BOUNDARY RIGIDITY WITH PARTIAL DATA

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1. INTRODUCTION AND MAIN RESULTS

Travel time tomography deals with the problem of determining the sound speed or index of refraction of a medium by measuring the travel times of waves going through the medium. This type of inverse problem, also called the inverse kinematic problem, arose in geophysics in an attempt to determine the substructure of the Earth by measuring the travel times of seismic waves at the surface. We consider an anisotropic index of refraction, i.e., a sound speed depending on the direction. The Earth is generally isotropic. However, more recently it has been realized, by measuring these travel times, that the inner core of the Earth exhibits anisotropic behavior with the fast direction parallel to the Earth's spin axis; see [5]. In the human body, muscle tissue is anisotropic. As a model of anisotropy, we consider a Riemannian metric $g = (g_{ij})$. The problem is to determine the metric from the lengths of geodesics joining points on the boundary.

This leads to the general question of whether given a compact Riemannian manifold with boundary (M, g) one can determine the Riemannian metric in the interior knowing the boundary distance function joining points on the boundary $d_g(x, y)$, with $x, y \in \partial M$. This is known as the *boundary rigidity problem*. Of course, isometries preserve distance, so that the boundary rigidity problem is whether two metrics that have the same boundary distance function are the same up to isometry fixing the boundary. Examples can be given of manifolds that are not boundary rigid. Such examples show that the boundary rigidity problem should be considered under some restrictions on the geometry of the manifold. The most usual of such restrictions is simplicity of the metric. A Riemannian manifold (M, g) (or the metric g) is called *simple* if the boundary ∂M is strictly convex (positive second fundamental form) and any two points $x, y \in M$ are joined by a unique minimizing geodesic. Michel conjectured [21] that every simple compact Riemannian manifold with boundary is boundary rigid.

Simple surfaces with boundary are boundary rigid [29]. In higher dimensions, simple Riemannian manifolds with boundary are boundary rigid under some a priori constant curvature on the manifold or special symmetries [2, 13]. Several local results near the Euclidean metric are known [3, 19, 39]. The most general result in this direction is the generic local (with respect to the metric) one proven in [41]. Surveys of some of the results can be found in [8, 16, 42].

In this paper, we consider the boundary rigidity problem in the class of metrics conformal to a given one and with partial (local) data; that is, we know the boundary distance function for points on the boundary near a given point. Partial data problems arise naturally in applications since in many cases one does not have access to the whole boundary. We prove the first result on the determination of the conformal factor locally near the boundary from partial data without assuming analyticity. We develop a novel method to attack partial data nonlinear problems that will likely have other applications.

We now describe the known results with full data on the boundary. Let us fix the metric g_0 and let c be a positive smooth function on the compact manifold with boundary M. The problem is whether we can determine c from $d_{c^{-2}g_0}(x,y), x, y \in \partial M$. Notice that in this case the problem is not invariant under changes of variables that are the identity at the boundary so that we expect to be able to recover c under appropriate a priori conditions. This was proven by Mukhometov in two dimensions [23], and in [24] in higher dimensions for the case of simple metrics. Of particular importance in applications is the case of an isotropic sound speed that is when we are in a bounded domain of Euclidean space and g_0 is the Euclidean metric. This is the isotropic case. This problem was considered by Herglotz [14] and Wieckert and Zoeppritz [51] for the case of a spherical symmetric sound speed. They found a formula to recover the sound speed from the boundary distance function assuming $\frac{\mathrm{d}}{\mathrm{d}r}(\frac{r}{c(r)}) > 0$. Notice that this condition is equivalent to the existence of a strictly convex foliation and is more general than simplicity; see Section 6.

From now on we will call d the function $d_{c^{-2}g_0}$. Below, $\tilde{d}$ is related to $d_{\tilde{c}^{-2}g_0}$.

The partial data problem, which we will also call the *local boundary rigidity problem*,[1] in this case is whether knowledge of the distance function on part of the boundary determines the sound speed c locally. Given another smooth $\tilde{c}$, here and below we define $\tilde{L}$, $\tilde{\ell}$, and $\tilde{d}$ in the same way but related to $\tilde{c}$. We prove the following uniqueness result.

Theorem 1.1. *Let $n = \dim M \geq 3$, let $c > 0$, $\tilde{c} > 0$ be smooth, and let ∂M be strictly convex with respect to both $g = c^{-2}g_0$ and $\tilde{g} = \tilde{c}^{-2}g_0$ near a fixed $p \in \partial M$. Let $d(p_1, p_2) = \tilde{d}(p_1, p_2)$ for p_1, p_2 on ∂M near p. Then $c = \tilde{c}$ in M near p.*

As mentioned earlier, this is the only known result for the boundary rigidity problem with partial data except in the case that the metrics are assumed to be real-analytic [19]. The latter follows from determination of the jet of the metric at a convex point from the distance function known near p.

The boundary rigidity problem is closely connected to the lens rigidity one. To define the latter, we first introduce the manifolds $\partial_{\pm}SM$, defined as the sets of all vectors (x, v) with $x \in \partial M$, v unit in the metric g, and pointing outside/inside M. We define the *scattering relation*

$$(1.1) \qquad\qquad L : \partial_-SM \longrightarrow \partial_+SM$$

in the following way: for each $(x, v) \in \partial_-SM$, $L(x, v) = (y, w)$, where (y, w) are the exit point and direction, if they exist, of the maximal unit speed geodesic $\gamma_{x,v}$ in the metric g, issued from (x, v). Let

$$\ell : \partial_-SM \longrightarrow \mathbb{R} \cup \infty$$

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[1]It is local in the sense that $d(x, y)$ is known locally and depends on c locally; the term *local* has been used before to indicate that the metric is a priori close to a fixed one.

be its length, possibly infinite. If $\ell < \infty$, we call M non-trapping. The maps (L, ℓ) together are called *lens relation* (or lens data).

The *lens rigidity* problem is whether the scattering relation L (and possibly, ℓ) determine g (and the topology of M) up to an isometry as above. The lens rigidity problem with partial data for a sound speed is whether we can determine the speed near some p from L known near the unit sphere $S_p \partial M$ considered as a subset of $\partial_- SM$, i.e., for vectors with base points close to p and directions pointing into M close to ones tangent to ∂M. For general metrics, we want to recover isometric copies of the metrics locally, as above.

We assume that ∂M is strictly convex at $p \in \partial M$ with respect to (w.r.t.) g. Then the boundary rigidity and the lens rigidity problems with partial data are equivalent: knowing d near (p, p) is equivalent to knowing L in some neighborhood of $S_p \partial M$. The size of that neighborhood, however, depends on a priori bounds of the derivatives of the metrics with which we work. This equivalence was first noted by Michel [21], since the tangential gradients of $d(x, y)$ on $\partial M \times \partial M$ give us the tangential projections of $-v$ and w; see also [38, Section 2]. Note that local knowledge of ℓ is not needed for the lens rigidity problem,[2] and in fact, ℓ can be recovered locally from either d or L; see, for example, the proof of Theorem 5.2.

Vargo [50] proved that real-analytic manifolds satisfying an additional mild condition are lens rigid. Croke has shown that if a manifold is lens rigid, a finite quotient of it is also lens rigid [8]. He has also shown that the torus is lens rigid [4]. Uhlmann and Stefanov have shown lens rigidity locally near a generic class of non-simple manifolds [44]. In a recent work, Guillarmou [12] proved that in two dimensions, one can determine from the lens relation the conformal class of the metric if the trapped set is hyperbolic and there are no conjugate points. He also proved deformation lens rigidity in higher dimensions under the same assumptions. The only result we know for the lens rigidity problem with incomplete (but not local) data is for real-analytic metric and metric close to them satisfying the micolocal condition in the next sentence [44]. While in [44], the lens relation is assumed to be known on a subset only, the geodesics issued from that subset cover the whole manifold, and their conormal bundle is required to cover $N^* M$. In contrast, in this paper, we have localized information.

We state below an immediate corollary of our main result for this problem. For the partial data problem instead of assuming $d = \tilde{d}$ locally, we can assume that $L = \tilde{L}$ in a neighborhood of $S_p \partial M$. To reduce this problem to Theorem 1.1 directly, we need to assume first that $c = \tilde{c}$ on ∂M near p to make the definition of $\partial_{\pm} SM$ independent of the choice of the speed but, in fact, one can redefine the lens relation in a way to remove that assumption; see [44].

Theorem 1.2. *Let M, c, $\tilde{c}$ be as in Theorem 1.1 with $c = \tilde{c}$ on ∂M near p. Let $L = \tilde{L}$ near $S_p \partial M$. Then $c = \tilde{c}$ in M near p.*

Remark 1.3. The theorem or its corollary does not preclude the existence of an infinite set of speeds c_j all having the same boundary distance function (or lens data) in $U \times U$, where $U \subset \partial M$ is some fixed small set but not coinciding in any fixed neighborhood of p. In principle, this may happen when the maximal neighborhood of U, which can be covered with strictly convex surfaces, which continuously deform U, shrinks when $j \to \infty$. Then the theorem does not imply existence of a fixed

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[2]If only L is given, then the problem is called *scattering rigidity* in some works

neighborhood of p, where all speeds are equal. If one assumes that a priori, the sound speeds have uniformly bounded derivatives of some finite order near p, this situation does not arise, and this case is covered by Theorem 5.2 below.

The linearization of the boundary rigidity and lens rigidity problem is the tensor tomography problem, i.e., recovery of a tensor field up to "potential fields" from integrals along geodesics joining points on the boundary. It has been extensively studied in the literature for both simple and non-simple manifolds [1, 11, 22, 26–28, 30, 32, 34, 36, 40, 43, 46]. See the book [33] and [28] for a recent survey. The *local* tensor tomography problem has been considered in [17] for functions and real-analytic metrics and in [18] for tensors of order two and real-analytic metrics. Those results can also be thought of as support theorems of Helgason type. The only known results for the local problem for smooth metrics and integrals of functions is [49].

Now we use a layer stripping type argument to obtain a global result which is different from Mukhometov's for simple manifolds.

Definition 1.1. Let (M, g) be a compact Riemannian manifold with boundary. We say that M satisfies the foliation condition by strictly convex hypersurfaces if M is equipped with a smooth function $\rho : M \to [0, \infty)$ with level sets $\Sigma_t = \rho^{-1}(t)$, $t < T$ with some $T > 0$ are strictly convex viewed from $\rho^{-1}((0, t))$ for g, $d\rho$ is non-zero on these level sets, and $\Sigma_0 = \partial M$ and $M \setminus \cup_{t \in [0, T)} \Sigma_t$ has an empty interior.

The statement of the global result on lens rigidity is as follows.

Theorem 1.4. *Let $n = \dim M \geq 3$, let $c > 0$, $\tilde{c} > 0$ be smooth and equal on ∂M, let ∂M be strictly convex with respect to both $g = c^{-2}g_0$ and $\tilde{g} = \tilde{c}^{-2}g_0$. Assume that M can be foliated by strictly convex hypersurfaces for g. Then if $L = \tilde{L}$ on $\partial_- SM$, we have $c = \tilde{c}$ in M.*

A more general foliation condition under which the theorem would still hold is formulated in [45]; see also Definition 5.1 below. In particular, Σ_0 does not need to be ∂M, and one can have several such foliations with the property that the closure of their union is M. If we can foliate only some connected neighborhood of ∂M, we would get $c = \tilde{c}$ there. Next, it is enough to require that $M \setminus \cup_{t \in [0, T)} \Sigma_t$ is simple (or that it is included in a simple submanifold; see the proof of Theorem 1.4 and Figure 2 in Section 4) to prove $c = \tilde{c}$ in $\cup_{t \in [0, T)} \Sigma_t$ first, and then use Mukhometov's results to complete the proof. The class of manifolds we get in this way is larger than the simple ones and can have conjugate points.

Speeds not necessarily radial (with g_0 the Euclidean metric) under the condition considered by Herglotz and Wieckert and Zoeppritz satisfy the foliation condition of the theorem; see also Section 6. Other examples of non-simple metrics that satisfy the condition are the tubular neighborhood of a closed geodesic in negative curvature. These have trapped geodesics. It follows from the result of [31], that manifolds with no focal points satisfy the foliation condition. It would be interesting to know whether this is also the case for simple manifolds. As it was mentioned earlier, manifolds satisfying the foliation condition are not necessarily simple.

The linearization of the non-linear problem with partial data considered in Theorem 1.1 was considered in [49], where uniqueness and stability were shown. This corresponds to integrating functions along geodesics joining points in a neighborhood of p. The method of proof of Theorem 1.1 relies on using an identity proven

in [39] to reduce the problem to a "pseudo-linear" one: to show uniqueness when one integrates the function $f = c^2 - \tilde{c}^2$ and its derivatives on the geodesics for the metric g joining points near p, with weight depending non-linearly on both g and $\tilde{g}$. Notice that this is not a proof by linearization, and unlike the problem with full data, an attempt to do such a proof by linearization is connected with essential difficulties. The proof of uniqueness for this linear transform follows the method of [49] introducing an artificial boundary and using Melrose's scattering calculus. In Section 2, we do the reduction to a "pseudo-linear problem," and in Section 3, we show uniqueness for the pseudo-linear problem. In Section 4, we finish the proofs of the main theorems.

We also prove Hölder conditional stability estimates related to the uniqueness theorems above. In the case of data on the whole boundary, such an estimate was proved in [41, Section 7] for simple manifolds and metrics not necessarily conformal to each other. Below, the C^k norm is defined in a fixed coordinate system. The next theorem is a local stability result, corresponding to the local uniqueness result in Theorem 1.1.

Theorem 1.5. *There exists $k > 0$ and $0 < \mu < 1$ with the following property. For any $0 < c_0 \in C^k(M)$, $p \in \partial M$, and $A > 0$, there exists $\varepsilon_0 > 0$ and $C > 0$ with the property that for any two positive c, $\tilde{c}$ with*

$$(1.2) \qquad \|c - c_0\|_{C^2} + \|\tilde{c} - c_0\|_{C^2} \leq \varepsilon_0, \quad and \quad \|c\|_{C^k} + \|\tilde{c}\|_{C^k} \leq A,$$

and for any neighborhood Γ of p on ∂M, we have the stability estimate

$$(1.3) \qquad \|c - \tilde{c}\|_{C^2(U)} \leq C\|d - \tilde{d}\|^\mu_{C(\Gamma \times \Gamma)}$$

for some neighborhood U of p in M.

In Theorem 5.2, we prove Hölder conditional stability estimates of global type as well, which can be considered as a "stable version" of Theorem 1.4.

The plan of the paper is as follows. The reduction to a pseudo-linear problem is done in Section 2. In Section 3, we present linear analysis using the scattering calculus. The main result there is Proposition 3.3, which is of its own interest as well. The proofs of the three uniqueness theorems are in Section 4. In Section 5, we prove the local stability result in Theorem 1.5 and the global Theorem 5.2. As an application of our results, we revisit the class of speeds studied by Herglotz [14] and Wieckert and Zoeppritz [51] in Section 6 without assuming that they are radial, and we prove that they are lens rigid. In particular, we show that their condition (6.1) is equivalent to the requirement that the Euclidean spheres are strictly convex for the metric $c^{-2}\mathrm{d}x^2$; therefore, it is a foliation condition.

2. Reducing the non-linear problem to a pseudo-linear one

We recall the known fact [19] that one can determine the jet of c at any boundary point p at which ∂M is convex (not necessarily strictly) from the distance function d known near (p, p). For a more general result not requiring convexity, see [44]. Since the result in [19] is formulated for general metrics, and the reconstruction of the jet is in boundary normal coordinates, we repeat the proof in this (simpler) situation of recovery of a conformal factor. As in [19], we say that ∂M is convex near $p \in \partial M$, if for any two distinct points $p_1, p_2 \in \partial M$, close enough to p, there exists a geodesic $\gamma : [0, 1] \to M$ joining them such that its length is $d(p_1, p_2)$ and all

the interior of γ belongs to the interior of M. Of course, strict convexity (positive second fundamental form at p) implies convexity.

Lemma 2.1. *Let c and $\tilde{c}$ be smooth and let ∂M be convex at p with respect to g and $\tilde{g}$. Let $d = \tilde{d}$ near (p,p). Then $\partial^\alpha c = \partial^\alpha \tilde{c}$ on ∂M near p for any multi-index α.*

Proof. Let $V \subset \partial M$ be a neighborhood of p on ∂M such that for any $p_1, p_2 \in V$, we have the property guaranteeing convexity at p. Let x^n be a boundary normal coordinate related to g; i.e., $x^n(q) = \mathrm{dist}(q, \partial M)$, and $x^n \geq 0$ in M. We can complete x^n to a local coordinate system (x', x^n), where x' parameterizes ∂M near p.

It is enough to prove

$$(2.1) \qquad \partial_{x^n}^k c = \partial_{x^n}^k \tilde{c} \quad \text{in } V, \, k = 0, 1, \dots.$$

For $k = 0$, this follows easily by taking the limit in $d(p,q) = \tilde{d}(p,q)$, $\partial M \ni q \to p$; and this can be done for any $p \in V$. Let l be the first value of k for which (2.1) fails. Without loss of generality, we may assume that it fails at p, and $\partial_{x^n}^l (c - \tilde{c}) > 0$ at p. Then $\partial_{x^n}^k (c - \tilde{c}) = 0$ in V, $k = 0, \dots, l-1$. Consider the Taylor expansion of $c - \tilde{c}$ w.r.t. x^n with x' close enough to $x'(p)$. We get $c - \tilde{c} > 0$ in some neighborhood of p in M minus the boundary.

Now, let $\gamma(p,q)$ be a minimizing geodesic in the metric g connecting p and q when $q \in \partial M$ as well, close enough to p; see also [19]. Let $If(p,q)$ be the geodesic ray transform of the tensor field f defined as an integral of $f_{ij}\dot{\gamma}^i \dot{\gamma}^j$ along $\gamma(p,q)$. All geodesics here are parameterized by a parameter in $[0,1]$ rather than being unit speed, and therefore the transform I is parameterized differently than the usual one. Then $I(g - \tilde{g}) > 0$ by what we proved above. On the other hand,

$$0 < I(g - \tilde{g}) = d^2(p,q) - I\tilde{g} \leq d^2(p,q) - \tilde{I}\tilde{g} = d^2(p,q) - \tilde{d}^2(p,q) = 0$$

because $\tilde{\gamma}(p,q)$ minimizes integrals of g along curves connecting p and q. This is a contradiction. $\qquad \square$

The starting point is an identity in [39]. We will repeat the proof.

Let V, $\tilde{V}$ be two vector fields on a manifold M (which will be replaced later with S^*M). Denote by $X(s, X^{(0)})$ the solution of $\dot{X} = V(X)$, $X(0) = X^{(0)}$, and we use the same notation for $\tilde{V}$ where the corresponding solutions are denoted by $\tilde{X}$. Then we have the following simple statement.

Lemma 2.2. *For any $t > 0$ and any initial condition $X^{(0)}$, if $\tilde{X}(\cdot, X^{(0)})$ and $X(\cdot, X^{(0)})$ exist on the interval $[0, t]$, then*

$$(2.2) \qquad \begin{aligned} &\tilde{X}\left(t, X^{(0)}\right) - X\left(t, X^{(0)}\right) \\ &= \int_0^t \frac{\partial \tilde{X}}{\partial X^{(0)}} \left(t - s, X(s, X^{(0)})\right) \left(\tilde{V} - V\right)\left(X(s, X^{(0)})\right) ds. \end{aligned}$$

Proof. Set

$$F(s) = \tilde{X}\left(t - s, X(s, X^{(0)})\right).$$

Then

$$F'(s) = -\tilde{V}\left(\tilde{X}(t-s, X(s, X^{(0)}))\right)$$
$$+ \frac{\partial \tilde{X}}{\partial X^{(0)}}\left(t-s, X(s, X^{(0)})\right) V\left(X(s, X^{(0)})\right).$$

The proof of the lemma would be complete by the fundamental theorem of calculus

$$F(t) - F(0) = \int_0^t F'(s)\, ds$$

if we show the following

$$(2.3) \qquad \tilde{V}\left(\tilde{X}(t-s, X(s, X^{(0)}))\right) = \frac{\partial \tilde{X}}{\partial X^{(0)}}\left(t-s, X(s, X^{(0)})\right) \tilde{V}\left(X(s, X^{(0)})\right).$$

Indeed, (2.3) follows from

$$0 = \frac{d}{d\tau}\bigg|_{\tau=0} X(T-\tau, X(\tau, Z)) = -V(X(T, Z)) + \frac{\partial X}{\partial X^{(0)}}(T, Z)V(Z), \quad \forall\ T,$$

after setting $T = t - s$, $Z = X(s, X^{(0)})$. $\qquad \square$

Let c, $\tilde{c}$ be two speeds. Then the corresponding metrics are $g = c^{-2}dx^2$, and $\tilde{g} = \tilde{c}^{-2}dx^2$. The corresponding Hamiltonians and Hamiltonian vector fields are

$$H = \frac{1}{2}c^2 g_0^{ij}\xi_i\xi_j, \qquad V = \left(c^2 g_0^{-1}\xi, -\frac{1}{2}\partial_x\left(c^2|\xi|^2_{g_0}\right)\right),$$

and the same ones related to $\tilde{c}$. We used the notation $|\xi|^2_{g_0} := g_0^{ij}\xi_i\xi_j$.

We change the notation at this point. We denote points in the phase space T^*M, in a fixed coordinate system, by $z = (x, \xi)$. We denote the bicharacteristic with initial point z by $Z(t, z) = (X(t, z), \Xi(t, z))$.

Then we get the identity already used in [39]

$$(2.4) \qquad \tilde{Z}(t, z) - Z(t, z) = \int_0^t \frac{\partial \tilde{Z}}{\partial z}(t-s, Z(s, z))\left(\tilde{V} - V\right)(Z(s, z))\, ds.$$

We can naturally think of the scattering relation L and the travel time ℓ as functions on the cotangent bundle instead of the tangent one. Then we get the following.

Proposition 2.1. *Assume*

$$(2.5) \qquad L(x_0, \xi^0) = \tilde{L}(x_0, \xi^0), \qquad \ell(x_0, \xi^0) = \tilde{\ell}(x_0, \xi^0)$$

for some $z_0 = (x_0, \xi^0) \in \partial_- S^* M$. *Then*

$$\int_0^{\ell(z_0)} \frac{\partial \tilde{Z}}{\partial z}(\ell(z_0) - s, Z(s, z_0))\left(V - \tilde{V}\right)(Z(s, z_0))\, ds = 0.$$

2.1. Linearization near $c = 1$ and g Euclidean.

As a simple exercise, let $c = 1$, $g_{ij} = \delta_{ij}$ and linearize for $\tilde{c}$ near 1 first under the assumption that $\tilde{c} = 1$ outside an open region $\Omega \subset \mathbb{R}^n$. Then

$$(2.6) \qquad Z(s, z) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} z, \qquad \frac{\partial Z(s, z)}{\partial z} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix},$$

and we get the following formal linearization of (2.4)

$$(2.7) \qquad \int \left(f\xi - \frac{1}{2}(t - s)(\partial_x f), -\frac{1}{2}\partial_x f \right)(x + s\xi, \xi)\, ds = 0,$$

where

$$(2.8) \qquad\qquad\qquad f := c^2 - \tilde{c}^2.$$

Notice that we would get the same thing if we replace $\partial \tilde{Z}/\partial z$ in (2.4) by $\partial Z/\partial z$. We integrate over the whole line $s \in \mathbb{R}$ because the integrand vanishes outside the interval $[0, \ell(x, \xi)]$. The last n components of (2.7) imply

$$(2.9) \qquad\qquad\qquad \int \partial_x f(x + s\xi)\, ds = 0.$$

Now, assume that this holds for all (x, ξ). Then $\partial_x f = 0$, and since $f = 0$ on ∂M, we get $f = 0$.

2.2. The general case.

We take the second n-dimensional component on (2.4) and use the fact that $c^2|\xi|^2_{g_0} = 1$ on the bicharacteristics related to c. We assume that both geodesics extend to $t \in [0, \ell(x, \xi)]$. We want to emphasize that the bicharacteristics on the energy level $H = 1/2$, related to c, do not necessarily stay on the same energy level for the Hamiltonian $\tilde{H}$. We get

$$
(2.10) \quad
\begin{aligned}
&\int \frac{\partial \tilde{\Xi}}{\partial x}(\ell(z) - s, Z(s, z))(fg_0^{-1}\xi)(Z(s, z))\, ds \\
&- \frac{1}{2}\int \frac{\partial \tilde{\Xi}}{\partial \xi}(\ell(z) - s, Z(s, z))(\partial_x(fg_0^{-1})\xi \cdot \xi)(Z(s, z))\, ds = 0
\end{aligned}
$$

for any $z \in \partial_- SM$ for which (2.5) holds. As before, we integrate over $s \in \mathbb{R}$ because the support of the integrand vanishes for $s \notin [0, \ell(x, \xi)]$ (for that, we extend the bicharacteristics formally outside so that they do not come back). Write

$$\partial_x(fg_0^{-1}) = g_0^{-1}\partial_x f + (\partial_x g_0^{-1})f$$

to get

$$
(2.11) \quad
\begin{aligned}
0 = &\int \frac{\partial \tilde{\Xi}}{\partial x}(\ell(z) - s, Z(s, z))(fg_0^{-1}\xi)(Z(s, z))\, ds \\
&- \frac{1}{2}\int \frac{\partial \tilde{\Xi}}{\partial \xi}(\ell(z) - s, Z(s, z))\left(\left(g_0^{-1}\partial_x f + f(\partial_x g_0^{-1})\right)\xi \cdot \xi \right)(Z(s, z))\, ds.
\end{aligned}
$$

One of the terms on the right-hand side (r.h.s.) above involves $g_0^{-1}\xi \cdot \xi$ which equals c^{-2} on the bicharacteristics of H on the level $1/2$.

Introduce the exit times $\tau(x, \xi)$ defined as the minimal (and the only) $t > 0$ so that $X(t, x, \xi) \in \partial M$. They are well defined near $S_p\partial M$, if ∂M is strictly convex

at p. We need to write $\frac{\partial \tilde{Z}}{\partial z}(\ell(z) - s, Z(s,z))$ as a function of $(x,\xi) = Z(s,z)$. We have

$$\frac{\partial \tilde{Z}}{\partial z}(\ell(z) - s, Z(s,z)) = \frac{\partial \tilde{Z}}{\partial z}(\tau(x,\xi), (x,\xi)).$$

Then we get, with f as in (2.8),

$$(2.12) \quad J_i f(\gamma) := \int \left(A_i^j(X(t), \Xi(t))(\partial_{x^j} f)(X(t)) + B_i(X(t), \Xi(t)) f(X(t)) \right) \mathrm{d}t = 0$$

for any bicharacteristic $\gamma = (X(t), \Xi(t))$ (related to the speed c) in our set, where

$$A_i^j(x,\xi) = -\frac{1}{2}\frac{\partial \tilde{\Xi}_i}{\partial \xi_j}(\tau(x,\xi), (x,\xi))c^{-2}(x),$$

$$(2.13) \qquad B_i(x,\xi) = \frac{\partial \tilde{\Xi}_i}{\partial x^j}(\tau(x,\xi), (x,\xi))g_0^{ik}(x)\xi_k$$

$$-\frac{1}{2}\frac{\partial \tilde{\Xi}_i}{\partial \xi_j}(\tau(x,\xi), (x,\xi))(\partial_{x^j} g_0^{-1}(x))\xi \cdot \xi.$$

A major inconvenience with this representation is that the exit time function $\tau(x,\xi)$ (recall that we assume strong convexity) becomes singular at $(x,\xi) \in T^*\partial M$. More precisely, the normal derivative w.r.t. x when ξ is tangent to ∂M has a square root type of singularity. On the other hand, we have some freedom to extend the flow outside M since we know that the jets of c and $\tilde{c}$ at ∂M are the same near p: therefore, any smooth local extension of c is also a smooth extension of $\tilde{c}$. Then for any $(x,\xi) \in \partial_- S^*M$ close enough to $S^*_{x_0} M$, the bicharacteristics originating from it will be identical once they exit T^*M but are still close enough to it. Similarly, instead of starting from $T^*\partial M$, we could start at points and codirections close to it, but away from $\bar{M}$.

With this in mind, we push the boundary away a bit. Let x_0 represent the point p near which we work, in a fixed coordinate system. Extend g_0 smoothly near x_0. Let $S(x_0, r)$ be the sphere in the metric $c^{-2}\mathrm{d}x^2$ centered at x_0 with radius $0 < r \ll 1$. For (x,ξ) with x in the geodesic ball $B(x_0, r)$, redefine $\tau(x,\xi)$ to be the travel time from (x,ξ) to $S(x_0, r)$. Let $U_- \subset \partial_- SB(x_0, r)$ be the set of all points on $S(x_0, r)$ and incoming unit directions so that the corresponding geodesic in the metric g is close enough to one tangent to ∂M at x_0. Similarly, let U_+ be the set of such pairs with outgoing directions. Redefine the scattering relation L locally to act from U_- to U_+, and redefine ℓ similarly; see Figure 1. Then under the assumptions of Theorem 1.2, $L = \tilde{L}$ and $\ell = \tilde{\ell}$ on U_-. We can apply the construction above by replacing $\partial_{\pm} SM$ locally by $U_{\pm}$. The advantage we have now is that on U_-, the travel time τ is nonsingular. Equalities (2.12) and (2.13) are preserved then.

We now have

$$A_i^j(x_0, \xi) = -\frac{1}{2}\frac{\partial \tilde{\Xi}_i}{\partial \xi_j}(r, (x_0, \xi))c^{-2}(x_0),$$

$$(2.14) \qquad B_i(x_0, \xi) = \frac{\partial \tilde{\Xi}_i}{\partial x^j}(r, (x_0, \xi))g_0^{ik}(x_0)\xi_k - \frac{1}{2}\frac{\partial \tilde{\Xi}_i}{\partial \xi_j}(r, (x, \xi))(\partial_{x^j} g_0^{-1}(x_0))\xi \cdot \xi.$$

Then by the strict convexity,

$$(2.15) \qquad A_i^j(x,\xi) = -\frac{1}{2}c^{-2}(x)\delta_i^j + O(r), \quad \text{for } (x,\xi) \in S^*M \text{ near } S_p^*\partial M.$$
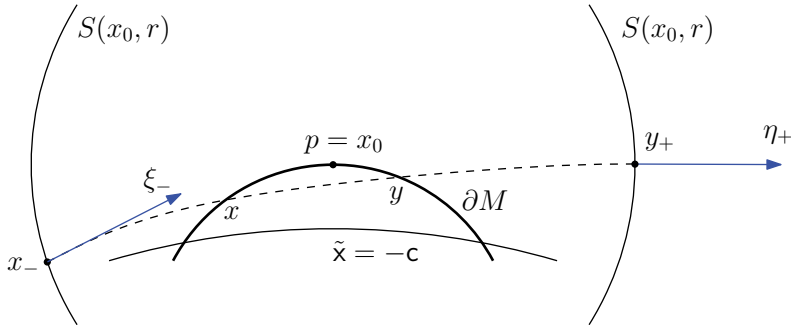
FIGURE 1. The redefined scattering relation $(x_-, \xi_-) \mapsto (y_+, \eta_+)$.

2.3. **A new linear problem.** The arguments above lead to the following linear problem.

Problem. Assume (2.12) holds with some f supported in M, for all geodesics close to the ones originating from $S^*_{x_0} \partial M$ (i.e., initial point x_0 and all unit initial codirections tangent to ∂M). Assume that ∂M is strictly convex at x_0 w.r.t. the speed c. Assume (2.15). Is it true that $f = 0$ near x_0?

We show below in Proposition 3.3 that the answer is affirmative. Note that this reduces the original non-linear problem to a linear one, but this is not a linearization. Then Theorem 1.2 follows from it. On the other hand, Theorem 1.2 is not equivalent to that problem because the weight there has a specific structure, and thus Proposition 3.3 is a more general statement.

3. LINEAR ANALYSIS

We first recall the setting introduced in [49] in our current notation. There the scalar x-ray transform along geodesics was considered, namely for $\beta \in SM$,

$$(If)(\beta) = \int_{\mathbb{R}} f(\gamma_\beta(t)) \, \mathrm{d}t,$$

where γ_β is the geodesic with lift to SM having starting point $\beta \in SM$. Here M is assumed to have a strictly convex boundary, which can be phrased as the statement that if ρ is a defining function for ∂M, then $-\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho \circ \gamma_\beta)|_{t=0} \geq C > 0$ whenever $\beta \in S\partial M$. One then considers a point $p \in \partial M$, and another function $\tilde{x}$, denoted in [49] by $\tilde{x}$, such that $\tilde{x}(p) = 0$, $\mathrm{d}\tilde{x}(p) = -\mathrm{d}\rho(p)$, and the level sets of $\tilde{x}$ near the 0 are strictly concave when viewed from the superlevel sets (which are on the side of p when talking about the c-level set with $\mathsf{c} < 0$), i.e., $\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\tilde{x} \circ \gamma_\beta)|_{t=0} \geq C > 0$ if $\beta \in S\{\tilde{x} = \mathsf{c}\}$, namely if $\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{x} \circ \gamma_\beta)|_{t=0} = 0$. For $\mathsf{c} > 0$, we denote $M \cap \{\tilde{x} > -\mathsf{c}\}$ by Ω_{c}; we assume that $\mathsf{c}_0 > 0$ is such that $\overline{\Omega_{\mathsf{c}_0}}$ is compact on M, and the concavity assumption holds on Ω_{c_0}. Then it was shown that the x-ray transform I restricted to $\beta \in S\overline{\Omega_{\mathsf{c}}}$ such that γ_β leaves Ω_{c} with both end points on ∂M (i.e., at $\rho = 0$) is injective if $\mathsf{c} > 0$ is sufficiently small, and indeed one has a stability estimate for f in terms of If on exponentially weighted spaces.

To explain this in detail, let $\mathsf{x} = \mathsf{x}_{\mathsf{c}} = \tilde{x} + \mathsf{c}$ be the boundary defining function of the artificial boundary, $\tilde{x} = -\mathsf{c}$, that we introduced; indeed, it is convenient to work in $\tilde{M}$, a C^∞ manifold extending M across the boundary, extending $\tilde{x}$ to $\tilde{M}$ smoothly,

and defining $\hat{\Omega} = \hat{\Omega}_{\mathsf{c}} = \{\mathsf{x}_{\mathsf{c}} > 0\}$ as the extension of Ω, so $\overline{\hat{\Omega}}$ is a smooth manifold with boundary, with only the artificial boundary being a boundary face. Then one writes $\beta = (\lambda, \omega) = \lambda\partial_{\mathsf{x}} + \omega\partial_{\mathsf{y}} \in S_{\mathsf{x},\mathsf{y}}\tilde{M}$ relative to a product decomposition $(-\mathsf{c}_0, \mathsf{c}_0)_{\mathsf{x}} \times U_{\mathsf{y}}$ of $\tilde{M}$ near p. The concavity condition becomes that for β whose λ-component vanishes,

$$2\alpha(\mathsf{x}, \mathsf{y}, 0, \omega) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathsf{x} \circ \gamma_\beta)|_{t=0} \geq 2C > 0,$$

with a new $C > 0$; see the discussion preceding Equation (3.1) in [49]. For $\chi \in C_0^\infty(\mathbb{R})$, $\chi \geq 0$, $\chi(0) > 0$, one considers the map

$$L_0 v(\mathsf{x}, \mathsf{y}) = \int_{\mathbb{R}} \int_{\mathbb{S}^{n-2}} \mathsf{x}^{-2}\chi(\lambda/\mathsf{x})v(\mathsf{x}, \mathsf{y}, \lambda, \omega)\,\mathrm{d}\lambda\,\mathrm{d}\omega$$

defined for v a function on $S_{\overline{\hat{\Omega}}}\tilde{M}$. This differs from [49] in that the weight x^{-2} differs by 1 from the weight x^{-1} used in [49]; this simply has the effect of removing an x^{-1} in [49, Proposition 3.3], as compared to the proposition stated below. If c is sufficiently small, or instead χ has sufficiently small support, for $(\mathsf{x}, \mathsf{y}) \in \overline{\Omega}$, I only integrates over points in $\beta \in S\overline{\hat{\Omega}}$ such that γ_β leaves Ω_{c} with both end points on ∂M, i.e., over β corresponding to Ω_{c}-*local geodesics*—the set of the latter is denoted by $\mathcal{M}_{\mathsf{c}}$. We refer to the discussion around [49, Equation (3.1)] for more detail, but roughly speaking the concavity of the level sets of x means that the geodesics that are close to being tangent to the foliation, with "close" measured by the distance from the artificial boundary, $\mathsf{x} = 0$, then they cannot reach $\mathsf{x} = 0$ (or reach again, in case they start there) without reaching $\mathsf{x} = \mathsf{c}'$ for some fixed $\mathsf{c}' > 0$; notice that the geodesics involved in the integration through a point on the level set $\mathsf{x} = \tilde{\mathsf{c}}$ make an angle $\lesssim \tilde{\mathsf{c}}$ with the tangent space of the level set due to the compact support of χ. Then we consider the map $P = L_0 \circ I$. The main technical result of [49], whose notation involving the so-called scattering Sobolev spaces $H_{\mathrm{sc}}^{s,r}(\overline{\hat{\Omega}})$ and scattering pseudo-differential operators $\Psi_{\mathrm{sc}}^{s,r}(\overline{\hat{\Omega}})$ is explained below, was the following proposition.

Proposition 3.1 (See [49, Proposition 3.3 and Lemma 3.6]). *For $F > 0$ let*

$$P_F = e^{-F/x}Pe^{F/x} : C_0^\infty(\overline{\hat{\Omega}}) \to C^\infty(\overline{\hat{\Omega}}).$$

Then $P_F \in \Psi_{\mathrm{sc}}^{-1,0}(\overline{\hat{\Omega}})$.

Further, if $\mathsf{c} > 0$ is sufficiently small, then for a suitable choice of $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(0) = 1$, $\chi \geq 0$, P is elliptic in $\Psi_{\mathrm{sc}}^{-1,0}(\overline{\hat{\Omega}})$ on a neighborhood of $\overline{\Omega}$.

Shrinking c further if needed, P_F satisfies the estimate

$$(3.1) \qquad \|v\|_{H_{\mathrm{sc}}^{s,r}(\overline{\hat{\Omega}})} \leq C\|P_F v\|_{H_{\mathrm{sc}}^{s+1,r}(\overline{\hat{\Omega}})}$$

for v supported in $M \cap \overline{\hat{\Omega}}$.

We now briefly explain the role of the so-called scattering pseudo-differential operators and the corresponding Sobolev spaces (which are typically used to study phenomena "at infinity") in our problem (where there is no obvious "infinity"); we refer to [49, Section 2] for a more thorough exposition. These concepts were introduced by Melrose [20], in a general geometric setting, but on $\mathbb{R}^n$ these operators actually correspond to a special case of Hörmander's Weyl calculus [15], also

studied earlier by Šubin [37] and Parenti [25]. So consider the reciprocal spherical coordinate map, $(0, \epsilon)_\mathsf{x} \times \mathbb{S}^{n-1}_\theta \to \mathbb{R}^n$, with $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ (unit sphere), and map $(\mathsf{x}, \theta) \mapsto \mathsf{x}^{-1}\theta \in \mathbb{R}^n$. This map is a diffeomorphism onto its range, and it provides a compactification of $\mathbb{R}^n$ (the so-called radial or geodesic compactification) by adding $\{0\}_\mathsf{x} \times \mathbb{S}^{n-1}_\theta$ as infinity to $\mathbb{R}^n$, to obtain $\overline{\mathbb{R}^n}$, which is now diffeomorphic to a ball. Now for general U_y above, we may regard, at least locally[3] U_y also as a coordinate chart in $\mathbb{S}^{n-1}$, and thus obtain an identification of $\overline{\Omega}$ with a region intersection $\overline{\mathbb{R}^n}$; thus our artificial boundary $\mathsf{x} = 0$ corresponds to infinity at $\mathbb{R}^n$. In particular, notions from $\mathbb{R}^n$ can now be transferred to a neighborhood of our artificial boundary. Since the relevant vector fields on $\mathbb{R}^n$ are generated by translation invariant vector fields, which are complete under the exponential map, the transferred analysis replaces the incomplete geometry of standard vector fields on $\overline{\hat{\Omega}}$ by a complete one. Concretely, these vector fields, when transferred, become linear combinations of $\mathsf{x}^2\partial_\mathsf{x}$ and $\mathsf{x}\partial_{\mathsf{y}_j}$, with smooth coefficients. In particular, these are the vector fields with respect to which Sobolev regularity is measured. Thus, $H^{s,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})$ is the so-called *scattering Sobolev space*, which is locally, under the above identification, just the standard weighted Sobolev space $H^{s,r}(\mathbb{R}^n)$ (see [49, Section 2]), while $\Psi^{s,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})$ is Melrose's *scattering pseudo-differential algebra*, which locally, again under this identification, simply corresponds to quantizing symbols a with $|D^\alpha_z D^\beta_\zeta a| \leq C\langle z \rangle^{r-|\alpha|}\langle \zeta \rangle^{s-|\beta|}$ on $T^*\mathbb{R}^n = \mathbb{R}^n_z \times \mathbb{R}^n_\zeta$ (see again [49, Section 2] for more detail). Note that ellipticity in this algebra, called full ellipticity, is in the sense as *both* $|z| \to \infty$ and $|\zeta| \to \infty$, i.e., modulo symbols one order lower *and* with an extra order of decay as $|z| \to \infty$.

Notice that (3.1) implies the estimate

$$\|f\|_{e^{F/\mathsf{x}}H^{s,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})} \leq C\|Pf\|_{e^{F/\mathsf{x}}H^{s+1,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})}$$

for the unconjugated operator, valid when f is supported in $M \cap \overline{\hat{\Omega}}$. Rewriting P as $L_0 \circ I$, this gives that for $\delta > 0$, $s \geq -1$,

$$\|f\|_{e^{(F+\delta)/\mathsf{x}}H^s(\overline{\hat{\Omega}})} \leq C\|If\|_{e^{F/\mathsf{x}}H^{s+1}(\mathcal{M}_\mathsf{c})};$$

see the discussion in [49] after Lemma 3.6.

After this recollection, we continue by generalizing (2.12) to regard the functions $\partial_{x_j}f$ and f entering into it as independent unknowns, while restricting the transform to the region of interest $\Omega = \Omega_\mathsf{c}$. So let $\tilde{J}_i$ be defined by

$$\tilde{J}_i(u_0, u_1, \ldots, u_n)(\beta)$$
$$:= \int_{\gamma_\beta} \left(A^j_i(X(t), \Xi(t))u_j(X(t)) + B_i(X(t), \Xi(t))u_0(X(t)) \right) \mathrm{d}t,$$

where γ_β is the geodesic with lift to $S\Omega$ having starting point $\beta \in S\Omega$. Let $\tilde{J} = (\tilde{J}_1, \ldots, \tilde{J}_n)$. This is a vector valued version of the geodesic x-ray transform considered in [49], and described above, sending functions on Ω with values in

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[3]That is, possibly at the cost of shrinking it; in fact, all concepts below are essentially local within $\mathbb{S}^{n-1}$; thus even in full generality one can reduce scattering objects to (conic regions near infinity in) $\mathbb{R}^n$ this way, much as standard Sobolev spaces and pseudo-differential operators are so reducible to subsets of $\mathbb{R}^n$ with compact closure.

$\mathbb{C}^{n+1}$ to functions with values in $\mathbb{C}^n$. We then define L as a map from $\mathbb{C}^n$-valued functions on $S\Omega$ to $\mathbb{C}^n$ valued functions on Ω by

$$Lv(\mathsf{x},\mathsf{y}) = \int_{\mathbb{R}} \int_{\mathbb{S}^{n-2}} \mathsf{x}^{-2} \chi(\lambda/\mathsf{x}) v(\mathsf{x},\mathsf{y},\lambda,\omega) \, d\lambda \, d\omega$$

as in [49]; this is a diagonal operator: $L = L_0 \otimes \mathrm{Id}$. Then we consider the map $P = L \circ \tilde{J}$, and in addition to the properties mentioned above in the scalar setting, we are also interested in continuity properties in terms of the background data, such as in the background metric as well as the function x. Recall that the map

(3.2) $\Gamma_+ : S\tilde{M} \times [0,\infty) \to [\tilde{M} \times \tilde{M}; \mathrm{diag}], \ \Gamma_+(\mathsf{x},\mathsf{y},\lambda,\omega,t) = ((\mathsf{x},\mathsf{y}), \gamma_{\mathsf{x},\mathsf{y},\lambda,\omega}(t))$

is a local diffeomorphism, and similarly for Γ_- in which $(-\infty,0]$ takes the place of $[0,\infty)$ (see the discussion around [49, Equations (3.2) and (3.3)]); indeed, this is true for more general curve families. Here $[\tilde{M} \times \tilde{M}; \mathrm{diag}]$ is the *blowup* of $\tilde{M}$ at the diagonal $z = z'$, which essentially means the introduction of spherical/polar coordinates, or often more conveniently projective coordinates, about it. Concretely, writing the (local) coordinates from the two factors of $\tilde{M}$ as (z, z'),

(3.3) $$z, |z - z'|, \frac{z - z'}{|z - z'|},$$

give (local) coordinates on this space. Note that when $\Gamma_\pm$ are given by geodesics of a metric g just C^1-near a fixed background metric g_0, as C^∞ maps, $\Gamma_\pm$ depend continuously on g in the C^∞ topology.

In order to consider continuity properties in $\tilde{\mathsf{x}}$ in a C^1-neighborhood of a fixed function $\tilde{\mathsf{x}}_0$, it is convenient to use the map $(\tilde{\mathsf{x}},\mathsf{y})$ to identify a neighborhood of p with a neighborhood O of the origin in $\mathbb{R} \times U$. Thus, for c fixed, but $\tilde{\mathsf{x}}$ being C^1-close to $\tilde{\mathsf{x}}_0$, on this fixed background $O \subset \mathbb{R} \times U$, the pulled back metrics $\underline{g}$ depend continuously on $(\tilde{\mathsf{x}}, g)$ as maps $C^\infty(\tilde{M}) \times C^\infty(\tilde{M}; S^2\tilde{M}) \to C^\infty(O)$; this normalizes $\tilde{\mathsf{x}}$ to be simply the first coordinate function $\underline{\tilde{\mathsf{x}}}$ on O. Correspondingly, below, the continuous dependence, of all objects discussed, on $\tilde{\mathsf{x}}$ (in the C^∞ topology) is automatic: what is meant always is that by pullback to O the resulting objects, living on fixed domains such as $\{\underline{\tilde{\mathsf{x}}} + \mathsf{c} > 0\}$, depend continuously on g and $\tilde{\mathsf{x}}$, which follows from the continuous dependence of these objects on $\underline{g}$. Since we do not want to overburden the notation, we do *not* write this pullback explicitly.

The main technical result here is the following proposition.

Proposition 3.2. *For $F > 0$, let*

$$P_F = e^{-F/\mathsf{x}} P e^{F/\mathsf{x}} : C_0^\infty(\overline{\tilde{\Omega}}; \mathbb{C}^{n+1}) \to C^\infty(\overline{\tilde{\Omega}}; \mathbb{C}^n).$$

Then $P_F \in \Psi_{\mathrm{sc}}^{-1,0}(\overline{\tilde{\Omega}}; \mathbb{C}^{n+1}, \mathbb{C}^n)$, and the map $\Gamma_\pm \mapsto P_F$ is continuous from the C^∞ topology to the Fréchet topology of $\Psi_{\mathrm{sc}}^{-1,0}(\overline{\tilde{\Omega}}; \mathbb{C}^{n+1}, \mathbb{C}^n)$.

Further, if c is sufficiently small and (2.15) holds, then for a suitable choice of $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(0) = 1$, $\chi \geq 0$, if we write $P_F = (P_0, \tilde{P})$, with P_0 corresponding to the first component, $\tilde{P}$ the last n components, in the domain space, then $\tilde{P}$ is elliptic in $\Psi_{\mathrm{sc}}^{-1,0}(\overline{\tilde{\Omega}}; \mathbb{C}^n, \mathbb{C}^n)$ in a neighborhood of $\overline{\Omega}$.

Remark 3.1. Notice that ellipticity being an open condition, this means that there exists $\mathsf{c}_0 > 0$ such that if $\mathsf{c} < \mathsf{c}_0$, then the same χ works for all g C^∞-close to g_0.

Further, in view of the paragraph preceding the proposition, the map $(\tilde{\mathsf{x}}, \Gamma_\pm) \mapsto$ P_Γ is continuous from the C^∞ topology to the Fréchet topology of $\Psi_{\mathrm{sc}}^{-1,0}(\overline{\tilde{\Omega}}; \mathbb{C}^{n+1},$ $\mathbb{C}^n)$, where the latter is understood to actually stand for $\Psi_{\mathrm{sc}}^{-1,0}(\{\tilde{\underline{\mathsf{x}}} + \mathsf{c} \geq 0\}; \mathbb{C}^{n+1}, \mathbb{C}^n)$ via the identifications discussed above.

Proof. This is simply a vector valued version of [49, Proposition 3.3] and [49, Lemma 3.6], recalled above in Proposition 3.1. In particular, to show $P \in \Psi_{\mathrm{sc}}^{-1,0}(\overline{\tilde{\Omega}};$ $\mathbb{C}^{n+1}, \mathbb{C}^n)$, it suffices to show that P is a matrix of pseudo-differential operators $P_{ij} \in \Psi_{\mathrm{sc}}^{-1,0}(\overline{\tilde{\Omega}})$, $i = 1, 2, \ldots, n$, $j = 0, 1, 2, \ldots, n$, depending continuously on $\Gamma_\pm$. But for $j > 0$, with $j = 0$ being completely analogous, $P_{ij}w$ has the form

$$\int_{\mathbb{R}} \int_{\mathbb{S}^{n-2}} \mathsf{x}^{-2} \chi(\lambda/\mathsf{x}) \int A_i^j(X_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \Xi_{\mathsf{x},\mathsf{y},\lambda,\omega}(t)) w(X_{\mathsf{x},\mathsf{y},\lambda,\omega}(t)) \, \mathrm{d}t \, \mathrm{d}\lambda \, \mathrm{d}\omega.$$

The only difference from [49, Proposition 3.3] then is the presence of the weight factor

$$A_i^j(X_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \Xi_{\mathsf{x},\mathsf{y},\lambda,\omega}(t)).$$

It is convenient to rewrite this via the metric identification, say by g_0, in terms of tangent vectors. Changing the notation for the flow, in our coordinates $(\mathsf{x}, \mathsf{y}, \lambda, \omega)$, writing now

$$(\gamma_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \gamma'_{\mathsf{x},\mathsf{y},\lambda,\omega}(t)) = (\mathsf{X}_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \mathsf{Y}_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \Lambda_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \Omega_{\mathsf{x},\mathsf{y},\lambda,\omega}(t))$$

for the lifted geodesic $\gamma_{\mathsf{x},\mathsf{y},\lambda,\omega}(t)$,

$$\tilde{A}_i^j(\mathsf{X}_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \mathsf{Y}_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \Lambda_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \Omega_{\mathsf{x},\mathsf{y},\lambda,\omega}(t))$$

replaces $A_i^j(X_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \Xi_{\mathsf{x},\mathsf{y},\lambda,\omega}(t))$. As in [49, Proposition 3.3] one rewrites the integral in terms of coordinates $(\mathsf{x}, \mathsf{y}, \mathsf{x}', \mathsf{y}')$ on the left and right factors of $\overline{\tilde{\Omega}}$ (i.e., one explicitly expresses the Schwartz kernel), using that the map Γ_+ of (3.2) is a local diffeomorphism, and similarly for Γ_-; we again refer to the discussion around [49, Equations (3.2) and (3.3)]. Further,

$$(\mathsf{x}, \mathsf{y}, \lambda, \omega, t) \mapsto \gamma'_{\mathsf{x},\mathsf{y},\lambda,\omega}(t)$$

is a smooth map $SM \times \mathbb{R} \to \mathbb{R}^n$ (depending continuously on $\Gamma_\pm$ in the respective C^∞ topologies) so composing it with $\Gamma_\pm^{-1}$ from the right, one can re-express the integral giving $P_{ij}w$ away from the boundary as

$$\int w(z') |z - z'|^{-n+1} b\left(z, \frac{z - z'}{|z - z'|}, |z - z'|\right) \mathrm{d}z'$$

as in [49, Equation (3.7)], with b a smooth function of the indicated variables (thus smooth on $[\tilde{M} \times \tilde{M}; \mathrm{diag}]$), depending continuously on $\Gamma_\pm$ in the respective C^∞ topologies, and with

$$b\left(z, \frac{z - z'}{|z - z'|}, 0\right) = \tilde{\chi}\left(z, \frac{z' - z}{|z' - z|}\right) \tilde{A}_i^j\left(z, \frac{z' - z}{|z' - z|}\right) \sigma\left(z, \frac{z' - z}{|z' - z|}\right)$$

with $\sigma > 0$, bounded below by a positive constant, by a weight factor, and where $\chi(\lambda/\mathsf{x})$ is written as $\tilde{\chi}(\mathsf{x}, \mathsf{y}, \lambda, \omega)$. Recall from [49, Section 2] that coordinates on Melrose's scattering double space, on which the Schwartz kernels of elements of $\Psi_{\mathrm{sc}}^{s,r}(\overline{\tilde{\Omega}})$ are conormal to the diagonal, near the lifted scattering diagonal, are (with $\mathsf{x} \geq 0$)

$$\mathsf{x}, \ \mathsf{y}, \ X = \frac{\mathsf{x} - \mathsf{x}'}{\mathsf{x}^2}, \ Y = \frac{\mathsf{y} - \mathsf{y}'}{\mathsf{x}}.$$

Further, it is convenient to write coordinates on $[\tilde{M} \times \tilde{M}; \text{diag}]$ in the region of interest (see the beginning of the paragraph of Equation (3.10) in [49]), namely (the lift of) $|\mathsf{x} - \mathsf{x}'| < C|\mathsf{y} - \mathsf{y}'|$, as

$$\mathsf{x}, \mathsf{y}, |\mathsf{y} - \mathsf{y}'|, \frac{\mathsf{x} - \mathsf{x}'}{|\mathsf{y} - \mathsf{y}'|}, \frac{\mathsf{y} - \mathsf{y}'}{|\mathsf{y} - \mathsf{y}'|},$$

with the norms being Euclidean norms,[4] instead of (3.3); we write $\Gamma_\pm$ in terms of these. Note that these are $\mathsf{x}, \mathsf{y}, \mathsf{x}|Y|, \frac{\mathsf{x}|X|}{|Y|}, \hat{Y}$. Then, similarly, near the boundary as in [49, Equation (3.13)], one obtains the Schwartz kernel

$$
\begin{aligned}
&K^\flat(\mathsf{x}, \mathsf{y}, X, Y) \\
(3.4) \qquad &= \sum_\pm e^{-FX/(1+\mathsf{x}X)} \chi\Big(\frac{X}{|Y|} + |Y|\tilde{\Lambda}_\pm\Big(\mathsf{x}, \mathsf{y}, \mathsf{x}|Y|, \frac{\mathsf{x}|X|}{|Y|}, \hat{Y}\Big)\Big) \\
&\qquad \times \tilde{A}_i^j\Big(\Gamma_\pm^{-1}\Big(\mathsf{x}, \mathsf{y}, \mathsf{x}|Y|, \frac{\mathsf{x}X}{|Y|}, \hat{Y}\Big)\Big)|Y|^{-n+1} J_\pm\Big(\mathsf{x}, \mathsf{y}, \frac{X}{|Y|}, |Y|, \hat{Y}\Big),
\end{aligned}
$$

with the density factor J smooth, positive, depending continuously on $\Gamma_\pm$ in the respective C^∞ topologies, $= 1$ at $\mathsf{x} = 0$. Here

$$\mathsf{x}, \mathsf{y}, |Y|, \frac{X}{|Y|}, \hat{Y}$$

are valid coordinates on the blowup of the scattering diagonal in[5] $|Y| > \epsilon|X|$, $\epsilon > 0$, which is the case automatically on the support of the kernel due to the argument of χ; cf. the discussion after [49, Equation (3.12)], so the argument of $\tilde{A}_i^j \circ \Gamma_\pm^{-1}$ is smooth *on this blown-up space* still depending continuously on $\Gamma_\pm$ in the respective C^∞ topologies. We can evaluate this argument: for instance, by [49, Equation (3.10)],

$$(\Lambda \circ \Gamma_\pm^{-1})\Big(\mathsf{x}, \mathsf{y}, \mathsf{x}|Y|, \frac{\mathsf{x}X}{|Y|}, \hat{Y}\Big) = \mathsf{x}\frac{X}{|Y|} + \mathsf{x}|Y|\tilde{\Lambda}_\pm\Big(\mathsf{x}, \mathsf{y}, \mathsf{x}|Y|, \frac{\mathsf{x}X}{|Y|}, \hat{Y}\Big)$$

with $\tilde{\Lambda}_\pm$ smooth, while the subsequent equation in the same location gives

$$(\Omega \circ \Gamma_\pm^{-1})\Big(\mathsf{x}, \mathsf{y}, \mathsf{x}|Y|, \frac{\mathsf{x}X}{|Y|}, \hat{Y}\Big) = \hat{Y} + \mathsf{x}|Y|\tilde{\Omega}_\pm\Big(\mathsf{x}, \mathsf{y}, \mathsf{x}|Y|, \frac{\mathsf{x}X}{|Y|}, \hat{Y}\Big)$$

with $\tilde{\Omega}_\pm$ smooth; here, both $\tilde{\Lambda}_\pm$ and $\tilde{\Omega}_\pm$ depend continuously on $\Gamma_\pm$ in the respective C^∞ topologies. This proves the first part of the proposition as in [49, Proposition 3.3].

To prove the second part, note that in view of (3.4) (which just needs to be evaluated at $\mathsf{x} = 0$), [49, Lemma 3.5] is replaced by the statement that the boundary principal symbol of P_{ij} in $\Psi_{\mathrm{sc}}^{-1,0}(\overline{\tilde{\Omega}})$ is twice the (X, Y)-Fourier transform of

$$(3.5) \qquad e^{-FX}\chi\Big(\frac{X - \alpha(0, \mathsf{y}, 0, \hat{Y})|Y|^2}{|Y|}\Big)\tilde{A}_i^j\Big(0, \mathsf{y}, 0, 0, \hat{Y}\Big)|Y|^{-n+1},$$

while for P_{i0} it is twice the (X, Y)-Fourier transform of

$$e^{-FX}\chi\Big(\frac{X - \alpha(0, \mathsf{y}, 0, \hat{Y})|Y|^2}{|Y|}\Big)\tilde{B}_i\Big(0, \mathsf{y}, 0, 0, \hat{Y}\Big)|Y|^{-n+1},$$

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[4]This is an example of partial projective coordinates for a blowup.

[5]This is another example of partial projective coordinates for a blowup.

with $\tilde{B}_i$ defined analogously to $\tilde{A}_i^j$. (Recall that $2\alpha(\mathsf{x},\mathsf{y},\lambda,\omega)$ is the x component of $\gamma''_{\mathsf{x},\mathsf{y},\lambda,\omega}(0)$, and the convexity assumption on x is that α is positive; see [49] above Equation (3.1).) For $A_i^j = -\frac{1}{2}c^{-2}(x_0)\delta_i^j$ (see (2.15)), the invertibility of the principal symbol, with values in $n \times n$ matrices, of the principal symbol of $\tilde{P}$ follows when χ is chosen as in [49, Lemma 3.6], for it is $-\frac{1}{2}c^{-2}(x_0)$ times the boundary symbol in [49, Lemma 3.6] times the $n \times n$ identity matrix. In general, due to the perturbation stability of the property of invertibility, the same follows for c sufficiently small. $\qquad\square$

Corollary 3.1. With the notation of Proposition 3.2, there is $\tilde{\mathsf{c}} > 0$ such that if $0 < \mathsf{c} < \tilde{\mathsf{c}}$, then P_F satisfies the estimate

$$(3.6) \qquad \|u\|_{H_{\mathrm{sc}}^{s,r}(\overline{\tilde{\Omega}})} \le C\|P_F u\|_{H_{\mathrm{sc}}^{s+1,r}(\overline{\tilde{\Omega}})} + C\|u_0\|_{H_{\mathrm{sc}}^{s,r}(\overline{\tilde{\Omega}})}$$

for u supported in $M \cap \overline{\tilde{\Omega}}$, with the constant C uniform in c. Further, fixing s, r, there exist $k, \epsilon > 0$, and $\rho_0 < 0$ such that C can be taken uniform for $\Gamma_\pm$ ϵ-close to a reference $\Gamma_\pm^0$ in C^k, and the estimate holds even for u supported in $\rho \ge \rho_0$ (in place of $\rho \ge 0$).

Remark 3.2. As in the case of the preceding proposition, the dependence on $\tilde{\mathsf{x}}$ is also continuous; i.e., by possibly increasing k, C can be taken uniform for $\Gamma_\pm$ C^k-close to a reference $\Gamma_\pm^0$ and $\tilde{\mathsf{x}}$ C^k-close to a reference $\tilde{\mathsf{x}}_0$.

Proof. By the density of elements of $\dot{C}^\infty(M \cap \overline{\tilde{\Omega}})$ in $H_{\mathrm{sc}}^{s,r}(\overline{\tilde{\Omega}})$ supported in $M \cap \overline{\tilde{\Omega}}$, it suffices to consider $u \in \dot{C}^\infty(M \cap \overline{\tilde{\Omega}})$ to prove (3.6).

Consider $s = 0$, $r = 0$. Let $\Lambda \in \Psi_{\mathrm{sc}}^{1,0}(\overline{\tilde{\Omega}})$ be elliptic and invertible (one can, e.g., locally identify Ω with $\mathbb{R}^n$; then on the Fourier transform side multiplication by $\langle\xi\rangle$ works). Thus, ΛP_F, with Λ acting diagonally on $\dot{C}^\infty(\overline{\tilde{\Omega}};\mathbb{C}^n)$, is in $\Psi_{\mathrm{sc}}^{0,0}(\overline{\tilde{\Omega}};\mathbb{C}^{n+1},\mathbb{C}^n)$, depending continuously, in the Fréchet topology of pseudo-differential operators, on $\Gamma_\pm$ in C^∞, and $\Lambda\tilde{P} \in \Psi_{\mathrm{sc}}^{0,0}(\overline{\tilde{\Omega}};\mathbb{C}^n,\mathbb{C}^n)$ is elliptic in $\Psi_{\mathrm{sc}}^{0,0}(\overline{\tilde{\Omega}}; \mathbb{C}^n,\mathbb{C}^n)$ *locally in a neighborhood of* Ω. This implies, as presented in [49] after Lemma 3.6 (without the uniform discussion on $\Gamma_\pm$), relying on the arguments at the end of Section 2 there, that there exist k and $\epsilon > 0$ such that if $\Gamma_\pm$ is ϵ-close to $\Gamma_\pm^0$ in C^k, then, if $\mathsf{c} > 0$ is sufficiently small, $\Lambda\tilde{P}$ satisfies

$$(3.7) \qquad \|w\|_{L_{\mathrm{sc}}^2(\overline{\tilde{\Omega}})} \le C_0\|\Lambda\tilde{P}w\|_{L_{\mathrm{sc}}^2(\overline{\tilde{\Omega}})}$$

for w supported in $\overline{\Omega}$. Here $L_{\mathrm{sc}}^2(\overline{\tilde{\Omega}}) = H_{\mathrm{sc}}^{0,0}(\overline{\tilde{\Omega}})$ is the L^2 space relative to a non-degenerate *scattering density*—the latter are equivalent to the lifted Lebesgue measure from $\mathbb{R}^n$, and thus are bounded multiples of $\frac{d\mathsf{x}\,d\mathsf{y}}{\mathsf{x}^{n+1}}$.

We recall the essential part of this argument briefly. One considers the whole family of domains $\overline{\tilde{\Omega}}_\mathsf{c}$, which can be identified with each other locally in the region of interest by the maps $\Phi_\mathsf{c}(\tilde{\mathsf{x}},\mathsf{y}) = (\tilde{\mathsf{x}} + \mathsf{c},\mathsf{y})$, i.e., simply translation in the $\tilde{\mathsf{x}}$-coordinate, so instead of considering a family of spaces with an operator on each of them, one can consider a fixed space, denote this by $\overline{\tilde{\Omega}}_0$, with a continuous (in c) family of operators, T_c, namely $T_\mathsf{c} = (\Phi_\mathsf{c}^{-1})^*\Lambda\tilde{P}\Phi_\mathsf{c}^*$; these depend continuously on $\Gamma_\pm$ in the Fréchet sense discussed above. Notice that we are interested in the

region[6] Ω_c, and that there is a continuous function f on $\mathbb{R}$ with $f(0) = 0$ and $x_c \leq f(c)$ on Ω_c. Correspondingly, in the translated space, $\tilde{x} \leq f(c)$ on the image of Ω_c; notice that this region shrinks as $c > 0$ goes to 0. On the other hand, there is a fixed open set $O \subset \overline{\hat{\Omega}}$, a neighborhood of x_0, on which the operators T_c are elliptic in $\Psi^{0,0}_{sc}(\overline{\hat{\Omega}})$ for $0 \leq |c| < c_0$. Let K_0 be a compact subset of O, still including a neighborhood of x_0, $\phi \in C^\infty_0(O)$ be identically 1 on a neighborhood of K. Then the elliptic parametrix construction (which is local, and uniform in c by the continuity) produces a parametrix family $G_c \in \Psi^{0,0}_{sc}(\overline{\hat{\Omega}}; \mathbb{C}^n, \mathbb{C}^n)$, $G_c T_c = \mathrm{Id} + E_c$, where $E_c \in \Psi^{0,0}_{sc}(\overline{\hat{\Omega}}; \mathbb{C}^n, \mathbb{C}^n)$ only, but $\phi E_c \phi \Psi^{-\infty,-\infty}_{sc}(\overline{\hat{\Omega}}; \mathbb{C}^n, \mathbb{C}^n)$, uniformly in c, depending continuously on $\Gamma_\pm$ in the Fréchet topologies. Then (multiplying the parametrix identity by ϕ from left and right and applying to v) for v supported in K, $(\mathrm{Id} + \phi E_c \phi)v = \phi G_c T_c v$. Now, the Schwartz kernel of $\phi E_c \phi$ is Schwartz, i.e., is bounded by $C_N (xx')^N$ for any N, uniformly in c. (Here we write, say, the Schwartz kernel relative to scattering densities, but as N is arbitrary, this makes little difference.) For $c > 0$, let ϕ_c be supported, say in $\tilde{x} \leq 2f(c)$, identically 1 near the region $\tilde{x} \leq f(c)$; one may assume that $\phi \equiv 1$ on $\mathrm{supp}\,\phi_c$ by making $c > 0$ small. Then, by Schur's lemma, $\phi_c \phi E_c \phi \phi_c$, acting say on $L^2_{sc}(\overline{\hat{\Omega}})$ (i.e., the L^2-space relative to scattering densities) is bounded by $C'_N f(c)^N$ for any N. Thus, there is $c_1 > 0$ such that $\mathrm{Id} + \phi_c \phi E_c \phi \phi_c = \mathrm{Id} + \phi_c E_c \phi_c$ is invertible for $0 < c < c_1$ on $L^2_{sc}(\overline{\hat{\Omega}})$. (Notice here that this requires the smallness of a seminorm of $\phi_c \phi E_c \phi \phi_c$ in the Fréchet topology of pseudo-differential operators; the continuity discussed above means that that this requires the C^k-closeness of $\Gamma_\pm$ to $\Gamma^0_\pm$ for some k.) In particular, for v supported in $\tilde{x} \leq f(c)$, so $\phi_c v = v$, $(\mathrm{Id} + \phi_c E_c \phi_c)v = \phi_c G_c T_c v$, so inverting the factor on the left and then undoing the transformation Φ_c gives the desired conclusion (3.7).

Thus, with $u = (u_0, \tilde{u})$, we have, with all norms being $L^2_{sc}(\overline{\hat{\Omega}})$-norms,

$$\|\Lambda P_F u\|^2 = \|\Lambda P_0 u_0 + \Lambda \tilde{P}\tilde{u}\|^2$$
$$= \|\Lambda \tilde{P}\tilde{u}\|^2 + \langle \Lambda \tilde{P}\tilde{u}, \Lambda P_0 u_0 \rangle + \langle \Lambda P_0 u_0, \Lambda \tilde{P}\tilde{u} \rangle + \langle \Lambda P_0 u_0, \Lambda P_0 u_0 \rangle.$$

By (3.7), $\|\Lambda \tilde{P}\tilde{u}\| \geq C_0 \|\tilde{u}\|$, $C_0 > 0$. On the other hand, $\|\Lambda P_0 u_0\| \leq C_1 \|u_0\|$, $\|\Lambda \tilde{P}\tilde{u}\| \leq C_1 \|\tilde{u}\|$ as elements of $\Psi^{0,0}_{sc}(\Omega)$ are L^2-bounded. (By the continuity of the map from $\Gamma_\pm$ to ΛP_0 in the appropriate Fréchet topologies, again there is k such that C_1 is uniform when $\Gamma_\pm$ is C^k-close to $\Gamma^0_\pm$.) Using the Cauchy-Schwartz inequality, for $\delta > 0$,

$$|\langle \Lambda P_0 u_0, \Lambda \tilde{P}\tilde{u} \rangle| \leq \frac{\delta}{2}\|\tilde{u}\|^2 + \frac{C_1^2}{2\delta}\|u_0\|^2.$$

Thus, the last three terms are bounded by $\delta\|\tilde{u}\|^2 + (1 + \delta^{-1})C_1^2\|u_0\|^2$ in absolute value, so we conclude that, with $\delta = C_0^2/2$,

$$(3.8) \qquad \frac{C_0^2}{2}\|\tilde{u}\|^2 \leq \|\Lambda P_F u\|^2 + C_1^2(1 + 2C_0^{-2})\|u_0\|^2,$$

completing the proof of the corollary if $s = r = 0$.

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[6]If we are interested in the region $\rho \geq \rho_0$, $\rho_0 < 0$, within $\overline{\hat{\Omega}}$, with ρ_0 to be specified, we take, e.g., $\rho_0 = -c$ in the discussion below, and the desired continuous f with $f(0) = 0$, and in this case with $x_c \leq f(c)$ on $\hat{\Omega}_c \cap \{\rho \geq \rho_0\}$, still exists.

The general case follows via conjugating P_F by an elliptic, invertible, element of $\Psi^{s,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})$, which is thus an isomorphism from $H^{s,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})$ to $L^2 = H^{0,0}_{\mathrm{sc}}(\overline{\hat{\Omega}})$. Note that such a conjugation does not change the principal symbol, and thus the ellipticity. $\qquad\square$

We now remark that the even simpler setting of the *scalar transform with a positive weight A on S^*M, I_A*, which was not considered in [49], and which can be considered a special case of $\tilde{J}$ with $(u_0, u_1, \ldots, u_n)$, is simply replaced by u. Thus, for consistency with the above notation, let $\tilde{A}(\mathsf{x}, \mathsf{y}, \lambda, \omega)$ be the weight on SM induced by the metric identification, and let

$$(I_A f)(\beta) = \int_{\mathbb{R}} \tilde{A}(\gamma_\beta(t), \gamma'_\beta(t)) f(\gamma_\beta(t)) \, \mathrm{d}t, \ \beta \in SM.$$

Then the above argument gives that $P_F \in \Psi^{-1,0}_{\mathrm{sc}}(\overline{\hat{\Omega}})$. If at $S_{\partial M}M$, the weight A is independent of the momentum variable ξ, it further gives that P_F is elliptic in a neighborhood of $\overline{\Omega}$. More generally, a modification of the argument of [49] due to Zhou [52] allows one to show that the principal symbol is fully elliptic in $\overline{\Omega}$ in the scattering sense merely assuming that A is positive (but *not* the independence condition just mentioned). To see this, one has to Fourier transform (3.5) in (X, Y) with $\tilde{A}^j_i$ replaced by $\tilde{A}$. The X-Fourier transform is unaffected by the presence of $\tilde{A}$ and gives, as in [49, Equation (3.16)],

$$|Y|^{2-n} e^{-\alpha(F+i\xi)|Y|^2} \hat{\chi}(\xi - iF) \tilde{A}(0, \mathsf{y}, 0, \hat{Y}).$$

Replacing χ with a Gaussian, $\chi(s) = e^{-s^2/(2\nu)}$, $\nu = F^{-1}\alpha$, which does not have compact support, but an approximation argument (in symbols of order -1) will give this desired property, and one can compute that this is, up to a constant factor,

$$F^{-1/2} \alpha^{1/2} |Y|^{2-n} e^{-F^{-1}(\xi^2+F^2)\alpha|Y|^2/2} \tilde{A}(0, \mathsf{y}, 0, \hat{Y}).$$

We need to compute the Fourier transform in Y. Following [52], one expresses this in polar coordinates in Y,

$$F^{-1/2} \int_{\mathbb{S}^{n-2}} \int_0^\infty e^{-i|Y|\hat{Y}\cdot\eta} \alpha^{1/2} e^{-F^{-1}(\xi^2+F^2)\alpha|Y|^2/2} \tilde{A}(0, \mathsf{y}, 0, \hat{Y}) \, \mathrm{d}|Y| \, \mathrm{d}\hat{Y}$$
$$= \frac{1}{2} F^{-1/2} \int_{\mathbb{S}^{n-2}} \int_{\mathbb{R}} e^{-it\hat{Y}\cdot\eta} \alpha^{1/2} e^{-F^{-1}(\xi^2+F^2)\alpha t^2/2} \tilde{A}(0, \mathsf{y}, 0, \hat{Y}) \, \mathrm{d}t \, \mathrm{d}\hat{Y},$$

which in turn becomes, up to a constant factor,

$$(\xi^2 + F^2)^{-1/2} \int_{\mathbb{S}^{n-2}} e^{-F|\hat{Y}\cdot\eta|^2/(2(\xi^2+F^2)\alpha)} \tilde{A}(0, \mathsf{y}, 0, \hat{Y}) \, \mathrm{d}\hat{Y}.$$

The integrand is now positive, which gives the desired ellipticity at $\mathsf{x} = 0$. (One also needs to check the ellipticity as $(\xi, \eta) \to \infty$; this is standard, see [49, 52].) One proceeds with an approximation argument as in [49, 52] to complete the proof of the ellipticity. Thus, the above argument gives the estimate

$$\|f\|_{H^{s,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})} \leq C \|P_F f\|_{H^{s+1,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})}.$$

As discussed in [49] after Lemma 3.6, this yields the following corollary.

Corollary 3.2. The weighted scalar transform with a positive weight A on S^*M, with $\tilde{A}$ the associated weight on SM,

$$(I_A f)(\beta) = \int_{\mathbb{R}} \tilde{A}(\gamma_\beta(t), \gamma_\beta'(t)) f(\gamma_\beta(t)) \, \mathrm{d}t,$$

satisfies that for $F > 0$ there is $c_0 > 0$ such that for $\delta > 0$, $0 < c < c_0$, $s \geq -1$, we have

$$\|f\|_{e^{(F+\delta)/x} H^s(\Omega_c)} \leq C \|I_A f\|_{e^{F/x} H^{s+1}(\mathcal{M}_c)}.$$

We now return to the actual case of interest and apply Corollary 3.1 with $u_0 = e^{-F/x} f$, $u_j = e^{-F/x} \partial_j f$. If we show that (uniformly in $\Gamma_\pm$, which are indeed irrelevant for this argument) given $\tilde{\delta} > 0$, there is $c_0 > 0$ such that $\|u_0\|_{L^2} \leq \tilde{\delta} \|\tilde{u}\|_{L^2}$; i.e., $e^{-F/x} f$ is bounded by a small multiple of a derivative of f times $e^{-F/x}$ in L^2, when f is supported in $\overline{\Omega_c}$, $0 < c < c_0$, and then for $\tilde{\delta} > 0$ sufficiently small (3.6) proves that if $P_F u$ vanishes, then so does u. That is, in this case so does f, for the u_0 term can then be absorbed into the left-hand side of (3.6),

$$(3.9) \qquad (1 - \tilde{\delta}C) \|u\|_{H^{0,0}_{\mathrm{sc}}(\overline{\Omega})} \leq C \|P_F u\|_{H^{1,0}_{\mathrm{sc}}(\overline{\Omega})}.$$

Further, rewriting this by removing the weights $e^{-F/x}$, and estimating the norms in terms of the standard L^2-based space (cf. the discussion after [49, Lemma 3.6] already referenced above),[7] gives, for $\delta > 0$,

$$(3.10) \qquad \|f\|_{e^{(F+\delta)/x} H^1(\Omega_c)} \leq C \|\tilde{J}(f, \partial_1 f, \ldots, \partial_n f)\|_{e^{F/x} H^1(\mathcal{M}_c)},$$

with C uniform for $\Gamma_\pm$ C^k-close to $\Gamma_\pm^0$.

But this can now easily be done: let V be a smooth vector field with $Vx = 0$, so V is tangent to the boundary of Ω_c for every c, and make the non-degeneracy assumption that, for some $c_1 > 0$, there is a continuous $T : [0, c_1) \to \mathbb{R}$ such that $T(0) = 0$ and the V-flow takes every point in $\overline{\Omega_c}$ to ∂M in time $\leq T(c)$ (i.e., outside the original manifold). Then the Poincaré inequality for V gives[8]

$$(3.11) \qquad \|v\|_{L^2_{\mathrm{sc}}} \leq C_2 T(c) \|V v\|_{L^2_{\mathrm{sc}}},$$

for v vanishing outside M; hence the constant is small if $T(c) > 0$ is small. (Here the L^2 space we need is the scattering L^2-space, L^2_{sc}, which is $x^{(n+1)/2}$ times the standard L^2-space, but the extra weight does not affect the argument, since V commutes with multiplication by powers of x.)

To see (3.11), we recall a standard proof of the local Poincaré inequality: in order to reduce confusion with the notation, let $(z_1, \ldots, z_n) = (z', z_n)$ be the coordinates, $z_1 = 0$ being the boundary of Ω (so x would be z_1), and assume that the flow of ∂_{z_n} flows from every point in Ω to outside the region in "time" $\leq \delta$. To normalize the argument, assume that $z_n \geq 0$ in Ω, and we want to estimate v in $z_n \leq \delta$. We assume that the L^2 space is given by a density $F(z') |dz|$. Then, for $v \in C^\infty(\mathbb{R}^n_+)$, $\mathbb{R}^n_+ = [0, \infty)_{z_1} \times \mathbb{R}^{n-1}$, with support in $z_n \geq 0$, by the fundamental theorem of

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[7] The δ loss is actually just the loss of a power of x, due to the change of the measure.

[8] Notice that our treatment of the $\tilde{x}$-dependence of the problem relies on reducing to a model, where $\tilde{x}$ is replaced by a fixed function $\underline{\tilde{x}}$, so the following argument is, in fact, uniform in $\tilde{x}$.

calculus and the Cauchy-Schwartz inequality,

$$|v(z', z_n)| = \left| \int_0^{z_n} \partial_n v(z', t) \, dt \right| \le \left(\int_0^{z_n} 1 \, dt \right)^{1/2} \left(\int_0^{z_n} |\partial_n v(z', t)|^2 \, dt \right)^{1/2}$$

$$\le \delta^{1/2} \left(\int_0^{\delta} |\partial_n v(z', t)|^2 \, dt \right)^{1/2}.$$

Squaring both sides, multiplying by $F(z')$, and integrating in z', z_n (to $z_n = \delta$) gives

$$\int_{z_n \le \delta} |v(z', z_n)|^2 \, F(z') \, dz \le \delta^2 \int_{t \le \delta} |\partial_n v(z', t)|^2 \, dt \, F(z') \, dz'.$$

This says that actually

$$\|v\|_{L^2(\mathbb{R}^n_+; F(z') \, |dz|)} \le \delta \|\partial_n v\|_{L^2(\mathbb{R}^n_+; F(z') \, |dz|)},$$

proving the claim (using $F(z') = z_1^{-n-1}$) in view of the quasi-isometry invariance (which gives a constant factor) of the bound (3.11). Even if there is more complicated topology, so there are no global coordinates and vector fields as stated, dividing up the problem into local pieces and adding them together gives the desired result: taking steps of size δ, one needs T/δ steps to cover the set, using cutoff functions to localize is easily seen to give a bound proportional to T.

On the other hand, in view of the strict convexity of the boundary, one can construct such a V and T. With $v = e^{-F/x} f$, this is exactly the desired conclusion since $V(e^{-F/x} f) = e^{-F/x} V f$, namely (3.9), and thus (3.10), hold.

In fact, we can prove the analogue of (3.9) on stronger spaces,

$$(3.12) \qquad \|u\|_{H^{s,0}_{\mathrm{sc}}(\overline{\Omega})} \le C \|P_F u\|_{H^{s+1,0}_{\mathrm{sc}}(\overline{\Omega})}, \qquad s \ge 0,$$

which in turn gives, for $s \ge 0$,

$$(3.13) \qquad \|f\|_{e^{(F+\delta)/x} H^{s+1}(\Omega_c)} \le C \|\tilde{J}(f, \partial_1 f, \ldots, \partial_n f)\|_{e^{F/x} H^{s+1}(\mathcal{M}_c)},$$

with C uniform for $\Gamma_\pm$ C^k-close to $\Gamma^0_\pm$. To see this, notice that (3.6) is an elliptic estimate when we have $u_0 = e^{-F/x} f$, $u_j = e^{-F/x} \partial_j f$, for

$$\|u\|_{H^{s,r}_{\mathrm{sc}}(\overline{\Omega})} \le C \|P_F u\|_{H^{s+1,r}_{\mathrm{sc}}(\overline{\Omega})} + C \|u_0\|_{H^{s,r}_{\mathrm{sc}}(\overline{\Omega})}$$

implies that[9]

$$\|e^{-F/x} f\|_{H^{s+1,r}_{\mathrm{sc}}(\overline{\Omega})} \le C \|P_F e^{-F/x}(f, df)\|_{H^{s+1,r}_{\mathrm{sc}}(\overline{\Omega})} + C \|e^{-F/x} f\|_{H^{s,r}_{\mathrm{sc}}(\overline{\Omega})}.$$

For $s = 1$, $r = 0$, we have the second term on the right-hand side controlled by $P_F e^{-F/x}(f, df)$ in $H^{1,0}_{\mathrm{sc}}$ in view of (3.9), so the $H^{2,0}_{\mathrm{sc}}$-norm of $e^{-F/x} f$ is, in fact, controlled by $\|P_F e^{-F/x}(f, df)\|_{H^{2,0}_{\mathrm{sc}}(\overline{\Omega})}$, which we can iterate further[10] to prove (3.12).

In summary we have proved that with $L^2(\Omega) = L^2(\Omega_c)$ the standard L^2-space now follows (as the exponential weight $e^{-F/x}$ maps such f to $L^2_{\mathrm{sc}}(\overline{\Omega})$; see also the discussion after Lemma 3.6 in [49]).

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[9]Notice that the scattering derivatives are actually weaker than the derivatives ∂_j entering u, and one can absorb the term given by commuting a scattering derivative through $e^{-F/x}$ into the last term on the right-hand side.

[10]We can easily allow $s \ge 0$ non-integer by slightly modifying the argument here.

Proposition 3.3. *There is* $\mathsf{c}_0 > 0$ *such that for* $0 < \mathsf{c} < \mathsf{c}_0$, *if* $f \in H^1(\Omega_\mathsf{c})$ *and* $\tilde{J}(f, \partial_1 f, \ldots, \partial_n f) = 0$, *then* $f = 0$.

In fact, for $\mathsf{F} > 0$, $s \geq 1$, *there exist* $\mathsf{c}_0 > 0$, k *and* $\epsilon > 0$ *such that the following holds. For* $\delta > 0$ *there is* $C > 0$ *such that if* $0 < \mathsf{c} < \mathsf{c}_0$, $\Gamma_\pm$ *is* ϵ-*close to* $\Gamma_\pm^0$ *in* C^k, $\tilde{\mathsf{x}}$ *is* ϵ-*close to* $\tilde{\mathsf{x}}_0$ *in* C^k, *then*

$$\|f\|_{e^{(\mathsf{F}+\delta)/\mathsf{x}}H^s(\Omega_\mathsf{c})} \leq C\|\tilde{J}(f, \partial_1 f, \ldots, \partial_n f)\|_{e^{\mathsf{F}/\mathsf{x}}H^s(\mathcal{M}_\mathsf{c})}.$$

Moreover, with $\Omega_{\mathsf{c},\rho_0} = \hat{\Omega}_\mathsf{c} \cap \{\rho \geq \rho_0\}$, *and* $\mathcal{M}_{\mathsf{c},\rho_0}$ *being defined analogously to* $\mathcal{M}_\mathsf{c}$ *with* $\partial M = \{\rho = 0\}$ *being replaced by* $\{\rho = \rho_0\}$, *we have the following: for* $\mathsf{F} > 0$ *and* $s \geq 1$ *there exist* $\mathsf{c}_0 > 0$, $\rho_0 < 0$, k, *and* $\epsilon > 0$ *such that the following holds. For* $\delta > 0$ *there is* $C > 0$ *such that if* $0 < \mathsf{c} < \mathsf{c}_0$, $\Gamma_\pm$ *is* ϵ-*close to* $\Gamma_\pm^0$ *in* C^k, $\tilde{\mathsf{x}}$ *is* ϵ-*close to* $\tilde{\mathsf{x}}_0$ *in* C^k, *then* $f \in H^{s+1}(\Omega_{\mathsf{c},\rho_0})$ *implies that*

$$\|f\|_{e^{(\mathsf{F}+\delta)/\mathsf{x}}H^s(\Omega_{\mathsf{c},\rho_0})} \leq C\|\tilde{J}(f, \partial_1 f, \ldots, \partial_n f)\|_{e^{\mathsf{F}/\mathsf{x}}H^s(\mathcal{M}_{\mathsf{c},\rho_0})}.$$

4. Proofs of Theorems 1.1–1.4

Proof of Theorem 1.2. Let c and $\tilde{c}$ be as in the theorem. Redefine the scattering relation $\mathcal{L}$ as in Figure 1. By Proposition 2.1, we get $J_j f(\gamma) = 0$; see (2.12) for all geodesics close enough to the ones tangent to ∂M at p. The weights are given by (2.14), in the new parameterization, with the ellipticity condition satisfied by (2.15). Then Proposition 3.3 implies $f = 0$ in a neighborhood of p, where $f = c^2 - \tilde{c}^2$ as in (2.8). □

Proof of Theorem 1.1. Note first that we can complete (M, c_0^{-2}), and similarly $(M, \tilde{c}^{-2}g_0)$ to compact Riemannian manifolds without boundary. Then we can choose a neighborhood of p small enough so that the exponential map based on any point of that neighborhood is a local diffeomorphism for short enough vectors, both for c and for $\tilde{c}$. This implies that there is a neighborhood U of p so that the distance $d(p_1, p_2)$ between any two points p_1, p_2 in U is realized by $|\exp_{p_1}^{-1}(p_2)|$, related to the first and the second metrics, respectively, where exp is the localized exponential map as above. We can easily recover c and $\tilde{c}$ on $\partial M \cap U$ by taking the limit $p_1 \to p_2$. As Michel proved [21], for simple manifolds, the scattering elation can be recovered by differentiating the distance function; see also [38, Section 2]. This applies to our case as well because U is small enough. Then the proof follows from Theorem 1.2. □

Proof of Theorem 1.4. Theorem 1.4 is now an easy consequence of Theorem 1.1 using a layer stripping argument. Let $f = c^2 - \tilde{c}^2$. Assume $f \neq 0$, then supp f has a non-empty interior. On the other hand, let $\tau = \inf_{\text{supp} f} \rho$; if $\tau = T$, we are done, for then supp $f \subset M \setminus \cup_{t \in [0,T)} \Sigma_t$. Thus, suppose $\tau < T$, so $f \equiv 0$ on Σ_t for $t < \tau$, but there exists $x \in \Sigma_\tau \cap \text{supp} f$ (since supp f is closed). We will show below how to use Theorem 1.2 on $M_\tau := \rho^{-1}(\tau, \infty)$ to conclude that a neighborhood of x is disjoint from supp f to obtain a contradiction.

All we need to show is that the scattering relations L_τ and $\tilde{L}_\tau$ on Σ_τ coincide. Note that $\Sigma_\tau = \partial M_\tau$ is strictly convex for $\tilde{g}$ as well because the second fundamental form for $\tilde{g}$ can be computed by taking derivatives from the exterior $\rho < \tau$, where $g = \tilde{g}$. Fix $(x_\tau, v_\tau) \in \partial_- SM_\tau$; see Figure 2. The geodesic $\gamma_{x_\tau, v_\tau}(s)$ cannot hit Σ_τ again for negative "times" s because, otherwise, we would get a contradiction with the strict convexity at Σ_t, where t corresponds to the smallest value of ρ on

that geodesic between two contacts with Σ_τ. Since $c = \tilde{c}$ outside M_τ, $\gamma_{x_\tau,v_\tau}(s)$ and $\tilde{\gamma}_{x_\tau,v_\tau}(s)$ coincide outside M_τ for $s < 0$. Proposition 5.1 below shows that this negative geodesic ray must be non-trapping; i.e., γ_{x_τ,v_τ} would hit ∂M for a finite negative time s at some point and direction $(x,v) \in \partial_- SM$. In the same way, we show that the same holds for the positive part, $s > 0$, of a geodesic issued from $L_\tau(x_\tau, y_\tau) =: (y_\tau, w_\tau) \in \partial_+ SM_\tau$; and the corresponding point on $\partial_+ SM$ will be denoted by (y,w). Then, since $L(x,v) = (y,w)$, we would also get $L_\tau(x_\tau, v_\tau) = (y_\tau, w_\tau) = \tilde{L}_\tau(x_\tau, v_\tau)$.   □
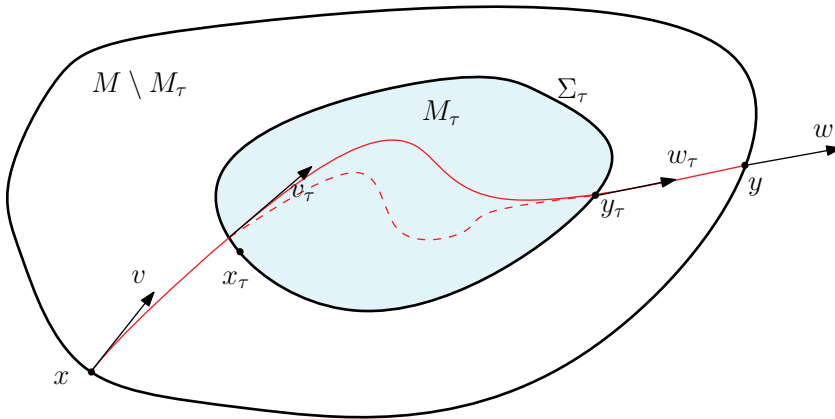
FIGURE 2. One can recover the scattering relation on Σ_τ knowing that on ∂M.

5. STABILITY ANALYSIS

In this section, we prove the stability estimate in Theorem 1.5 and a global stability estimate, see Theorem 5.2 below. We follow some of the ideas in [41, section 7].

5.1. Boundary stability.
We start with stability at the boundary.

Theorem 5.1. *Let c and $\tilde{c}$ be such that ∂M is strictly convex w.r.t. both $c^{-2}g_0$ and $\tilde{c}^{-2}g_0$, and let $\Gamma \subset\subset \Gamma' \subset \partial M$ be two sufficiently small open subsets of the boundary. Then*

$$(5.1) \qquad \left\| \partial_{x^n}^k (c - \tilde{c}) \right\|_{C^m(\bar{\Gamma})} \leq C_{k,m} \left\| d^2 - \tilde{d}^2 \right\|_{C^{m+2k+2}\left(\overline{\Gamma' \times \Gamma'}\right)},$$

where $C_{k,m}$ depends only on Ω and on an upper bound of c, c^{-1}, $\tilde{c}$, $\tilde{c}^{-1}$ in $C^{m+2k+5}(M)$.

Proof. We know from [41, Section 7] that it is true for the metrics in boundary normal coordinates. More precisely, let (x', x^n) be boundary local coordinates for g, i.e., locally,

$$c^{-2}g_0 = h_{\alpha\beta}dx^\alpha dx^\beta + (dx^n)^2 =: h_{ij}dx^i dx^j.$$

Note that the notational convention is different than the one in Section 3; (x, y) is now replaced by (x', x^n), and x and y are generic points in (the local chart) in $\mathbb{R}^n$. We use the convention that Greek indices run from 1 to $n-1$, while Latin ones

run from 1 to n. Let ψ be the diffeomorpishm fixing the boundary pointwise near p, i.e., $\psi(x', 0) = (x', 0)$, so that

$$(5.2) \qquad \psi^*(\tilde{c}^{-2} g_0) = \tilde{h}_{\alpha\beta} dx^\alpha dx^\beta + (dx^n)^2$$

near p. Then (5.1) holds for $h - \tilde{h}$.

We have, with $y = \psi(x)$,

$$(5.3) \qquad \psi^*(\tilde{c}^{-2} g_0)(x) = (c^2 \tilde{c}^{-2})(y(x)) h_{i'j'}(y(x)) \frac{\partial y^{i'}}{\partial x^i} \frac{\partial y^{j'}}{\partial x^j} dx^i dx^j.$$

In particular, by (5.2) and (5.3),

$$(5.4) \qquad (c^2 \tilde{c}^{-2})(y(x)) h_{i'j'}(y(x)) \frac{\partial y^{i'}}{\partial x^\alpha} \frac{\partial y^{j'}}{\partial x^\beta} = \tilde{h}_{\alpha\beta}(x),$$

which can be written as

$$(5.5) \quad \left((c^2 \tilde{c}^{-2})(y(x)) - 1\right) h_{i'j'}(y(x)) \frac{\partial y^{i'}}{\partial x^\alpha} \frac{\partial y^{j'}}{\partial x^\beta} = \tilde{h}_{\alpha\beta}(x) - h_{i'j'}(y(x)) \frac{\partial y^{i'}}{\partial x^\alpha} \frac{\partial y^{j'}}{\partial x^\beta}.$$

We have

$$\partial y^i / \partial x^\beta = \delta^i_\beta \quad \text{for } x^n = 0,$$

which, of course, can be further differentiated w.r.t. tangential variables x^α. Therefore,

$$(5.6) \qquad \left(\tilde{c}^{-2} c^2 - 1\right) h_{\alpha\beta}|_{x^n=0} = \left(\tilde{h}_{\alpha\beta} - h_{\alpha\beta}\right)|_{x^n=0}.$$

Since $\{h_{\alpha\beta}\}$ is invertible (at least for one multi-index is enough), we get (5.1) for $\tilde{c}^{-2} c^2 - 1$, and therefore for $\tilde{c} - c$ for $k = 0$.

In order to do the same for $k = 1$, we need to estimate $\partial(y - x)/\partial x^n$ at $x^n = 0$ first. The diffeomorphism ψ identifies boundary normal coordinates for g and $\tilde{g}$. The normal $\nu = (0, 1)$ (here, 0 is $n - 1$ dimensional) is the unit for $g = c^{-2} g_0$, but it has length $\tilde{c}^{-1} c$ in the metric $\tilde{g}$. The inner unit vector in the metric $\tilde{g}$ is therefore $\tilde{\nu} = \tilde{c} c^{-1}(0, 1)$; hence

$$(5.7) \qquad y = \tilde{\gamma}_{(x', 0),(0, \tilde{c} c^{-1})}(x^n).$$

Differentiate this at $x^n = 0$ to get that it depends only on $\tilde{c} c^{-1}$ on $x^n = 0$ but not on its derivatives; in fact,

$$(5.8) \qquad \left.\frac{\partial y}{\partial x^n}\right|_{x^n=0} = (0, \tilde{c} c^{-1}),$$

and this can be differentiated w.r.t. x'. Note that we can get the same result by comparing the αn metric elements in (5.2) and (5.3). Then $\partial(y - x)/\partial x^n|_{x^n=0} = (0, \tilde{c} c^{-1} - 1)$. By what we proved, the latter satisfies (5.1).

Differentiate (5.5) with respect to x^n at $x^n = 0$. Using (5.8), we get

$$
\begin{aligned}
\tilde{c} c^{-1} & h_{\alpha\beta} \left.\frac{\partial(c^2 \tilde{c}^{-2} - 1)}{\partial x^n}\right|_{x^n=0} \\
(5.9) \qquad & = -\left(c^2 \tilde{c}^{-2}(x', 0) - 1\right) \left.\frac{\partial}{\partial x^n}\right|_{x^n=0} \left(h_{i'j'}(y(x)) \frac{\partial y^{i'}}{\partial x^\alpha} \frac{\partial y^{j'}}{\partial x^\beta}\right) \\
& \quad + \left.\frac{\partial}{\partial x^n}\right|_{x^n=0} \left(\tilde{h}_{\alpha\beta}(x) - h_{i'j'}(y(x)) \frac{\partial y^{i'}}{\partial x^\alpha} \frac{\partial y^{j'}}{\partial x^\beta}\right).
\end{aligned}
$$

To estimate the last derivative, notice that by (5.8),

$$\frac{\partial^2 y}{\partial x^n \partial x^\alpha}\bigg|_{x^n=0} = \frac{\partial}{\partial x^\alpha}\bigg|_{x^n=0}(0,\tilde{c}c^{-1}),$$

and we proved the estimate for the tangential derivatives of $\tilde{c}c^{-1}$ (same as the tangential derivatives of $\tilde{c}c^{-1} - 1$) already. Therefore, the second summand on the r.h.s. of (5.9) can be written as a sum of a term involving the normal derivative of $h - \tilde{h}$, which estimate is known, plus terms involving the first tangential derivatives of $\tilde{c}c^{-1} - 1$, which we estimated already. We can now deduce the desired estimate for $\partial(c^2\tilde{c}^{-2} - 1)/\partial x^n$ at $x^n = 0$. Differentiate again (5.5) and (5.7) with respect to x^n at $x^n = 0$, to prove the estimate for $k = 2$, etc. $\square$

5.2. Local interior stability, proof of Theorem 1.5. Set

$$(5.10) \qquad\qquad \delta := \|d^2 - \tilde{d}^2\|_{C(\overline{\Gamma'\times\Gamma'})}.$$

We use below interpolation estimates in the C^k spaces; see, e.g., [48]. If $m < k$ (see (1.2)), we have

$$(5.11) \qquad\qquad \|c - \tilde{c}\|_{C^m(U)} \leq C\varepsilon_0^\mu,$$

with some $0 < \mu = \mu(m) \leq 1$. Also, by Theorem 5.1, for any m,

$$(5.12) \qquad\qquad \|d^2 - \tilde{d}^2\|_{C^m(\overline{\Gamma'\times\Gamma'})} \leq C\delta^\mu,$$

with another $0 < \mu = \mu(m) \leq 1$, if $k \gg 1$, under the a priori estimate (1.2) of the theorem. We will use the smallest μ above, and in the proof below, we will not specify the values of k and μ which are guaranteed to work, even though this can be done. In principle, increasing k in the theorem (assuming a stronger a priori bound) increases μ (makes the Hölder stability estimate stronger), and vice versa.

We extend c and $\tilde{c}$ outside M by preserving the δ^μ-closeness on Γ. Extend c in a smooth way first. As above, let (x', x^n) be boundary normal coordinates so that x^n is the distance in the metric g_0 to ∂M, positive outside M. Given an integer k, let $E_k(\tilde{c} - c)$ be the truncated Taylor series of $\tilde{c} - c$ w.r.t. x^n at $x^n = 0$ of degree m, for $x^n > 0$.

Let $0 < -\rho_0 \ll 1$, $\mathsf{c} > 0$ be such that the estimate in Proposition 3.3 holds in $\Omega_{\mathsf{c},\rho_0}$, with the choice of a boundary defining function $\rho = -x^n$. We can also assume that Proposition 3.3 holds for all sound speeds c as in (1.2), with A fixed, and $\varepsilon_0 \ll 1$. Set

$$\tilde{c}_1 = \begin{cases} \tilde{c}, & x^n < 0, \\ c + \chi(x^n/|\rho_0|)E_k(\tilde{c} - c), & x^n > 0, \end{cases}$$

where $\chi(t) = 1$ for $t < 1/4$, and $\chi(t) = 0$ for $t \geq 1/2$. Then $\tilde{c}_1 = \tilde{c}$ in M, and $\tilde{c}_1(x) = c(x)$ when $d(x, M) > |\rho_0|/2$. Moreover, for any small enough neighborhood U of p,

$$(5.13) \qquad\qquad \|c - \tilde{c}_1\|_{C^m(U\setminus M)} \leq C\|c - \tilde{c}\|_{C^m(\partial M)} \leq C\delta^\mu.$$

We drop the subscript 1 and denote the $\tilde{c}_1$ by $\tilde{c}$ below.

Next, we compare L and $\tilde{L}$ pushed to $\partial SB(x_0, r)$; see the Figure 1. Recall that in Section 2.2, we redefined L and $\tilde{L}$ to act from U_- to U_+, where $U_\pm \subset \partial_\pm SB(x_0, r)$.

It is easy to see that for x, y near x_0, $d(x,y)/C \leq \tilde{d}(x,y) \leq Cd(x,y)$ with C depending on A in (1.2) only. Then writing $d - \tilde{d} = (d^2 - \tilde{d}^2)/(d + \tilde{d})$, we get from (5.12),

$$(5.14) \qquad |\partial_{x',y'}^\alpha (d(x,y) - \tilde{d}(x,y))| \leq C\delta^\mu / d(x,y)^{|\alpha|+1} \quad \text{on } \Gamma' \times \Gamma'.$$

We are only going to need this for $|\alpha| \leq 1$. Set

$$\xi(x,y) = -\mathrm{d}_x d(x,y), \qquad \eta(x,y) = \mathrm{d}_y d(x,y),$$

which are just the tangent unit codirections at x and y of the geodesic connecting those two points. We define $\tilde{\xi}(x,y)$ and $\tilde{\eta}(x,y)$ in a similar way. By the strict convexity,

$$|\xi_n| \geq d(x,y)/C.$$

Since ξ and $\tilde{\xi}$ are units,

$$\xi_n(x,y) = -\sqrt{c^{-2}(x) - g_0^{\alpha\beta}(x)\xi_\alpha \xi_\beta}, \qquad \tilde{\xi}_n(x,y) = -\sqrt{\tilde{c}^{-2}(x) - g_0^{\alpha\beta}(x)\tilde{\xi}_\alpha \tilde{\xi}_\beta}.$$

Then, using (5.14) with $|\alpha| = 1$, we get

$$|\xi_n(x,y) - \tilde{\xi}_n(x,y)| \leq C\delta^\mu / d(x,y)^2.$$

This, combined with (5.14) for $\alpha = 0$, yields

$$(5.15) \qquad |\xi(x,y) - \tilde{\xi}(x,y)| \leq C\delta^\mu / d(x,y)^2.$$

In the same way, we get

$$(5.16) \qquad \|\eta(x,y) - \tilde{\eta}(x,y)\|_{C^{k-1}} \leq C\delta^\mu / d(x,y)^2.$$

For every $(x,y) \in \partial M \times \partial M$, both close to x_0, consider the shortest geodesics $\gamma_{[x,y]}$ and $\tilde{\gamma}_{[x,y]}$ connecting them, in the metrics g and $\tilde{g}$, respectively. By the a priori estimate (1.2), those two geodesics lie in a neighborhood of x_0 of size which shrinks to zero, as $d(x,x_0)$ and $d(y,x_0)$ tend to zero, uniformly for all c and $\tilde{c}$ satisfying (1.2). Next, we extend each one of them, lifted to the cotangent bundle (i.e., to the bicharacteristics, called "rays" below), in both directions until they hit $\partial SB(x_0, r)$. We denote by $z_- = (x_-, \xi_-)$ (respectively, $\tilde{z}_- = (\tilde{x}_-, \tilde{\xi}_-)$) the common point with $\partial_- SB(x_0, r)$, and by $z_+ = (y_+, \eta_+)$ (respectively, $\tilde{z}_+ = (\tilde{y}_+, \tilde{\eta}_+)$) the common point with $\partial_+ SB(x_0, r)$; see Figure 3.

Let Σ_- be the submanifold of $\partial_- SB(x_0, r)$ consisting of all z there such that the bicharacteristic $Z(t,z)$ through it, in the metric g, is tangent to ∂M at some point close enough to x_0. Let U_0 be the connected neighborhood of it consisting of those points staying at distance (with respect to fixed coordinates) at most $O(\delta^{\mu'})$ with $0 < \mu' < \mu$ which we chose below. Then the small but δ-independent neighborhood U_- of Σ_- admits the decomposition $U_- = U_0 + U_{\mathrm{in}} + U_{\mathrm{out}}$, where U_{in} and U_{out} are disconnected components, such that the rays from U_{in} in the metric g hit ∂M, and the rays from U_{out} do not. All of them are extended until they hit $\partial_+ SB(x_0, r)$. We denote the corresponding travel times by $\ell(z)$, as in Section 2.2, but now ∂M is replaced by the geodesic sphere $\partial B(x_0, r)$.

The rays issued from U_{out} would hit $\partial_+ SB(x_0, r)$ and miss ∂M by definition. They will stay at a distance at least $\delta^{\mu'}/C$ from ∂M. The same is true for the rays issued from U_{out} related to the metric $\tilde{g} = \tilde{c}^{-2} g_0$ for $\delta \ll 1$, because by (5.13) the Hamiltonian vector fields of g and $\tilde{g}$ are $O(\delta^\mu)$ close, and $\mu' < \mu$. In particular, the rays related to $\tilde{g}$ will not hit ∂M for $\delta \ll 1$. Then

$$(5.17) \qquad |Z(z,t) - \tilde{Z}(z,t)| \leq C\delta^\mu, \qquad z \in U_{\mathrm{out}},$$
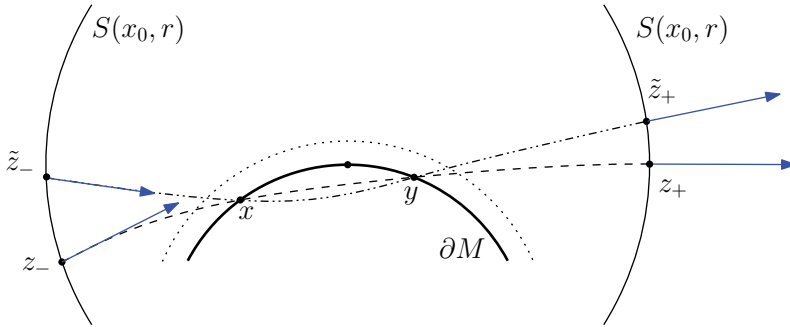
FIGURE 3. Illustration to the proof of Theorem 1.5. Here, $z_\pm$ and $\tilde{z}_\pm$ are actually points and codirections.

for t over a compact interval. Recall that $Z(z,t)$ is the bicharacteristic issued from z.

The rays issued from U_{in} will hit ∂M at an angle at least $\delta^{\mu'/2}/C$, by the strict convexity. This is true, for $\delta \ll 1$, for the rays related to $\tilde{g}$ as well by the closeness of the Hamiltonians (outside M) argument. The travel times $\tilde{\ell}(z)$ of the rays in the metric $\tilde{g}$ are in general different from $\ell(z)$, but the points $\tilde{Z}(\ell(z),z)$ lie at a distance from $Z(t,\ell(z)) \in \partial_+ SM$ which decreases with ε in (1.2). Therefore, for $0 < \varepsilon \ll 1$, $\tilde{Z}(t,z)$ with $z \in U_{\text{in}}$ would enter M and leave it for $t = \ell(z)$, and it will also leave the $O(\delta^\mu)$ neighborhood of M where we modified $\tilde{c}$. This is needed for the argument below.

By (5.15) and (5.16),

$$(5.18) \qquad |z_- - \tilde{z}_-| \le C\delta^\mu/d^2, \qquad |z_+ - \tilde{z}_+| \le C\delta^\mu/d^2, z \in U_{\text{in}}.$$

Here, $d = d(x,y)$ with (x,y) uniquely determined as the contacts of the ray trough z with ∂M. Also, by the closeness of the Hamiltonians outside M, and by (5.16), the lengths of the segments of $Z(z,t)$ and $\tilde{Z}(\tilde{z},t)$ from z_- and $\tilde{z}_-$, respectively, to their common point x, differ by $O(\delta^\mu/d^2)$; see Figure 3. The segments in M differ by $O(\delta^\mu/d)$; see (5.14). The remaining segments, after they exit M and hit $S(x_0,r)$, differ by $O(\delta^\mu/d^2)$ by the above argument. Therefore,

$$(5.19) \qquad |\ell(z) - \tilde{\ell}(\tilde{z})| \le C\delta^\mu/d^2, \qquad z \in U_{\text{in}}.$$

By (2.4),

$$(5.20) \qquad \begin{aligned} &\tilde{Z}(\ell(z_-),z_-) - Z(\ell(z_-),z_-) \\ &\qquad = \int_0^{\ell(z_-)} \frac{\partial \tilde{Z}}{\partial z}(t-s, Z(s,z_-))(\tilde{V}-V)(Z(s,z_-))\,\mathrm{d}s. \end{aligned}$$

We replace next $\tilde{Z}(\ell(z_-),z_-)$ by $\tilde{Z}(\tilde{\ell}(\tilde{z}_-),\tilde{z}_-) = \tilde{z}_+$ modulo errors controlled by (5.18) and (5.19), so that we can get $z_+ - z_-$ on the left and use the fact that the latter satisfies (5.18). This would allow us to conclude that the integral is "small" in U_{in}. Indeed, since $Z(z,t)$ is a smooth function with uniformly bounded derivatives for z and t bounded, we get by (5.18) and (5.19)

$$(5.21) \qquad \left| \tilde{Z}(\ell(z_-),z_-) - \tilde{Z}(\tilde{\ell}(\tilde{z}_-),\tilde{z}_-) \right| \le C\delta^\mu/d^2, \qquad z \in U_{\text{in}}.$$

This and (5.20) yield

$$(5.22) \qquad \int_0^{\ell(z_-)} \frac{\partial \tilde{Z}}{\partial z}(t - s, Z(s, z_-))(\tilde{V} - V)(Z(s, z_-)) \, ds$$
$$= \tilde{z}_+ - z_+ + O(\delta^\mu/d^2) = O(\delta^\mu/d^2), \qquad z \in U_{\text{in}},$$

where, in the last step, we used the second inequality in (5.18). The factor $d(x, y)$ can be replaced by the square root of the distance between U_{in} and the tangent manifold Σ_-, and that root is $\delta^{\mu'/2}$ by construction. We choose μ' so that $\mu - 2\mu'/2 > \mu/2$ to obtain $O(\delta^{\mu/2})$ in (5.22).

In U_{out}, the even better estimate $O(\delta^\mu)$ holds, without the d^{-2} term, by (5.17). In U_0, we used the argument explained before to extend the estimate from U_{out} to get an $O(\delta^\mu) + O(\delta^{\mu'}) = O(\delta^{\mu'})$ estimate there. By interpolation, we get

$$(5.23) \qquad \int_0^{\ell(z_-)} \frac{\partial \tilde{Z}}{\partial z}(t - s, Z(s, z_-))(\tilde{V} - V)(Z(s, z_-)) \, ds = O(\delta^{\mu/2}) \quad \text{in } C^1(U).$$

Now, by (2.4), (2.11), and (2.12) we can write the left-hand side of (5.23) in the form $\tilde{J}(f, \partial f)$ and complete the proof of Theorem 5.1 by applying Proposition 3.3.

5.3. **Global stability.** For the purpose of the next theorem, we will extend and slightly generalize the foliation condition to compact submanifolds M_0 of M. Let, as before, $\tilde{M}$ be a neighborhood of M, and extend c smoothly there. Note that the tilde over M is not an indication that it is related to $\tilde{c}$.

Definition 5.1. Let $M_0 \subset M$ be compact. We say that M_0 can be foliated by strictly convex hypersurfaces if there exists a smooth function $\rho : \tilde{M} \to [0, \infty)$ which level sets $\Sigma_t = \rho^{-1}(t)$, $t \leq T$ with some $T > 0$, restricted to M_0, are strictly convex viewed from $\rho^{-1}((0, t))$ for g; $d\rho$ is non-zero on these level sets, $\Sigma_0 \cap M = \emptyset$ and $M_0 \subset \rho^{-1}([0, T])$.

Note that this definition is not equivalent to Definition 1.1 when $M_0 = M$ because in Definition 1.1, we allow $M \setminus \rho^{-1}([0, T])$ to be non-empty (but with an empty interior). Indeed, for uniqueness, proving $c = \tilde{c}$ outside such a set suffices since c and $\tilde{c}$ are at least continuous. For stability, however, it is convenient to assume that this set is empty.

We show next that the foliation condition implies non-trapping.

Proposition 5.1. *If M_0 can be foliated by strictly convex hypersurfaces, then any maximal geodesic in M_0 is of finite length.*

Proof. Assume that $s \mapsto \gamma(s)$ exists for $s \in (0, \infty)$ and stays in M_0; and in particular, it never reaches ∂M_0 for $s > 0$. The function $f(s) := \rho \circ \gamma(s)$ cannot have more than one critical point as a consequence of the implication $f'(s) = 0 \implies f''(s) < 0$. By shifting the s variable, we can always assume that the possible critical point is negative. Then f is a strictly decreasing function for $s > 0$. On the other hand, it is bounded by below, so it has a limit $\hat{\tau} \geq 0$, as $s \to \infty$, which is also its infimum. By compactness, there exists a sequence $s_j \to \infty$ (we can start with $s_j = j$ and take a subsequence) so that $(x_j, v_j) := (\gamma(s), \dot{\gamma}(s))$ converges to some $(\hat{x}, \hat{v}) \in SM_0$. Next, $\rho(x_j) \searrow \hat{\tau}$. The limit $\hat{v}$ must be tangent to $\Sigma_{\hat{\tau}}$ at $\hat{x}$, because we can easily obtain a contradiction with the strict convexity if it is not. Now, by the strict convexity of $\Sigma_{\hat{\tau}}$ again, there exists $\delta > 0$ so that $\gamma_{\hat{x}, \hat{v}}(s)$ would hit $\Sigma_{\hat{\tau} - \delta}$ for some positive

time. This property is preserved under a small perturbation of $(\hat{x}, \hat{v})$ and therefore applies to (x_j, v_j) for $j \gg 1$. This, however, contradicts the choice of $\hat{\tau}$. □

In particular, we get the following.

Corollary 5.1. Assume that $M = M_0 \cup M_1$, where M_0 can be foliated by strictly convex hypersurfaces, and M_1 is non-trapping. Then M is non-trapping as well.

Note that M_1 can be a point, or a small neighborhood of a point, which happens if the level surfaces of $\rho = c > 0$ are diffeomorphic to spheres but $\rho = 0$ degenerates into a point. Another example is when M_1 is simple; see also the remarks following Theorem 1.4.

The C^k norm below is defined in a fixed finite atlas of local coordinate charts. In the same way we define $\mathrm{dist}(L, \tilde{L})$ and its $C(D)$ norm: in any coordinate system we can just take the supremum of $L - \tilde{L}$ and then the maximum over all charts. They can be defined in an invariant way, in principle but we do not do that for the sake of simplicity.

Theorem 5.2. *Assume that $M_0 \subset M$ can be foliated by strictly convex hypersurfaces for $g = c^{-2}g_0$. Let $D \subset \partial_- SM$ be a neighborhood of the compact set of all $\beta \in \partial_- SM \cap \partial_- SM_0$ consisting of the initial points of all geodesics γ_β tangent to the intersections of the strictly convex hypersurfaces with M_0. Then with k, μ, c_0, c, $\tilde{c}$, ε_0, and A as in Theorem 1.5, we have the stability estimate*

$$(5.24) \qquad \|c - \tilde{c}\|_{C^2(M_0)} \leq C \|\mathrm{dist}(L, \tilde{L})\|_{C(D)}^\mu$$

for c, $\tilde{c}$ satisfying (1.2).

Proof. We will show first that the estimate in Theorem 1.5, near a boundary point p, can be written in the form (5.24). For x and y on ∂M, both close to the fixed point p, let $\gamma_0 : [0, 1] \mapsto \partial M$ be the boundary geodesic connecting x (for $t = 0$) and y (for $t = 1$). Set $v(s) = \exp_x^{-1} \gamma_0(s)$, where exp is the interior exponential map. Then $(y, w) = L(x, v(1)/|v(1)|_g)$; set $(\tilde{y}, \tilde{w}) = \tilde{L}(x, v(1)/|v(1)|_{\tilde{g}})$. Set

$$(5.25) \qquad \delta = \|\mathrm{dist}(L, \tilde{L})\|_{C(D)}.$$

Then $|y - \tilde{y}| \leq \delta$. On the other hand, in the notation following (1.1), given by $L(x, v) = (y, w)$,

$$(5.26) \qquad \frac{\mathrm{d}}{\mathrm{d}s} d(x, \gamma_0(s)) = \langle w'(s), \dot{\gamma}_0(s) \rangle_g,$$

where $w'(s)$ is the tangential component of $w(s)$, the second component of $L(x, v(s)/|v(s)|_g)$. We also have

$$(5.27) \qquad \frac{\mathrm{d}}{\mathrm{d}s} \tilde{d}(x, \gamma_0(s)) = \langle \tilde{w}'(s), \dot{\gamma}_0(s) \rangle_{\tilde{g}},$$

where $\tilde{w}'(s)$ is the second component of $\tilde{L}(x, v(s)/|v(s)|_{\tilde{g}})$. Integrate the difference of (5.26) and (5.27) w.r.t. s from 0 to 1, and use (5.25) and the fact that $c = \tilde{c}$ on ∂M near x to get $d(x, y) - \tilde{d}(x, \tilde{y}) = O(\delta)$ (actually, we get $O(\delta)d(x, y)$). Note that this also proves that $\ell - \tilde{\ell} = O(\delta)$ near $S_{x_0}M$. We remark that in the inductive step below, we will only have that c and $\tilde{c}$ are $O(\delta^\mu)$ close on ∂M near x, instead of being equal. Then we would get $O(\delta^\mu)$ instead of $O(\delta)$. So we get now

$d^2(x, y) - \tilde{d}^2(x, \tilde{y}) = O(\delta)$ as well. Since d^2 and $\tilde{d}^2$ are smooth and $|y - \tilde{y}| \leq \delta$, we get

$$d^2(x, y) - \tilde{d}^2(x, y) = O(\delta).$$

The constant in this estimate depends on the a priori bound A in (1.2). We can therefore apply Theorem 5.1 to estimate the jet of $c - \tilde{c}$ at ∂M of any finite order. We then extend c and $\tilde{c}$ in a neighborhood of $\partial M_0 \cap \partial M$ so that c and $\tilde{c}$ are $O(\delta^\mu)$ there, as in (5.13).

For any geodesic in M_0 issued from D, we extend it to $\tilde{M}$ and push the scattering relations L and $\tilde{L}$ to $\partial \tilde{M}$. We show below that we have dist$(L, \tilde{L}) = O(\delta^\mu)$ for the so extended lens relations, and a similar estimate for $\ell - \tilde{\ell}$. We choose U_0, U_{in}, and U_{out} as the proof of Theorem 5.1 but this time U_0 is an $O(\delta^\mu)$ neighborhood of Σ (rather than $O(\delta^{\mu'})$, because the singular factor $1/d^2$ is not present anymore). For any $(x, v) \in U_{\text{in}}$, let z_-, $\tilde{z}_-$ be the points (in the phase space) of contact with the geodesic issued from (x, v) for negative "times" with $\partial \tilde{M}$ w.r.t. the metrics g and $\tilde{g}$, respectively. We define z_+, $\tilde{z}_+$ in a similar way (see Figure 4), where, in this case, $\Sigma_i = \partial M$ and $\Omega_i = M_0$. Then $z_- - \tilde{z}_- = O(\delta^\mu)$ (the difference makes sense in local coordinates only) because we have the same for $c - \tilde{c}$ in C^2. We have an $O(\delta)$ bound of $y - \tilde{y}$, and of the difference of the corresponding directions there, by the closeness of the scattering relations. Next, $z_+ - \tilde{z}_+ = O(\delta^\mu)$ as well. Therefore, dist$(L(z_-), \tilde{L}(\tilde{z}_-)) = O(\delta^\mu)$. Since L and $\tilde{L}$ are smooth with uniformly bounded first derivatives, we also get dist$(L(z_-), \tilde{L}(z_-)) = O(\delta^\mu)$, for all $z_- \in U_{\text{in}}$. We also have $\ell(z_-) - \tilde{\ell}(z_-) = O(\delta^\mu)$ as shown in the paragraph following (5.27). In this step, we are no longer working near a fixed point on ∂M, and for almost tangential directions, the requirement for U_{in} is that the geodesic issued from it are in the set required by the theorem but that set is a distance at least δ^μ from the tangential set Σ. In U_{out} and U_0, we argue as above.

Therefore, we have now that $\| \text{dist}(L, \tilde{L}) \|_{C(D_0)} = O(\delta^\mu)$, where L, $\tilde{L}$, and D_0 are defined as L, $\tilde{L}$, and D but with ∂M replaced by $\partial \tilde{M}$. We have the same for $\ell - \tilde{\ell}$ as well. The advantage of this is that the geodesics issued from D_0 hitting supp$(c - \tilde{c})$ are never tangent to $\partial \tilde{M}$, and we actually have a uniform lower bound on the angle they make with $\partial \tilde{M}$. Thus we reduced the analysis to the case when $c - \tilde{c}$ is a priori known to be supported in the interior of M.
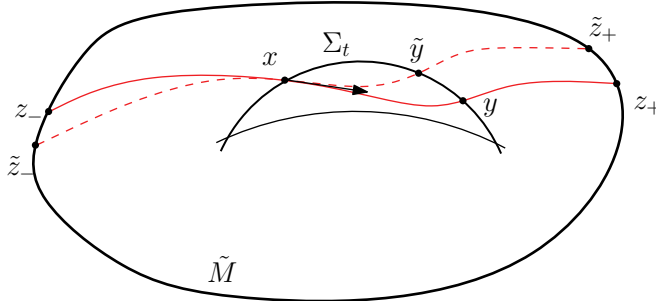

FIGURE 4. Illustration to the proof of Theorem 5.2.

We then apply a layer stripping argument finite number of times; see also the introduction in [49]. For each Σ_t, assuming the estimate outside Σ_t, we can choose

an appropriate lens shaped domain near each point on Σ_t and the artificial boundary close enough to it; see Figure 4. The $O(\delta^\mu)$ closeness of L and $\tilde{L}$, and ℓ and $\tilde{\ell}$, implies the same for c and $\tilde{c}$ by the arguments using (5.23). We do this for a finite number of points on Σ_t by a compactness argument to prove the estimate in $\rho^{-1}(0, t+\epsilon)$ for some $\epsilon > 0$. We then cover M_0 with a finite set $\Sigma_i := \rho^{-1}(t_i)$, $M_i := \rho^{-1}(-\infty, t_i)$, and finish the proof with finitely many steps. $\qquad\square$

6. Herglotz and Wieckert and Zoeppritz speeds are lens rigid

We revisit the Herglotz [14] and Wieckert and Zoeppritz [51] class of speeds. Let $M = B(0, R)$, $R > 0$, be the ball in $\mathbb{R}^n$, $n \geq 3$, centered at the origin with radius $R > 0$. The background metric g_0 in this section is the Euclidean one. Let $0 < c(x)$ be smooth in $B(0, R)$. Assume that c satisfies the Herglotz and Wieckert and Zoeppritz condition

$$(6.1) \qquad \frac{\partial}{\partial r} \frac{r}{c(r)} > 0, \quad \text{for } 0 < r = |x| \leq R,$$

where $\frac{\partial}{\partial r} = \frac{x}{|x|} \cdot \partial_x$ is the radial derivative. We do not assume that c is radial, i.e., that it depends on $r = |x|$ only. We show below that (6.1) is, in fact a foliation condition.

Proposition 6.1. *The Herglotz and Wieckert and Zoeppritz condition (6.1) is equivalent to the condition that the Euclidean spheres $S_r = \{|x| = r\}$ are strictly convex in the metric $c^{-2}\mathrm{d}x^2$ for $0 < r \leq R$.*

Proof. This proposition is essentially proved in [47, Proposition 7.1]. We will show first that the strict convexity condition of an oriented hypersurface S (positive second fundamental form) is equivalent to the following: (A) if G is the generator of the geodesic flow, and if ρ is a defining function of S positive on the "positive" side of S, then $G^2\rho > 0$ when $G\rho = \rho = 0$ on a non-zero energy level. In semigeodesic coordinates (x', x^n) with $x^n > 0$ on the "positive" side of S, we have $\rho = x^n\rho'$ with $\rho' > 0$ on S. Since

$$G = \xi^j \partial_{x^j} - \Gamma_{ij}^k \xi^i \xi^j \partial_{\xi_k},$$

we have $G\rho = \xi^n + x^n G\rho'$. On S, $G\rho = \xi^n$; therefore, (A) can be reformulated as follows: $x^n = \xi^n = 0$ and $\xi \neq 0$ imply $G^2\rho > 0$. Differentiating $G\rho$ again, we see that under those conditions, $G^2\rho = -\Gamma_{\alpha\beta}^n \xi^\alpha \xi^\beta$, where the Greek indices run from 1 to $n - 1$. This is well known and follows directly from the definition to be the second fundamental form corresponding to the orientation $x^n > 0$. Then strict convexity, viewed from $\rho > 0$ is equivalent to a negative second fundamental form. For manifolds with a boundary we study $\rho > 0$ in the interior of M, and therefore convexity viewed from the exterior means a positive second fundamental form.

We can take $\rho = r^2/2 - r_0^2/2 = |x|^2/2 - r_0^2/2$ as a defining function of the sphere S_{r_0} for $x \neq 0$. Then $G\rho = \xi \cdot x$. Next, $G^2\rho = |\xi|^2 - \Gamma_{ij}^k \xi^i \xi^j x^k$ and on the energy level $c^{-2}|\xi|^2 = 1$, we have $G^2\rho = c^2 - \Gamma_{ij}^k \xi^i \xi^j x^k$. Here and below, we still have summation w.r.t. k even though both indices are upper. A direct computation shows that

$$\Gamma_{ij}^k = \frac{1}{2} c^2 \left(\delta_j^k \partial_{x^i} + \delta_i^k \partial_{x^j} - \delta_{ij} \partial_{x^k} \right) c^{-2},$$

with $c = c(|x|)$, and therefore,

$$\Gamma_{ij}^k \xi^i \xi^j x^k = -c^{-1} \left(2\xi^k x^k \xi^i \partial_{x^i} - |\xi|^2 x^k \partial_{x^k} \right) c$$
$$= -c^{-1} \left(2(\xi \cdot x)\xi \cdot \partial_x - |\xi|^2 x \cdot \partial_x \right) c.$$

Therefore, on the unit energy level and for $G\rho = 0$, strict convexity is equivalent to

$$c - x \cdot \partial_x c > 0,$$

i.e., $r\partial_r c < c$, which is equivalent to (6.1). Note that the computations in [47, Proposition 7.1] are done in the cotangent bundle and are somewhat shorter. □

Corollary 6.1. Speeds satisfying (6.1) are lens rigid in $B(0, R)$ in the sense of Theorem 1.4.

Note that we only require c in Theorem 1.4 to satisfy (6.1). Then any other speed $\tilde{c}$ for which $\partial B(0, R)$ is still strictly convex and $\tilde{c} = c$ on $\partial B(0, R)$ with the same scattering relation is equal to c. This extends the Herglotz [14] and the Wieckert and Zoeppritz [51] results to not necessarily radial speeds c satisfying (6.1).

If (6.1) holds in the shell $R_0 < |x| < R$ only, with some $0 < R_0 < R$, then we get lens rigidity in the shell and we only need to use the scattering relation for geodesics staying in it. The speed c does not even need to be defined in $|x| < R_0$. We will skip the details.

Acknowledgments

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