M)$ vanishing outside Ω if $s + 1/2$ is not an integer, and only then.

***** Complex Interpolation *****

***** Trace Thm, Embedding thms *****

2. Riesz potentials and the Fourier Transform of some homogeneous distributions

2.1. Riesz potentials.

LEMMA 2.1. *The distribution* $|x|^\mu$ *extends to a meromorphic function of* μ *from* $\Re\mu > -n$ *to* $\mathbf{C}$ *with simple poles at* $\mu = -n - 2k$, $k \in \mathbf{Z}_+$.

LEMMA 2.2.

$$\mathcal{F}|x|^\mu = 2^{n+\mu}\pi^{n/2}\frac{\Gamma\left(\frac{n+\mu}{2}\right)}{\Gamma\left(-\frac{\mu}{2}\right)}|\xi|^{-\mu-n}, \quad \mu, -n-\mu \notin 2\mathbf{Z}_+, \tag{2.1}$$

where $|x|^\mu$, $|\xi|^{-\mu-n}$ *are the meromorphic extensions in Lemma 2.2.*

In particular,

$$\mathcal{F}|x|^{-n+1} = \pi|S^{n-2}||\xi|^{-1}, \quad \mathcal{F}|x|^{-1} = \frac{2(2\pi)^{n-1}}{|S^{n-2}|}|\xi|^{-n+1} \tag{2.2}$$

We used the fact that

$$|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

and that $\Gamma(1/2) = \sqrt{\pi}$.

Here is one way to compute $\mathcal{F}|x|^{-n+1}$. ***** This should not be included ***** Write, formally,

$$\begin{aligned}\hat{h}(\xi) &= \int e^{-\mathrm{i}x\cdot\xi}|x|^{-n+1}\,\mathrm{d}x = \int_{\mathbf{R}_+\times S^{n-1}} e^{-\mathrm{i}r\omega\cdot\xi}\mathrm{d}r\,\mathrm{d}\omega\\ &= \frac{1}{2}\int_{\mathbf{R}\times S^{n-1}} e^{-\mathrm{i}r\omega\cdot\xi}\mathrm{d}r\,\mathrm{d}\omega\\ &= \pi\int_{S^{n-1}}\delta(\omega\cdot\xi)\,\mathrm{d}\omega = \frac{\pi|S^{n-2}|}{|\xi|}.\end{aligned} \tag{2.3}$$

The second line can be explained by the fact that the integrand $e^{-\mathrm{i}x\cdot\xi}$ is an even function of (r, ω). The first two integrals above are not convergent (but still make sense as certain Fourier transforms), and those calculations can be justified with distribution theory.

2.2. Homogeneous distributions and their Fourier Transforms. Riesz potentials are a special case of homogeneous distributions. We only sketch a few relevant theorems from [**18**] without proofs because we will not use them with their full strenght. If $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, f is homogeneous of order μ if $f(tx) = t^\mu f(x)$ for $t > 0$. If we think of f as a distribution, then

$$\langle f, \phi \rangle = \int_{\mathbf{R}^n} f(x)\phi(x)\,\mathrm{d}x = t^\mu \int_{\mathbf{R}^n} f(x)\phi_t(x)\,\mathrm{d}x$$

with $\phi_t(x) = t^n \phi(tx)$ as can be seen by a change of variables. This motivates the following.

Definition 2.3. *The distribution $f \in \mathcal{D}'(\mathbf{R}^n)$ is called homogeneous of order μ if for every $\phi \in C_0^\infty(\mathbf{R}^n)$,*

$$\langle f, \phi \rangle = t^\mu \langle f, \phi_t \rangle, \quad \forall t > 0, \quad \textit{where } \phi_t(x) = t^n \phi(tx).$$

We define homogeneous distributions in $\mathbf{R}^n \setminus 0$ in the same way but with $\phi \in C_0^\infty(\mathbf{R}^n \setminus 0)$.

Theorem 2.4. *Let f be a homogeneous distribution in $\mathbf{R}^n \setminus 0$ of order μ and let μ is not an integer less or equal than $-n$. Then there exists unique extension $\dot{f}$ of f to a homogeneous distribution in $\mathbf{R}^n \setminus 0$ of order μ. Moreover the map $f \mapsto \dot{f}$ is continuous.*

For a proof, we refer to [**18**, Theorem 3.2.3]. We will only comment on the uniqueness part. If $\dot{f}$ is such an extension, then $f - \dot{f}$ is supported at the origin, and is therefore a linear combination of δ and its derivatives. Since $\partial^\alpha \delta$ is homogeneous of order $-n-\alpha$ and none of those numbers can be equal to μ, we get $f - \dot{f} = 0$.

If $\mu < -n$ and f is locally integrable away from the origin, then $f(x) = |x|^\mu f(x/|x|)$ there, and the extension if given by the same function, which is locally L^1 even near the origin. Those two observations is all that we will need for our purposes.

Theorem 2.5. *Let f be a homogeneous distribution in $\mathbf{R}^n$ of order μ. Then $f \in \mathcal{S}'(\mathbf{R}^n)$ and $\hat{f}$ is a homogeneous distribution in $\mathbf{R}^n$ of order $-\mu - n$. Moreover, if f is smooth away from the origin, then so is $\hat{f}$.*

For a proof, we refer to [**18**, Theorem 7.1.16, 7.1.18]. In particular, the proof that $\hat{f}$ is homogeneous as well follows easily from Definition 2.3.

***** more *****

3. The Hilbert Transform

$$Hf(p) = \frac{1}{\pi}\,\mathrm{pv}\!\int_{\mathbf{R}} \frac{f(s)}{p - s}\,\mathrm{d}s,$$

where "pv$\int$" stands for an integral in a principal value sense.

$$H = -\mathrm{i}\,\mathrm{sgn}(D) = -\mathrm{i}\mathcal{F}^{-1}\mathrm{sgn}(\xi)\mathcal{F}$$

***** more *****

4. Duality

Throughout this section, $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^m$ are open sets.

DEFINITION 4.1. *A linear map*

$$A : C_0^\infty(Y) \to C_0^\infty(X)$$

is called continuous, if

(i) *For any compact $K_Y \subset Y$ there is a compact $K_X \subset X$ so that* $\operatorname{supp} A\phi \subset K_X$ *if* $\operatorname{supp}\phi \subset K_Y$*,*

(ii) *For any compact $K \subset Y$ and any $N \geq 0$, there exist constants C and M so that*

$$\sum_{|\alpha|\leq N} \sup |\partial^\alpha A\phi| \leq C \sum_{|\beta|\leq M} \sup |\partial^\beta \phi|, \quad \forall \phi \in C_0^\infty(Y),\ \operatorname{supp}\phi \subset K.$$

For such a map, define the transpose $A' : \mathcal{D}'(X) \to \mathcal{D}'(Y)$ by

$$\langle A' f, \phi\rangle = \langle f, A\phi\rangle, \quad f \in \mathcal{D}'(X), \quad \phi \in C_0^\infty(Y). \tag{4.1}$$

Then $A' : \mathcal{D}'(X) \to \mathcal{D}'(Y)$ is a sequentially continuous map.

If the map A preserves the smooth functions but not necessarily the compactness of the support (in a uniform way, as in Definition 4.1(i)), we can give the following definition.

DEFINITION 4.2. *A linear map*

$$A : C_0^\infty(Y) \to C^\infty(X)$$

is called continuous, if for any compacts $K_X \subset X$, $K_Y \subset Y$, and any multi-index α, there exist constants C and M so that

$$\sup_{K_X} |\partial^\alpha A\phi| \leq C \sum_{|\beta|\leq M} \sup |\partial^\beta \phi|, \quad \forall \phi \in C_0^\infty(Y),\ \operatorname{supp}\phi \subset K_Y.$$

Then A' can be defined on $\mathcal{E}'(X)$ only, i.e., we have $A' : \mathcal{E}'(X) \to \mathcal{D}'(X)$ defined by

$$\langle A' f, \phi\rangle = \langle f, A\phi\rangle, \quad f \in \mathcal{E}'(X), \quad \phi \in C_0^\infty(Y). \tag{4.2}$$

The operator $A' : \mathcal{E}'(X) \to \mathcal{D}'(Y)$ is sequentially continuous.

In applications, this is often applied to A being the transpose of an operator that we want to extend, i.e., $A = B'$. More precisely, let $B : C_0^\infty(Y) \to \mathcal{D}'(X)$ be a sequentially continuous operator. Suppose that we can verify directly that B' maps $C_0^\infty(X)$ to $C^\infty(Y)$, and that $B' : C_0^\infty(X) \to C^\infty(Y)$ is continuous. Then B extends to a map $B : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ that is sequentially continuous. If we have the stronger property: $B' : C_0^\infty(X) \to C_0^\infty(Y)$ is continuous, then by the results above, B extends to a map $B : \mathcal{D}'(Y) \to \mathcal{D}'(X)$ that is sequentially continuous.

explain: there is a support condition, as well

APPENDIX B

Wave Front Sets and Pseudo-Differential Operators (ΨDOs)

1. Introduction

Microlocal analysis, loosely speaking, is analysis near points and directions, i.e., in the "phase space" rather than in the base space, as the classical analysis. We introduce first the notion of the wave front set of a distribution which gives the location of the "singularities" in the phase space. Next, we review the basic theory of ΨDOs needed for our exposition.

2. Wave front sets

The phase space in $\mathbf{R}^n$ is the cotangent bundle $T^*\mathbf{R}^n$ that can be identified with $\mathbf{R}^n \times \mathbf{R}^n$. Given a distribution $f \in \mathcal{D}'(\mathbf{R}^n)$, a fundamental object to study is the wave front set $\mathrm{WF}(f) \subset T^*\mathbf{R}^n \setminus 0$ that we define below.

2.1. Definition. The basic idea goes back to the properties of the Fourier transform. If f is an integrable compactly supported function, one can tell whether f is smooth by looking at the behavior of $\hat{f}(\xi)$ (that is smooth, even analytic) when $|\xi| \to \infty$. It is known that f is smooth if and only if for any N, $|\hat{f}(\xi)| \le C_N|\xi|^{-N}$ for some C_N. If we localize this requirement to a conic neighborhood V of some $\xi_0 \neq 0$ (V is conic if $\xi \in V \Rightarrow t\xi \in V, \forall t > 0$), then we can think of this as a smoothness in the cone V. To localize in the base x variable however, we first have to cut smoothly near a fixed x_0.

We say that $(x_0, \xi_0) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus 0)$ is *not* in the wave front set $\mathrm{WF}(f)$ of $f \in \mathcal{D}'(\mathbf{R}^n)$ if there exists $\phi \in C_0^\infty(\mathbf{R}^n)$ with $\phi(x_0) \neq 0$ so that for any N, there exists C_N so that

$$|\widehat{\phi f}(\xi)| \le C_N|\xi|^{-N}$$

for ξ in some conic neighborhood of ξ_0. This definition is independent of the choice of ϕ. If $f \in \mathcal{D}'(\Omega)$ with some open $\Omega \subset \mathbf{R}^n$, to define $\mathrm{WF}(f) \subset \Omega \times (\mathbf{R}^n \setminus 0)$, we need to choose $\phi \in C_0^\infty(\Omega)$. Clearly, the wave front set is a closed conic subset of $\mathbf{R}^n \times (\mathbf{R}^n \setminus 0)$. Next, multiplication by a smooth function cannot enlarge the wave front set. The transformation law under coordinate changes is that of covectors making it natural to think of $\mathrm{WF}(f)$ as a subset of $T^*\mathbf{R}^n \setminus 0$, or $T^*\Omega \setminus 0$, respectively.

The wave front set $\mathrm{WF}(f)$ generalizes the notion singsupp(f) — the complement of the largest open set where f is smooth. The points (x, ξ) in $\mathrm{WF}(f)$ are referred to as *singularities* of f. Its projection onto the base is singsupp(f), i.e.,

$$\mathrm{singsupp}(f) = \{x;\ \exists\xi, (x, \xi) \in \mathrm{WF}(f)\}.$$

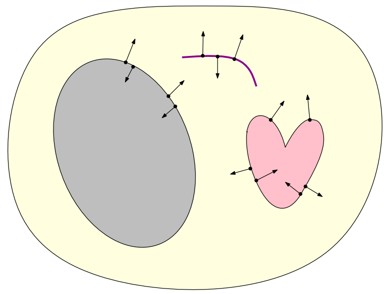

FIGURE B.1. Example of wave front set

Examples.

(a) $\mathrm{WF}(\delta) = \{(0,\xi);\ \xi \neq 0\}$. In other words, the Dirac delta function is singular at $x = 0$, in all directions.

(b) Let $x = (x', x'')$, where $x' = (x_1, \dots, x_k)$, $x'' = (x_{k+1}, \dots, x_n)$ with some k. Then $\mathrm{WF}(\delta(x')) = \{(0, x'', \xi', 0), \xi' \neq 0\}$, where $\delta(x')$ is the Dirac delta function on the plane $x' = 0$, defined by $\langle \delta(x'), \phi\rangle = \int \phi(0, x'')\, \mathrm{d}x''$. In other words, $\mathrm{WF}(\delta(x'))$ consists of all (co)vectors with a base point on that plane, perpendicular to it.

(c) Let f be a piecewise smooth function that has a non-zero jump across some smooth hypersurface S. Then $\mathrm{WF}(f)$ consists of all (co)vectors at points of S, normal to it. This follows from a change of variables that flattens S locally. This also applies to submanifolds S of codimension lower than one. Delta functions supported on such submanifolds have the same wave front sets. Those are examples of *conormal singularities*.

(d) Let $f = \mathrm{pv}\frac{1}{x} - \pi \mathrm{i}\delta(x)$ in $\mathbf{R}$. Then $\mathrm{WF}(f) = \{(0,\xi);\ \xi > 0\}$.

In example (d) we see a distribution with a wave front set that is not symmetric under the change $\xi \mapsto -\xi$. In fact, wave front sets (of complex-valued distributions) do not have a special structure except for the requirement to be closed conic sets; given any such set, there is a distribution with a wave front set exactly that set.

We see in (c) that the points in $\mathrm{WF}(f)$ cannot be characterized as (x,ξ) where the directional derivative at x in the direction of ξ does not exist in classical sense. In that example, the directional derivative at points on S does not exist for all ξ not tangent to S, at least; while only (co)normal ξ are in the wave front set.

Two distributions cannot be multiplied in general. However, if $\mathrm{WF}(f)$ and $\mathrm{WF}'(g)$ do not intersect, there is a "natural way" to define a product. Here, $\mathrm{WF}'(g) = \{(x, -\xi);\ (x,\xi) \in \mathrm{WF}(g)\}$.

3. Pseudodifferential Operators

3.1. Definition. We first define the symbol class $S^m(\Omega)$, $m \in \mathbf{R}$, as the set of all smooth functions $p(x,\xi)$, $(x,\xi) \in \Omega \times \mathbf{R}^n$, called symbols, satisfying the following symbol estimates: for any compact $K \subset \Omega$, and any multi-indices α, β, there is a constant $C_{K,\alpha,\beta} > 0$ so that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le C_{K,\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}, \quad \forall (x,\xi) \in K \times \mathbf{R}^n. \tag{3.1}$$

More generally, one can define the class $S^m_{\rho,\delta}(\Omega)$ with $0 \le \rho$, $\delta \le 1$ by replacing $m - |\alpha|$ there by $m - \rho|\alpha| + \delta|\beta|$. Then $S^m(\Omega) = S^m_{1,0}(\Omega)$. Often, we omit Ω and simply write S^m. There are other classes in the literature, for example $\Omega = \mathbf{R}^n$, and (3.1) is required to hold for all $x \in \mathbf{R}^n$.

The least constant $C_{K,\alpha,\beta}$ we can choose in (3.1) defines a seminorm in $S^m(\Omega)$. This makes the latter a Fréchet space.

The estimates (3.1) do not provide any control of p when x approaches boundary points of Ω, or ∞.

Given $p \in S^m(\Omega)$, we define the pseudodifferential operator (ΨDO) with symbol p, denoted by $p(x, D)$, by

$$p(x, D)f = (2\pi)^{-n} \int e^{\mathrm{i}x\cdot\xi} p(x, \xi)\hat{f}(\xi)\,\mathrm{d}\xi, \quad f \in C_0^\infty(\Omega). \tag{3.2}$$

The definition is inspired by the following. If $P = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$ is a differential operator, where $D = -\mathrm{i}\partial$, then using the Fourier inversion formula we can write P as in (3.2) with a symbol $p = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ that is a polynomial in ξ with x-dependent coefficients. The symbol class S^m allows for more general functions. The class of the pseudo-differential operators with symbols in S^m is denoted usually by Ψ^m. The operator P is called a ΨDO if it belongs to Ψ^m for some m. By definition, $S^{-\infty} = \cap_m S^m$, and $\Psi^{-\infty} = \cap_m \Psi^m$.

An important subclass is the set of the *classical symbols* that have an asymptotic expansion of the form

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x, \xi), \tag{3.3}$$

where $m \in \mathbf{R}$, and p_{m-j} are smooth and positively homogeneous in ξ of order $m-j$ for $|\xi| > 1$, i.e., $p_{m-j}(x, \lambda\xi) = \lambda^{m-j} p_{m-j}(x, \xi)$ for $|\xi| > 1$, $\lambda > 1$; and the sign $\sim$ means that

$$p(x, \xi) - \sum_{j=0}^{N} p_{m-j}(x, \xi) \in S^{m-N-1}, \quad \forall N \ge 0. \tag{3.4}$$

Any ΨDO $p(x, D)$ is continuous from $C_0^\infty(\Omega)$ to $C^\infty(\Omega)$, and can be extended by duality as a continuous map from $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$.

3.2. Principal symbol. The principal symbol of a ΨDO given by (3.2) is the equivalence class $S^m(\Omega)/S^{m-1}(\Omega)$, and any its representative is called a principal symbol as well. In case of classical ΨDOs, the convention is to choose the principal symbol to be the first term p_m, that in particular is positively homogeneous in ξ.

3.3. Smoothing Operators. Those are operators than map continuously $\mathcal{E}'(\Omega)$ into $C^\infty(\Omega)$. They coincide with operators with smooth Schwartz kernels in $\Omega \times \Omega$. They can always be written as ΨDOs with symbols in $S^{-\infty}$, and vice versa — all operators in $\Psi^{-\infty}$ are smoothing. Smoothing operators are viewed in this calculus as negligible and ΨDOs are typically defined modulo smoothing operators, i.e., $A = B$ if and only if $A - B$ is smoothing. Smoothing operators are not "small".

3.4. The pseudolocal property. For any ΨDO P and any $f \in \mathcal{E}'(\Omega)$,

$$\operatorname{singsupp}(Pf) \subset \operatorname{singsupp} f. \tag{3.5}$$

In other words, a ΨDO cannot increase the singular support. This property is preserved if we replace singsupp by WF, see (4.1).

3.5. Symbols defined by an asymptotic expansion. In many applications, a symbol is defined by consecutively constructing symbols $p_j \in S^{m_j}$, $j = 0, 1, \dots$, where $m_j \searrow -\infty$, and setting

$$p(x,\xi) \sim \sum_j p_j(x,\xi). \tag{3.6}$$

The series on the right may not converge but we can make it convergent by using our freedom to modify each p_j for ξ in expanding compact sets without changing the large ξ behavior of each term. This extends the Borel idea of constructing a smooth function with prescribed derivatives at a fixed point. The asymptotic (3.6) then is understood in a sense similar to (3.4). This shows that there exists a symbol $p \in S^{m_0}$ satisfying (3.6). That symbol is not unique but the difference of two such symbols is always in $S^{-\infty}$.

3.6. Amplitudes. A seemingly larger class of ΨDOs is defined by

$$Af = (2\pi)^{-n} \int e^{\mathrm{i}(x-y)\cdot\xi} a(x,y,\xi) f(y)\,\mathrm{d}y\,\mathrm{d}\xi, \quad f \in C_0^\infty(\Omega), \tag{3.7}$$

where the amplitude a satisfies

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x,y,\xi)| \le C_{K,\alpha,\beta,\gamma}(1+|\xi|)^{m-|\alpha|}, \quad \forall (x,y,\xi) \in K \times \mathbf{R}^n \tag{3.8}$$

for any compact $K \subset \Omega \times \Omega$, and any α, β, γ. In fact, any such ΨDO A is a ΨDO with a symbol $p(x,\xi)$ (independent of y) with the formal asymptotic expansion

$$p(x,\xi) \sim \sum_{\alpha \ge 0} D_\xi^\alpha \partial_y^\alpha a(x,x,\xi). \tag{3.9}$$

In particular, the principal symbol of that operator can be taken to be $a(x,x,\xi)$. We often use the notation $\mathrm{Op}(a)$ or $\mathrm{Op}(p)$ to denote operators of the kind (3.9) and (3.2).

3.7. Transpose and adjoint operators to a ΨDO. The mapping properties of any ΨDO A indicate that it has a well defined transpose A', and a complex adjoint A^* with the same mapping properties. They satisfy

$$\langle Au, v\rangle = \langle u, A'v\rangle, \quad \langle Au, \bar v\rangle = \langle u, \overline{A^* v}\rangle, \quad \forall u, v \in C_0^\infty$$

where $\langle \cdot,\cdot\rangle$ is the pairing in distribution sense; and in this particular case just an integral of uv. In particular, $A^*u = \overline{A'\bar u}$, and if A maps L^2 to L^2 in a bounded way, then A^* is the adjoint of A in L^2 sense.

The transpose and the adjoint are ΨDOs in the same class with amplitudes $a(y,x,-\xi)$ and $\bar a(y,x,\xi)$, respectively; and symbols

$$\sum_{\alpha\ge 0} (-1)^{|\alpha|} \frac{1}{\alpha!} (\partial_\xi^\alpha D_x^\alpha p)(x,-\xi), \quad \sum_{\alpha \ge 0} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \bar p(x,\xi),$$

if $a(x,y,\xi)$ and $p(x,\xi)$ are the amplitude and/or the symbol of that ΨDO. In particular, the principal symbols are $p_0(x,-\xi)$ and $\bar p_0(x,\xi)$, respectively, where p_0 is (any representative of) the principal symbol.

3.8. Composition of ΨDOs and ΨDOs with properly supported kernels. Given two ΨDOs A and B, their composition may not be defined even if they are smoothing ones because each one maps C_0^∞ to C^∞ but may not preserve the compactness of the support. For example, if $A(x,y)$, and $B(x,y)$ are their Schwartz kernels, the candidate for the kernel of AB given by $\int A(x,z)B(z,y)\,\mathrm{d}z$ may be a divergent integral. On the the hand, for any ΨDO A, one can find a smoothing correction R, so that $A+R$ has properly supported kernel, i.e., the kernel of $A+R$, has a compact intersection with $K\times\Omega$ and $\Omega\times K$ for any compact $K\subset\Omega$. The proof of this uses the fact that the Schwartz kernel of a ΨDO is smooth away from the diagonal $\{x=y\}$ and one can always cut there in a smooth way to make the kernel properly supported at the price of a smoothing error. ΨDOs with properly supported kernels preserve $C_0^\infty(\Omega)$, and also $\mathcal{E}'(\Omega)$, and therefore can be composed in either of those spaces. Moreover, they map $C^\infty(\Omega)$ to itself, and can be extended from $\mathcal{D}'(\Omega)$ to itself. The property of the kernel to be properly supported is often assumed, and it is justified by considering each ΨDO as an equivalence class.

If $A\in\Psi^m(\Omega)$ and $B\in\Psi^k(\Omega)$ are properly supported ΨDOs with symbols a and b, respectively, then AB is again a ΨDO in $\Psi^{m+k}(\Omega)$ and its symbol is given by

$$\sum_{\alpha\geq 0}(-1)^{|\alpha|}\frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)D_x^\alpha b(x,\xi).$$

In particular, the principal symbol can be taken to be ab.

3.9. Change of variables and ΨDOs on manifolds. Let Ω' be another domain, and let $\phi:\Omega\to\tilde\Omega$ be a diffeomorphism. For any $P\in\Psi^m(\Omega)$, $\tilde Pf := (P(f\circ\phi))\circ\phi^{-1}$ maps $C_0^\infty(\tilde\Omega)$ into $C^\infty(\tilde\Omega)$. It is a ΨDO in $\Psi^m(\tilde\Omega)$ with principal symbol

$$p(\phi^{-1}(y),(\mathrm{d}\phi)'\eta) \tag{3.10}$$

where p is the symbol of P, $\mathrm{d}\phi$ is the Jacobi matrix $\{\partial\phi_i/\partial x_j\}$ evaluated at $x=\phi^{-1}(y)$, and $(\mathrm{d}\phi)'$ stands for the transpose of that matrix. We can also write $(\mathrm{d}\phi)'=((\mathrm{d}\phi^{-1})^{-1})'$. An asymptotic expansion for the whole symbol can be written down as well.

Relation (3.10) shows that the transformation law under coordinate changes is that of a covector. Therefore, the principal symbol is a correctly defined function on the cotangent bundle $T^*\Omega$. The full symbol is not invariantly defined there in general.

Let M be a smooth manifold, and $A:C_0^\infty(M)\to C^\infty(M)$ be a linear operator. We say that $A\in\Psi^m(M)$, if its kernel is smooth away from the diagonal in $M\times M$, and if in any coordinate chart (A,χ), where $\chi:U\to\Omega\subset\mathbf{R}^n$, we have $(A(u\circ\chi))\circ\chi^{-1}\in\Psi^m(\Omega)$. As before, the principal symbol of A, defined in any local chart, is an invariantly defined function on T^*M.

3.10. Mapping properties in Sobolev Spaces. Any $P\in\Psi^m(\Omega)$ is a continuous map from $H^s_{\mathrm{comp}}(\Omega)$ to $H^{s-m}_{\mathrm{loc}}(\Omega)$. If the symbols estimates (3.1) are satisfied in the whole $\mathbf{R}^n\times\mathbf{R}^n$, then $P:H^s(\mathbf{R}^n)\to H^{s-m}(\mathbf{R}^n)$.

3.11. Elliptic ΨDOs and their parametrices. The operator $P\in\Psi^m(\Omega)$ with symbol p is called elliptic of order m, if for any compact $K\subset\Omega$, there exists constants $C > 0$ and $R > 0$ so that

$$C|\xi|^m \le |p(x,\xi)| \quad \text{for } x \in K, \text{ and } |\xi| > R. \tag{3.11}$$

Then the symbol p is called also elliptic of order m. It is enough to require the principal symbol only to be elliptic (of order m). For classical ΨDOs, see (3.3), the requirement can be written as $p_m(x,\xi) \neq 0$ for $\xi \neq 0$. A fundamental property of elliptic operators is that they have parametrices. In other words, given an elliptic ΨDO P of order m, there exists $Q \in \Psi^{-m}(\Omega)$, so that

$$QP - \mathrm{Id} \in \Psi^{-\infty}, \quad PQ - \mathrm{Id} \in \Psi^{-\infty}.$$

The proof of this is to construct a left parametrix first by choosing a symbol $q_0 = 1/p$, cut off near the possible zeros of p, that form a compact any time when x is restricted to a compact as well. The corresponding ΨDO Q_0 will then satisfy $Q_0P = \mathrm{Id} + R$, $R \in \Psi^{-1}$. Then we take a ΨDO E with asymptotic expansion $E \sim \mathrm{Id} - R + R^2 - R^3 + \dots$, that would be the formal Neumann series expansion of $(\mathrm{Id} + R)^{-1}$, if the latter existed. Then EQ_0 is a left parametrix that is also a right parametrix.

An important consequence is the following elliptic regularity statement. If P is elliptic (and properly supported), then

$$\operatorname{singsupp}(PF) = \operatorname{singsupp}(f), \quad \forall f \in \mathcal{D}'(\Omega).$$

In particular, $Pf \in C^\infty$ implies $f \in C^\infty$.

Elliptic ΨDOs are not invertible or even injective in general. For example, the Laplace-Beltrami operator Δ on the sphere S^{n-1} is elliptic but $-\Delta - z$ which is also elliptic, is not injective when z is an eigenvalue — and there are infinitely many of them. On small sets, they are injective, however. In the next theorem, we consider $L^2(U)$ as a subspace of $L^2(\Omega)$ consisting of functions vanishing outside $\bar\Omega$.

THEOREM 3.1. *Let $P \in \Psi^m$ be elliptic properly supported in a neighborhood Ω_1 of $\bar\Omega$. Then there exists $\varepsilon > 0$ so that for any open $U \subset \bar\Omega$ with measure $|U| < \varepsilon$, P is injective on distributions supported in U and*

$$\|f\|_{L^2(U)} \le C\|Pf\|_{H^{-m}(\Omega_1)} \quad \forall f \in L^2(U), \tag{3.12}$$

PROOF. Let $Q \in \Psi^{-m}$ be a properly supported parametrix; then $QP = \mathrm{Id} + K$, where K is smoothing in a neighborhood of K, i.e., it has a smooth Schwartz kernel $\mathcal{K}(x,y)$ near $\bar\Omega \times \bar\Omega$. Then

$$\|Kf(x)\|_{L^2(U)} \le \Big\| \int_U \mathcal{K}(x,y)f(y)\,\mathrm{d}y \Big\|_{L^2(U)} \le \|\mathcal{K}\|_{\mathrm{HS}}\|f\|_{L^2(U)},$$

where $\|\mathcal{K}\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm of $\mathcal{K}$. Since $\mathcal{K}$ is uniformly bounded in $\bar\Omega \times \bar\Omega$, we have $\|\mathcal{K}\|_{\mathrm{HS}} \le C|U|$. Therefore, for $|U| \ll 1$, $K : L^2(U) \to L^2(U)$ has a norm less than one; and therefore, $\mathrm{Id} + K$ is invertible there. Write $(\mathrm{Id} + K)^{-1}QP = \mathrm{Id}$ on $L^2(U)$ to get (3.12). This estimate holds for every $f \in L^2$ which vanishes outside $\bar\Omega$. Any distribution $f \in \operatorname{Ker} P$ would be C^∞ by ellipticity, therefore we get injectivity for distributions as well, as stated. □

Note that the constants C and ε can be estimated in terms of some of the seminorms of the symbol of P and in particular can be chosen uniform for a class of P's with that seminorm uniformly bounded.

4. ΨDOs and wave front sets

The microlocal version of the pseudo-local property is given by the following:

$$\mathrm{WF}(Pf) \subset \mathrm{WF}(f) \tag{4.1}$$

for any (properly supported) ΨDO P and $f \in \mathcal{D}'(\Omega)$. In other words, a ΨDO cannot increase the wave front set. If P is elliptic for some m, it follows from the existence of a parametrix that there is equality above, i.e., $\mathrm{WF}(Pf) = \mathrm{WF}(f)$.

We say that the ΨDO P is of order $-\infty$ in the open conic set $U \subset T^*\Omega\setminus 0$, if for any closed conic set $K \subset U$ with a compact projection on the the base "x-space", (3.1) is fulfilled for any m. The *essential support* $\mathrm{ES}(P)$, sometimes also called the *microsupport* of P, is defined as the smallest closed conic set on the complement of which the symbol p is of order $-\infty$. Then

$$\mathrm{WF}(Pf) \subset \mathrm{WF}(f) \cap \mathrm{ES}(P).$$

Let P have a homogeneous principal symbol p_m. The characteristic set $\mathrm{Char}\, P$ is defined by

$$\mathrm{Char}\, P = \{(x, \xi) \in T^*\Omega \setminus 0;\ p_m(x, \xi) = 0\}.$$

$\mathrm{Char}\, P$ can be defined also for general ΨDOs that may not have homogeneous principal symbols. For any ΨDO P, we have

$$\mathrm{WF}(f) \subset \mathrm{WF}(Pf) \cup \mathrm{Char}\, P, \quad \forall f \in \mathcal{E}'(\Omega). \tag{4.2}$$

P is called *microlocally elliptic* in the open conic set U, if (3.11) is satisfied in all compact subsets, similarly to the definition of $\mathrm{ES}(P)$ above. If it has a homogeneous principal symbol p_m, ellipticity is equivalent to $p_m \neq 0$ in U. If P is elliptic in U, then Pf and f have the same wave front set restricted to U, as follows from (4.2) and (4.1).

4.1. The Hamilton flow and propagation of singularities. Let $P \in \Psi^m(M)$ be properly supported, where M is a smooth manifold, and suppose that P has a real homogeneous principal symbol p_m. The Hamiltonian vector field of p_m on $T^*M \setminus 0$ is defined by

$$H_{p_m} = \sum_{j=1}^{n} \left(\frac{\partial p_m}{\partial x_j} \frac{\partial}{\partial \xi_j} - \frac{\partial p_m}{\partial \xi_j} \frac{\partial}{\partial x_j} \right).$$

The integral curves of H_{p_m} are called *bicharacteristics* of P. Clearly, $H_{p_m} p_m = 0$, thus p_m is constant along each bicharacteristics. The bicharacteristics along which $p_m = 0$ are called *zero bicharacteristics.*

Hörmander's theorem about propagation of singularities is one of the fundamental results in the theory. It states that if P is an operator as above, and $Pu = f$ with $u \in \mathcal{D}'(M)$, then

$$\mathrm{WF}(u) \setminus \mathrm{WF}(f) \subset \mathrm{Char}\, P,$$

and is invariant under the flow of H_{p_m}.

An important special case is the wave operator $P = \partial_t^2 - \Delta_g$, where Δ_g is the Laplace Beltrami operator associated with a Riemannian metric g. We may add lower order terms without changing the bicharacteristics. Let (τ, ξ) be the dual variables to (t, x). The principal symbol is $p_2 = -\tau^2 + |\xi|_g^2$, where $|\xi|_g^2 := \sum g^{ij}(x)\xi_i\xi_j$, and $(g^{ij}) = (g_{ij})^{-1}$. The bicharacteristics equations then are $\dot\tau = 0$, $\dot t = -2\tau$, $\dot x^j = 2\sum g^{ij}\xi_i$, $\dot\xi_j = -2\partial_{x^j} \sum g^{ij}(x)\xi_i\xi_j$, and they are null one if $\tau^2 = |\xi|_g^2$.

Here, $\dot{x} = \mathrm{d}x/\mathrm{d}s$, etc. The latter two equations are the Hamiltonian curves of $\tilde{H} := \sum g^{ij}(x)\xi_i\xi_j$ and they are known to coincide with the geodesics $(\gamma, \dot{\gamma})$ on TM when identifying vectors and covectors by the metric. They lie on the energy surface $\tilde{H} = \text{const}$. The first two equations imply that τ is a constant, positive or negative, and up to rescaling, one can choose the parameter along the geodesics to be t. That rescaling forces the speed along the geodesic to be 1. The null condition $\tau^2 = |\xi|_g^2$ defines two smooth surfaces away from $(\tau, \xi) = (0, 0)$: $\tau = \pm|\xi|_g$. This corresponds to geodesics starting from x in direction either ξ or $-\xi$. To summarize, for the homogeneous equation $Pu = 0$, we get that each singularity (x, ξ) of the initial conditions at $t = 0$ starts to propagate from x in direction either ξ or $-\xi$ or both (depending on the initial conditiosn) along the unit speed geodesic. In fact, we get this first for the singularities in $T^*(\mathbf{R}_t \times \mathbf{R}_x^n)$ first, but since they lie in $\operatorname{Char} P$, one can see that they project to $T^*\mathbf{R}_x^n$ as singularities again.

5. Schwartz Kernels of ΨDOs

Since ΨDOs satisfy the continuity requirement of the Schwartz kernel theorem, they have kernels, which in general are distributions. One can characterize ΨDOs completely by their kernels. In fact, in the applications we consider in this book, typical ΨDOs are $X'X$, $R'R$, etc. They are naturally described by their Schwartz kernels and the oscillatory representations (3.2) and (3.7) are foreign to them. We will review some of the relationships between a symbol and an amplitude on one hand; and a Schwartz kernel, on the other.

By (3.7), we formally get that $A = a(x, y, D)$ has a Schwartz kernel $\mathcal{A}(x, y)$ given by

$$\mathcal{A}(x, y) = (2\pi)^{-n} \int e^{\mathrm{i}(x-y)\cdot\xi} a(x, y, \xi)\, \mathrm{d}\xi.$$

The oscillatory integral above may not be convergent but it is always well defined as a distribution. If we denote by $\check{a}(x, y, z)$ the inverse Fourier transform of a with respect to ξ (with dual variable z), we get

$$\mathcal{A}(x, y) = \check{a}(x, y, x - y). \tag{5.1}$$

Any amplitude a is a smooth function of (x, y) with values in tempered distributions by the amplitude estimates (3.8), hence $\check{a}(x, y, z)$ is well defined as distribution of z smoothly depending on z. Then it is a simple exercise to show that $\check{a}(x, y, x - y)$ is also well defined in the same class.

Proposition 5.1. *$\mathcal{A}(x, y)$ is C^∞ away from the diagonal $x = y$.*

Proof. For any multi-index we have

$$(x - y)^\alpha \mathcal{A}(x, y) = (2\pi)^{-n} \int e^{\mathrm{i}(x-y)\cdot\xi} D_\xi^\alpha a(x, y, \xi)\, \mathrm{d}\xi.$$

When $m - |\alpha| < -n$, where m is the order of a, the integral is absolutely convergent and therefore, the left-hand side is a continuous function of (x, y). Apply $\partial_{x,y}^\beta$ to that to get $\partial_{x,y}^\beta (x - y)^\alpha \mathcal{A}(x, y) \in C(\mathbf{R}^n \times \mathbf{R}^n)$ when $m - |\alpha| < -n$. This proves the claim. □

This gives a simple proof of the pseudo-local property 3.5.

The following immediate corollary worth formulating.

COROLLARY 5.2. *If the Schwartz kernel of the linear operator A has singularities away from the diagonal, it is not a ΨDO.*

For example, for the geodesic X-ray transform X, $X'X$ is a ΨDO in an open convex Ω if and only if there are no conjugate points there.

An important question is how to tell that A is a ΨDO if we know its Schwartz kernel, and how to find it symbol, when it is. One interesting example, covering $X'X$ and $R'R$ is Riesz potentials; by Lemma A.2.2, convolutions with $|x|^{-k}$, $0 < k < n$, i.e., operators with kernels $|x-y|^{-k}$, are Fourier multipliers with symbols $C|\xi|^{-n+k}$, and therefore, ΨDOs. Note that the singularity at $\xi = 0$ can be dealt with, see (II.3.9).

Going back to ΨDOs in Ψ^m, it can be shown that [**42**]

$$|\partial^\beta_{x,y}\mathcal{A}(x,y)| \le C|x-y|^{-n-m-|\beta|}$$

if $n+m+|\beta| > 0$. This is consistent with the x-independent symbol case above. Based on this, we would suspect that a linear operator with a Schwartz kernel satisfying the estimate above might be a ΨDO of order $-m$. Such a theorem (with additional assumptions) can be found in [**42**].

We will consider the following special case, see also Theorem 7.1.24 in [**19**].

THEOREM 5.3. *Let*

$$\mathcal{A}(x,y) = \frac{\alpha\left(x, y, \frac{x-y}{|x-y|}\right)}{|x-y|^k}, \quad 0 < k < n,$$

where $\alpha(x,y,\theta)$ is smooth and has the parity of the integer $n-1-k$. Then $\mathcal{A}$ is the Schwartz kernel of a ΨDO of order $-n+k$ with amplitude

$$a(x,t,\xi) = \pi \mathrm{i}^{-n+k+1}\int_{S^{n-1}} \alpha(x,y,\theta)\delta^{(n-k-1)}(\xi\cdot\theta)\,\mathrm{d}\theta,$$

where $\delta^{(k)}$ the the k-the derivative of δ.

PROOF. Write $\mathcal{A}(x,y) = \alpha(x,y,z/|z|)|z|^{-k}$, $z = x-y$. If the operator A with kernel $\mathcal{A}$ is a ΨDO, it has to have an amplitude

$$a(x,y,\xi) = \int e^{-\mathrm{i}z\cdot\xi}\frac{\alpha(x,y,z/|z|)}{|z|^k}\,\mathrm{d}z,$$

see (5.1). Pass to polar coordinates $z = r\theta$ to get

$$\begin{aligned} a(x,y,\xi) &= \int_{S^{n-1}}\int_0^\infty e^{-\mathrm{i}r\xi\cdot\theta}\alpha(x,y,\theta)r^{n-1-k}\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \frac{1}{2}\int_{S^{n-1}}\int_{-\infty}^\infty e^{-\mathrm{i}r\xi\cdot\theta}\alpha(x,y,\theta)r^{n-1-k}\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \pi\mathrm{i}^{-n+k+1}\int_{S^{n-1}}\alpha(x,y,\theta)\delta^{(n-k-1)}(\xi\cdot\theta)\,\mathrm{d}\theta. \end{aligned}$$

We have an amplitude of order $-n+k$, with a singularity at $\xi = 0$. Cutting off the singularity at $\xi = 0$ in a smooth way (contributing to a smoothing operator), we get an amplitude in the symbol class S^{-n+k}.

$$\begin{aligned}\langle a(x,y,\cdot),\phi\rangle &= \int \frac{\alpha\left(x,y,z/|z|\right))}{|z|^k}\hat\phi(z)\,\mathrm{d}z\\ &= \int_{S^{n-1}}\int_0^\infty \alpha\left(x,y,\theta)\right)r^{n-1-k}\hat\phi(r\theta)\,\mathrm{d}r\,\mathrm{d}\theta\\ &= \frac12\int_{S^{n-1}}\int_{-\infty}^\infty \alpha(x,y,\theta)r^{n-1-k}\hat\phi(r\theta)\,\mathrm{d}r\,\mathrm{d}\theta.\end{aligned}$$

By the Fourier Slice Theorem for R (Theorem II.1.11, $(2\pi)^{-n}\int e^{\mathrm{i}rp}r^j\hat\phi(r\theta)\,\mathrm{d}r = D_p^jRf(p,\theta)$. Set $p=0$ to get

$$\langle a(x,y,\cdot),\phi\rangle = \pi\int_{S^{n-1}}\alpha(x,y,\theta)D_p^{n-1-k}R\phi(p,\theta)|_{p=0}\,\mathrm{d}\theta.$$

Since $\phi(\xi)\mapsto R\phi(p,\theta)$ has a Schwartz kernel $\delta(p-\theta\cdot\xi)$, this proves the theorem. □

6. ΨDOs acting on tensor fields

ΨDOs acting on tensor fields are a part of the more general theory of ΨDOs acting on smooth sections of vector bundles. We will consider covariant tensor fields only. In local coordinates in $\mathbf{R}^n$, we say that the linear operator A mapping k-fields to l ones $(Af)_{i_1\dots i_l} = a^{j_1\dots j_k}_{i_1\dots i_l}f_{j_1\dots j_k}$ belongs to Ψ^m if each $a^{j_1\dots j_k}_{i_1\dots i_l}$ belongs to Ψ^m. Under a change of variables $y=y(x)$ (the same for f and Af), the principal symbol of each $a^{j_1\dots j_k}_{i_1\dots i_l}$ would change as a function on the cotangent bundle but f and Af change as well by the laws of transformations of covariant tensors, see (2.1) in Appendix D. We get

$$(Af)_{i_1\dots i_l}(y) = a^{j_1\dots j_k}_{i_1\dots i_l}\big(x(y),(\partial y/\partial x)^TD_y\big)f_{j_1\dots j_k}(y)+(Rf)_{i_1\dots i_l}(y),$$

where R is of order -1. To convert this formula to a coordinate representation in the y-basis, we need to multiply it by $\frac{\partial x^{i_1}}{\partial y^{i'_1}}\dots\frac{\partial x^{i_l}}{\partial y^{i'_l}}$ and replace f by $f_{j_1\dots j_k}\frac{\partial x^{j_1}}{\partial y^{j'_1}}\dots\frac{\partial x^{j_k}}{\partial y^{j'_l}}$, i.e., to apply that linear transformation to f above and the inverse one. This give us the following transformation law for the principal symbol $\sigma_p(\tilde a)$ of A in the y-coordinates

$$\sigma_p(\tilde a^{j'_1\dots j'_k}_{i'_1\dots i'_l}) = \sigma_p(a^{j_1\dots j_k}_{i_1\dots i_l})\frac{\partial x^{i_1}}{\partial y^{i'_1}}\dots\frac{\partial x^{i_l}}{\partial y^{i'_l}}\frac{\partial y^{j'_1}}{\partial x^{j_1}}\dots\frac{\partial y^{j'_l}}{\partial x^{j_k}}.$$

Therefore, the principal symbol of A transforms as a tensor field of order (k,l), see (2.1) in Appendix D again, and we could have derived this fact by treating $\sigma_p(a^{j_1\dots j_k}_{i_1\dots i_l})$ as a tensor field itself.

7. Analytic ΨDOs

We review here a part of the analytic ΨDOs theory, following [**43**]. We adopt a very minimalistic approach — to present this part of the theory that we really need.

We start with the notion of an analytic function (sometimes called real analytic) in a domain $\Omega\subset\mathbf{R}^n$. This class is denoted by $\mathcal{A}(\Omega)$ and consists of all smooth functions satisfying either of the three equivalent conditions:

(i) The Taylor series of f about any point $a\in\Omega$ converges to f near $x=a$. Note that this does not require existence of a point a so that the Taylor series about a converges in the whole Ω. Example: $f(x)=\log x$ on $\Omega=(0,\infty)$.

(ii) For every compact set $K \subset \Omega$ there exits constants $A > 0$ and $C > 0$ so that

$$|\partial^\alpha f(x)| \le AC^{|\alpha|}\alpha!, \quad \forall x \in K. \tag{7.1}$$

(iii) The function f can be extended to a holomorphic function in some complex neighborhood $\Omega^{\mathbf{C}}$ of Ω in $\mathbf{C}^n$.

Holomorphic functions in open sets in $\mathbf{C}^n$ are those smooth functions solving the Cauchy-Riemann system

$$\partial_{\bar z} h := \bar\partial h := \frac{1}{2}\left(\frac{\partial h}{\partial x^j} + \mathrm{i}\frac{\partial h}{\partial y^j}\right) = 0, \quad j = 1, \dots, n,$$

where $z^j = x^j + \mathrm{i}y^j$.

Sums, products and compositions of analytic functions are analytic. If f has no zeros is some domain, then $1/f$ is analytic there. If f is an analytic diffeomorphism, its inverse is analytic, too.

One of the essential difficulties when working with analytic functions is the lack of non-trivial analytic functions with compact support. To overcome this difficulty, we use sequences of cut-off functions instead of a single one. Let $\Subset V$ be open. We want to construct an "analytic" function g_N, $N = 1, 2, \dots$ so that $g_n = 1$ on U and $g_n = 0$ outside V. Of course, there is no such analytic function. Instead, we require an estimate of the type (7.1) to hold for $|\alpha| = N$ only.

One such choice is a smooth function g_N with the properties: for every $d > 0$, $0 \le g_N \le 1$, $g_N = 1$ on U and $g_N(x) = 0$ when $\mathrm{dist}(x, U) > d$, and

$$|\partial^\alpha g_N| \le (CN/d)^{|\alpha|}, \quad |\alpha| \le N.$$

Note that $(CN)^{|\alpha|} \le e^{CNn}\alpha!$, see also (7.1).

Cutoffs in the phase variable ξ are done with a different type of sequences $g^R(\xi)$ dependent on a large parameter $R > 0$, see [**43**]. We would really like to have analytic cutoffs $g(\xi)$ (equal to 1 in some cone and to 0 away from a large one) which do not destroy the analyticity in the x variable, i.e., for which $g(D)$ would be analytically pseudolocal. We also want them to be symbols of order 0. Such cutoffs do not exist but one can build a sequence $g^R(\xi)$ that does this asymptotically, in some sense. For the analytic pseudolocal property it is needed $\check g(x)$ to be analytic away from the origin. We will require then $\check g^R$ to be analytic for $|x| > C/R$, which is true for g^R introduced below.

Given two open conic sets $\Gamma \Subset \Gamma^*$ in $\mathbf{R}^n \setminus 0$, there exists $C > 0$ so that for for every $R > 0$ one can find a smooth function g^R in $\mathbf{C}^n \setminus \mathrm{i}\mathbf{R}^n$ so that

$$\begin{aligned} 0 \le g^R \le 1 &\quad \text{on } \mathbf{R}^n \setminus 0,\\ g^R = 1 &\quad \text{on } \Gamma,\\ g^R = 0 &\quad \text{outside } \Gamma^*, \end{aligned} \tag{7.2}$$

and $|g(\zeta)| \le C\exp(C|\Im\zeta|/R)$, $|\bar\partial g^R(\zeta)| \le C\exp\{C(|\Im\zeta| - |\Re\zeta|)/R\}$. The proof of this statement in [**43**] is not trivial.

The cutoffs functions g^R will be used later.

7.1. Analytic wave front sets. One can tell the analyticity of a distribution by its localized Fourier transform in the following way. The distribution $u \in \mathcal{D}'(\Omega)$

is analytic in Ω if and only if for every $x_0 \in \Omega$, there exists $\varphi_N \in C_0^\infty$ with $\varphi(x_0) = 1$ near x_0 so that

$$|\widehat{\varphi_N u}(\xi)| \le C^{N+1} N!(1+|\xi|)^{-N} \tag{7.3}$$

with C independent of N. To prove the necessity of this condition, we just choose $\varphi_N = g_N$ as above.

If estimate (7.3) holds in a conic neighborhood of some $(x_0, \xi_0) \in T^*\mathbf{R}^n \setminus 0$, we say that u is microlocally analytic at (and also near) (x_0, ξ_0). The complement of all such points is the *analytic wave front set* $\mathrm{WF_A}(u)$. The definition is independent on the choice of the cut-off function, and the proof of that actually uses the cutoffs g^R described below. The projection of $\mathrm{WF_A}(u)$ on the base (where x lives) coincides with the analytic singular support — the complement to the largest open set where u is analytic, see also [**18**].

An equivalent definition of an analytic wave front set is given by Bros-Iagolnitzer [**10**]. A distribution u is called microlocally analytic at (x_0, ξ^0), $\xi^0 \neq 0$ if

$$\int e^{\mathrm{i}\lambda(x-y)\cdot\xi - \lambda|x-y|^2/2} \chi(x) u(x)\, \mathrm{d}x = O(e^{-\lambda/C}) \tag{7.4}$$

for some $\chi \in C_0^\infty$ with $\chi(x_0) \neq 0$ and for ξ near ξ^0. This is necessarily a conic (open) set. The complement of the points in $T^*\mathbf{R}^n \setminus 0$ where u is microlocally analytic is the analytic wave front set of u. This definition is equivalent to the Hörmander's one in (7.3) as shown by Bony [**9**], see also [**34**].

The following theorem is essential for proving support theorems in this book.

Theorem 7.1 (Sato-Kawai-Kashiwara [**29**]). *Let u be a distribution such that $0 \in \operatorname{supp} u$ and $\operatorname{supp} u \subset \{x^n \ge 0\}$. Then $(0, \pm e^n) \in \mathrm{WF_A}(u)$, where $e^n := (0, \dots, 0, 1)$.*

Under the conditions of the theorem, u cannot be analytic near 0, therefore, there will be $\xi \neq 0$ so that $(0, \xi) \in \mathrm{WF_A}(u)$. The proof is then completed by the following result by Kashiwara.

Theorem 7.2 (The "Watermelon theorem"). *Let u be a distribution satisfying $\operatorname{supp} u \subset \{x^n \ge 0\}$. If $(0, \eta) \notin \mathrm{WF_A}(u)$ for some $\eta \neq 0$, then $(0, (\eta', t)) \notin \mathrm{WF_A}(u)$ for all $t \in \mathbf{R}$.*

Here, $\eta = (\eta', \eta_n)$. The Watermelon theorem says that we have two alternatives for such an u at $x = 0$: either $\mathrm{WF_A}(u)$ consists of $(0, \lambda e^n)$, $\lambda \neq 0$ only; or there is ξ with $\xi' \neq 0$ so that $(0, \xi) \in \mathrm{WF_A}(u)$; and then all (x, ξ) not collinear with e^n will be in $\mathrm{WF_A}(u)$. In the latter case, by the closedness of $\mathrm{WF_A}(u)$ as a conic set, $(0, \xi) \in \mathrm{WF_A}(u)$ for all $\xi \neq 0$. Therefore, the alternatives are $\xi = \lambda e^n$ or all ξ. In either case, we get Theorem 7.1 as a corollary of Theorem 7.2.

The next lemma shows that independently of the behavior of the amplitude in the ξ variable, vanishing in a certain open cone means the corresponding ΨDO is analytically regularizing there.

Lemma 7.3. *Let $a(x, y, \xi)$ be a smooth function on $\Omega \times \Omega \times (\mathbf{R}^n \setminus 0)$so that for every $K \Subset \Omega \times \Omega$, there is $C > 0$, so that*

$$|\partial_x^\alpha a(x, y, \xi)| \le C^{|\alpha|+1} \alpha! (1 + |\xi|)^m, \quad \forall (x, y) \in K,\ \forall \xi \in \mathbf{R}^n,\ \forall \alpha$$

for some m. Let $a = 0$ for ξ in some open cone $\Gamma \subset \mathbf{R}^n \setminus 0$. Then for every $u \in \mathcal{E}'(\Omega)$, then $\mathrm{Op}(a)u$ (defined by (3.7), which makes sense as an oscillatory integral) is microlocally analytic in $\Omega \times \Gamma$.

Note that only analyticity in x is needed, and even though a may not be a (smooth) symbol in the lemma; in our applications, it will always be. On the other hand, the amplitude may not be of the kind $\tilde{a}(x,y,\xi)g^R(\xi)$ described below. Actually, our main use of this lemma is the following. Localizing our integral transform and forming the corresponding normal operator, we do get an amplitude supported in a cone, but if is not of the type ag^R. Then we use this lemma to complete the argument, see ... reference

The following "obvious" statements hold (but the proofs are not trivial): the analytic wave front set of $u|_Y$, where Y is open, is the intersection of $\mathrm{WF_A}(u)$ and T^*Y. Also, the projection of $\mathrm{WF_A}(u)$ on the base is the analytic singular support $\mathrm{supp_A}\ u$ of u, i.e. the complement of the largest open set where u is analytic.

7.2. Analytic ΨDOs. The continuous linear map $A : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$ is called analytically regularizing if its image is in $\mathcal{A}(\Omega)$. Those are the negligible operators in the analytic ΨDO calculus.

Examples of analytically regularizing operators are operators with analytic kernels, of course; and ΨDOs with amplitudes decaying exponentially in the following sense. If $a(z,w,\xi)$ is smooth in $\Omega^{\mathbf{C}} \times \Omega^{\mathbf{C}} \times \mathbf{R}^n$, holomorphic in (z,w), and if for every compact set $K \subset \Omega^{\mathbf{C}} \times \Omega^{\mathbf{C}}$ we have

$$|a(x,w,\xi)| \le C^{-|\xi|/C}, \quad \forall (z,w) \in K,\ \forall \xi \in \mathbf{R}^n, \tag{7.5}$$

then $\mathrm{Op}(a)$ is analytically regularizing. In particular, if $a(x,y,\xi)$ is analytic in (x,y) and has a compact support in the ξ variable, then $\mathrm{Op}(a)$ is analytically regularizing. Note that analyticity in ξ is not required in those examples.

Ideally, one would require analytic amplitudes to be analytic in all variables. This would exclude such important cases as $|\xi|^{-2}$, for example, or the operators $X'X$ and $R'R$. It would prevent elliptic analytic ΨDOs from having parametrices. The examples above show that analyticity in the ξ variable in compact sets contributes an analytically regularizing operator. This explains why we do not require analytic regularity near $\xi = 0$ below.

Definition 7.4 (pseudoanalytic amplitudes). *The smooth function $a(x,y,\xi)$ on $\Omega \times \mathbf{R}^n$ is called a pseudoanalytic amplitude of order m in $\Omega \times \Omega$ if there exists a complex neighborhood $\Omega^{\mathbf{C}}$ of Ω so that a it extends to a holomorphic function $a(z,w,\xi)$ in $\Omega^{\mathbf{C}} \times \Omega^{\mathbf{C}} \times \mathbf{R}^n$, so that for any compact set $K \subset \Omega^{\mathbf{C}} \times \Omega^{\mathbf{C}}$, there exists C, $R_0 > 0$ so that for all $(z,w) \in K$, $\xi \in \mathbf{R}^n$, and multiindices α,*

$$|\partial_\xi^\alpha a(z,w,\xi)| \le C^{|\alpha|+1}\alpha!|\xi|^{m-|\alpha|} \quad \textit{for } |\xi| > R_0 \max(|\alpha|, 1). \tag{7.6}$$

Definition 7.5 (analytic amplitudes). *The pseudoanalytic amplitude $a(x,y,\xi)$ in $\Omega \times \Omega$ is called an analytic amplitude if there exists a complex neighborhood $\Omega^{\mathbf{C}}$ of Ω and $\delta_0 > 0$ so that a it extends to a holomorphic function $a(z,w,\zeta)$ in* same estimate?

$$\left\{(z,w,\zeta) \in \Omega^{\mathbf{C}} \times \Omega^{\mathbf{C}} \times \mathbf{C}^n \,|\, 1 + |\Im\zeta| < \delta_0 |\Re\zeta|\right\}. \tag{7.7}$$

so that for any compact set $K \subset \Omega^{\mathbf{C}} \times \Omega^{\mathbf{C}}$, there exists C, $R_0 > 0$ so that for all $(z,w) \in K$, $\xi \in \mathbf{R}^n$, and multiindices α,

$$|\partial_\xi^\alpha a(z,w,\xi)| \le C^{|\alpha|+1}\alpha!|\xi|^{m-|\alpha|} \quad \textit{for } |x| > R_0 \max(|\alpha|, 1). \tag{7.8}$$

In fact, given a pseudoanalytic amplitude, one can find an analytic one so that the difference of the corresponding ΨDOs is analytically regularizing.

If we have a holomorphic a function on the set (7.7), it is enough that for some m, and for any $K^{\mathbf{C}} \Subset \Omega^{\mathbf{C}} \times \Omega^{\mathbf{C}}$ to have

$$|a(z,w,\zeta)| \le C|\zeta|^m, \quad \forall (z,w) \in K,\ \zeta \in \mathbf{C}^n,\ 1 + |\Im\zeta| < |\Re\zeta|/R. \tag{7.9}$$

Then the estimates (7.6) can be obtained by the Cauchy integral formula in the ball $|\zeta - \xi| \le |\xi|/(2R)$.

DEFINITION 7.6 (analytic ΨDO). *The ΨDO A in Ω is called an analytic ΨDO if for every $\tilde\Omega \Subset \Omega$ there exists a pseudoanalytic amplitude $\tilde a$ in $\tilde\Omega \times \tilde\Omega$ so that $A - \mathrm{Op}(\tilde a)$ is analytically regularizing in Ω.*

Analytic ΨDOs are analytic pseudolocal similarly to (3.5).

There is a calculus of the analytic ΨDOs similar to the smooth calculus but the proofs are mode delicate since in the asymptotic expansions, we need to control all constants. In particular, each analytic ΨDO $a(x,y,D)$ is equivalent (on any compact subset), modulo analytically regularizing operators, to an operator of the type $p(x,D)$, called symbol of A; and the symbol is unique, modulo analytically regularizing ones.

Microlocally equivalent symbols are defined in the following way. Let $p(x,\xi)$ be a pseudoanalytic symbol in Ω and if $U \subset \mathbf{R}^n$ is an open set, and let Γ^0 be an open comic subset of $\mathbf{R}^n \setminus 0$. Then we say that a is equivalent to 0 (microlocally) on $U \times \Gamma^0$, if its extends to a smooth function of $(z,\xi) \in U^{\mathbf{C}} \times \Gamma^0$ holomorphic in z and satisfying the following. For every conic-compact subset $\mathcal{K}$ of $U^{\mathbf{C}} \times \Gamma^0$,

$$|p(z,\xi) \le Ce^{-|\xi|/C}, \quad \forall (z,\xi) \in \mathcal{K}.$$

Here, $U^{\mathbf{C}}$ is a complex neighborhood of U, as usual. This allows us to define equivalent symbols near a fixed (x_0,ξ^0) in a natural way; and equivalent symbols in a conic set not necessarily of product type (by the requirement to be equivalent near every point).

One defines $\mathrm{Char}(A)$ in a natural way now: as the complement of all points, at which it symbol is equivalent to zero. Then (4.2) holds for the analytic wave front sets as well.

Symbols microlocally equivalent to zero in a conic set correspond to Ψ*DOs analytically regularizing in that open conic set*; i.e., operators for which $\mathrm{WF_A}(u)$ is not contained there for every $u \in \mathcal{E}'(\Omega)$. Then one can show that microlocally, every analytic ΨDO is equivalent to $\mathrm{Op}(a)g^R(D)$ with some g^R as in (7.2).

*** ... ***

7.3. Parametrices of elliptic analytic ΨDOs. Elliptic analytic ΨDOs are those analytic ΨDOs which are elliptic as standard ΨDOs. They have parametrices in standard sense but they also have parametrices in the analytic calculus.

THEOREM 7.7. *Let A be an analytic ΨDO in Ω, elliptic of order m. Then for every $\tilde\Omega \Subset \Omega$ there exist a ΨDO B in Ω of order $-m$ so that*

(i) B extends to $B : \mathcal{D}'(\Omega) \to \mathcal{E}'(\Omega)$ continuously;

(ii) B restricted to $\tilde\Omega$ is an analytic ΨDO (of order $-m$) in $\tilde\Omega$;

(iii) $AB - \mathrm{Id}$ and $BA - \mathrm{Id}$ are analytically regularizing operators in $\tilde\Omega$.

The theorem extends to matrix-valued operators which symbols are elliptic in the sense that they have a left inverse for $|\xi| \gg 1$; then we can claim that $BA - \mathrm{Id}$ is analytically regularizing only.

In particular, if A is an elliptic analytic ΨDO in Ω, then

$$\mathrm{WF_A}(Au) = \mathrm{WF_A}(u).$$

Standard elliptic ΨDOs are Fredholm but not invertible in general. On the othet hand, analytic elliptic ΨDOs are injective *on a priori compactly supported functions.*

COROLLARY 7.8. *Let A be an elliptic analytic ΨDO in Ω and let $Af = 0$ for some $f \in \mathcal{E}'(\Omega)$. Then $f = 0$.*

PROOF. Let $\tilde\Omega \Subset \Omega$ be such that $\operatorname{supp} f \subset \tilde\Omega$. Let B be a parametrix for A related to $\tilde\Omega$. Then $f = Rf$, where R is analytically regularizing in $\tilde\Omega$. Therefore, f is analytic in $\tilde\Omega$. Since it is compactly supported there, it vanishes by analytic continuation. □

REMARK 7.1. Note that it was enough to assume $Af \in \mathcal{A}(\Omega)$. Also, the assumption that f was a priori compactly supported was essential. Such a statement is clearly wrong on analytic compact manifolds (to which the analytic calculus extends) or in domains without the compactness assumption. For example $f \equiv 1$ is in the kernel of the Laplacian in the latter case; and also in the former one, if we define an analytic Riemannian metric on the manifold first.

7.4. Microlocally defined analytic ΨDOs. We would like to have a way to construct an analytic ΨDO of a fixed order m with a certain prescribed pseudoanalytic symbol when the phase variable ξ belongs to a fixed conic set V. In the standard ΨDO calculus, we can just extend the symbol smoothly by keeping it a symbol of order m, and we can set to to be zero outside a larger conic set. In the analytic calculus, such an extension has to be analytic (for large $|\xi|$) and may not exist. This looks as a technical problem only because we only want to do our analysis in V but possible lack of analytic extension outside V would prevent us form doing it. Fortunately, we can use the cutoffs g^R introduced earlier to solve the problem.

DEFINITION 7.9 (analytic amplitude in a conic set). *Fix $(x_0, \xi^0) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus 0)$. An analytic amplitude near (x_0, ξ^0) of order m us any holomorphic function $a(z, , w, \zeta)$ on the set*

$$\left\{(z, w, \zeta) \in \Omega^{\mathbf{C}} \times \Omega^{\mathbf{C}} \times \mathbf{C}^n |\, \Re\zeta \in \Gamma^0,\ 1 + |\Im\zeta| < \delta_0 |\Re\zeta|\right\}, \tag{7.10}$$

where $U^{\mathbf{C}}$ is an open neighborhood of x^0 in $\mathbf{C}^n$, Γ^0 is an open cone in $\mathbf{R}^n \setminus 0$ containing ξ^0 and $\delta_0 > 0$ is such that

$$|a(z, w, \zeta)| \le C|\zeta|^m \quad \textit{on the set (7.10).}$$

Let now $U = U^{\mathbf{C}} \cap \mathbf{R}^n$ be the real part of $U^{\mathbf{C}}$. Let $\xi^0 \in \Gamma \Subset \Gamma^* \Subset \Gamma^0$ be open cones. Let g^R be the function as in (7.2) related to Γ and Γ^*. Then we construct the ΨDO (but not an analytic ΨDO)

$$A^R u(x) = (2\pi)^{-n} \iint e^{\mathrm{i}(x-y)\cdot\xi} a(x, y, \xi) g^R(\xi) u(y)\, \mathrm{d}y\, \mathrm{d}\xi. \tag{7.11}$$

It follows from Lemma 7.3 that if we choose another $g'^{R'}$, then the difference of the resulting operators will be analytically regularizing near (x_0, ξ^0). In that sense, A^R is independent of the choice of g^R near (x_0, ξ^0).

LEMMA 7.10. *Let $V' \Subset V \subset U \subset \mathbf{R}^n$ be all open. If $u \in \mathcal{E}'(U)$ is microlocally analytic in $V \times \Gamma^0$, then $A^R u$ is analytic in V' for $R \gg 1$.*

The lemma implies the following, in particular. Let u be any distribution and let $\chi \in C_0^\infty$ be equal to one near x_0. Then $A^R \chi u$ is well defined. If we chose another such function $\tilde{\chi}$, then $A^R(\chi - \tilde{\chi})u$ is analytic near x_0 for $R \gg 1$ by the lemma. Indeed, if $V \ni x_0$ is small enough, then $(\chi - \tilde{\chi})u = 0$ in V, and by the lemma, $A^R(\chi - \tilde{\chi})u$ is analytic in a smaller neighborhood for $R \gg 1$. Therefore, by changing χ in $A^R \chi u$, we are just adding an analytic function to the result near x_0. Therefore, we have the freedom to chose different g^R and χ and the resulting $A^R \chi$ will be all microlocally equivalent near (x_0, ξ^0).

One can build now a symbolic calculus of the analytic ΨDOs in a conic neighborhood of (x_0, ξ^0). Every such operator has a symbol $\tilde{(}x, \xi)$ independent of y. We can compose such operators using the same composition rules. Elliptic operators are defined as in the global case but the elliptic estimate is only required in a conic neighborhood of (x_0, ξ^0). They have parametrices. In particular, we get that in such a conic neighborhood, for an elliptic A, Au is microlocally analytic there if and only if u is.

8. The complex stationary phase method of Sjöstrand

As we saw, the analytic ΨDO calculus requires very special cut-offs when we need to localize. In applications to the geodesic ray transform, the cut-offs are imposed naturally on the manifold of the geodesics. Let us say that we want to recover the analytic wave front set $\mathrm{WF_A}(f)$ of f knowing the weighted geodesic X-ray transform $X_w f$ of f with $w > 0$ and real analytic on some open set $\mathcal{U}$ of geodesics. This is how we prove a support theorem for X_w. Then one would want to study $N := X'\chi X$ where χ is a smooth cutoff restricting to $\mathcal{U}$. If there are no conjugate points, N is a ΨDO elliptic on the conormals of the geodesics in $\mathcal{U}$ on which $\chi > 0$, which recovers the smooth $\mathrm{WF}(f)$. Using the same kind of arguments to recover $\mathrm{WF_A}(f)$ is problematic because χ cannot be analytic and having support localized in some set unless it is zero. The resulting ΨDO N would not be in an analytic ΨDO; the symbol loses analyticity with respect to all variables x, y, ξ. We cannot have the luxury of choosing the allowed cutoffs $\chi_N(x) g^R(\xi)$; they are what they are. One can try to micro-localize in a smaller cone using the allowed cutoffs and analyze the difference, as done in ... in the Euclidean case but that approach has its own problems in the geodesic one.

A more direct approach to recover the analytic wave front set is to use the complex stationary phase method of Sjöstrand [**34**] as done in [**20**]. The idea is to multiply $X_w f = 0$ by $e^{\mathrm{i}\lambda\phi}$, $\lambda \gg 1$ with some properly chosen phase ϕ having a real and an imaginary parts, integrate with respect of some of teh variables, and then apply the complex stationary phase method in Sjöstrand [**34**] to obtain an FBI type of transform as in (7.4) which can be used to resolve $\mathrm{WF_A}(f)$. This argument is local in nature and the localization problems are easier to resolve.

The main tool is the following.

THEOREM 8.1. *Let $U \subset \mathbf{C}^n$ be a neighborhood of 0 and let ϕ be a holomorphic function on U satisfying the following: $z = 0$ is its only unique point, $\phi(0) = 0$ and $\det D^2\phi(0) \neq 0$. Let $V \Subset U$ be another neighborhood of 0 and assume that $\Re\phi(x) \geq 0$ for each $x \in V_{\mathbf{R}} := V \cap \mathbf{R}^n$ and $\Re\phi > 0$ on $\partial V_{\mathbf{R}}$. Then, there exist*

$C > 0$ and $\epsilon > 0$, so that for every bounded holomorphic function u on U we have

$$\begin{aligned} I(h) :&= \int_{V_{\mathbf{R}}} e^{-\lambda\phi(x)} u(x)\,\mathrm{d}x \\ &= (2\pi)^{n/2} \sum_{0\le k\le \lambda/C} \frac{1}{k!}\lambda^{-n/2-k} \left(\frac{1}{2}\tilde{\Delta}\right)^k \left(\frac{u}{J}\right)(0) + R(h), \end{aligned} \tag{8.1}$$

***** to be completed *****

APPENDIX C

Fourier Integral Operators

1. Conormal ditributions

2. FIOs with conormal kernels

3. More...

APPENDIX D

Elements of Riemannian Geometry and Tensor Analysis

We recall some basic facts about Riemannian geometry here. Similarly to the other appendices, this is not meant to be a self-contained exposition; we assume that the reader is familiar with the main concepts. We follow the local coordinates approach which should not be considered as a sign of disrespect for the invariant way to introduce the main concepts. We work in some domain $\Omega \subset \mathbf{R}^n$ which, with any coordinate system there can be considered as a coordinate chart of some manifold M. For this reason, when we have an invariantly defined object (say, a vector field) we consider it as such an object on M.

1. Vectors and covectors

We denote local coordinates by x^1, x^2, $\dots$. We use the Einstein summation convention all the time. By smooth functions, we mean C^∞ ones. Unless mentioned otherwise, all objects below are smooth. The most naive definition of a *vector* v at a point x is just an element of $\mathbf{R}^n$, "attached" at x, which, under a change of variables, transforms by the first law in (1.1). We think of the vectors as the pair (x, v). All vectors at a point x form a linear space $T_x\Omega$ called the *tangent space* at x. The collection of all $T_x\Omega$ is the *tangent bundle* $T\Omega$.

A *covector* w at x, i.e., (x, w), is defined similarly but we postulate the second change of variables law in (1.1). We define the *cotangent space* $T_x^*\Omega$ and the *cotangent bundle* $T^*\Omega$ in a similar way. If at any point x we are given a vector or a covector smoothly dependent on the point, we have vector/covector fields. We denote vector fields either by $(v^1, \dots, v^n)$ or by $v^i\partial_{x^i}$. The later reflects the fact that we can associate vectors to first order differential operators which share the same transformation laws, of course. Covector fields $(w_1, \dots, w_n)$ can be distinguished by their lower indices, and can be also represented by $w_i \mathrm{d}x^i$ because we can think of covectors as linear forms acting on vectors by $\langle w, v\rangle = w_i v^i$. This is clearly coordinate independent and can be used to define covectors once we have defined vectors.

A word of warning: vectors/covectors at different points cannot be compared. By definition, they are associated with its "base points". Even if they are "equal", a generic change of coordinates will change this. This makes vectors different from vectors in the Euclidean space in elementary geometry, where they can be translated freely without changing them. The harmony can be restored if in that particular case, we think of the Euclidean space as the tangent space $T_{x_0}\Omega$ at a fixed point.

The notations $v^i\partial_{x^i}$ and $w_i\mathrm{d}x^i$ make it easy to remember the coordinate change laws. If $(x'^1, \dots, x'^n)$ is another coordinate systems (i.e., $x \to x'$ is a diffeomorphism between two open sets), then v and w have components v'^i and w'_i computed by the chain rule

$$v^i \frac{\partial}{\partial x^i} = v^i \frac{\partial x'^j}{\partial x^i}\frac{\partial}{\partial x'^j} \quad w_i \mathrm{d}x^i = w_i \frac{\partial x^i}{\partial x'^j}\mathrm{d}x'^j,$$

thus

$$v'^j = v^i \frac{\partial x'^j}{\partial x^i}, \quad w'_j = w_i \frac{\partial x^i}{\partial x'^j}. \tag{1.1}$$

Given a function f, the *differential* $\mathrm{d}f$ is a covector defined by $\mathrm{d}f = f_{x^i}\mathrm{d}x^i$. If $t \mapsto c(t) \in M$ is a smooth curve, then $\dot{c}(t) = \mathrm{d}c/\mathrm{d}t$ is a vector field (along c) with components $c^i(t)$ at any point of the curve.

2. Tensor fields

A *tensor* $h^{i_1\dots i_r}_{j_1\dots j_s}$ (in a fixed coordinate system) of type (r,s), where r, s are non-negative integers and all indices vary from 1 to n is just a map from the set of indices to $\mathbf{R}$. We also associate to each tensor a multi-linear map with "matrix" given by that tensor:

$$h^{i_1\dots i_r}_{j_1\dots j_s}\mathrm{d}x^{j_1}\otimes\cdots\otimes\mathrm{d}x^{j_s}\otimes\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_r}}.$$

When we have a tensor at each point x, smoothly depending on it, we call it a *tensor field*. Similarly to vectors and covectors, we postulate the following transformation law under change of coordinates:

$$h'^{i_1\dots i_r}_{j_1\dots j_s} = \frac{\partial x'^{i_1}}{\partial x^{k_1}}\cdots\frac{\partial x'^{i_r}}{\partial x^{k_r}}\frac{\partial x^{l_1}}{\partial x'^{j_1}}\cdots\frac{\partial x^{l_s}}{\partial x'^{j_s}}h^{k_1\dots k_r}_{l_1\dots l_s}. \tag{2.1}$$

Tensors/tensor fields of the type (r,s) are called r times contravariant and s time covariant. Tensor fields of type $(0,0)$ are just functions; those of type $(1,0)$ are vector fields, type $(0,1)$ are covector fields. The change of variables makes the associated multilinear map on s vectors and r covectors, invariant under coordinate changes.

3. Riemannian metrics

A *Riemannian metric* on an open domain Ω is defined by a positive definite quadratic form depending on the base point x, i.e., by a symmetric matrix $g_{ij}(x)$ with the property

$$g_{ij}(x)v^iv^j \ge |v|^2/C$$

on any compact subset. We will not study metrics degenerating at the boundary, and by a metric in $\bar{\Omega}$, we mean $\{g_{ij}\}$ satisfying the estimate above uniformly on Ω. Metrics are tensors of type $(0,2)$ and we often use the notation

$$g_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

to emphasize that we thing of $g = \{g_{ij}\}$ as a bilinear form.

We use the convention of raising and lowering the indices freely. For example, given a vector field v as above, v_i stands for $v_i = g_{ij}(x)v^j$. This allows to identify vector fields and covector ones.

The norm in the metric is denoted by $|v| = (g_{ij}(x)v^iv^j)^{1/2}$ for vectors and by the same notation $|w| = (g^{ij}(x)w_iw_j)^{1/2}$ for covectors. Here, $\{g^{ij}\}$ is the matrix inverse to g which is consistent with the raising of the indices convention. Clearly, if we identify the vector v with a covector w, they would have the same norms. The inner product of vectors is denoted sometimes by $\langle u, v\rangle = g_{ij}u^i v^j$, and we use the same notation for covectors.

A metric allows us to measure magnitudes/lengths of vectors, and therefore lengths of curves. If $[a,b] \ni t \to c(t) \in M$ is such a curve, then

$$\text{length}(c) = \int_a^b |\dot c(t)|\, \mathrm{d}t.$$

As usual, here $|\dot c|$ is the norm of the vector $\dot c$ in the metric. One can define naturally lengths of piecewise smooth curves. Then one can define the distance between two points x and y as the infimum of the lengths of all piecewise smooth curves connecting them. If we work in a domain (or in a closure of such), then we need to restrict those curves to the domain.

4. Volume forms

The metric defines an invariant *volume form* $\mathrm{d}\,\mathrm{Vol} = (\det g)^{1/2}\mathrm{d}x$. This allows us to define naturally L^2 spaces of functions with an invariant norm. The natural invariant volume form on TM is $(\det g)\mathrm{d}x\,\mathrm{d}v$; and on T^*M, it is $\mathrm{d}x\,\mathrm{d}\xi$.

For a fixed x, the natural volume form on T_xM is given by $(\det g)^{1/2}\mathrm{d}v$. Then the volume form on TM is just $\mathrm{d}\,\mathrm{Vol}(x)\,\mathrm{d}\sigma_x(v)$. Similarly, the natural volume form on T^*M is $(\det g)^{-1/2}\mathrm{d}\xi$. On the tangent bundle, the natural choice of the area form is

$$\mathrm{d}\sigma_x(v) = (\det g)^{1/2}\,\mathrm{d}\sigma_0(v), \quad \mathrm{d}\sigma_0(v) := \frac{1}{|v|}\sum_{i=1}^n (-1)^i v^i \mathrm{d}v^1 \wedge \dots \wedge \widehat{\mathrm{d}v^i} \wedge \dots \wedge \mathrm{d}v^n,$$

where the hat over a term means that this term is ommiteed.

5. Geodesics

For every x, if y is close enough to x, there is a unique minimizing curve $\gamma(t)$ connecting x and y, called a geodesic. It satisfies the following geodesic differential equation which can be derived by variation of the curve:

$$\ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = 0. \tag{5.1}$$

Here,

$$\Gamma^k_{ij} = \frac{1}{2}g^{kp}\left(\frac{\partial g_{jp}}{\partial x^i} + \frac{\partial g_{ip}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^p}\right) \tag{5.2}$$

are the *Christoffel symbols*. It is easy to check that Christoffel symbols are not tensor fields because they do not satisfy (2.1) under coordinate changes.

Any curve satisfying the geodesic equation is called a geodesic (defined in an interval, finite or infinite). If $\gamma(t)$ is geodesic, then so is $\gamma(kt)$ for every constant $k \neq 0$. It can be computed directly, and it will be seen below from a different point of view, that $|\dot\gamma|$ remains constant along the geodesic. Therefore, one can choose unique parameterization up to reversing the direction from t to $-t$, and shifts, so that $|\dot\gamma| = 1$. Such geodesics are called unit speed geodesics.

Since the geodesic equation is a second order ODE, one way to determine unique solution is determined is to prescribe an initial point and an initial direction:

$$\gamma(0) = x, \quad \dot\gamma(0) = v.$$

It is unit speed if and only if v is unit (at x). We denote the corresponding geodesic by $\gamma_{x,v}(t)$. We call it maximal, if it is defined in the maximal interval containing $t = 0$.

The geodesic equation (5.1) can be written as a system on the tangent bundle: $\dot\gamma = v$, $\dot v^k = -\Gamma^k_{ij} v^i v^j$. The solution defines a *geodesic flow* on TM with a generator

$$G := v^i \frac{\partial}{\partial x^i} - \Gamma^k_{ij}(x) v^i v^j. \tag{5.3}$$

The flow preserves the the speed (i.e., $|v| = 1$ along the flow).

Liouville

[**4**]

The *exponential map* $v \mapsto \exp_x v$ is defined by

$$\exp_x v = \gamma_{x,v}(1).$$

Then $\gamma_{x,v}(t) = \exp_x(tv)$. The exponential map is jointly smooth in the x and in the v variables. This is less than obvious at $v = 0$. Indeed, it is obvious that it is smooth in polar coordinates for v, i.e., $(x, t, \theta) \mapsto \exp_x(t\theta)$ is smooth, where $|\theta| = 1$ by standard regularity results for ODEs. Since polar coordinates are singular at the origin, this does not prove smoothness at $v = 0$. Actually, the same map for magnetic dynamical systems is smooth in polar coordinates but C^1 only in the v variable, see, e.g., [**12**]. The reason for the smoothness in the non-magnetic case is that the geodesic equation (5.1) contains a quadratic form of $\dot\gamma$ versus a quadratic plus a linear one for the magnetic geodesic equation.

Since the proof of the smoothness of $\exp_x$ in some presentations is not done properly, we will do it here.

Theorem 5.1. *The map $(x, v) \mapsto \exp_x v$ is smooth on its domain of definition.*

Proof. We will study the Taylor's expansion at $v = 0$, which is the only "suspicious" point. Write $v = t\theta$, $|\theta| = 1$, $t \geq 0$. Then

$$\exp_x(t\theta)|_{t=0} = x, \quad \frac{\mathrm{d}}{\mathrm{d}t}\exp_x(t\theta)|_{t=0} = \theta.$$

To compute the higher order terms, we use the geodesic equation. Thus

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\exp_x(t\theta)|_{t=0} = -\Gamma^k_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)|_{t=0} = -\Gamma^k_{ij}(x)\theta^i\theta^j.$$

Therefore,

$$\exp_x(t\theta) = x + t\theta - \frac{t^2}{2}\Gamma^k_{ij}(x)\theta^i\theta^j + O(t^3). \tag{5.4}$$

The terms in this expansions are homogeneous polynomials of θ of dergrees 0, 1 and 2. Since $v = t\theta$, we get

$$\exp_x(v) = x + v - \frac{1}{2}\Gamma^k_{ij}(x)v^i v^j + O(|v|^3).$$

This shows that $\exp_x v$ is a twice differentiable at $v = 0$. In fact, using Taylor's theorem, see, e.g., [**1**], and a similar expansion in the x variable, we show that $\exp_x v$ is a C^2 function in all variables.

To prove C^3 regularity, we compute the third order term in (5.4) by differentiating the geodesic equation and then setting $t = 0$. That term then is

$$-\frac{t^3}{3!}\frac{\mathrm{d}}{\mathrm{d}t}\Gamma^k_{ij}(\gamma(t))\dot\gamma^i(t)\gamma^j(t)|_{t=0}.$$

It is easy to see that we get a homogeneous polynomial of $\dot\gamma$ of degree three (we have to use the geodesic equation again for two of the terms). We then complete the proof by induction. $\square$

We denote by $\exp_x^{-1}$ inverse map, i.e., $v = \exp_x^{-1} y$ if and only if $y = \exp_x v$. Such a v may not even exist for every y (given x) and there might be more than one solution in other cases. If y is close enough to x, then there is a unique solution with v close enough to 0 as the implicit function theorem easily implies. Even if y is in a "small" neighborhood of x, there might be other v not guaranteed by the implicit function theorem! For example, on the sphere, a point p close enough to the South Pole S (just a fixed point) is connected to S by a unique geodesic (a part of a meridian) with length equal to the distance $\operatorname{dist}(p, S)$. There is another, much longer geodesic (the remainder of the "grand circle"/meridian determined by the short one) which also connects p and S. Those two different geodesics determine two different vectors v so that $\exp_p v = S$, and it does not matter how close to S the point p is for this example; we can even take $p = S$. If we want to have an example in $\mathbf{R}^2$, we can just map a neighborhood of a fixed meridian to an open set in $\mathbf{R}^2$.

Given x and y, if there exists a unique geodesic γ connecting them, so that its length is $\operatorname{dist}(x, y)$, we say that there exists a unique minimizing geodesic connecting x and y. On the sphere, for example, this is true if and only if x and y are not antipodal. On the other hand, a minimizing geodesic on the sphere always exists. If M is *complete* (every geodesics extends to the whole real line), then for every two points there is a (possibly non-unique) minimizing geodesic connecting them.

6. Covariant Derivatives

Differential of a function is invariantly defined as a covector field. Coordinate derivatives of tensor fields however, even of order $(0, 1)$ or $(1, 0)$, are nor tensor fields because they do not transform the right way under coordinate changes. This calls for an invariant definition of taking derivatives of tensors called a *covariant derivatie*.

We postulate a few properties first. Among them: we want the derivative ∇h of a tensor field h of type (r, s) to be a tensor field of type $(r, s + 1)$; we want $\nabla f = \mathrm{d}f$ for every function; and we want to preserve the product rule in an appropriate sense. Then it turns out that there is a unique such operator ∇ (the reader is strongly advised to consult a Riemannian geometry book for a complete and precise statement, and proof, which we intentionally omit here). If h is a tensor field of type (s, r), in local coordinates, ∇h is the usual differential plus multiplication terms involving the Christoffel symbols:

$$\begin{aligned} \nabla_k h^{i_1\dots i_r}_{j_1\dots j_s} = {} & \frac{\partial}{\partial x^k} (\nabla u)^{i_1\dots i_r}_{j_1\dots j_s} \\ & + \sum_{m=1}^{\infty} \Gamma^{i_m}_{kp} h^{i_1\dots i_{m-1} p i_{m+1}\dots i_r}_{j_1\dots j_s} - \sum_{m=1}^{s} \Gamma^{p}_{k j_m} h^{i_1\dots i_r}_{j_1\dots j_{m-1} p j_{m+1}\dots j_s}. \end{aligned} \tag{6.1}$$

The expression on the left should be considered as a tensor of type $(r, s+1)$, not as an operator, depending on k, applied to $h^{i_1\dots i_r}_{j_1\dots j_s}$ for a fixed choice of the indices! For example, $\nabla_1 w_2$ depends on the other components of w, as well; and its meaning is just the $\{1, 2\}$-th component of the tensor ∇w. The most interesting cases for us

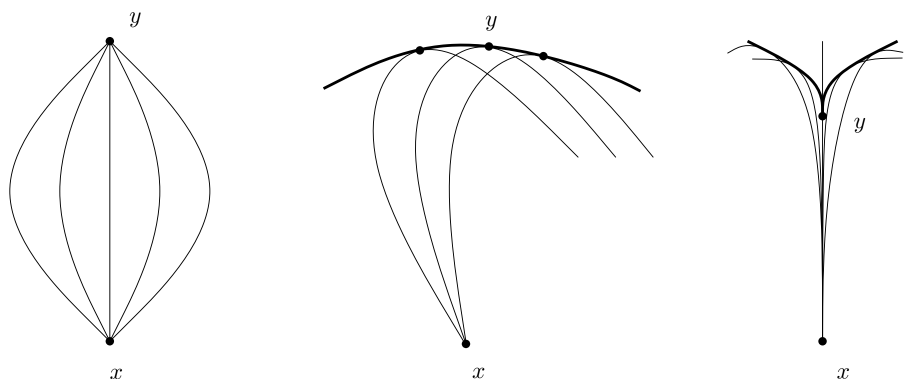

FIGURE D.1. Three types of conjugate point in the plane: blow down (like on the sphere), a fold, and a cusp.

are

$$\nabla_k f_{ij} = \partial_{x^k} f_{ij} - \Gamma^p_{ki} f_{pj} - \Gamma^p_{kj} f_{ip}, \tag{6.2}$$

and

$$\nabla_k w_i = \partial_{x^k} w_i - \Gamma^p_{ki} w_p, \quad \nabla_k v^i = \partial_{x^k} v^i + \Gamma^i_{kp} v^p. \tag{6.3}$$

Note that the operation of lowering or raising an index commutes with taking a covariant derivative.

Given a vector field v, one denotes by ∇_v the covariant derivative along X given in local coordinates by $\nabla_v = v^i \nabla_i$. The metric g is constant with respect to covariant differentiation. If $c(t)$ is a smooth curve, then the covariant $\nabla_{\dot c}$ derivative defined on that curve is denoted often by D_t. Then for every two fields on $c(t)$, we have

$$D_t\langle u, v\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t} u, v\right\rangle + \left\langle u, \frac{\mathrm{d}}{\mathrm{d}t} v\right\rangle.$$

Using covariant derivatives, the geodesic equation then reads

$$\nabla_{\dot\gamma}\dot\gamma = 0, \quad \text{or } D_t\dot\gamma = 0, \quad \text{or } D_t^2\gamma = 0,$$

which is equivalent to (5.1) in local coordinates. It can also be interpreted as requiring the tangent $\dot\gamma$ to be a parallel vector field along γ.

7. Conjugate points

Assume that x and y are connected by a geodesic γ. Then $y = \exp_x v_0$, where v_0 is the tangent vector at x of that geodesic with length $\operatorname{dist}(x, y)$. The points x and y are called *conjugate* along γ if the map $v \mapsto \exp_x v$ fails to be a local diffeomorphism for v near v_0. This is equivalent to saying that $\mathrm{d}_v \exp_x$ is degenerate at $v = v_0$. An example is any pair of antipodal points on the sphere. This is a rather symmetric example, however. A more generic example is a fold type of a conjugate point, where varying the direction v at x, one gets geodesics having an envelope curve, consisting of conjugate points to x. A cusp point is a another example, see Figure D.1.

Conjugate points can be described by the *Jacobi equation* which is the variation (the linearization) of the geodesic equation for a family of geodesics depening on a parameter:

$$D_t^2 J(t) + R(\dot\gamma(t), J(t))\dot\gamma(t) = 0, \tag{7.1}$$

where D_t^2 is the second covariant derivative along γ, i.e., $D_t^2 J(t) = \nabla_{\dot\gamma}\nabla_{\dot\gamma} J(t)$ and R is the *curvature*. The Jacobi equation is a second order ODE in local coordinates, and therefore for a unique solution, it is enough to prescribe $J(0)$ and $D_t J(0)$.

The points $\gamma(0)$ and $\gamma(1)$ are conjugate along the geodesic γ if and only if there exists a non-trivial Jacobi field J on $[0,1]$ so that $J(0) = J(1) = 0$. Then $D_t J(0)$ is a variation of v for which $\mathrm{d}_v \exp_p v$ vanishes.

***** introduce the trivial Jacobi fields *****

8. Hypersurfaces: Semigeodesic (boundary normal) coordinates

Let S be a smooth hypersurface near a fixed point. Let $x' = x'(p)$ be local coordinates on S, and set $x^n = \pm\,\mathrm{dist}(p, \partial M)$ with the sign depending on which side of S the point p lies (we can always choose an orientation locally). We accept the convention that $x^n > 0$ defines the "interior", even when S is not a closed surface. Then $x = (x', x^n)$ are called semigeodesic, or boundary normal coordinates. In those coordinates, $g_{in} = 0$, $\forall i$. This is easy to see on S given by $x^n = 0$ because $\partial/\partial x^n$ is orthogonal to it. To prove it for x close to ∂M, notice first that the lines $x' = 0$, $x^n = s$ are unit speed geodesics. Therefore, $g_{nn} = 1$ and by the geodesic equation (5.1), $\Gamma^k_{nn} = 0$ for every k. By (5.2), this implies $\partial g_{in}/\partial x^n = 0$ for every i. Since $g_{in} = 0$ for $x^n = 0$, we have the same for $0 \le x^n \ll 1$.

For future reference, in semigeodesic coordinates we have

$$g_{in} = 0, \quad \Gamma^i_{nn} = \Gamma^n_{in} = 0, \quad \forall i. \tag{8.1}$$

Those coordinates cannot be extended too far from S, in principle. If $S = \partial M$ is the boundary of a compact manifold, we can extend then by compactness to some neighbohood of ∂M. to the whole M, but not to the whole M. In those coordinates, the lines $x' =$ const. are geodesics, normal to the surfaces $x^n =$ const., and in particular to ∂M.

8.1. The second fundamental form and strict convexity of a hypersurface. Let S be locally defined hypersurface. The *second fundamental form* measures how curved S is in M. For every two vector fields X, Y on S, the second fundamental form $\mathrm{II}(X,Y)$ is defined as

$$\mathrm{II}(X,Y) = \langle \nabla_X \nu, Y\rangle,$$

where ν is the unit normal field corresponding to a chosen orientation. If $S = \partial M$ is the boundary of a compact manifold, the standard convention is to choose ν to point to the exterior. This formula shows that in particular, the second fundamental form is an extrinsic property of S, i.e., idepends on the induced metric on S only (which is just $g|_{TS}$).

Let $x = (x', x^n)$ be boundary normal coordinates to S. Then $\nu = -\partial/\partial x^n$. Then by (6.3), $\nabla_k \nu^i = -\Gamma^i_{kn}$; and therefore,

$$\mathrm{II}(X,Y) = \Gamma^n_{\alpha\beta} X^\alpha Y^\beta = -\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^n} X^\alpha Y^\beta. \tag{8.2}$$

Definition 8.1. *∂M is called strictly convex if the second fundamental form on ∂M is strictly positive.*

Combining (8.2) with the geodesic equation (5.1), we see that for any geodesic $\gamma_{x,v}$ tangent to S at $x \in S$, we have

$$\gamma^k_{x,v}(t) = x^k + tv^k - \frac{t^2}{2}\Gamma^k_{ij}(x)v^i v^j + O(t^3).$$

In particular, since $x^n = 0$ and $v^n = 0$,

$$\gamma^n_{x,v}(t) = -\frac{t^2}{2}\Gamma^k_{ij}(x)v^\alpha v^\beta + O(t^3),$$

where the Greek indices run from 0 to $n-1$. Since the form on the right is negative, we see that the geodesic is in the exterior of S for $0 < |t| \ll 1$. Moreover, $\mathrm{dist}(S, \gamma_{x,v}(t)) \sim t^2$. By continuity, geodesics close to tangent to S will have the same behavior of their signed distance to S see Figure D.2. This argument shows that strict convexity corresponds to our expectation what the later should mean, and can in fact be defined by it.

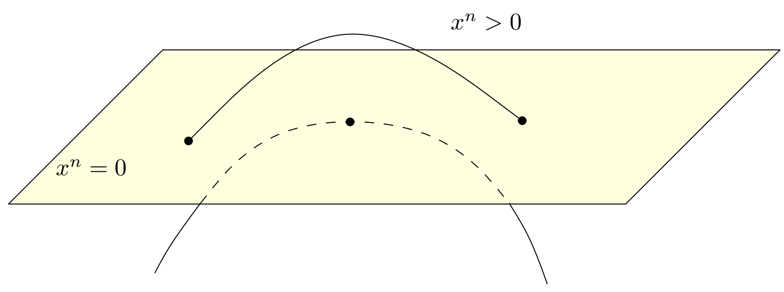

FIGURE D.2. Geodesics near a strictly convex surface $S = \{x^n = 0\}$. The interior is above the surface.

8.2. Area form of a hypersurface. Let S be a locally defined hypersurface. In boundary normal coordinates, the area (volume) form on S is given by $\mathrm{d}S = (\det g)^{1/2}\mathrm{d}x'$. Recall that $g_{jn} = \delta_{jn}$, and that $g_{\alpha\beta}$ is the induced metric on $S = \{x^n = 0\}$ (here, as always, $1 \le \alpha \le n-1$). Then $\det g = \det\{g_{\alpha\beta}\}$ as well. The reason that $\mathrm{d}S$ is exactly that form is that this the only form with $\mathrm{d}S\,\mathrm{d}x^n = \mathrm{d}\,\mathrm{Vol}$, and x^n is the signed geodesic distance to S. On the other hand, x' are arbitrary coordinates on S; therefore the area form on S is just the volume form on S considered as a Riemannian submanifold itself, with the induced metric.

8.3. Relation to Hamiltonian mechanics. Let $H(x,\xi)$ be a smooth function on $T^*\mathbf{R}^n$ called *Hamiltonian*. The corresponding dynamical system is given by

$$\dot{x}^j = \frac{\partial H}{\partial \xi_j}, \quad \dot{\xi}_j = -\frac{\partial H}{\partial x^j}. \tag{8.3}$$

It is straightforward to check that $H = \mathrm{const.}$ on the integral curves of H. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}H(x(t),\xi(t)) = \frac{\partial H}{\partial x^j}\dot{x}^j + \frac{\partial H}{\partial \xi_j}\dot{\xi}_j = 0.$$

Assume that $\mathrm{d}H = 0$ locally; then $\{H = E\}$, with the "energy" E fixed is called an energy surface of H. The Hamiltonian curves stay on that surface then.

The variables x and ξ naturally "live" on the cotangent bundle under change of variables.

Let $g_{ij}(x)$ be a Riemannian metric. Consider the Hamiltonian

$$H(x,\xi) := \frac{1}{2} g^{ij}(x)\xi_i\xi_j,$$

where, as usual, $\{g^{ij}\}$ is the inverse of $\{g_{ij}\}$. Then the integral curves of H on the energy level $H = 1/2$ coincide with the unit speed geodesics $(\gamma(t), \dot{\gamma}(t))$ by the identification of vectors and covectors provided by the metric g. This can be checked directly.

8.4. Liouville's Theorem. Given a Hamiltonian system, let Ψ^t be the flow on $T^*\mathbf{R}^n$. Then Liouville's Theorem says that for any measurable subset D, $\mathrm{Vol}(\Psi^t(G))$ is independent of t (for those t for which the dynamics is still defined). In other words, the volume form $\mathrm{d}x\,\mathrm{d}\xi$ is invariant under the flow. This follows, for example, from the fact that the Hamiltonian vector field $\left(\frac{\partial H}{\partial \xi_j}, -\frac{\partial H}{\partial \xi_j}\right)$ is divergence free; or from the fact that the flow preserves the symplectic form $\mathrm{d}x \wedge \mathrm{d}\xi$ and its powers, see, e.g., [**3**]. From the relation between the geodesic and the Hamiltonian flow, we get that the volume form $(\det g)\mathrm{d}x\,\mathrm{d}v$ is preserved under the geodesic flow on $T\mathbf{R}^n$.

Remarks: Petrov [**27**]

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