THE IDENTIFICATION PROBLEM FOR THE ATTENUATED X-RAY TRANSFORM

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ABSTRACT. We study the problem of recovery both the attenuation a and the source f in the attenuated X-ray transform in the plane. We study the linearization as well. It turns out that there is a natural Hamiltonian flow that determines which singularities we can recover. If the perturbation δa is supported in a compact set that is non-trapping for that flow, then the problem is well posed. Otherwise, it may not be, and least in the case of radial a, f, it is not. We present uniqueness and non-uniqueness results for both the linearized and the non-linear problem; as well as a Hölder stability estimate.

1. INTRODUCTION

We study the attenuated X-ray transform

$$X_a f(x,\theta) = \int e^{-Ba(x+t\theta,\theta)} f(x+t\theta)\,\mathrm{d}t, \quad x\in \mathbf{R}^2,\ \theta\in S^1, \tag{1.1}$$

in the plane with a source f and an attenuation a that we want to recover. We denote by

$$Ba(x,\theta) = \int_0^\infty a(x+t\theta)\,\mathrm{d}t \tag{1.2}$$

the "beam transform" of a, usually denoted by Da. We will assume that both a and f are compactly supported. In applications, a constant attenuation a is also considered but when observations are made on the boundary of a compact domain, one can replace that constant by a constant multiple of the characteristic function of that domain.

The problem that we study is: can we recover both a and f from knowledge of $X_a f$? Sometimes this is called the *Identification Problem* (for SPECT).

This problem arises in Single Photon Emission Computerized Tomography (SPECT). Radioactive markers are injected into a patient's body and the emitted X-rays, attenuated by the body, are detected outside of it. The problem is to recover the source with a unknown attenuated coefficient.

When a is known, it is known that f can be reconstructed uniquely, even by means of explicit formulas. In this connection, the first analytic reconstruction method was developed in [1] and the first Radon type explicit inversion formula was given in [23]. For more information and related results, see [6, 23, 24]. For this reason, some of the numerical attempts to do a reconstruction are focused on recovery, or getting a good approximation of a first, instead of treating (a, f) as a pair. Sometimes this is called *attenuation correction*, see e.g., [37, 26]. In clinical applications, additional X-rays are taken to reconstruct a first. Eliminating or reducing those additional X-rays remains an important problem.

There has not been much progress in the mathematical understanding of the identification problem so far. A related but not identical problem for finding both a constant attenuation and the source in the exponential X-ray transform has been solved in [27], see also [15]. The main result in [27] is, roughly speaking, that specific pairs of constant a and radial f cannot be distinguished but all other pairs can. The identification problem with f a finite sum of delta sources has been studied in [20, 21], see also [3], but the results there

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do not and cannot imply uniqueness. Natterer also viewed the problem as a range characterization problem: if the ranges of X_{a_1} and X_{a_2} happen to be the same, for example, then there cannot be uniqueness. Range conditions, see e.g., [24], have been viewed as a possible tool for solving the problem, both numerically, see e.g., [5] and analytically, as in the recent work [2]. Numerical reconstructions have been tried, too, with variable success, in [8, 19, 37, 26, 4, 5, 12, 38], for example. Some of them use clinical data. A. L. Bukhgeim [7] recently outlined a recovery algorithm if a is a priori known to be a constant multiple of the characteristic function of a star-shaped domain.

Our approach is based on the following. The attenuated X-ray transform, and its linearization, carry information about f and a along each line twice because we integrate both in θ and $-\theta$ directions. From a microlocal point of view, those two lines determine the wave front sets at covectors normal to them. So we have two equations for two unknowns. We study first a linear problem that appears as a linearization of X_af near some fixed (a_0, f_0). Also, the non-linear map $X_{a_2}f_2 - X_{a_1}f_1$ is of that form, see (5.3). This problem can be formulated as the inversion of $\mathcal{I}g := I_{w_1}g_1 + I_{w_2}g_2$, $g = (g_1, g_2)$, where I_w is the weighted X-ray transform with a weight $w(x, \theta)$, see (2.4). The weights $w_{1,2}$ are of specific type in the case of the Identification Problem but we study general weights first. The operator $\mathcal{I}$ is a Fourier Integral Operator but we do not study it directly. Instead, to analyze the equation $\mathcal{I}g = h$, we apply an explicit operator Q to convert the equations

$$(I_{w_1}g_1 + I_{w_2}g_2)\,(z, \pm\theta) = h(z, \pm\theta)$$

to equivalent pseudo-differential ones of the type

$$\Big(w_1(x, \pm D^\perp/|D|) + \text{l.o.t.}\Big)\, g_1 + \Big(w_2(x, \pm D^\perp/|D|) + \text{l.o.t.}\Big)\, g_2 = Qh, \tag{1.3}$$

see Proposition 6.1. Here, “l.o.t.” stands for “lower order terms”, and $w_j(x, \pm D^\perp/|D|)$ are pseudo-differential operators (ΨDOs) with symbols $w_j(x, \pm\xi^\perp/|\xi|)$. We view this as a 2×2 system of ΨDO equations. The determinant of the principal symbol is given by $p_0(x, \xi) = W(x, \xi^\perp/|\xi|)$, where

$$W(x, \theta) = w_1(x, \theta)w_2(x, -\theta) - w_1(x, -\theta)w_2(x, \theta).$$

Since W is an odd function of ξ, p_0 is not elliptic over any x, and has a non-trivial characteristic variety regardless of what $w_{1,2}$ are, in the cotangent bundle of any domain. Then p_0 is a Hamiltonian of fundamental importance for this system. The singularities of g that may not be recoverable lie on zero bicharacteristics of that Hamiltonian; moreover each zero bicharacteristic either consists of singularities only, or there is none on it. This brings us to the following condition, well known in the theory of ΨDOs of real principal type: if g has a support in a compact set K that is non-trapping for the Hamiltonian flow, then the singularities of g can be recovered, with a loss of one derivative. Otherwise they may not be but the non-trapping condition is known not to be “if and only if”. In sections 4 and 5, based on the microlocal understanding on the problem explained above, we prove actual injectivity and stability of $\mathcal{I}$ for f supported in non-trapping K for generic (w_1, w_2), including ones satisfying some analyticity assumptions; or for small K. We then apply this analysis to the non-linear Identification Problem in section 5.3 to get local uniqueness and Hölder stability in a neighborhood of generic (a, f), including those with analytic Ba, u, see the next section, under the a priori assumption that the perturbation of a is supported in a non-trapping set. There is no need to assume the same thing for the perturbation of f because the partial derivative of X_af w.r.t. f is just X_a, that is elliptic. Another explicit condition for local solvability of the Indentification Problem is (4.25), see Corollary 5.2.

The microlocal consequences of (1.3) are analyzed in more detail in Section 6. In particular, we describe the “null eigenspace” at the characteristic points. In non-degenerate cases, (1.3) is of rank one on the characteristic variety $p_0 = 0$. As a consequence of this, a certain linear combination of δa, δf is more stably reconstructed than either term, see Proposition 6.1.

We also study the case of radial a and f in Section 7. A thorough study of the radial case is behind the scope of this work however. The reason we include it is to present an example where the Hamiltonian flow can be explicitly computed, is trapping in any disk containing the origin, and the non-uniqueness set (for radial perturbations) is infinite dimensional. The projections of the zero bicharacteristics happen to be the circles $|x| = R$, $R \geq 0$. Then $K \subset \mathbf{R}^2$ is non-trapping if and only if it does not contain an entire circles centered at 0, including the origin. In case K is trapping, and contains a ball $|x| < R$, then the uniqueness fails and there is an infinite dimensional family of pairs (a, f) with the same data. They consist of radial a and f. This fact agrees with the microlocal analysis that we present because the latter implies we may not be able to recover radial singularities. In this case actually, the non-trapping condition is also necessary for the problem to be well posed.

In [25], Qian, Luo and the author present reconstruction methods and numerical examples confirming the theory developed in this paper.

2. Preliminaries

The attenuated X-ray transform results from the following transport equation model. Let $f(x)$ be a compactly supported source of particles (or a signal propagating along lines with unit speed) propagating in a medium with attenuation coefficient $a(x)$. Then at the point $x \in \mathbf{R}^n$ and direction $\theta \in S^{n-1}$ (the dimension n can be arbitrary), the total number of $u(x,\theta)$ of particles originating from the source solves the transport equation

$$(\theta \cdot \partial_x + a)u = f, \quad u|_{\theta\cdot x \ll 0} = 0. \tag{2.1}$$

This is a linear ODE along the lines $t \mapsto (x + t\theta, \theta)$ and its solution is given by

$$u(x,\theta) = \int_{-\infty}^{0} e^{-\int_t^0 a(x+s\theta)\,\mathrm{d}s} f(x+t\theta)\,\mathrm{d}t. \tag{2.2}$$

This formula can be interpreted as the superposition of all attenuated signals at (x,θ) coming from the source. Then at points x so that $\theta \cdot x \gg 0$, one has $u = X_a f$. We regard u as an attenuated beam transform of f (in the direction $-\theta$ instead of θ).

It is useful to extend the definition of B, see (1.2), to functions a depending on both x and θ:

$$Ba(x,\theta) = \int_0^{\infty} a(x+t\theta,\theta)\,\mathrm{d}t. \tag{2.3}$$

For such a, the solution to (2.1) is given by (2.2) again, with $a(x + t\theta)$ replaced by $a(x + t\theta, \theta)$.

We introduce also the notation

$$I_w f(x,\theta) = \int w(x+t\theta,\theta) f(x+t\theta)\,\mathrm{d}t, \tag{2.4}$$

for the weighted X-ray transform with weight $w(x,\theta)$. Then $I_w = X_a$ for $w = e^{-Ba}$ but we will allow more general weights in I_w. Also, $I_1 = X_0$.

We will also denote

$$v^{\perp} := (-v_2, v_1), \quad v = (v_1, v_2) \in \mathbf{R}^2.$$

2.1. **A Radon transform type of parameterization of X_a and I_w.** Since for a fixed direction θ, x and $x + s\theta$ parametrize the same (directed) line, we will think of $X_a f$ and $I_w f$ as parameterized by (z,θ), $z \in \theta_\perp := \{z;\ z\cdot\theta = 0\}$. We denote by Z the variety

$$Z = \{(z,\theta);\ \theta \in S^1,\ z \in \theta^{\perp}\},$$

which is essentially the tangent bundle of S^1. Then we can set $z = p\theta^\perp$, and write $X_a f$ as

$$X_a f(p\theta^\perp, \theta) = \int e^{-Ba(p\theta^\perp + t\theta, \theta)} f(p\theta^\perp + t\theta)\, \mathrm{d}t, \quad (p, \theta) \in \mathbf{R} \times S^1, \tag{2.5}$$

and similarly for $I_w f$. We think of $(p, \theta) \in \mathbf{R} \times S^1$ as a parameterization of Z. We also define a measure on Z by $\mathrm{d}z := \mathrm{d}p\, \mathrm{d}\theta$, where $\mathrm{d}\theta$ is the natural measure on S^1 given by $\mathrm{d}\vartheta$, with ϑ being the polar angle of θ.

2.2. **Functional spaces.** We will assume throughout the paper that $\operatorname{supp} f$ is contained in a fixed compact set; and we can always assume that this compact set is included in $(-\pi, \pi)^2$. We can therefore assume that f is defined on the torus $\mathbf{T}^2$. For any compact set $K \subset \mathbf{T}^2$, we define $H^s(K)$ to be the closed subspace of $H^s(\mathbf{T}^2)$ of functions supported in K. In other words, the Sobolev norm in K is defined through Fourier series. We define the Sobolev spaces $H^s(Z)$ in a similar way. Since $|p| < \pi$ in (2.5), we can assume that p belongs to the unit circle represented by $[-\pi, \pi]$ with both ends identified. Then $(p, \theta) \in S^1_p \times S^1_\theta$. The space $H^s(Z)$ is then defined by the norm

$$\|g\|_{H^s(Z)} = \left\|(1 - \partial_p^2)^{s/2} g\right\|_{L^2(Z)}, \tag{2.6}$$

where ∂_p^2 is the second derivative w.r.t. p on the compact manifold S^1. Notice that there are no θ derivatives in this definition, see also [22, Theorem II.5.2] for involving the θ derivatives when $a = 0$. In other words, $H^s(Z)$ is defined through Fourier Series in the p variable.

3. LINEARIZATION

We are going to compute the linearization of the identification problem starting from formula (1.1). Another way to do this, based on the transport equation, is presented in section 5.

Assume that a and f are smooth enough so that the calculations below make sense. Denote by $G = \theta \cdot \partial_x$ be the generator of the geodesic flow on $T\mathbf{R}^2$ w.r.t. the Euclidean metric. Since a has compact support, then $GBa = -a$, and $Ba = 0$ for $x \cdot \theta \gg 0$; and $Ba = I_1 a$ for $x \cdot \theta \ll 0$. Here, $I_1 = I_a$ for $a = 1$.

Since the problem is linear w.r.t. f, we linearize near some a first, with f fixed. Let $a_s = a + s\delta a$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} X_{a_s} f = -\int e^{-Ba(x+t\theta,\theta)} f(x + t\theta) B\delta a(x + t\theta, \theta)\, \mathrm{d}t.$$

Write

$$e^{-Ba} f = -GBe^{-Ba} f$$

and plug this into the formula above. Integrate by parts to get

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} X_{a_s} f = \int \left[\left(Be^{-Ba} f\right) \delta a\right] (x + t\theta, \theta)\, \mathrm{d}t - X_a f . X_0 \delta a.$$

The linearization of $X_a f$ w.r.t. a is therefore a weighted X-ray transform of the perturbation δa of the form

$$\int w(x + t\theta, \theta) \delta a(x + t\theta)\, \mathrm{d}t$$

with a weight function

$$w = Be^{-Ba} f - X_a f. \tag{3.1}$$

The second term on the right is constant along each line. The weight can also be expressed as

$$w(x, \theta) = -\int_{-\infty}^{0} e^{-Ba(x+t\theta,\theta)} f(x + t\theta)\, \mathrm{d}t. \tag{3.2}$$

A direct calculation yields

$$w = -e^{-Ba} u, \tag{3.3}$$

where u is the solution (2.2) of (2.1).

Let $\delta X_{a,f}(\delta a, \delta f)$ denote the linearization of $X_a f$ near fixed a, f. We just proved the following, see also (5.3).

Proposition 3.1.

$$\delta X_{a,f}(\delta a, \delta f) = I_w \delta a + X_a \delta f,$$

where w is as in (3.2) or (3.3).

4. A more general linear problem: invertibility of a sum of two weighted X-ray transforms

4.1. **Formulation and preliminaries.** Consider a more general problem. Let $\mathcal{I}(g_1, g_2) = I_{w_1} g_1 + I_{w_2} g_2$, where $w_{1,2}$ are two weight functions, i.e.,

$$\mathcal{I}(g_1, g_2)(x, \theta) = \int w_1(x + t\theta, \theta) g_1(x + t\theta)\, \mathrm{d}t + \int w_2(x + t\theta, \theta) g_2(x + t\theta)\, \mathrm{d}t. \tag{4.1}$$

We will compute $\mathcal{I}^*\mathcal{I}$, where the star denotes adjoint w.r.t. the inner product in $L^2(Z)$. Clearly,

$$\mathcal{I}^*\mathcal{I} = \begin{pmatrix} I^*_{w_1} I_{w_1} & I^*_{w_1} I_{w_2} \\ I^*_{w_2} I_{w_1} & I^*_{w_2} I_{w_2} \end{pmatrix} \tag{4.2}$$

By Proposition A.2, $\mathcal{I}^*\mathcal{I}$ is a ΨDO of order -1 with principal symbol

$$\delta a_p(\mathcal{I}^*\mathcal{I}) = \frac{\pi}{|\xi|} \begin{pmatrix} |w_{1,+}|^2 + |w_{1,-}|^2 & \bar{w}_{1,+} w_{2,+} + \bar{w}_{1,-} w_{2,-} \\ w_{1,+} \bar{w}_{2,+} + w_{1,-} \bar{w}_{2,-} & |w_{2,+}|^2 + |w_{2,-}|^2 \end{pmatrix} \tag{4.3}$$

where

$$w_{j,\pm} = w_j(x, \pm\xi^\perp/|\xi|), \quad j = 1, 2.$$

A direct calculation yields

$$\det \delta a_p(\mathcal{I}^*\mathcal{I}) = |w_{1,+} w_{2,-} - w_{2,+} w_{1,-}|^2 = \left| \det \begin{pmatrix} w_{1,+} & w_{2,+} \\ w_{1,-} & w_{2,-} \end{pmatrix} \right|^2. \tag{4.4}$$

That determinant not being zero is a microlocal ellipticity condition. As we see below, it vanishes over any point x; therefore, $\mathcal{I}^*\mathcal{I}$ cannot be elliptic over (i.e., in the cotangent bundle of) any domain.

Set

$$W(x, \theta) = w_1(x, \theta) w_2(x, -\theta) - w_1(x, -\theta) w_2(x, \theta). \tag{4.5}$$

Then $\det \delta a_p(\mathcal{I}^*\mathcal{I}) = |W(x, \xi^\perp)|^2$. The function W is odd in θ, and therefore, for any x it has zeros for some vectors θ. The inconvenience of working with (4.4) however is that it has double characteristics.

Instead of studying the invertibility of $\mathcal{I}^*\mathcal{I}$, we will approach the problem in a more direct way, slightly different (but equivalent) than what we do in Section 6, see also (1.3). Set

$$Jh(x, \xi) = h(x, -\xi). \tag{4.6}$$

Let $\alpha(x, \theta)$ be any smooth function, odd on S^1 w.r.t. θ. Let I'_w be the transpose of I_w, see the Appendix. Apply $I'_{\alpha J w_2}$ to the equation

$$I_{w_1} g_1 + I_{w_2} g_2 = h \tag{4.7}$$

to get

$$I'_{\alpha J w_2} I_{w_1} g_1 + I'_{\alpha J w_2} I_{w_2} g_2 = I'_{\alpha J w_2} h. \tag{4.8}$$

By Proposition A.2, both operators on the left are ΨDOs of order -1. The principal symbol of $I'_{\alpha J w_2} I_{w_1}$ is given by $2\pi/|\xi|$ times the even part of $(\alpha w_1 J w_2)(x, \xi^\perp/|\xi|)$, i.e., by $2\pi\alpha(x, \xi^\perp/|\xi|)/|\xi|$ times the odd part of $(w_1 J w_2)(x, \xi^\perp/|\xi|)$. Thus

$$\sigma_p(I'_{\alpha J w_2} I_{w_1}) = \frac{\pi}{|\xi|} \alpha W\big|_{\theta=\xi^\perp/|\xi|}. \tag{4.9}$$

Notice that W is the determinant in the r.h.s. of (4.4) but not squared. It has the same zeros as (4.4) but they are simple. In the same way, we get that the principal symbol of $I'_{\alpha J w_2} I_{w_2}$ is as above but with w_1 replaced by w_2, i.e., it is zero; and therefore, $I'_{\alpha J w_2} I_{w_2}$ is of order -2.

Choose $\alpha = \theta_1$ first. Then $|\xi|\alpha(\xi^\perp/|\xi|) = -\xi_2$, which is the symbol of $-D_2 = \mathrm{i}\partial_2$, and we get

$$\sigma_p(I'_{\theta_1 J w_2} I_{w_1}) = \frac{\pi}{|\xi|^2}(-\xi_2) W(x,\theta)\big|_{\theta=\xi^\perp/|\xi|}. \tag{4.10}$$

Modulo lower order terms, (4.8) becomes

$$\frac{\pi}{|D|^2}(-D_2) W\left(x, D^\perp/|D|\right) g_1 \cong I'_{\theta_1 J w_2} h,$$

where the meaning of $1/|D|$ is given by the ΨDO calculus. Similarly, taking $\alpha = \theta_2$, we get

$$\frac{\pi}{|D|^2} D_1 W\left(x, D^\perp/|D|\right) g_1 \cong I'_{\theta_2 J w_2} h.$$

Apply $-D_2$ to the first identity, D_1 to the second, and add them together to get

$$\pi W\Big(x, D^\perp/|D|\Big) g_1 \cong \left(-D_2 I'_{\theta_1 J w_2} + D_1 I'_{\theta_2 J w_2}\right) h.$$

Notice that the lower order terms on the left involve g_2 as well. In a similar way we get

$$\pi W\Big(x, D^\perp/|D|\Big) g_2 \cong \left(D_2 I'_{\theta_1 J w_1} - D_1 I'_{\theta_2 J w_1}\right) h.$$

We therefore proved the following.

Proposition 4.1. *For all compactly supported g_1, g_2 we have*

$$Pg = (\pi \mathrm{i})^{-1}(\partial_1 I'_{\theta_2 J w_2} - \partial_2 I'_{\theta_1 J w_2}, -\partial_1 I'_{\theta_2 J w_1} + \partial_2 I'_{\theta_1 J w_1})\mathcal{I} g, \tag{4.11}$$

where $g = (g_1, g_2)$, and P is a matrix valued classical ΨDO of order 0 with a scalar principal symbol given by

$$p_0(x,\xi) := W\Big(x, \xi^\perp/|\xi|\Big). \tag{4.12}$$

In particular, this means that P is a matrix operator of real principal type, see [9].

We notice that (4.11) can also be written in the form

$$\pi \mathrm{i} Pg$$
$$= \left(-\int_{S^1} (\theta^\perp\cdot\partial_x) w_2(x,-\theta)\mathcal{I}g(x-(\theta\cdot x)\theta,\theta)\,\mathrm{d}\theta, \int_{S^1} (\theta^\perp\cdot\partial_x) w_1(x,-\theta)\mathcal{I}g(x-(\theta\cdot x)\theta,\theta)\,\mathrm{d}\theta\right).$$

Let

$$\Sigma = \{(x,\xi)\in T^*\mathbf{R}^n\setminus 0;\ p_0(x,\xi)=0\} = \{W=0\}^\perp \tag{4.13}$$

be the characteristic variety of p_0, where the sign $\perp$ applies to the second variable θ only.

There are several definitions of real principal type ΨDOs in the literature, including or not the differential condition below, or the non-trapping one, in a fixed domain. We will use the following one. We say that the ΨDO $P\in\Psi^m$ is of real principal type, if its principal symbol p_m is real, scalar, homogeneous in ξ, and $\mathrm{d}p_m$ is not collinear to $\xi\mathrm{d}x$ on $\{p_m=0\}$ for $\xi\neq 0$. The latter condition says that if we identify covectors of different length by their direction, then the Hamiltonian vector field never vanishes, and in particular, the flow does not have stationary points. Such operators are microlocally equivalent to ∂_{x_1} modulo lower order terms. We also note that this condition makes $\{W = 0\}$ a codimension one (dimension 2) smooth submanifold. The same applies to Σ, considered as part of the unit cotangent bundle.

Below, $\partial_{\theta^\perp}$ is the angular derivative in the θ variable, i.e., the derivative $\partial/\partial\vartheta$ w.r.t. to the polar angle ϑ.

Proposition 4.2. *The ΨDO P is of real principal type in some domain $\Omega \subset \mathbf{R}^2$, if and only if*

$$W,\ \theta\cdot\partial_x W,\ \partial_{\theta^\perp} W \text{ cannot be all zero at the same time, for any point in } \Omega\times S^1. \tag{4.14}$$

Proof. Extend W to $\theta \neq 0$ as a homogeneous function of order 0. We have

$$\begin{aligned} \mathrm{d}p_0 &= \partial_x W(x,\xi^\perp)\,\mathrm{d}x + (\partial/\partial\xi)W(x,\xi^\perp)\,\mathrm{d}\xi \\ &= W_x(x,\xi^\perp)\,\mathrm{d}x + (W_{\theta_2}, -W_{\theta_1})(x,\xi^\perp)\,\mathrm{d}\xi. \end{aligned}$$

Then P is of real principal type if an only if $W(x,\xi^\perp) = 0$ and $(\partial/\partial\xi)W(x,\xi^\perp) = 0$ imply that $\partial_x W(x,\xi^\perp)$ is not collinear with ξ. The latter is equivalent to the requirement that $\partial_x W(x,\xi^\perp)$ is not normal to $\xi^\perp$, i.e., $\xi^\perp\cdot\partial_x W(x,\xi^\perp) \neq 0$. Set $\theta = \xi^\perp$. Then the requirement is the following: if $W(x,\theta) = 0$ and $(\partial_{\theta^\perp} W)(x,\theta) = 0$, then $\theta\cdot\partial_x W(x,\theta) \neq 0$. Notice next that the radial derivative $\partial/\partial|\theta|$ of W vanishes on $W = 0$, therefore the requirement on $\partial_{\theta^\perp} W$ is actually a requirement on the angular derivative only. □

The following condition plays a critical role in the theory of local solvability of ΨDOs of real principal type. Let K be a compact subset of $\mathbf{R}^2$.

Definition 4.1. *We say that K is* non-trapping *(for p_0) if there is no maximally extended zero bicharacteristic that lies entirely over K. We call the projections of the zero bicharacteristics of p_0 to the x space* rays.

Notice that the rays are continuous but not necessarily smooth curves. They may even degenerate to a point, see Example 4.1.

Hörmander's propagation of singularities theorem (see, e.g., [34, VI.2.1] or [18, Theorem 26.1.4]) implies that if g is supported in a non-trapping K, and $Pg \in H^s$, then $g \in H^{s-1}$. In other words, we have non-local "hypoellipticity", with a loss of one derivative. As a consequence, by the open mapping theorem, for any s and ℓ, there is $C > 0$ so that

$$\|g\|_{H^{s-1}} \le C\|Pg\|_{H^s} + C\|g\|_{H^\ell}, \quad \forall g \in C_0^\infty(K), \tag{4.15}$$

see eqn. (VI.3.3) in [34].

Example 4.1. Let $w_1 = \frac{1}{2}\theta\cdot x$ and $w_2 = 1$. Notice that w_1 is not non-vanishing. In this case, $W = \theta\cdot x$. Then $|\xi|p_0 = x\cdot\xi^\perp = x_2\xi_1 - x_1\xi_2$ and Σ consists of (x,ξ), $\xi \neq 0$ that are collinear. In other words, all singularities that may not be recoverable are the radial ones. The Hamiltonian equations are given by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1, \quad \dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = -\xi_1.$$

The zero bicharacteristics then are given by

$$x = R(\sin t, \cos t), \quad \xi = \lambda(\sin t, \cos t), \quad R \ge 0,\ \lambda \neq 0. \tag{4.16}$$

Their projections on the base (the rays) are given by the circles $x = R(\sin t, \cos t)$, $R \ge 0$. If $R = 0$, then that projection is a point. The whole bicharacteristic is not stationary however and is given by $x = 0$, $\xi = \lambda(\sin t, \cos t)$, $\lambda \neq 0$. We then see that a compact set K is non-trapping if and only if K contains no entire circle $|x| = R$, $R \ge 0$ (including the origin), see Figure 2. Then $\mathcal{I}g$ recovers the singularities of $g = (g_1, g_2)$. If K is trapping, the singularities that may not be possible to be recovered are the radial ones.

Inverting $\mathcal{I}g = I_{w_1}g_1 + I_1 g_2$ is easy. The first term is odd w.r.t. θ, and the second one is even. The equation $\mathcal{I}g = h$ then decouples into two equations $I_{w_1}g_1 = h_{\mathrm{odd}}$, $I_1 g_2 = h_{\mathrm{even}}$. The kernel of $\mathcal{I}$ (on

$\mathcal{E}'(\mathbf{R}^2)$) then consists of pairs $(g_1, 0)$, where $g_1 \in \operatorname{Ker} I_{w_1}$. Using arguments similar to the Fourier Slice theorem, see Section 6, we can see easily that $I_{w_1} g_1 = 0$ if and only if $(x_2\partial_1 - x_1\partial_2)g_1 = 0$ (the operator in the parentheses is $p_0(x, D)$, up to an elliptic factor) and the solutions of the latter in $\mathcal{E}'(\mathbf{R}^2)$ are given by all compactly supported radial distributions. In particular, on $L^2_{\text{comp}}(B(0,1))$, the kernel of I_{w_1} consists of all radial functions in that space. We therefore get

$$\operatorname{Ker} \mathcal{I} = \{(g_1, 0);\ g_1 \text{ is radial}\}\,. \tag{4.17}$$

Since radial functions can have (radial) singularities at all points, we get that no radial singularity of g_1, i.e., a singularity of the type $(x = \mu\xi, \xi)$, $\mu \in \mathbf{R}$, $\xi \neq 0$, can be recovered in general. On the other hand, if K is non-trapping for p_0, and $\operatorname{supp} g \subset K$, then they can. In that case, I_{w_1} is microlocally equivalent to the derivative w.r.t. to the polar angle in $\mathbf{R}^2$; and since on each circle there is an open arc where $g = 0$, that circle cannot support a singularity of g, by the propagation of the singularities theorem. In fact, by (4.17), if K is non-trapping, we have more: $g_1 = g_2 = 0$.

This example also reveals that $\mathcal{I}g \in C^\infty$ is not microlocally equivalent to (4.11), see Section 6 for more details. Indeed, only the radial singularities of g_1 are not recoverable, while those of g_2 are recoverable. Moreover, assume that $\mathcal{I}g \in H^s(Z)$. Then $g_2 \in H^{s-1/2}$ by the usual inversion results. We can think of $I_{w_1} g_1$ as the Doppler X-ray transform of the vector field $g_1(x)x$. It is well know that we can only reconstruct the curl of $g_1(x)x$ that is $(x_1\partial_2 - x_2\partial_1)g_1$, and the latter is in $H^{s-3/2}$. Let $\operatorname{supp} g_1 \in K$, with K non-trapping, for example, assume that the ray $\{x_1 \geq 0,\ x_2 = 0\}$ does not intersect K. Then, in polar coordinates (r, ϕ),

$$g_1(r, \phi) = \int_0^\phi \left[(x_1\partial_2 - x_2\partial_1)g_1\right](r, \psi)\, \mathrm{d}\psi.$$

This integration is not smoothing (not in all directions), and we still have $g_1 \in H^{s-3/2}$, with an improved regularity in angular directions. This is consistent with (4.20) below but as we see, the one derivative loss is only in g_1, and that estimate does not reveal the extra regularity in characteristic directions. The latter is however reflected by the fact that $\operatorname{WF}(g_1)$ can have radial directions only.

It is interesting to know when the rays are smooth curves. The projection of a bicharacteristic to the x variables, with its parameterization determined by the Hamiltonian equation, at some x, has a tangent vector $\partial_\xi p_0$ evaluated at some $(x, \xi) \in \Sigma$ (i.e., $p_0(x, \xi) = 0$). This projection is non-degenerate, and therefore, that ray is a smooth curve, if $\partial_\xi p_0 \neq 0$. There might be more than one ξ with that property but there is at least one ξ (and the whole line spanned by it) because p_0 is odd in ξ. On the other hand, $\xi \cdot \partial_\xi p_0 = 0$ on Σ, therefore a tangent vector is actually $\xi^\perp$ and the whole line that it spans.

Translating this in terms of W, see (4.12), we get the following. If for some (x_0, θ_0), we have

$$W(x_0, \theta_0) = 0 \text{ and } \partial_{\theta^\perp} W(x_0, \theta_0) \neq 0, \tag{4.18}$$

then there is a smooth ray through it. Moreover, starting from (x_0, θ_0) with that property, by the implicit function theorem, we can solve $W(x, \theta) = 0$ locally for θ. This gives us a smooth unit vector field, with integral curves that are rays. Then $W = 0 \Rightarrow \partial_{\theta^\perp} W \neq 0$ on $K \times S^1$ is a sufficient condition for all rays through K to be smooth.

4.2. **Basic Properties of $\mathcal{I}$.** Below, $\mathcal{E}'(K)$ stands for the space of distributions supported in K, and we similarly define $H^s_0(K)$. Also, since g is vector valued, $L^2(K)$, $H^S(K)$ below are spaces of vector valued functions.

Theorem 4.1. *Let $\mathcal{I}$ be as in (4.1) with w_1 and w_2 smooth. Let $K \subset \mathbf{R}^2$ be a non-trapping compact set. Then for any $s \geq 0$ we have the following.*

(a) For any ℓ, there exist a constant $C>0$ so that

$$\|g\|_{H^s(K)} \le C\|\mathcal{I}g\|_{H^{s+3/2}(Z)} + C\|g\|_{H^\ell(\mathbf{R}^2)}, \quad \forall g\in C_0^\infty(K). \tag{4.19}$$

(b) The kernel of $\mathcal{I}$ on $\mathcal{E}'(K)$ is finite dimensional, and consist of $C_0^\infty(K)$ functions g.

(c) On the orthogonal complement of $\operatorname{Ker}\mathcal{I}$ *in $H_0^s(K)$, we have*

$$\|g\|_{H^s(K)} \le C\|\mathcal{I}g\|_{H^{s+3/2}(Z)} \tag{4.20}$$

with another constant $C>0$. In particular, if $\mathcal{I}$ is injective on $C_0^\infty(K)$, then (4.20) holds.

Proof. To prove (a), we will apply (4.15). To this end, replace P by $\chi P\chi$, where $\chi\in C_0^\infty$ equals 1 near K (this makes P properly supported, in particular). Let Ω be a bounded domain containing $\operatorname{supp}\chi$. Use estimate (4.15) combined with (4.11) to get for any fixed ℓ,

$$\begin{aligned}\|g\|_{H_0^s(K)} &\le C\sum_{i,j=1}^{2}\|I'_{\theta_i Jw_j}\mathcal{I}g\|_{H^{s+2}(\Omega)} + C\|g\|_{H^\ell}\\ &\le C\|\mathcal{I}g\|_{H^{s+3/2}(Z)} + C\|g\|_{H^\ell}, \quad \forall g\in C_0^\infty(K),\end{aligned} \tag{4.21}$$

see also Proposition A.3. Notice the one derivative loss in this estimate since $\mathcal{I}$ is of order $-1/2$, see the Appendix. If we replace $\mathcal{I}$ by a single weighted X-ray transform I_w with a non-vanishing weight w, then one has the same estimate but with $H^{1/2}(Z)$. We also note that $\mathcal{I}g$ has compact support.

Consider (b). Every $g\in\mathcal{E}'(K)$ in the kernel of $\mathcal{I}$ must be smooth by propagation of singularities and by the assumption that K is non-trapping (it also follows from (4.21)). Apply then (4.19) to get

$$\|g\|_{H^1(K)} \le C\|g\|_{L^2(K)}, \quad \forall g\in \operatorname{Ker}\mathcal{I}\cap\mathcal{E}'(K) = \operatorname{Ker}\mathcal{I}\cap C_0^\infty(K).$$

Since the inclusion $H_0^1(K)\hookrightarrow L^2(K)$ is compact, we get the finite dimensionality of $\operatorname{Ker}\mathcal{I}$ on K.

Consider (c). Let $\mathcal{D}$ be the closure of $C_0^\infty(K)$ under the graph norm $\|g\|_{H_0^s(K)} + \|\mathcal{I}g\|_{H^{s+3/2}(Z)}$. We consider now $\mathcal{I}$ as an operator from $\mathcal{D}$ to $H^{s+3/2}(Z)$. Then $\mathcal{I}$ is a well defined bounded operator. Indeed, $\mathcal{D}$ is a subspace of the space of the compactly distributions, together with the topology. Then $\mathcal{I}$ can be considered as an operator originally defined as $\mathcal{I}:\mathcal{E}'(\mathbf{R}^2)\to\mathcal{E}'(Z)$, and then restricted to $\mathcal{D}$. We then get

$$\|g\|_{\mathcal{D}} \le C\|\mathcal{I}g\|_{H^{s+3/2}(Z)} + C\|g\|_{H^\ell}, \quad \forall g\in\mathcal{D}.$$

By (a), $\mathcal{I}$ is injective on $\mathcal{D}\cap(\operatorname{Ker}\mathcal{I})^\perp$. Then by [34, Proposition V.3.1], for $\ell<s$, we have the same inequality as above on $\mathcal{D}\cap(\operatorname{Ker}\mathcal{I})^\perp$ but without the last term. We refer also to [30, Lemma 3] as well for similar arguments, or to inequality (26.1.6) in [18]. □

In the applications to the linearized Identification Problem, one of the weights is non-vanishing. This allows us to weaken the non-trapping requirements.

Corollary 4.1. *Let*

$$\operatorname{supp} g_1\subset K_1, \quad \operatorname{supp} g_2\subset K_2 \tag{4.22}$$

where $K_{1,2}$ are compact sets. Assume that $w_2\neq 0$ on K_2. Then the conclusions of Theorem 4.1 hold if only K_1 is non-trapping.

Proof. Let K_1 be non-trapping. Choose an open set $U\supset K_1$ so that $\bar U$ is non-trapping as well. Let $\chi\in C^\infty$ be such that $\chi=1$ in a neighborhood of $K_2\setminus U$, and $\chi=0$ near K_1. Apply I'_{w_2} to $\mathcal{I}g$ first to get

$$I'_{w_2}\mathcal{I}g = I'_{w_2}I_{w_1}g_1 + I'_{w_2}I_{w_2}g_2. \tag{4.23}$$

Let Q be a properly supported parametrix of the elliptic ΨDO $I'_{w_2}I_{w_2}$. We get

$$\chi Q I'_{w_2}\mathcal{I}g = \chi g_2 + R_1 g_1 + R_2 g_2, \tag{4.24}$$

where R_1 and R_2 are smoothing operators; the first one by the pseudo-local property; and the second one, by the parametrix construction. Then we get the estimate (4.21) for χg_2; and actually, we can replace $s+3/2$ by $s+1/2$ there.

We now write

$$\mathcal{I}\left(g_1,(1-\chi)g_2)\right)=\mathcal{I}g-I_{w_2}\chi g_2.$$

Use (4.21) to estimate $(g_1,(1-\chi)g_2)$ through $\mathcal{I}g$ and χg_2, which we estimated already. This proves (4.21) for g. The rest of the proof is the same. $\square$

Remark 4.1. If, in addition, $w_1\neq 0$ on K_1, then it is enough $K_1\cap K_2$ to be non-trapping.

4.3. **Conditions for injectivity of** $\mathcal{I}$**.**

Corollary 4.2. *Let w_1 and w_2 be smooth. Let $x_0\in\mathbf{R}^2$ be such that*

$$W(x_0,\theta)=0 \quad\Longrightarrow\quad \partial_{\theta^\perp}W(x_0,\theta)\neq 0,\quad \forall\theta\in S^1. \tag{4.25}$$

Then if $0<\varepsilon\ll 1$, $\mathcal{I}$ is injective on distributions supported in the ball $B(x_0,\varepsilon)$, and in particular, (4.20) holds for $K=B(x_0,\varepsilon)$.

Proof. Condition (4.25) guarantees that any ray through x_0 is smooth at x_0, and there are finite number of such rays. There is $\varepsilon_0>0$ so that $B(x_0,\varepsilon_0)$ is non-trapping.

Assume the opposite. Then for any $\varepsilon=1/j$, $j\geq 1$, there is a non-trivial C_0^∞ function supported in $B(x_0,1/j)$ in the kernel of $\mathcal{I}$. Then we get an infinite number of non-trivial functions ϕ_j in the finitely dimensional space $V=\operatorname{Ker}\mathcal{I}\cap C_0^\infty(B(x_0,\varepsilon_0))$, see Theorem 4.1(b), with supports shrinking to the point x_0. This is a contradiction. Indeed, $-\Delta: V\to-\Delta V$ must be a bounded operator. On the other hand, $-\Delta$ is bounded below on $H_0^1(B(x_0,1/j))\cap H^2$ by its first eigenvalue μ_j, that tends to infinity as $j\to\infty$. Therefore, $(-\Delta\phi_j,\phi_j)/\|\phi_j\|^2\to\infty$, which is a contradiction. $\square$

Clearly, under the assumptions of Corollary 4.1, (4.25) is not needed if $x_0\in K_2\setminus K_1$.

Theorem 4.2. *Let w_1, w_2 be analytic in $\Omega\times S^1$, where Ω is an open set containing a non-trapping compact set $K\subset\mathbf{R}^2$. Then the operator $\mathcal{I}$, restricted to $\mathcal{E}'(K)$, is injective.*

Proof. We use a result about propagation of analytic singularities, see [13], for analytic ΨDOs with real principal symbols. The result in [13] covers in fact a more general class of operators with complex-valued principal symbols that have real bicharacteristics and carries over to operators with matrix lower order terms.

The operator P is an analytic ΨDO in Ω of order 0. Indeed, to prove that, it is enough to prove that operators of the kind $I_b'I_a$, see the Appendix, are analytic ΨDOs ([36]) of order -1 when a, b are analytic in $\Omega\times S^1$. The amplitude of such an operator is given by (A.2), and it is clearly an analytic one, see also the proof of [31, Proposition 1].

The propagation of singularities result in [13] then implies that each zero bicharacteristic of P in K either consists of (analytic) singular points only, or does not intersect the analytic wave front set of g. Since K is non-trapping, we have the latter alternative. Therefore, the analytic wave front set of g is empty. Then g is analytic. Since g is of compact support, we get $g=0$. $\square$

Corollary 4.3. *Under the assumptions of Corollary 4.1, Theorem 4.2 holds if only K_1 is non-trapping.*

Proof. The operators $I_{w_2}'I_{w_{1,2}}$ are then an analytic ΨDOs in Ω, and the first one is elliptic in a neighborhood of K_2, see also (4.23). Using the property of analytic elliptic ΨDOs to resolve analytic singularities, we conclude that for $g\in\operatorname{Ker}\mathcal{I}$, g_2 is analytic in the interior of $K_2\setminus K_1$. By analytic propagation of singularities, g must be analytic near K_1 because the latter is non-trapping. As above, we conclude $g=0$. $\square$

4.4. **Generic injectivity of $\mathcal{I}$.** Let K be a non-trapping compact set. Then any small enough compact neighborhood K' of K is still non-trapping, see the proof of [18, Theorem 26.1.7]. Therefore, there exists an open $\Omega \supset K$ so that every compact subset of Ω is non-trapping. Then P is of real principal type in Ω, by the definition in [18].

Definition 4.2. *The set Ω is said to be pseudo-convex w.r.t. P, if any compact subset is non-trapping, and for any compact set $K \subset \Omega$, there exists a compact set $K' \subset \Omega$ so that every bicharacteristic interval in Ω having endpoints over K, lies entirely over K'.*

In particular, if K is convex w.r.t. the bicharacteritics (i.e., one can choose $K' = K$), then K is pseudo-convex.

Pseudo-convexity is a condition that guarantees existence of a global parametrix of P, see [10] and [18, Theorem 26.1.14]. Under that condition, we show below that injectivity of $\mathcal{I}$ is preserved under small perturbations of the weights.

Theorem 4.3. *Let K be non-trapping for P and assume that there exists a pseudo-convex neighborhood $\Omega \supset K$ of K. Assume that $\mathcal{I}$ is injective on $\mathcal{E}'(K)$. Then there exist $k > 0$ and $\varepsilon > 0$ so that for any $\tilde{w}_1$, $\tilde{w}_2$, ε-close to w_1 and w_2 in $C^k(\bar{\Omega})$, the corresponding operator $\tilde{\mathcal{I}}$ is still injective, and the estimate (4.20) holds with a constant C independent of the particular choice of $\tilde{w}_1$, $\tilde{w}_2$.*

Proof. By [10], see also [18, Theorem 26.1.14], under the assumptions of the theorem one can construct a parametrix E so that

$$EP = \mathrm{Id} + R,$$

where R has a smooth kernel. The parametrix E is not unique, even modulo smoothing operators. Loosely speaking, it is unique modulo smoothing operators if we fix an orientation on each connected set of bicharacteristics through K. The operator E has the mapping property $E : H^s_0 \to H^{s-1}$. If we make P of order 1, then P would be microlocally equivalent to $\partial/\partial x_1$; and then roughly speaking, E is integration w.r.t. x_1 in that representation in the direction of the chosen orientation.

By (4.11), we have

$$EQ\mathcal{I} = \mathrm{Id} + R, \tag{4.26}$$

where Q is of order $1/2$. Notice that E is of order 1, and $\mathcal{I}$ is of order $-1/2$. While the composition $EQ\mathcal{I}$ a priori is of order 1 just based on the individual terms, it is actually of order 0 as (4.26) shows.

The construction of the Fourier Integral Operator (FIO) E is described in [18]. In order to get R above to be just of order -1, all microlocal constructions need to be done up to finite order only in order to satisfy finitely many symbol estimates, see, e.g., [17, Theorem 18.1.11'] and [28]. In each step, finitely many derivatives of the symbols are needed; therefore, finitely many derivatives of w_1 and w_2 are needed. Therefore, for some k, $C^k \ni (w_1, w_2) \to R$ is continuous, where $R : H^s \to H^{s+1}$ for a fixed s.

The arguments below follow the proof of [31, Proposition 5.1]. The idea is to correct the parametrix EQ by a finite rank operator so that the new $\mathrm{Id} + R$ would be injective. We should be able to do this because $\mathcal{I}$ is injective.

Restrict equation (4.26) to K. In this stage of the proof, we will indicate the dependence on $w := (w_1, w_2)$ by a subscript w. We can always assume that R_w is self-adjoint because we can apply $\mathrm{Id} + R_w^*$ to both sides of (4.26). The operator $\mathrm{Id}+R_w$ has at most a finite-dimensional kernel V on $L^2(K)$. Since $\mathcal{I}_w$ is injective on $L^2(K)$, $\mathcal{I}_w : V \to \mathcal{I}_w V$ is an isomorphism; let B_w be its inverse. Let also Π_w be the orthogonal projection to $\mathcal{I}_w V$. For $\tilde{w}$ close to w as in the theorem, set $B^\sharp_{\tilde{w}} := E_{\tilde{w}} Q_{\tilde{w}} + B_w \Pi_w$. Then

$$B^\sharp_{\tilde{w}} \mathcal{I}_{\tilde{w}} g = (\mathrm{Id} + R^\sharp_{\tilde{w}}) g, \tag{4.27}$$

where $R^\sharp_{\tilde w} := R_{\tilde w} + B_w\Pi_w\mathcal{I}_{\tilde w}$ is compact. We claim that $\mathrm{Id} + R^\sharp_{\tilde w}$ is injective for $\tilde w = w$. Indeed, assume $(\mathrm{Id} + R^\sharp_w)g = 0$. Then $(\mathrm{Id} + R_w)g + B_w\Pi_w\mathcal{I}_w g = 0$. The first term is in $V^\perp$; the second one is in V, therefore they are both zero. Thus $g \in V$, and $B_w\Pi_w\mathcal{I}_w g = 0$. By the definition of B_w and Π_w, this implies $g = 0$. Therefore, $\mathrm{Id} + R^\sharp_w$ is injective, and actually invertible in $L^2(K)$. This property is preserved under small C^k perturbations of w, $k \gg 1$, as discussed above, with a uniformly bounded norm. The statement of the theorem now follows directly from (4.27). □

Theorem 4.3 and Theorem 4.2 imply the following generic uniqueness result.

Corollary 4.4. *Let K and Ω be as in Theorem 4.3. For some $k \gg 1$, there is an open dense set of pairs (w_1, w_2) in $C^k(\Omega)$ so that the corresponding operator $\mathcal{I}$ is injective on $\mathcal{E}'(\Omega)$, and satisfies the stability estimate (4.20) with a locally uniform constant.*

Remark 4.2. If $w_2 \neq 0$ we have corollaries similar to Corollary 4.1 and Corollary 4.3 that we will not formulate.

5. The non-linear Identification Problem

Let (a, f) and $(\tilde a, \tilde f)$ be two attenuation-source pairs. We will denote functions and operators related to $(\tilde a, \tilde f)$ by placing a tilde over them. The difference $v := \tilde u - u$ of the solutions of (2.1) solves

$$(\theta \cdot \partial_x + \tilde a)v = \delta f - u\delta a, \quad v|_{\theta\cdot x \ll 0} = 0, \tag{5.1}$$

where

$$\delta a := \tilde a - a, \quad \delta f := \tilde f - f. \tag{5.2}$$

Therefore,

$$X_{\tilde a}\tilde f - X_a f = I_w \delta a + X_{\tilde a}\delta f, \tag{5.3}$$

where I_w is the weighted X-ray transform with weight

$$w = -e^{-B\tilde a}u. \tag{5.4}$$

We used here the obvious generalization of (2.2) for sources f dependent on θ as well, see the remark following (2.3). If we replace $\tilde a$ on the right with a, then we get the linearization formula of Proposition 3.1, as we should.

5.1. A summary of the properties of the linearization $\delta X_{a,f}$.

We are in the situation of the previous section with

$$g_1 = \delta a, \quad g_2 = \delta f, \quad w_1 = -e^{-B\tilde a}u, \quad w_2 = e^{-B\tilde a}. \tag{5.5}$$

If δa, δf are given by (5.2), then (4.7) is a non-linear equation, of course. If we treat them as independent (of a, $\tilde a$, f, $\tilde f$) functions then we have the linear problem that we analyzed above. Then

$$W = e^{-B\tilde a}e^{-JB\tilde a}W_0, \quad W_0 := (u - Ju), \tag{5.6}$$

see (4.5). The characteristic variety Σ in this case is given by

$$\Sigma = \left\{u(x, \xi^\perp/|\xi|) = u(x, -\xi^\perp/|\xi|)\right\}. \tag{5.7}$$

The Hamiltonian p_0 is then given by (4.12). Since an elliptic factor does not change the zero bicharacteritics, just their parameterization, the zero bicharacteristics are then given by the following Hamiltonian

$$H(x, \xi) = (u(x, \theta) - u(x, -\theta))\,\big|_{\theta = \xi^\perp/|\xi|}. \tag{5.8}$$

Recall that u is the solution of (2.1).

One important improvement in this case is due to the fact that $w_2 > 0$. Let $\operatorname{supp}\delta a \subset K_1$, $\operatorname{supp}\delta f \subset K_2$, with $K_{1,2}$ compact sets. We can therefore use Corollary 4.1 and Corollary 4.3 to weaken the non-trapping condition to the requirement that only K_1 is non-trapping. If, in addition, $u \neq 0$ on $\overline{K_1 \setminus K_2}$, see (2.1), then it is enough to ask $K_1 \cap K_2$ to be non-trapping.

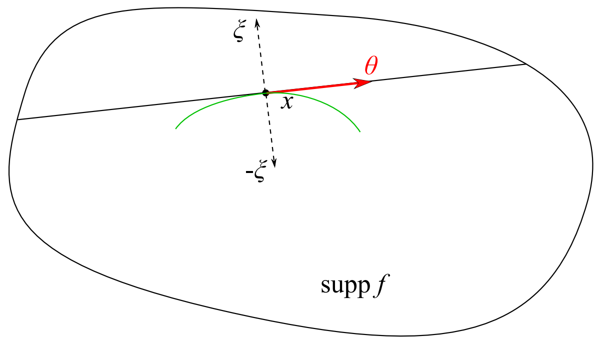

FIGURE 1. The zeros (x, θ) of W are characterized by the property that the attenuated integrals of f from x in the directions θ and $-\theta$ are equal. The conormals ξ to such θ are the characteristic ones. If $\partial_{\theta^\perp} W_0(x, \theta) \neq 0$, then there is a smooth ray through x tangent to θ.

We will summarize the properties of the rays, see Definition 4.1, in this case. Let $\mathcal{I}$ be the linear operator defined in (4.1) with weights w_1 and w_2 as in (5.5) but $g_1 = \delta a$ and $g_2 = \delta f$ considered as independent functions. Notice that the rays depend on a and f only. On the other hand, $\tilde{a}$, $\tilde{f}$ affect the weights in $\mathcal{I}$.

- For any x there is at least one ray through it which might be a point.
- The rays may not be smooth. Given $(x, \theta) \in \mathbf{R}^n \times S^1$, there is a smooth ray through x in the direction of θ if and only if $W_0(x, \theta) = 0$ and $\partial_{\theta^\perp} W_0(x, \theta) \neq 0$.
- When $a = 0$, we have $\theta \cdot \partial_x W_0 = 2f$, and then the condition $f(x) \neq 0$ is sufficient for P to be of real principal type at (x, ξ), $\forall \xi$.
- A compact set $K \subset \mathbf{R}^2$ is called non-trapping, if all rays eventually leave K.
- Assume here and below that $\mathcal{I}$ is restricted to pairs such that $\operatorname{supp}\delta a \subset K_1$, $\operatorname{supp}\delta f \subset K_2$, with $K_{1,2}$ compact sets. If K_1 is non-trapping, then $\mathcal{I}$, has a finite dimensional kernel, smooth enough if $B\tilde{a}$ and u are smooth enough near K. Also, (4.19) holds.
- If $\mathcal{I}$ is injective (on $\mathcal{E}'(K_1) \times \mathcal{E}'(K_2)$), then it is stable, as well, with a loss of one derivative, i.e., (4.20) holds. If in addition K_1 has a pseudo-convex neighborhood, then the injectivity is preserved under a small enough perturbation with a uniform stability estimate (4.20).
- If $W_0(x_0, \theta) = 0$ implies $\partial_{\theta^\perp} W_0(x_0, \theta) \neq 0$ for all θ, then $\mathcal{I}$ is injective (and stable) restricted to functions supported in some neighborhood of x_0.
- If K_1 is non-trapping, and $B\tilde{a}$ and u are analytic in a neighborhood of $K_1 \cup K_2$, then $\mathcal{I}$ is injective (and stable).

5.2. **Uniqueness and stability results.** Our first main result about the identification problem is the following theorem. Recall that the requirement on Ω to be pseudo-convex implies that K is non-trapping. We also recall that u is defined by (2.2) as an attenuated integral of f. Given (a_j, f_j), we denote by u_j the corresponding u, $j = 0, 1, 2$.

Theorem 5.1. *Let $K_{1,2} \subset \mathbf{R}^2$ be compact sets and let $\Omega \supset K_1 \cup K_2$ be open. Let a_0, f_0 be of compact support so that Ba_0 and u_0 are in $C^k(\Omega \times S^1)$. Assume that $K_1 \subset \Omega_1$, and the latter is an open set pseudoconvex w.r.t. the Hamiltonian H defined in (5.8), related to a_0 and f_0. Let a_0, f_0 be such that*

$\delta X_{a_0,f_0}$, see Proposition 3.1, is injective on $K_1 \times K_2$. Then if $k \gg 1$, there exists $\varepsilon > 0$ so that for any (a_1, f_1), (a_2, f_2) with $a_j - a_0 \in C^k$ and $f_j - f_0 \in C^k$ supported in K_1 and K_2, respectively, satisfying

$$\|B(a_j - a_0)\|_{C^k(\bar\Omega\times S^1)} + \|u_j - u_0\|_{C^k(\bar\Omega\times S^1)} \le \varepsilon, \quad j = 1, 2, \tag{5.9}$$

there exist constants $C > 0$, $\mu \in (0, 1)$ so that

$$\|a_1 - a_2\|_{L^\infty(K_1)} + \|f_1 - f_2\|_{L^\infty(K_2)} \le C\|X_{a_1}f_1 - X_{a_2}f_2\|^{\mu}_{L^\infty(Z)}. \tag{5.10}$$

Proof. By (5.3), we have

$$\begin{aligned} X_{a_2}f_2 - X_{a_1}f_1 &= -I_{e^{-Ba_1}u_1}\delta a + X_{a_1}\delta f + R \\ &= \delta X_{a_1,f_1}(\delta a, \delta f) + R, \end{aligned} \tag{5.11}$$

where

$$R = I_{(e^{-Ba_1} - e^{-Ba_2})u_1}\delta a + (X_{a_2} - X_{a_1})\delta f, \tag{5.12}$$

and $\delta a = a_2 - a_1$, $\delta f = f_2 - f_1$. Next,

$$\begin{aligned} \|R\|_{L^\infty} &\le C\|B(a_2 - a_1)\|_{L^\infty(\Omega\times S^1)}\|a_2 - a_1\|_{L^\infty(K_1)} \\ &\quad + C\|B(a_2 - a_1)\|_{L^\infty(\Omega\times S^1)}\|f_2 - f_1\|_{L^\infty(K_2)} \\ &\le C'\left(\|\delta a\|^2_{L^\infty(K_1)} + \|\delta f\|^2_{L^\infty(K_2)}\right), \end{aligned} \tag{5.13}$$

where C' depends on an a priori bound of $\|f_1\|_{L^\infty(\Omega)}$ which can always be found depending on a_0, f_0, ε; by (5.9).

We will apply [32, Theorem 2]. Set $\mathcal{A}(a, f) = X_a f$. Set also $\mathcal{B}_1 = L^\infty(K_1) \times L^\infty(K_2)$, $\mathcal{B}_2 = L^\infty(Z)$. Then $\mathcal{A} : \mathcal{B}_1 \to \mathcal{B}_2$ is continuous. By (5.11) and (5.13), $\mathcal{A}$ is differentiable at (a_1, f_1) with a quadratic estimate of the remainder.

By assumption, $Ba_0|_{\bar\Omega\times S^1}$ and the solution u_0 of (2.1) related to a_0, f_0 belong to C^k. Since $a_j - a_0 \in C_0^k(K_1)$ and $f_j - f_0 \in C_0^k(K_2)$, we also get the same for Ba_j and u_j, $j = 1, 2$. Moreover, by (5.9), Ba_j and u_j, $j = 1, 2$ are $O(\varepsilon)$ perturbations of Ba_0 and u_0 in $C^k(\bar\Omega \times S^1)$. For $k \gg 1$, we apply Theorem 4.3, see also Remark 4.2, to conclude that $\delta X_{a_1,f_1}$ is still injective, satisfying a stability estimate (4.20) with a constant C independent of a_1, f_1. Take $s > 1$ in (4.20), for example, $s = 3/2$, to get

$$\|\delta a\|_{L^\infty(K_1)} + \|\delta f\|_{L^\infty(K_2)} \le C\|\delta X_{a_1,f_1}(\delta a, \delta f)\|_{H^3(Z)}.$$

Based on that, we set

$$\mathcal{B}_1' = \mathcal{B}_1 = L^\infty(K_1) \times L^\infty(K_2), \quad \mathcal{B}_2' = H^3(Z).$$

Then we have the following interpolation estimate

$$\|h\|_{\mathcal{B}_2'} \le C\|h\|^{\mu_2}_{L^2}\|h\|^{1-\mu_2}_{H^s} \le C'\|h\|^{\mu_2}_{\mathcal{B}_2}\|h\|^{1-\mu_2}_{\mathcal{B}_2''},$$

where $\mathcal{B}_2'' = H^s(Z)$ and $s = 3/(1 - \mu_2)$, $\mu_2 \in (0, 1)$. If we take $\mu_2 \in (1/2, 1)$, we have all the conditions met to apply [29, Theorem 2]. We therefore get that if $k \gg 6$ (k needs to satisfy both $k > 6$ and the requirements of Theorem 4.3), the stability estimate (5.10) holds with $\mu = 1/2$. □

Remark 5.1. It is enough to assume that a_j and f_j, $j = 0, 1, 2$, satisfy the regularity assumptions, instead of Ba_j, u_j but that would be more restrictive.

Remark 5.2. The value for μ that we got is $\mu = 1/2$ but that was based on specific, and a bit arbitrary choice of the interpolation space H^6. As shown in [29, Theorem 2], and as can be easily seen from the proof, we can choose any $\mu < 1$ in (5.10), as close to 1 as we wish, at the expense of increasing k.

Remark 5.3. It may seem strange that we use L^∞ norms in (5.10) instead of more natural ones as in (4.20). In fact, since we used interpolation estimates in the proof, we have some freedom which norms to use in (5.10).

5.3. **Explicit sufficient condition for local solvability and stability of the Identification Problem.** There are two simple cases where Theorem 5.1 hold under easy to verify conditions.

The first one is when certain analyticity conditions are satisfied.

Corollary 5.1. *Let $K_{1,2} \subset \Omega$ be as in Theorem 5.1. Let a_0, f_0 be of compact support so that Ba_0 and u_0 are analytic in $\Omega \times S^1$. Then the conclusions of Theorem 5.1 hold.*

This corollary implies in a trivial way also local uniqueness, in K, near a generic (dense and open in C^k, $k \gg 1$) set of (a, f). The proof follows immediately from Theorem 4.2.

The second corollary below states local uniqueness and stability in a small enough non-trapping set.

Corollary 5.2. *Let $x_0 \in R^2$ is such that Ba_0, u_0 are smooth near x_0, and either $x_0 \in K_2 \setminus K_1$, or W_0 satisfies (4.25). Then there exists an open set $U \ni x_0$, so that for any $K \subset U$ the conclusions of Theorem 5.1 hold.*

Proof. By Corollary 4.2 and the remark after it, if U is small enough, $\delta X_{a,f}$ is injective on any compact set $K \subset U$. Then we apply Theorem 5.1. □

5.4. **Conditions for smoothness and analyticity of Ba and u.** The results above require Ba and u to be either smooth enough or analytic in some open set Ω. The smoothness, for example, certainly hold if a and f are smooth enough in Ω but this is too restrictive. The following condition is sufficient.

Proposition 5.1. *Let $\Omega \subset \mathbf{R}^2$ be open. Let $\{c_j\}_{j=1}^N$ be a finite number of C^k (respectively analytic) non-intersecting curves in $\mathbf{R}^2 \setminus \Omega$ so that a and f are C^k/analytic in $\mathbf{R}^2 \setminus \{c_j\}$, up to the boundary on either side of each curve. Assume that each line through Ω intersects every c_j transversely. Then Ba, u are in $C^k(\Omega \times S^1)$, respectively analytic in $\Omega \times S^1$.*

Proof. Near each $(x_0, \theta_0) \in \Omega \times S^1$, Ba, and similarly u, is given by

$$Ba(x,\theta) = \sum_{j=1}^{N} \int_{\alpha_j(x,\theta)}^{\alpha_{j+1}(x,\theta)} a(x + t\theta, \theta)\, \mathrm{d}t,$$

where $\alpha_0 = 0$, $\alpha_{N+1} = \infty$, and the rest of the α_j's are determined by the intersection points of the ray $x + t\theta$ with the curves c_j. The statement now follows directly from this representation. □

Remark 5.4. We presented the condition above in a form suitable for applications. For C^∞, respectively, analytic regularity of a, f in $\Omega \times S^1$, it is necessary and sufficient to assume that a and f have the same regularity in Ω; and a, f, have no C^∞, respectively analytic singularities, conormal to some line through Ω. The necessity follows from standard properties of the Radon transform to recover conormal smooth or analytic singularities. This condition is sufficient, because of the standard relation between the smooth/analytic wave front set of Ba or u on one side; and the Schwartz kernel of B and a or f, on the other. We sill skip the details.

6. FURTHER MICROLOCAL PROPERTIES OF $\mathcal{I}$

Take the Fourier transform of $I_w(p\theta^\perp, \theta)$ w.r.t. p to get

$$\int e^{-\mathrm{i}\lambda p} I_w f(p\theta^\perp, \pm\theta)\, \mathrm{d}p = \int e^{-\mathrm{i}\lambda\theta^\perp \cdot y} w(y, \pm\theta) f(y)\, \mathrm{d}y. \tag{6.1}$$

Set $\xi = \lambda\theta^\perp$, $\lambda \ge 0$, to get

$$\int_{\mathbf{R}} e^{-\mathrm{i}p|\xi|}(I_w f)(p\theta^\perp, \pm\xi^\perp/|\xi|)\,\mathrm{d}p = \int e^{-\mathrm{i}y\cdot\xi} w(y, \pm\xi^\perp/|\xi|) f(y)\,\mathrm{d}y. \tag{6.2}$$

Take the inverse Fourier transform of both sides to get

$$\bar{w}^*(x, \pm D^\perp/|D|)f = (2\pi)^{-2}\int_{\mathbf{R}\times\mathbf{R}^2} e^{\mathrm{i}(x\cdot\xi - p|\xi|)}(I_w f)(p\xi/|\xi|, \pm\xi^\perp/|\xi|)\,\mathrm{d}p\,\mathrm{d}\xi, \tag{6.3}$$

where $\bar{w}^*(x, \pm D^\perp/|D|)$ is the ΨDO with amplitude $\alpha(x,y,\xi) = w(y, \pm\xi^\perp/|\xi|)$. The principal symbol is $w(x, \pm\xi^\perp/|\xi|)$. If $w = 1$, one can see that we get $C|D|I_1' I_1 f$ on the right, and f on the left, which is just one of the inversion formulas for I_1.

Apply the described operation to the equation

$$I_{w_1} g_1 + I_{w_2} g_2 = 0, \tag{6.4}$$

compare with (4.7). We get

$$\bar{w}_1^*(x, \pm D^\perp/|D|)g_1 + \bar{w}_2^*(x, \pm D^\perp/|D|)g_2 = 0. \tag{6.5}$$

This is actually a system, see also (1.3).

Proposition 6.1. *Let $w_{1,2}$ be two smooth weight functions, and let $g = (g_1, g_2) \in \mathcal{E}'(\mathbf{R}^n)$. Then*

$$\mathcal{I}g \in H^s(Z) \tag{6.6}$$

if and only if

$$\begin{pmatrix} \bar{w}_1^*(x, D^\perp/|D|) & \bar{w}_2^*(x, D^\perp/|D|) \\ \bar{w}_1^*(x, -D^\perp/|D|) & \bar{w}_2^*(x, -D^\perp/|D|) \end{pmatrix} g \in H^{s-1/2}(\mathbf{R}^2). \tag{6.7}$$

Proof. Assume that the l.h.s. of (6.7) is in $H^{s-1/2}$. Then the r.h.s. of (6.3) with $I_w g$ replaced by $\mathcal{I}g := I_{w_1}g_1 + I_{w_2}g_2$ belongs to the same space. Take the Fourier transform of that to get, see also (6.1),

$$\langle\lambda\rangle^{s-1/2}\int_{\mathbf{R}} e^{-\mathrm{i}\lambda p}\mathcal{I}g(p\theta^\perp, \pm\theta)\,\mathrm{d}p \quad \in \quad L^2\left(\mathbf{R}_+ \times S^1,\ \lambda\,\mathrm{d}\lambda\,\mathrm{d}\theta\right). \tag{6.8}$$

Since the relation above holds with either choice of the $\pm$ sign, we can fix the positive one, and allow λ to be negative, as well. Therefore, $\langle\lambda\rangle^{s-1/2}|\lambda|^{1/2}\mathcal{F}_{p\mapsto\lambda}\mathcal{I}(p\theta^\perp, \theta) \in L^2\left(\mathbf{R}\times S^1,\ \mathrm{d}\lambda\,\mathrm{d}\theta\right)$. This easily implies, see e.g., the proof of [22, Theorem II.5.1], that $\langle\lambda\rangle^s\mathcal{F}_{p\mapsto\lambda}\mathcal{I}(p\theta^\perp, \theta) \in L^2(\mathbf{R}\times S^1)$, which yields (6.6).

Now, assume (6.6). Reversing the arguments above, we get (6.8). Take inverse Fourier transform w.r.t. $\xi = \lambda\theta^\perp$ to get (6.7). $\square$

Proposition 6.1 reduces the problem of the microlocal invertibility of the FIO $\mathcal{I}$ to that of the matrix valued ΨDO in (6.7) with a principal symbol

$$\left.\begin{pmatrix} w_1(x,\theta) & w_2(x,\theta) \\ w_1(x,-\theta) & w_2(x,-\theta) \end{pmatrix}\right|_{\theta=\xi^\perp/|\xi|}. \tag{6.9}$$

The determinant of the latter is $W(x, \xi^\perp/|\xi|)$, see (4.4) and (4.5). An immediate consequence of (6.7) is the following. For some matrix valued classical ΨDO $\tilde{P}$ with a scalar principal symbol $p_0(x,\xi) = W(x, \xi^\perp/|\xi|)$, see (4.12), relation (6.6) implies

$$\tilde{P}g \in H^{s-1/2}(\mathbf{R}^2). \tag{6.10}$$

This also follows from Proposition 4.1.

Assume now that (4.18) is satisfied for some (x_0, θ_0). Then we can solve the equation $W(x,\theta) = 0$ for $\theta \in S^1$ locally to get a smooth function $\theta(x)$. Since W is an odd function of θ, the same thing applies near the point $(x_0, -\theta_0)$, as well, with a solution $-\theta(x)$. This implies that in a conic neighborhood of $(x_0, \pm\theta_0^\perp) \in \Sigma$, the characteristic manifold Σ is given by $\xi^\perp/|\xi^\perp| = \pm\theta(x)$. Set $v_j^\pm(x) = w_j(x, \pm\theta(x))$, $j = 1, 2$. Then $w_j(x, \pm\xi^\perp/|\xi|) - v_j^\pm(x)$ vanishes on Σ, and is therefore locally given by $p_0(x, \xi)$ times a smooth function, homogeneous of order 0 in ξ, hence a symbol. This implies that (6.7) can be written as

$$\begin{pmatrix} v_1^+(x) & v_2^+(x) \\ v_1^-(x) & v_2^-(x) \end{pmatrix} g + (Q_0\tilde{P} + Q_{-1})g \in H^{s-1/2}(x_0, \xi_0), \tag{6.11}$$

where Q_0 and Q_{-1} are classical ΨDOs of order 0 and -1, respectively. Using (6.10), we get

$$v_1^\pm g_1 + v_2^\pm g_2 + Q_{-1}^\pm g \in H^{s-1/2}(x_0, \xi_0), \tag{6.12}$$

with $Q_{-1}^\pm$ of order -1, and the equations with the $+$ and the $-$ sign are actually linearly dependent up to the lower order term (including the possibility that one of them has zero coefficients). Now, if the assumptions of Theorem 4.1 are satisfied, (6.6) yields $g \in H^{s-3/2}$. Then $Q_{-1}^\pm g \in H^{s-1/2}$, and we get

$$v_1^\pm g_1 + v_2^\pm g_2 \in H^{s-1/2}(x_0, \xi_0). \tag{6.13}$$

Since the matrix in (6.11) has rank 1, only one of the equations (6.13) is relevant (see also the notion of polarization set in [9]). This is an improvement over the estimate (4.20), that asserts that (6.6) implies $g_{1,2} \in H^{s-3/2}$, if $\operatorname{supp} g$ is supported in a non-trapping compact set. This improvement applies to the linear combination (6.13) only.

6.1. **Applications to the linearized Identification Problem.** Let $\mathcal{I} = \delta X_{a,f}$ be the linearization of $X_a f$, see Proposition 3.1. Then w_j are given by (5.5). The determinant W can be replaced in the analysis by W_0, see (5.6). Notice that $w_2 > 0$. The discussion above yields the following.

Proposition 6.2. *Fix $(x_0, \theta_0) \in \mathbf{R}^2 \times S^1$. Let $W_0(x_0, \theta_0) = 0$ and $\partial_{\theta^\perp} W_0(x_0, \theta_0) \neq 0$. Let*

$$v(x) = u(x, \theta(x)), \quad \textit{for } x \textit{ near } x_0, \tag{6.14}$$

where $\theta(x)$ is the unique local solution of $W_0(x, \theta) = 0$ with $\theta(x_0, \theta_0) = \theta_0$, and u is defined by (2.2). Then if $\delta X_{a,f}(\delta a, \delta f) \in H^s$, we have

$$v\delta a - \delta f \in H^{s-1/2}(x_0, \pm\theta_0^\perp). \tag{6.15}$$

Proof. In this particular case, $w_1 = -e^{-Ba}u$, $w_2 = e^{-Ba}$. Under the non-degeneracy assumption on W_0, $w_2 > 0$, and $w_1 = -uw_2$. Divide by the elliptic factor w_2 in either of the two relations (6.13) to get (6.15). □

Remark 6.1. Theorem 4.1 says that under the non-trapping condition we can recover $WF(f_{1,2})$ with a loss of one derivative, compared to the standard X-ray transform. On the other hand, Proposition 6.2 says that under the additional mild condition on W, one can recover the wave front set of the linear combination (6.15) without loss. This has the following implications for the recovery of a and f: we can expect $v\delta a - \delta f$ to be recoverable in a more stable way than either δa or δf.

Remark 6.2. We need to assume that the assumptions of Theorem 4.1 are satisfied just to conclude that $f \in H^{s-3/2}$; and then to deduce that $Q_{-1}^\pm f \in H^{s-1/2}$, see (6.12) and (6.13). If we know a priori that f has certain regularity, then we can use that fact instead. In applications, it would be natural to assume that $(\delta a, \delta f) \in L^2$. Let us assume that the measurements show that $\delta X_{a,f}(\delta a, \delta f) \in H^{3/2}$ (or better). Then we conclude that $v\delta a - \delta f \in H^1$, that in particular excludes jump types of singularities at smooth surfaces of that particular linear combination. There is no need to assume the trapping condition for this argument.

7. THE RADIAL CASE

As explained in the Introduction, the thorough study of the case of radial a and f is behind the scope of this work. The purpose of this section is to present a case, where the rays can be easily computed, when both the linearized map, and the non-linear one have huge kernels if the non-trapping assumption is not satisfied. So at least in the cases described below, the non-trapping assumptions is not only sufficient but also necessary for the problem to be "well-behaved".

7.1. **The linearized map for a simple radially symmetric example.** We start with perhaps the simplest example. Let $\mathbf{1}_{B(0,1)}$ be the characteristic function of the unit disk. We study the linearization δX w.r.t. (a, f) near

$$a = 0, \quad f = \mathbf{1}_{B(0,1)}. \tag{7.1}$$

We will choose perturbations of those a and f supported in $B(0, 1)$ only. The weight w, see (3.2) or (3.3), restricted to the unit disk, is given by

$$w(x, \theta) = -\sqrt{1 - (\theta^\perp \cdot x)^2} - \theta \cdot x. \tag{7.2}$$

Then, see (5.6),

$$W_0 = -2\theta \cdot x. \tag{7.3}$$

The Hamiltonian H, up to a constant factor, is as in Example 4.1. Indeed, by (5.8), $H = -2x \cdot \xi^\perp/|\xi| = 2(x_1\xi_2 - x_2\xi_1)/|\xi|$. Therefore, $|\xi|H/2$ is the symbol of

$$x_1 D_2 - x_2 D_1 = -\mathrm{i}\partial/\partial\phi,$$

where ϕ is the polar angle in the x space. The bicharacteristics are given by (4.16). In particular, the rays are the concentric circles $|x| = R$, $R \geq 0$, including the degenerate case $x = 0$. As before, $K \subset B(0, 1)$ is non-trapping, if and only if K does not contain an entire circle of that kind, see Figure 2.

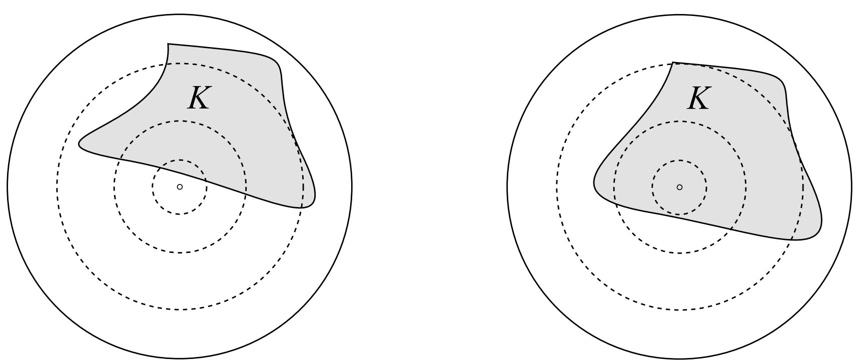

FIGURE 2. The rays of Example 4.1 in the unit disk and an example of a non-trapping K, left; and a trapping K, right.

The equation $\delta X(\delta a, \delta f) = 0$ can then be written as

$$-\int_{\ell_{z,\theta}} \left(\sqrt{1 - (\theta^\perp \cdot x)^2} + \theta \cdot x\right) \delta a \, \mathrm{d}s + \int_{\ell_{z,\theta}} \delta f \, \mathrm{d}s = 0,$$

where $\ell_{z,\theta}$ is the line through $z \in \theta^\perp$ in the direction of θ, and $\mathrm{d}s$ is the natural measure on it. The integral over the line $\ell_{z,-\theta}$ would produce the same term with $\theta \cdot x$ replaced by $-\theta \cdot x$. Therefore, both the even and the odd part w.r.t. θ above vanish:

$$-\int_{\ell_{z,\theta}} \sqrt{1-(\theta^\perp\cdot x)^2}\delta a \,\mathrm{d}s + \int_{\ell_{z,\theta}} \delta f\,\mathrm{d}s = 0, \qquad \int_{\ell_{z,\theta}} \theta\cdot x\delta a\,\mathrm{d}s = 0. \tag{7.4}$$

The third integral is the X-ray transform of the vector field $(\delta a)x$. It is well known that we can only determine the curl of that, i.e.,

$$(x_1\partial_2 - x_2\partial_1)\delta a = 0.$$

In other words, δa needs to be radial. Then the first term in (7.4) is invariant under rotations of (x,θ), i.e., when we consider (x,θ) as points in the unit tangent bundle. Then so is the second term. Apply I_1' to it, and we get that $|D|^{-1}\delta f$ is radial, as well. Then so is δf.

Therefore, the kernel of $\delta X(\delta a,\delta f)$ consists of radial δa and δf that are connected by the first identity in (7.4). Since the weight there is constant along the lines, using Radon transform notation, $Rh(p,\omega)$, we get

$$\sqrt{1-p^2}R\delta a - R\delta f = 0. \tag{7.5}$$

It follows from the analysis below that there exists an infinite dimensional space of pairs $(\delta a,\delta f)$ satisfying (7.5). Indeed, for any radial $\delta a\in C_0^\infty(B(0,1))$, we can solve (7.5) for δf, and vice versa.

Going back to (7.4), the arguments in the proof of Proposition 6.2 (or its conclusion directly, together with Remark 6.2) show that $\delta X(\delta a,\delta f)\in H^s$ and $(\delta a,\delta f)\in H^{s-3/2}$ imply

$$\sqrt{1-|x|^2}\delta a - \delta f \in H^{s-1/2}$$

in the annulus $U := \{0<|x|<1\}$; i.e., the singularities of that particular linear combination in U can be recovered without a derivative loss. Note that for any $x\in U$, the characteristic directions (zeros of W) are given by $\theta = \pm x^\perp/|x|$, and the characteristic codirections — by $\xi = \pm x/|x|$. Then the integral of f, starting from x, in a characteristic direction θ is exactly $\sqrt{1-|x|^2}$. This is the value of u for characteristic directions, see (5.8) and (6.14), and confirms (6.15).

7.2. **The linearized map for $a=0$ and f radial has an infinite dimensional kernel.** Let now a and f be general radial smooth functions of compact support. Then the characteristic variety of Example 4.1 and Section 7

$$\Sigma_0 = \{(x,\xi);\ x \text{ and } \xi \text{ are collinear}\}$$

is included in the characteristic variety Σ in this case but the latter can be larger.

We study now $\delta X_{a,f}$ for

$$a=0,\quad f \text{ radial}. \tag{7.6}$$

We also assume that f is smooth and has compact support. With some abuse of notation, we replace f by $f=f(|x|)$, where f has even smooth extension. By Proposition 3.1,

$$\delta X_{0,f}(\delta a,\delta f) = -I_{JBf}\delta a + I_0\delta f, \tag{7.7}$$

see (4.6). We restrict $\delta X_{0,f}$ to radial δa, δf, as well.

We will use Radon type of parameterization for $I_{JBf}\delta a$ by setting $\omega=\theta^\perp$. Write

$$R_{JBf}\delta a(p,\omega) = I_{JBf}\delta a(p\omega,-\omega^\perp) = \int \delta(p-\omega\cdot x)Bf(x,\omega^\perp)\delta a(x)\,\mathrm{d}x.$$

Here δ is the Dirac Delta function, not to be confused with the variation symbol in δa, δf. Since f is radial, for any rotation U, we have $Bf(Ux,U\omega^\perp) = Bf(x,\omega^\perp)$. Since δa is radial as well, we easily get that

$I_{JBf}\delta a$ is independent of ω, i.e., $I_{JBf}\delta a = I_{JBf}\delta a(p)$. We claim that $I_{JBf}\delta a(p)$ is an even function of p. Indeed, set $\omega = (1,0)$. Then

$$\begin{aligned}
R_{JBf}\delta a(-p) &= \int \delta(-p-x_1)Bf(x,(0,1))\delta a(x)\,\mathrm{d}x \\
&= \int \delta(p+x_1)Bf(x,(0,1))\delta a(x)\,\mathrm{d}x \qquad \text{because } \delta \text{ is even} \\
&= \int \delta(p-x_1)Bf(x,(0,1))\delta a(x)\,\mathrm{d}x \qquad \text{after the change } x_1 \mapsto -x_1 \\
&= R_{JBf}\delta a(p).
\end{aligned}$$

In the last equation, we also used the fact that f is radial.

To study the kernel of $\delta X_{0,f}$, we write, see (7.7),

$$R_{JBf}\delta a - R\delta f = 0, \tag{7.8}$$

where, with some change of notation again, R is the classical Radon transform acting on radial functions, i.e., considered as a map on functions of a single variable. It is easy to see that, see also [14],

$$Rg(p) = 2\int_0^\infty g\left(\sqrt{p^2+t^2}\right)\mathrm{d}t, \quad p \ge 0.$$

It is known, see [22], and can be easily seen that this equation can be written in the form

$$Rg(p) = 2\int_p^\infty \left(1-\frac{p^2}{r^2}\right)^{-1/2} g(r)\,\mathrm{d}r, \quad p \ge 0.$$

This an equation of Abel type with explicit inversion given by (see [11, 22])

$$g(r) = -\frac{1}{\pi}\int_r^\infty (p^2-r^2)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}p}Rg(p)\,\mathrm{d}p. \tag{7.9}$$

Moreover, the Abel transform R is given by a composition of the cosine Fourier transform F_c and the zero order Hankel one H_0 (see [11]), with a proper normalization, i.e., $R = F_cH_0$. If $h \in C^\infty(\mathbf{R}_+)$ is of compact support, and admits a smooth even extension, then we get a direct confirmation that the equation $Rg = h$ has a (unique) solution given by $g = H_0F_ch$. Indeed, for such h, F_ch has smooth even extension in the Schwartz class, and then H_0F_ch is well defined and solves $Rg = h$.

This shows that the function δ in (7.9) is given by

$$\delta f(r) = -\frac{1}{\pi}\int_r^\infty (p^2-r^2)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}p}I_{JBf}\delta a(p\omega,\omega^\perp)\,\mathrm{d}p, \tag{7.10}$$

see also (7.7). We recall that $I_{JBf}\delta a$ is independent of ω. We summarize this into the following.

Proposition 7.1. *Let $f \in C_0^\infty(\mathbf{R}^2)$ be radial. Then the linearized map $\delta X_{0,f}$ (with $a = 0$) has an infinite dimensional kernel, including all radial pairs $(\delta a, \delta f)$ with δa smooth function of compact support, and δf given by (7.10).*

In other words, besides the inability to recover the singularities (without support restrictions), we actually have an infinite dimensional kernel. Therefore, in this case, the non-trapping condition is a necessary condition for the problem to be well posed, as well.

7.3. **Non-uniqueness for the Identification Problem for radial a, f near $a = 0$.** We show next that not only does the linearized map $\delta X_{a,f}$ can have an infinite dimensional kernel in the case above, but the non-linear map $(a, f) \mapsto X_a f$ has a rich set of radial pairs with the same image.

Theorem 7.1. *Let $a \in C_0^\infty$ and $f \in C_0^\infty$ be radial. Then there exists a radial $f_0 \in C_0^\infty$ so that*

$$X_a f = X_0 f_0. \tag{7.11}$$

Proof. We will work again with the Radon transform parameterization $R_a f(p, \omega) = X_a f(p\omega, -\omega^\perp)$ instead, see (2.5). As above, it is straightforward to check that

$$R_a f(p, \omega) = R_a f(-p, -\omega).$$

We saw above that $R_a f(p, \omega)$ is actually independent of ω, and an even function of p. Then for any $k = 0, 1, \dots,$

$$\int R_a f(p, \omega) p^k \, \mathrm{d}p = C_k = \text{const.},$$

and $C_k = 0$ if k is odd. Therefore, the integral above is a restriction of the homogeneous polynomial $C_k|\xi|^k$ to the unit sphere. Therefore, $R_a f \in \mathcal{S}_H$, and by the Helgason range characterization theorem, see [14], (7.11) holds with some $f_0 \in \mathcal{S}(\mathbf{R}^2)$. By the support theorem, f_0 is compactly supported. Since $X_a f$ is independent of ω, we get that f_0 must be radial. □

We can actually make this constructive. By (7.9), writing $f_0 = f_0(r)$, we get

$$f_0(r) = -\frac{1}{\pi} \int_r^\infty (p^2 - r^2)^{-1/2} \frac{\mathrm{d}}{\mathrm{d}p} X_a f(p\omega, -\omega^\perp) \, \mathrm{d}p,$$

recall that $X_a f(p\omega, \omega^\perp)$ is independent of ω.

Appendix A. $I_b^* I_a$ as a ΨDO

As explained in Section 2, we view $X_a f$ and $I_w f$ as functions on Z, with a natural measure $\mathrm{d}z$ there. Then X_a, and more generally, I_w have well defined transpose (w.r.t. the distribution pairing) and conjugate (w.r.t. the L^2 product) operators X_a' and X_a^*; and I_w', I_w^*, respectively. Below, we use the notation $\theta_\perp$ for the line given by $s \mapsto p\theta^\perp$.

Proposition A.1.

$$I_w' \psi(x) = \int_{S^1} w(x, \theta) \psi(x - (x \cdot \theta)\theta, \theta) \, \mathrm{d}\theta.$$

Proof. For $\phi \in C_0^\infty(\mathbf{R}^2)$, $\psi \in C_0^\infty(Z)$, we have

$$\int_Z (I_w \phi) \psi \, \mathrm{d}z = \int_Z \int_{\mathbf{R}} w(z + s\theta, \theta) \phi(z + s\theta) \psi(z, \theta) \, \mathrm{d}s \, \mathrm{d}z.$$

Set $x = z + s\theta$, $z \in \theta_\perp$. For any fixed θ, $(z, s) \mapsto x$ is a diffeomorphism with a Jacobian equal to 1. Its inverse is given by

$$z = x - (x \cdot \theta)\theta, \quad s = x \cdot \theta.$$

Therefore,

$$\int_Z (I_w \phi) \psi \, \mathrm{d}z = \int_{S^1} \int_{\mathbf{R}^2} w(x, \theta) \phi(x) \psi(x - (x \cdot \theta)\theta, \theta) \, \mathrm{d}x \, \mathrm{d}\theta,$$

and this proves the proposition. □

Proposition A.2. *For any two smooth functions a and b,*

$$I_b' I_a f(x) = \int \frac{A\big(x, y, \frac{x-y}{|x-y|}\big)}{|x-y|} f(y)\, \mathrm{d}y,$$

where

$$A(x, y, \theta) = b(x, \theta)a(y, \theta) + b(x, -\theta)a(y, -\theta). \tag{A.1}$$

Moreover, $I_b' I_a$ is a classical ΨDO of order -1 with amplitude

$$\frac{\pi}{|\xi|}\Big(b(x, \xi^\perp/|\xi|)a(y, \xi^\perp/|\xi|) + b(x, -\xi^\perp/|\xi|)\, a(y, -\xi^\perp/|\xi|)\Big), \tag{A.2}$$

and principal symbol

$$\frac{\pi}{|\xi|}\Big(a(x, \xi^\perp/|\xi|)b(x, \xi^\perp/|\xi|) + a(x, -\xi^\perp/|\xi|)b(x, -\xi^\perp/|\xi|)\Big).$$

Proof. By Proposition A.1,

$$\begin{aligned} I_b' I_a f(x) &= \int_{S^1} b(x, \theta) \int a(x - (x\cdot\theta)\theta + t\theta, \theta) f(x - (x\cdot\theta)\theta + t\theta)\, \mathrm{d}t\, \mathrm{d}\theta \\ &= \int_{S^1} b(x, \theta) \int a(x + t\theta, \theta) f(x + t\theta)\, \mathrm{d}t\, \mathrm{d}\theta. \end{aligned}$$

Split the t-integral in two parts: for $t > 0$ and for $t < 0$, and replace t by $-t$ in the second one to get

$$\begin{aligned} I_b' I_a f(x) &= \int_{S^1} b(x, \theta) \int a(x + t\theta, \theta) f(x + t\theta)\, \mathrm{d}t\, \mathrm{d}\theta \\ &= \int_{S^1} b(x, \theta) \int_0^\infty a(x + t\theta, \theta) f(x + t\theta)\, \mathrm{d}t\, \mathrm{d}\theta \\ &\quad + \int_{S^1} b(x, \theta) \int_0^\infty a(x - t\theta, \theta) f(x - t\theta)\, \mathrm{d}t\, \mathrm{d}\theta. \end{aligned} \tag{A.3}$$

Replace $-\theta$ by θ in the second integral to get

$$I_b' I_a f(x) = \int_{S^1} \int_0^\infty \left[b(x, \theta)a(x + t\theta, \theta) + b(x, -\theta)a(x + t\theta, -\theta)\right] f(x + t\theta)\, \mathrm{d}t\, \mathrm{d}\theta. \tag{A.4}$$

Pass to polar coordinates $y = x + t\theta$, centered at x to finish the proof.

To write $I_b' I_a$ as a ΨDO, recall that if the Schwartz kernel of a linear operator is given by $K(x, y, (x - y)/|x-y|)$, then it is a formal ΨDO with an amplitude given by the Fourier transform of K w.r.t. the third variable. Therefore, $I_b' I_a$ is a formal ΨDO with amplitude

$$\begin{aligned} \int e^{\mathrm{i}z\cdot\xi}|z|^{-1} A(x, y, z/|z|)\, \mathrm{d}z &= \int_{\mathbf{R}_+\times S^1} e^{\mathrm{i}r\theta\cdot\xi} A(x, y, \theta)\, \mathrm{d}r\, \mathrm{d}\theta = \pi \int_{S^1} A(x, y, \theta)\delta(\theta\cdot\xi)\, \mathrm{d}\theta \\ &= \frac{\pi}{|\xi|}\Big(A(x, y, \xi^\perp/|\xi|) + A(x, y, -\xi^\perp/|\xi|)\Big). \end{aligned}$$

We used here the fact that A is an even function of θ and that the inverse Fourier transform of 1 is δ, see also [16, Theorem 7.1.24]. Since this is a homogeneous function of ξ, with an integrable singularity that can be cut-off resulting in a smoothing operator, this completes the proof. $\square$

The mapping properties of those operators are well understood even in the more general setting of the weighted geodesic transform. We summarize those properties below. Recall the definition of the Sobolev space $H^s(Z)$ in (2.6) first. Given a compact set $K \subset \mathbf{R}^2$, we also use the notation $H^s(K)$ to denote the closed subspace of the distributions in $H^s(\mathbf{R}^2)$ supported in K, see [35, Chapter 4.5], where those spaces are denoted by $H^s_K(M)$.

Proposition A.3. *For any compact set $K \subset \mathbf{R}^2$, and any $s \geq 0$,*

$$I_w : H^{s-1/2}(K) \mapsto H^s_{\text{comp}}(Z), \quad I'_w : H^{s-1/2}_{\text{comp}}(Z) \mapsto H^s_{\text{loc}}(\mathbf{R}^2) \tag{A.5}$$

are continuous.

Proof. We follow the proof of Proposition 5.1 in [33]. We can always assume that w is extended smoothly for x outside K so that it vanishes outside a small neighborhood of K. Then we can replace $\mathbf{R}^2$ and Z by compact manifolds, as explained in Section 2, and work with $f \in C^\infty(\mathbf{T}^2)$.

Note first that $I'_b I_a : H^s \to H^{s+1}$. Next, if $s \geq 0$ is an integer, for f supported in K,

$$\|I_w f\|^2_{H^s(Z)} \leq C \sum_{j \leq 2s} \left| \left(\partial_p^j I_w f, I_w f\right)_{L^2(Z)} \right| = C \sum_{j \leq 2s} \left| \left(I_w^* \partial_p^j I_w f, f\right)_{L^2(K)} \right|. \tag{A.6}$$

The term $\partial_p^j I_w f$ is a sum of weighted X-ray transforms of derivatives of f up to order $2s$, and therefore, $I_w^* \partial_p^j I_w$ a ΨDO of order $2s-1$. This easily implies that for $f \in C^\infty(\mathbf{T}^2)$,

$$\|I_w f\|^2_{H^s(Z)} \leq C \|f\|^2_{H^{s-1/2}(K)}.$$

The case of general $s \geq 0$ follows by interpolation. The estimate then holds for any $f \in H^{s-1/2}(\mathbf{T}^2)$, and therefore, for any $f \in H^{s-1/2}(K)$, as well.

To prove the second estimate, notice first that $\partial^\alpha I_w^* \psi$ is a sum of operators of the kind I_w^* but with different weights applied to p-derivatives of ψ up to order $|\alpha|$. Then for any integer $j \geq 0$,

$$|(f, I_a^* \partial_p^j \psi)_{L^2}| = |(I_a f, \partial_p^j \psi)_{L^2(Z)}| \leq C \|f\|_{L^2} \|\psi\|_{H^{j-1/2}}.$$

This proves the second estimate for $s = 0, 1, \dots$. For general $s \geq 0$ we use interpolation. $\square$

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