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Stability estimates for the hyperbolic Dirichlet to Neumann map in anisotropic media

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1 Introduction and statement of the results

One of the basic inverse problems in anisotropic media is the determination of a Riemannian metric in a domain by measuring the Dirichlet to Neumann map at the boundary of the domain.

In this paper we consider the question of stability, that is, whether if two Dirichlet to Neumann maps associated to two metrics are close enough in an appropriate topology then the Riemannian metrics are close enough in an appropriate topology.

We now describe the problem and the main results.

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary. Given a Riemannian metric $g(x) = (g_{ij}(x))$ in Ω, consider the Laplace-Beltrami operator

$$\Delta_g = (\det g)^{-\frac{1}{2}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} (\det g)^{\frac{1}{2}} g^{ij} \frac{\partial}{\partial x_j}$$

in Ω. Here $(g^{ij}) = (g_{ij})^{-1}$, $\det g = \det(g_{ij})$. Consider the following problem

$$\left\{ \begin{array}{rcll} (\partial_t^2 - \Delta_g)u &=& 0 & \text{in } (0,\infty)\times\Omega, \\ u|_{t=0} = \partial_t u|_{t=0} &=& 0 & \text{in } \Omega, \\ u|_{(0,\infty)\times\partial\Omega} &=& f, & \end{array} \right. \tag{1.1}$$

where $f \in H^2_{\mathrm{loc}}$, $f = 0$ for $t < 0$. Denote by $\nu = \nu(x)$ the outer normal to $\partial\Omega$ at $x \in \partial\Omega$. We define the hyperbolic Dirichlet-to-Neumann (DN) map Λ_g by

$$\Lambda_g f := (\det g)^{\frac{1}{2}} \sum_{i,j=1}^{3} \nu_i g^{ij} \frac{\partial u}{\partial x_j}\Big|_{(0,\infty)\times\partial\Omega}.$$

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It is easy to see [S-U] that if

$$\psi : \overline{\Omega} \to \overline{\Omega}$$

is a diffeomorphism with $\psi|_{\partial\Omega} =$ identity, then $\Lambda_{\psi^* g} = \Lambda_g$, where $\psi^* g$ denotes the pull back of the metric g. Therefore the best one can do is determine the metric up to isometries that leave the boundary fixed.

In this paper we prove that the hyperbolic DN map Λ_g determines in a stable way g up to isometries that leave the boundary fixed, provided that g is sufficiently close to the euclidean metric e.

Let $\|\Lambda\|_*$ denote the norm of Λ considered as an operator

$$\Lambda : H^1((0,T) \times \partial\Omega) \longrightarrow L^2((0,T) \times \partial\Omega)$$

with T large enough (see (4.14) for a more precise estimate of T depending on the metric g). Next, let $\|\Lambda\|_{**}$ denote the operator norm of

$$\Lambda : e^{\sqrt{\varepsilon}t} H_0^2(\mathbf{R}_+ \times \partial\Omega) \longrightarrow e^{\sqrt{\varepsilon}t} L^2(\mathbf{R}_+ \times \partial\Omega).$$

It is easy to see that $\|\Lambda_g\|_{**}$ is finite applying the trace theorem and standard energy estimates. It follows from [CP] that $\|\Lambda_g\|_*$ is finite as well.

Theorem 1.1 *Let $g_k \in C^{10}(\bar{\Omega})$, $k = 1, 2$ be two Riemannian metrics and denote as above by Λ_{g_1}, Λ_{g_2} the corresponding DN maps. Then there exists $\varepsilon > 0$ such that if g_1, g_2 satisfy*

$$\|g_k - e\|_{C^{10,\mu}(\bar{\Omega})} < \varepsilon, \qquad k = 1, 2 \tag{1.2}$$

with some $\mu > 0$, one can find a C^{11} diffeomorphism $\psi : \bar{\Omega} \to \bar{\Omega}$ with $\psi|_{\partial\Omega} = Id$, such that

$$\|\psi^* g_1 - g_2\|_{L^2(\Omega)} \leq C \left(\|\Lambda_{g_1} - \Lambda_{g_2}\|_*^\sigma + \|\Lambda_{g_1} - \Lambda_{g_2}\|_{**}^\sigma \right) \tag{1.3}$$

for any $\sigma < 1/5$ with $C = C(\varepsilon, \sigma)$.

Remark. We note that one can also obtain an L^∞ estimate instead of L^2 estimates with $\sigma < 1/6$ by using interpolation techniques as in [Su].

Of course Theorem 1.1 implies identifiability of the metric from the hyperbolic DN map up to isometries that leave the boundary fixed. We have

Corollary 1.1 *Let $g_k \in C^{10}(\bar{\Omega})$, $k = 1, 2$ be two Riemannian metrics and denote by Λ_{g_1}, Λ_{g_2} the corresponding DN maps. Then there exists a constant $\varepsilon > 0$, such that if g_1, g_2 satisfy (1.2) with some $\mu > 0$ and if*

$$\Lambda_{g_1} = \Lambda_{g_2},$$

then there exists a C^{11}-diffeomorphism $\psi : \bar{\Omega} \to \bar{\Omega}$ with $\psi|_{\partial\Omega} = Id$, such that $\psi^* g_1 = g_2$.

Corollary 1.1 is known under more general conditions. For smooth metrics (without a smallness condition on the metric) it is a consequence of [B-K] and [T]. The paper [B-K] uses the boundary control method introduced by Belishev [B]. This method requires that the so-called observation operator is injective (see [B-K]). This, in turn, is a consequence of the unique continuation theorem of Tataru [T] (see also [H II] and [R-Z]). Because of the use of unique continuation in the proof, it seems unlikely that stable estimates of the form (1.3) can be obtained using this method. We also mention that a linearized version of Corollary 1.1 was discussed in [S-U] and [C-M]. See also the survey paper [U] for connections between this problem and other inverse problems.

In this paper we give a proof of Corollary 1.1 first since the method used can be easily extended to give the estimate (1.3). The Corollary is proven in Sections 2–4. The stability estimate is proven in Section 5.

We remark that the condition that the metrics are close to the euclidean metric is used in several places. First of all, to prove, say Corollary 1.1, we reduce the problem to an inversion of a Fourier integral operator, similar to a generalized Radon transform, which we can invert if the metric is close to the euclidean metric. Second, the diffeomorphism ψ is constructed using harmonic coordinates, i.e. if g denotes a Riemannian metric we solve

$$\Delta_g \psi = 0, \quad \psi|_{\partial\Omega} = Id,$$

where Id denotes the identity. If g is close to the euclidean metric, then ψ is a diffeomorphism. Moreover one can use the condition that the hyperbolic Dirichlet to Neumann maps associated to two metrics are the same to conclude that the harmonic coordinates can be extended to be equal outside the domain.

We also mention that stability estimates for the Dirichlet to Neumann map associated to the wave equation plus potential were proven in [A-S], [I-S], [Su].

2 Construction of the singular solution

Proposition 2.1 *Let u_1, u_2 solve the following problems in $(0,T)\times\Omega$ with some $T>0$:*

$$\left\{\begin{array}{rcl} (\partial_t^2-\Delta_{g_1})u_1 &=& 0,\\ u_1|_{t=0}=\partial_t u_1|_{t=0} &=& 0,\\ u_1|_{(0,T)\times\partial\Omega} &=& f_1, \end{array}\right. \qquad \left\{\begin{array}{rcl} (\partial_t^2-\Delta_{g_2})u_2 &=& 0,\\ u_2|_{t=T}=\partial_t u_2|_{t=T} &=& 0,\\ u_2|_{(0,T)\times\partial\Omega} &=& f_2, \end{array}\right. \tag{2.1}$$

where $f_j\in H^2$, $j=1,2$. Then

$$\begin{aligned}\int_0^T\!\!\int_{\partial\Omega} f_2(\Lambda_{g_1}-\Lambda_{g_2})f_1\,dS_x\,dt &= \int_0^T\!\!\int_\Omega \sum_{i,j=1}^3\left[(\det g_1)^{\frac12}g_1^{ij}-(\det g_2)^{\frac12}g_2^{ij}\right]\frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j}\,dx\,dt\\ &\quad -\int_0^T\!\!\int_\Omega\left[(\det g_1)^{\frac12}-(\det g_2)^{\frac12}\right]\frac{\partial u_1}{\partial t}\frac{\partial u_2}{\partial t}\,dx\,dt.\end{aligned}$$

Proof. We have

$$
\begin{aligned}
0 &= \int_0^T\int_\Omega \Big((\partial_t^2-\Delta_{g_1})u_1\Big)(\det g_1)^{\frac12}u_2\,dx\,dt \\
&= \int_0^T\int_\Omega(\partial_t^2u_1)u_2(\det g_1)^{\frac12}\,dx\,dt-\int_0^T\int_\Omega\sum_{i,j=1}^3\left(\frac{\partial}{\partial x_i}(\det g_1)^{\frac12}g_1^{ij}\frac{\partial}{\partial x_j}u_1\right)u_2\,dx\,dt \\
&= -\int_0^T\int_\Omega(\partial_tu_1)(\partial_tu_2)(\det g_1)^{\frac12}\,dx\,dt+\int_0^T\int_\Omega\sum_{i,j=1}^3(\det g_1)^{\frac12}g_1^{ij}\frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i}\,dx\,dt \\
&\qquad -\int_0^T\int_{\partial\Omega}(\Lambda_{g_1}f_1)f_2\,dS_x\,dt. \qquad (2.2)
\end{aligned}
$$

In the same way we get

$$
\begin{aligned}
0 &= -\int_0^T\int_\Omega(\partial_tu_2)(\partial_tu_1)(\det g_2)^{\frac12}\,dx\,dt+\int_0^T\int_\Omega\sum_{i,j=1}^3(\det g_2)^{\frac12}g_2^{ij}\frac{\partial u_2}{\partial x_j}\frac{\partial u_1}{\partial x_i}\,dx\,dt \\
&\qquad -\int_0^T\int_{\partial\Omega}f_1(\Lambda_{g_2}^*f_2)\,dS_x\,dt. \qquad (2.3)
\end{aligned}
$$

Here Λ_g^* is defined by the same formula as Λ_g the only difference being that $u|_{t=0}=\partial_tu|_{t=0}=0$ is replaced by $u|_{t=T}=\partial_tu|_{t=T}=0$. By (2.2), (2.3) for $g=g_1=g_2$ we see that Λ_g^* is the adjoint to Λ_g (in fact, the adjoint to its restriction to $t\in(0,T)$), in other words, $\int_0^T\int_{\partial\Omega}f_1(\Lambda_{g_2}^*f_2)\,dS_x\,dt=\int_0^T\int_{\partial\Omega}(\Lambda_{g_2}f_1)f_2\,dS_x\,dt$. After subtracting (2.2), (2.3), we complete the proof of the proposition. □

Assume that we are given a Riemannian metric $g\in C^{k+1}(\bar\Omega)$ satisfying

$$
\|g-e\|_{C^{k+1}(\bar\Omega)}<\varepsilon \qquad (2.4)
$$

with some $k\geq 2$ (compare with (1.2)). Let us extend it to a C^k-metric in the whole $\mathbf{R}^3$ (which we will continue to denote by g) such that $g=e$ outside B_ρ. One can arrange that the extended metric satisfies

$$
\|g-e\|_{C^{k+1}(\mathbf{R}^3)}<C\varepsilon \qquad (2.5)
$$

with $C>0$ depending on Ω, ρ and $\mathrm{dist}(\Omega,\partial B_\rho)$. We construct a phase function $\phi(x,\theta)$, $\theta\in S^2$ associated to g as the solution to the following eikonal equation

$$
\left\{\begin{array}{rcl}\sum_{i,j=1}^3 g^{ij}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j} &=& 1,\\ \phi|_{x\cdot\theta\leq-\rho} &=& x\cdot\theta.\end{array}\right. \qquad (2.6)
$$

The Hamiltonian related to (2.6) is $H=\sum_{i,j=1}^3 g^{ij}(x)\xi_i\xi_j-1$. Let $\theta\in S^2$ be fixed. Then one can assume that $\theta=(1,0,0)$. Therefore, we get the following Hamiltonian system

$$
\left\{\begin{array}{rclrcll}\frac{d}{ds}x_m &=& 2\sum_{j=1}^3 g^{mj}\xi_j, & \frac{d}{ds}\xi_m &=& -\sum_{i,j=1}^3\frac{\partial g^{ij}}{\partial x_m}\xi_i\xi_j, & m=1,2,3,\\ x|_{s=0} &=& (-\rho,\eta), & \xi|_{s=0} &=& (1,0,0), & \end{array}\right. \qquad (2.7)
$$

where $\eta \in \mathbf{R}^2$ parameterizes the plane $x_3 = -\rho$. If $g = e$, then the solution to (2.7) is given by $x = (2s-\rho, \eta)$, $\xi = (1,0,0)$. It is easy to see that for general g the solution exists for all s. Estimate (2.5) implies immediately the following.

Lemma 2.1 *Fix $a > 0$. Then there exists $C > 0$ such that for the solution $x = x(s,\eta)$, $\xi = \xi(s,\eta)$ of (2.7) we have*

$$\|x - (2s-\rho,\eta)\|_{C^k([0,a]\times\mathbf{R}^2)} + \|\xi - (1,0,0)\|_{C^k([0,a]\times\mathbf{R}^2)} \le C\varepsilon.$$

In particular, Lemma 2.1 implies that under the smallness assumption (4.6) the Hamiltonian flow is non-trapping for small ε, more precisely, $x(s,\eta) \notin B_\rho = \{x;\ |x| < \rho\}$ for $s > a$ with some $a > 0$. Moreover, the mapping $(s,\eta) \mapsto x(s,\eta)$ is a C^k–diffeomorphism on $[0,a]\times\{\eta \in \mathbf{R}^2;\ |\eta| \le 2\rho\}$ and its range covers B_ρ provided that ε is small enough. For technical reasons in the proof of Proposition 2.2 we will need in fact to work in a larger domain, so let us assume that ε and a are such that $(s,\eta) \mapsto x(s,\eta)$ maps $[0,a]\times\{\eta \in \mathbf{R}^2;\ |\eta| \le 5\rho\}$ into a compact covering $B_{4\rho}$.

The phase function satisfies $\frac{d}{ds}\phi = \xi \cdot H'_\xi = 2\sum_{i,j=1}^3 g^{ij}\xi_i\xi_j$. Therefore,

$$\phi(x) = -\rho + 2\int \sum_{i,j=1}^{3} g^{ij}(x)\xi_i\xi_j\, ds, \tag{2.8}$$

where we integrate along the bicharacteristic joining $\{x_1 = -\rho,\ \xi = (1,0,0)\}$ and (x,ξ). Since $H = 0$ along the solutions of (2.7), we get from (2.8)

$$\phi(x) = -\rho + 2s. \tag{2.9}$$

The change of coordinates $x \to (s,\eta)$ is ε-close to $x = (2s-\rho,\eta)$ in C^k, which implies that ϕ must be close to $\phi = x_1$. So far $\theta \in S^2$ was fixed. One can easily investigate the dependence of ϕ on θ. As a consequence of Lemma 2.1 and (2.9) we get the following.

Lemma 2.2 *Assume that (2.5) holds with $\varepsilon > 0$ sufficiently small. Then there exists $C_0 > 0$ such that*

$$\|\phi(x,\theta) - x\cdot\theta\|_{C^k(B_{4\rho}\times S^2)} \le C_0\varepsilon.$$

We are going next to construct a singular solution to $(\partial_t^2 - \Delta_g)u = 0$. Given $\theta \in S^2$ denote by $v(t,x,\theta)$ the solution (in distribution sense) of the following problem

$$\left\{\begin{array}{rcll} (\partial_t^2 - \Delta_g)v &=& 0 & \text{in } \mathbf{R}\times\mathbf{R}^3,\\ v|_{t\le-\rho} &=& \delta(t - x\cdot\theta). & \end{array}\right. \tag{2.10}$$

One can easily solve (2.10). Given $j = 0, 1, \ldots$, denote

$$h_j(s) = \left\{\begin{array}{ll} s^j/j!, & \text{if } s \ge 0\\ 0, & \text{otherwise.}\end{array}\right. \tag{2.11}$$

Then the following problem has unique solution $w \in H^2_{\rm loc}$ such that $\partial_t w \in H^1_{\rm loc}$.

$$\left\{ \begin{array}{rcll} (\partial_t^2 - \Delta_g) w & = & 0 & \text{in } \mathbf{R} \times \mathbf{R}^3, \\ w|_{t \le -\rho} & = & h_2(t - x\cdot\theta). & \end{array} \right. \tag{2.12}$$

The solution to (2.10) is then given by $v = \partial_t^3 w$. Denote

$$\tau_g = \rho + C_0 \varepsilon, \tag{2.13}$$

where C_0 is the constant in Lemma 2.2.

Proposition 2.2 *Assume that (2.5) holds with $k \ge 9$ and $\varepsilon > 0$ sufficiently small. Then there exists a constant $C > 0$, such that for $|t| < 3\tau_g$, and for any $\theta \in S^2$ we have*

$$v(t,x,\theta) = \alpha(x,\theta)\delta(t - \phi(x,\theta)) + \beta(x,\theta) h_0(t - \phi(x,\theta)) + r(t,x,\theta),$$

where

$$\|\alpha - 1\|_{C^{k-2}(B_{4\rho}\times S^2)} \le C\varepsilon, \qquad \|\beta\|_{C^{k-4}(B_{4\rho}\times S^2)} \le C\varepsilon, \tag{2.14}$$

and

$$\|r(t,\cdot,\theta)\|_{L^\infty} + \|\partial_t r(t,\cdot,\theta)\|_{L^2} \le C\varepsilon. \tag{2.15}$$

Moreover, for $R(t,x,\theta) := \int_{-\infty}^t r(s,x,\theta)ds$ we have

$$\|\nabla R(t,\cdot,\theta)\|_{L^\infty} \le C\varepsilon. \tag{2.16}$$

Proof. We look for a solution v of the form

$$v(t,x,\theta) = \alpha(x,\theta)\delta(t-\phi(x,\theta)) + \beta(x,\theta)h_0(t-\phi(x,\theta)) + \gamma(x,\theta)h_1(t-\phi(x,\theta)) + \tilde r(t,x,\theta).$$

Then $\alpha = 1 + \tilde\alpha$, β, γ solve the transport equations

$$\Big(2\sum_{i,j=1}^3 g^{ij}\frac{\partial\phi}{\partial x_i}\frac{\partial}{\partial x_j} + \Delta_g\phi\Big)\tilde\alpha = -\Delta_g\phi, \qquad \tilde\alpha|_{x\cdot\theta=-\rho} = 0, \tag{2.17}$$

$$\Big(2\sum_{i,j=1}^3 g^{ij}\frac{\partial\phi}{\partial x_i}\frac{\partial}{\partial x_j} + \Delta_g\phi\Big)\beta = \Delta_g\alpha, \qquad \beta|_{x\cdot\theta=-\rho} = 0, \tag{2.18}$$

$$\Big(2\sum_{i,j=1}^3 g^{ij}\frac{\partial\phi}{\partial x_i}\frac{\partial}{\partial x_j} + \Delta_g\phi\Big)\gamma = \Delta_g\beta, \qquad \gamma|_{x\cdot\theta=-\rho} = 0, \tag{2.19}$$

while $\tilde r$ solves

$$(\partial_t^2 - \Delta_g)\tilde r = (\Delta_g\gamma)h_1(t-\phi), \qquad \tilde r|_{t\ll 0} = 0. \tag{2.20}$$

Note that we need to solve (2.17) — (2.19) in the compact $x\cdot\theta \ge -\rho$, $\phi(x,\theta) \le 3\tau_g$, $|\eta| < \rho$ (η is determined by $x = x(s,\eta)$) and for ε sufficiently small this compact is contained in $B_{4\rho}$.

It is easy to see by Lemma 2.1 and Lemma 2.2 that the solutions exist and α, β, γ satisfy the required estimates if $k \geq 6$. Applying standard hyperbolic estimates we see that $\tilde{r}$ is compactly supported with respect to x (uniformly in $\varepsilon < 1$, $|t| < 3\tau_g$) and satisfies

$$\|\tilde{r}\|_{H^2} + \|\partial_t \tilde{r}\|_{H^1} \leq C\varepsilon. \tag{2.21}$$

By the Sobolev embedding theorem this proves (2.15) for $r = \gamma h_1(t-\phi) + \tilde{r}$. In order to prove (2.16) note that (2.21) implies $\|\partial_t^2 R\|_{H^1} \leq C\varepsilon$. Since $\partial_t^2 R = \Delta_g R + (\Delta_g \gamma) h_2(t-\phi)$ and $\|\Delta_g \gamma\|_{C^1(B_{4\rho})} \leq C\varepsilon$ ($k \geq 9$), we get $\Delta_g R \in H^1$ and $\|\Delta_g R\|_{H^1} \leq C\varepsilon$, which implies (2.16). □

3 Moding out the group of diffeomorphisms

Recall that we have the freedom to change the metric $g \to \psi^* g$ without changing the DN map as long as ψ is a diffeomorphism that leaves the boundary fixed pointwise. In particular we shall construct the diffeomorphism as a harmonic function with respect to the Laplace-Beltrami operator Δ_g.

Proposition 3.1 *Suppose $\psi : \bar{\Omega} \to \bar{\Omega}$ solves the problem*

$$\begin{cases} \Delta_g \psi &= 0 \quad \text{in } \Omega, \\ \psi|_{\partial\Omega} &= Id. \end{cases} \tag{3.1}$$

Then if g satisfies (2.4) with ε sufficiently small and $k \geq 2$, ψ is a diffeomorphism and

$$\|\psi - Id\|_{C^{k+2,\mu}(\bar{\Omega})} \leq C\varepsilon$$

with some $C > 0$. Moreover, for $\tilde{g} := \psi^* g$ we have

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} (\det \tilde{g})^{\frac{1}{2}} \tilde{g}^{i\alpha} = 0 \quad \text{in } \Omega, \quad \alpha = 1, 2, 3. \tag{3.2}$$

Proof. For the components ψ_α of ψ we have $\Delta_g \psi_\alpha = 0$, $\psi_\alpha|_{\partial\Omega} = x_\alpha$. Clearly, $\Phi := \psi - Id$ solves

$$\begin{cases} \Delta_g \Phi_\alpha &= -(\det g)^{-\frac{1}{2}} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (\det g)^{\frac{1}{2}} g^{i\alpha} \quad \text{in } \Omega, \\ \Phi_\alpha|_{\partial\Omega} &= 0. \end{cases}$$

Condition (2.4) implies that $\|\Phi\|_{C^{k+2,\mu}(\bar{\Omega})} \leq C\varepsilon$ with some $C > 0$. This in particular implies that for ε small enough the map $\psi = Id + \Phi$ is a diffeomorphism. Let $\tilde{g} := \psi^* g$, where ψ solves (3.1). Under the change of coordinates $x \to \psi(x)$ the operator Δ_g transforms into $\Delta_{\tilde{g}}$, the function ψ transforms into x, therefore $\Delta_{\tilde{g}} x = 0$, or $\Delta_{\tilde{g}} x_\alpha = 0$ for any $\alpha = 1, 2, 3$, which is precisely (3.2). □

Proposition 3.2 *Let g_1, g_2 be two metrics satisfying (2.4) with $\Lambda_{g_1} = \Lambda_{g_2}$. Then there exists a $C^{k+2,\mu}$ diffeomorphism $\psi : \bar{\Omega} \to \bar{\Omega}$ with $\psi|_{\partial\Omega} = Id$, so that $\psi^* g_1 = g_2$ on $\partial\Omega$. Moreover, $\psi = Id + O(\varepsilon)$ in C^{k+2}.*

Proof. This proposition has been proven in [S-U] under the assumption that g_1 and g_2 belong to C^∞ and it is in fact shown that $g_1 = g_2$ of infinite order at the boundary. Under the finite smoothness assumption made here, the proof in [S-U] still works to show that $g_1 = g_2$ on $\partial\Omega$. Indeed, one can construct highly oscillating solutions as in [S-U], not as an infinite series but as a sum of two leading terms plus a remainder that is easy to estimate (very similarly to our construction in Proposition 2.2). Then one gets $g_1 = g_2$ on $\partial\Omega$ by comparing the action of the DN map on the leading terms of those oscillating solutions as in [S-U]. $\square$

Proposition 3.3 *Let g_i, $i = 1, 2$ satisfy the assumptions of Theorem 1.1. Let $\tilde{g}_i = \psi_i^* g$, where ψ_i solves (3.1) with $g = g_i$, $i = 1, 2$. Then if $g_1|_{\partial\Omega} = g_2|_{\partial\Omega}$, we have $\tilde{g}_1|_{\partial\Omega} = \tilde{g}_2|_{\partial\Omega}$.*

Proof. Let $w_i(t, x)$, $i = 1, 2$ solve

$$\begin{cases} (\partial_t^2 - \Delta_{g_i}) w_i &= 0 \qquad \text{in } (0,\infty) \times \Omega, \\ w_i|_{t=0} = \partial_t w_i|_{t=0} &= 0 \qquad \text{in } \Omega, \\ w_i|_{(0,\infty)\times\partial\Omega} &= \chi(t) Id_x, \end{cases} \tag{3.3}$$

where $\chi \in C_0^\infty(\mathbf{R}_+)$, $\int \chi(t) dt = 1$. Since $\Lambda_{g_1} = \Lambda_{g_2}$, we have

$$(\det g)^{\frac{1}{2}} \sum_{i,j=1}^{3} g^{ij} \nu^i \frac{\partial w_1}{\partial x_j} = (\det g)^{\frac{1}{2}} \sum_{i,j=1}^{3} g^{ij} \nu^i \frac{\partial w_2}{\partial x_j} \quad \text{on } (0,\infty) \times \partial\Omega,$$

where $g := g_1 = g_2$ on the boundary. Since for any $t > 0$ the tangential derivatives (with respect to x) of w_i coincide, $i = 1, 2$, we conclude that

$$\nabla_x w_1(t, x) = \nabla_x w_2(t, x), \quad \forall t \geq 0, \quad x \in \partial\Omega. \tag{3.4}$$

Set

$$\Psi_i(x, \lambda) = \int_0^\infty e^{i\lambda t} w_i(t, x)\, dt. \tag{3.5}$$

Since the energy $\|\nabla_x w_i\|_{L^2(\Omega)} + \|\partial_t w_i\|_{L^2(\Omega)}$ is bounded as $t \to \infty$ (in fact it is constant for large t), the distribution Ψ_i is well defined as the Fourier transform of w_i extended as zero for $t < 0$. By (3.1) we get that away from the square roots of the Dirichlet eigenvalues of $-\Delta_{g_i}$ in Ω, the distribution Ψ_i is a smooth (analytic) function of λ solving

$$\begin{cases} (\Delta_{g_i} + \lambda^2)\Psi_i &= 0 \qquad \text{in } \Omega, \\ \Psi_i|_{\partial\Omega} &= \hat{\chi}(\lambda) Id_x, \end{cases}$$

where $\hat{\chi}(\lambda) = \int e^{i\lambda t} \chi(t) dt$. Since $\lambda^2 = 0$ is not a Dirichlet eigenvalue of $-\Delta_{g_i}$, we get that $\Psi_i(\lambda, x)$ is smooth near $\lambda = 0$ and in particular $\psi_i(x) := \Psi(x, 0)$ is well defined and solves (3.1). By (3.4), $\nabla_x \psi_1 = \nabla_x \psi_2$ on $\partial\Omega$ which directly implies that $\tilde{g}_1 = \tilde{g}_2$ on $\partial\Omega$. We would like to mention here that in fact we can deduce that $\tilde{g}_1 = \tilde{g}_2$ on $\partial\Omega$ of order 10. $\square$

4 Proof of Corollary 1.1

Assume that we have two metrics g_1 and g_2 satisfying (2.4) with $\Lambda_{g_1} = \Lambda_{g_2}$. We first apply the results of Section 3. First, according to Proposition 3.2, there exist a diffeomorphism φ which is identity on the boundary, such that $\tilde{g}_1 := \varphi^* g_1$ and $\tilde{g}_2 := g_2$ coincide on the boundary. Next, according to Proposition 3.1, $\tilde{\tilde{g}}_i := \psi_i^* \tilde{g}_i$ satisfy (3.2), where ψ_i solve (3.1), $i = 1, 2$. And finally, since $\tilde{g}_1 = \tilde{g}_2$ on $\partial\Omega$, by Proposition 3.3 we get $\tilde{\tilde{g}}_1 = \tilde{\tilde{g}}_2$ on $\partial\Omega$. Notice that $\tilde{\tilde{g}}_i$ and g_i, $i = 1, 2$ have the same DN maps. Moreover, they satisfy (2.4). In what follows we denote $\tilde{\tilde{g}}_i$ again by g_i, $i = 1, 2$ and we have therefore

$$
\begin{aligned}
\sum_{j=1}^{3} \frac{\partial}{\partial x_i} (\det g_\alpha)^{\frac{1}{2}} g_\alpha^{ij} &= 0 \quad \text{in } \Omega,\ j = 1, 2, 3,\ \alpha = 1, 2, && (4.1)\\
g_1 - g_2 &= 0 \quad \text{on } \partial\Omega. && (4.2)
\end{aligned}
$$

By Proposition 2.1, given $T > 0$ we have

$$
\begin{aligned}
0 = & \int_0^T \int_\Omega \sum_{i,j=1}^{3} \left[(\det g_1)^{\frac{1}{2}} g_1^{ij} - (\det g_2)^{\frac{1}{2}} g_2^{ij} \right] \frac{\partial u_1}{\partial x_i} \frac{\partial u_2}{\partial x_j} \, dx \, dt \\
& - \int_0^T \int_\Omega \left[(\det g_1)^{\frac{1}{2}} - (\det g_2)^{\frac{1}{2}} \right] \frac{\partial u_1}{\partial t} \frac{\partial u_2}{\partial t} \, dx \, dt && (4.3)
\end{aligned}
$$

for any two solutions u_1, u_2 of (2.1). Denote

$$
m_{ij} = (\det g_1)^{\frac{1}{2}} g_1^{ij} - (\det g_2)^{\frac{1}{2}} g_2^{ij} = \gamma_1^{ij} - \gamma_2^{ij}, \tag{4.4}
$$

where $\gamma_\alpha^{ij} := (\det g_\alpha)^{\frac{1}{2}} g_\alpha^{ij}$, $\alpha = 1, 2$. We aim to show that $m = 0$ which would easily imply $g_1 = g_2$. By (4.1), (4.2),

$$
\sum_{i=1}^{3} \frac{\partial m_{ij}}{\partial x_i} = 0, \quad j = 1, 2, 3 \qquad \text{and} \qquad m|_{\partial\Omega} = 0. \tag{4.5}
$$

We have

$$
\det(\gamma_\alpha^{ij}) = (\det g_\alpha)^{\frac{3}{2}} \det(g_\alpha^{ij}) = (\det g_\alpha)^{\frac{1}{2}}.
$$

Thus, $\det g_\alpha = (\det(\gamma_\alpha^{ij}))^2$. For the second integrand in (4.3) we therefore have

$$
(\det g_1)^{\frac{1}{2}} - (\det g_2)^{\frac{1}{2}} = \det(\gamma_1^{ij}) - \det(\gamma_2^{ij}).
$$

Let us denote $\gamma = (\gamma^{ij})$, $\det \gamma = \det(\gamma^{ij})$.

Lemma 4.1

$$
\det \gamma_1 - \det \gamma_2 = \operatorname{tr}(\gamma_1 - \gamma_2) + \sum_{i,j=1}^{3} d_{ij} (\gamma_1^{ij} - \gamma_2^{ij}),
$$

where d_{ij} are polynomials of degree 2 of the entries of $\gamma_1 - Id$, $\gamma_2 - Id$ with no zero-degree terms.

Proof. Denote $\gamma_1^{ij} = \delta_{ij} + a_{ij}$, $\gamma_2^{ij} = \delta_{ij} + b_{ij}$. Let $F(x) := \det(Id + x)$, x being a 3×3 matrix which we can consider as a 9-dimensional vector. Then

$$F(x) - F(y) = (x - y) \cdot \int_0^1 \nabla_x F(tx + (1-t)y)\, dt. \tag{4.6}$$

For $\nabla_x F = \nabla_x \det(Id + x)$ we have

$$\frac{\partial}{\partial x_{i_0,j_0}} \det(Id + x) = (-1)^{i_0+j_0} \det\left((Id + x_{ij})_{i\neq i_0, j\neq j_0}\right).$$

If $i_0 = j_0$, then $\partial \det(Id + x)/\partial x_{i_0,j_0} = 1 + O(|x|)$, where $O(|x|)$ denotes a polynomial containing only linear and quadratic terms, while for $i_0 \neq j_0$ we get $\partial \det(Id + x)/\partial x_{i_0,j_0} = O(|x|)$. Therefore, $\nabla_x F(x) = (\delta_{ij}) + O(|x|)$. By plugging this into (4.6), we get

$$\det(Id + a) - \det(Id + b) = \operatorname{tr}(a - b) + \sum_{i,j=1}^{3} d_{ij}(a_{ij} - b_{ij}),$$

where $d_{ij} = O(|a| + |b|)$. This completes the proof. $\square$

By Lemma 4.1 we see that (4.3) can be rewritten as

$$\int_0^T \int_\Omega \left(\sum_{i,j=1}^{3} m_{ij} \frac{\partial u_1}{\partial x_i} \frac{\partial u_2}{\partial x_j} - \operatorname{tr} m \frac{\partial u_1}{\partial t} \frac{\partial u_2}{\partial t} - \sum_{i,j=1}^{3} d_{ij} m_{ij} \frac{\partial u_1}{\partial t} \frac{\partial u_2}{\partial t} \right) dx\, dt = 0 \tag{4.7}$$

with

$$\|d_{ij}\|_{C^k} = O(\varepsilon). \tag{4.8}$$

We are going to use in (4.7) the solutions u_1 and u_2 to the following problems:

$$\begin{cases} (\partial_t^2 - \Delta_{g_1})u_1 &= 0 \quad \text{in } \mathbf{R} \times \mathbf{R}^3, \\ u_1|_{t\le 0} &= \delta(t - \rho - x\cdot\theta_1), \end{cases} \quad \begin{cases} (\partial_t^2 - \Delta_{g_2})u_2 &= 0 \quad \text{in } \mathbf{R} \times \mathbf{R}^3, \\ u_2|_{t\ge s+2\rho} &= h_0(s - t + \rho - x\cdot\theta_2), \end{cases} \tag{4.9}$$

where g_1 and g_2 are the extended metrics satisfying (2.5). Here $\theta_j \in S^2$, $j = 1, 2$, s are parameters and

$$-2\rho \le s \le T - 2\rho, \tag{4.10}$$

where $T > 0$ will be chosen later. In other words, if v_j denotes the solution to (2.10) with $g = g_j$, $j = 1, 2$, then

$$u_1(t, x, \theta_1) = v_1(t - \rho, x, \theta_1), \qquad u_2(t, x, \theta_2) = V_2(s - t + \rho, x, \theta_2), \tag{4.11}$$

where $V_2(t, x, \theta) = \int_{-\infty}^t v_2(s, x, \theta)ds$. Note that $u_1|_{t=0}$, $\partial_t u_1|_{t=0}$ vanish in B_ρ. Similarly, $u_2|_{t=T}$, $\partial_t u_2|_{t=T}$ vanish in B_ρ, too, provided that (4.10) holds. Therefore, u_1 and u_2 solve

(2.1) with some f_1 and f_2 and we can plug them into (4.7). Since f_1 and f_2 are not H^2-functions as required, we could first integrate sufficient number of times u_1 and u_2 with respect to t and then differentiate back (4.7) with respect to s, thus substituting u_1 and u_2 in (4.7) is correct. From now on, we assume that u_1 and u_2 in (4.7) solve (4.9).

By Proposition 2.2,

$$\begin{aligned} u_1 &= \alpha_1\delta(t-\rho-\phi_1(x,\theta_1)) + \beta_1 h_0(t-\rho-\phi_1(x,\theta_1)) + r_1(t-\rho,x,\theta_1), \\ u_2 &= \alpha_2 h_0(s-t+\rho-\phi_2(x,\theta_2)) + \beta_2 h_1(s-t+\rho-\phi_2(x,\theta_2)) + R_2(s-t+\rho,x,\theta_2), \end{aligned}$$

where $R_2(t,\cdot,\cdot) := \int_{-\infty}^t r_2(s,\cdot,\cdot)ds$. For the first term in (4.7) we get

$$\begin{aligned} \int_0^T\!\!\int_\Omega \sum_{i,j=1}^3 m_{ij}\frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j}\,dx\,dt \;&=\; \int_\Omega \sum_{i,j=1}^3 m_{ij}\Big[\frac{\partial\phi_1}{\partial x_i}\frac{\partial\phi_2}{\partial x_j}\alpha_1\alpha_2\delta'(s-\phi_1-\phi_2) + B_{ij}\delta(s-\phi_1-\phi_2) \\ &\quad + C_{ij} + \int_0^T \partial_{x_i} r_1(t-\rho)\partial_{x_j}R_2(s-t+\rho)\,dt\Big]\,dx. \end{aligned} \tag{4.12}$$

Here $\alpha = \alpha_1(x,\theta_1)$, $\alpha_2 = \alpha_2(x,\theta_2)$, $\phi_1 = \phi_1(x,\theta_1)$, $\phi_2 = \phi_2(x,\theta_2)$, $C_{ij} = C_{ij}(x,s,\theta_1,\theta_2)$, $r_1(t) = r_1(t,x,\theta_1)$, $R_2(t) = R_2(t,x,\theta_2)$. According to (2.14) – (2.16), $\|\alpha_1\alpha_2 - 1\|_{C^{k-2}} = O(\varepsilon)$, $B_{ij} = O(\varepsilon)$ uniformly in θ_1, θ_2 and $\int C_{ij}^2(x,s,\theta_1,\theta_2)dx = O(\varepsilon^2)$ uniformly in s, θ_1, θ_2. Similarly, the last term in (4.12) involving r_1 and R_2 is also an L^2-function of x with norm $O(\varepsilon)$ uniformly in s, θ_1, θ_2.

For the second and the third term in (4.7) we get analogously

$$\begin{aligned} &\int_0^T\!\!\int_\Omega \Big(-\operatorname{tr} m - \sum_{i,j=1}^3 d_{ij}m_{ij}\Big)\frac{\partial u_1}{\partial t}\frac{\partial u_2}{\partial t}\,dx\,dt \\ &\qquad = \int_\Omega \Big(\operatorname{tr} m + \sum_{i,j=1}^3 d_{ij}m_{ij}\Big)\Big[\alpha_1\alpha_2\delta'(s-\phi_1-\phi_2) + B\delta(s-\phi_1-\phi_2) \\ &\qquad\quad + C + \int_0^T \partial_t r_1(t-\rho) r_2(s-t+\rho)\,dt\Big]\,dx, \end{aligned} \tag{4.13}$$

where B, C and the last term in (4.13) have similar properties as above.

Recall the definition (2.13) of τ_g. It is easy to see that $\operatorname{diam}_{g_j}(B_\rho) \le \rho + \tau_{g_j}$, $j = 1,2$. Here g_j denotes the extended metric. Notice that the s-support of $\delta'(s-\phi_1-\phi_2)$ is contained in $s \in [-2\rho, \tau]$, where $\tau := \tau_{g_1} + \tau_{g_2}$. We will choose T so that the latter interval is included in the interval (4.10). To this end we set

$$T_0 = 2\rho + \tau, \tag{4.14}$$

and from now on we assume that $T > T_0$. Notice that $T_0 = 4\rho + O(\varepsilon)$.

By (4.12), (4.13) we see that (4.7) can be rewritten as

$$I_0 = I_1 + I_2, \qquad s \in [-2\rho, \tau], \quad \theta_1 \in S^2, \quad \theta_2 \in S^2, \tag{4.15}$$

where $I_j = I_j(s, \theta_1, \theta_2)$, $j = 0, 1, 2$ are given by

$$I_0 = \int_\Omega \alpha_1\alpha_2\delta'(s - \phi_1 - \phi_2) \sum_{i,j=1}^{3} m_{ij}\Big(\frac{\partial\phi_1}{\partial x_i}\frac{\partial\phi_2}{\partial x_j} + \delta_{ij} + d_{ij}\Big)\, dx, \tag{4.16}$$

$$I_1 = \int_\Omega \sum_{i,j=1}^{3} \tilde{B}_{ij} m_{ij}\delta(s - \phi_1 - \phi_2)\, dx, \tag{4.17}$$

$$I_2 = \int_\Omega \sum_{i,j=1}^{3} \tilde{C}_{ij}(x, s, \theta_1, \theta_2) m_{ij}(x)\, dx \tag{4.18}$$

with $\|d_{ij}\|_{C^k} = O(\varepsilon)$, $\|\tilde{B}_{ij}\|_{C^0} = O(\varepsilon)$, and $\|\tilde{C}_{ij}(\cdot, s, \theta_1, \theta_2)\|_{L^2} = O(\varepsilon)$ uniformly in s, θ_1, θ_2. Notice that I_0 and I_1 are defined for all s but vanish outside $[-2\rho, \tau]$. Therefore, the same is true for I_2.

Let us take the Fourier transform $\hat{I}_0 := \int e^{i\lambda s} I_0 ds$ of I_0 where we have denoted the dual variable of s by λ. Then

$$\hat{I}_0 = -i\lambda F, \tag{4.19}$$

with

$$F = \int_\Omega e^{i\lambda\phi}\alpha_1\alpha_2 \sum_{i,j=1}^{3} \Big(\frac{\partial\phi_1}{\partial x_i}\frac{\partial\phi_2}{\partial x_j} + \delta_{ij} + d_{ij}\Big) m_{ij}\, dx, \tag{4.20}$$

where $\phi := \phi_1(x, \theta_1) + \phi_2(x, \theta_2)$. Notice that ϕ is close to $x \cdot (\theta_1 + \theta_2)$. Given $\xi \in \mathbf{R}^3 \setminus \{0\}$, we are going to choose $\lambda = \lambda(\xi)$, $\theta_1 = \theta_1(\xi)$, $\theta_2 = \theta_2(\xi)$ so that $\lambda(\theta_1 + \theta_2) = \xi$. Then the phase function $\lambda\phi$ will be close to $x \cdot \xi$. Denote by

$$\omega = \frac{\xi}{|\xi|} \in S^2, \qquad r = |\xi| \geq 0$$

the polar coordinates related to ξ. Let $p \in S^2$ be a parameter. Set

$$\theta_1 = \frac{\omega + (-p + (p\cdot\omega)\omega)}{|\omega + (-p + (p\cdot\omega)\omega)|} \in S^2, \qquad \theta_2 = \frac{\omega - (-p + (p\cdot\omega)\omega)}{|\omega - (-p + (p\cdot\omega)\omega)|} \in S^2. \tag{4.21}$$

Notice that $-p + (p\cdot\omega)\omega$ is perpendicular to ω. Further,

$$|\omega \pm (-p + (p\cdot\omega)\omega)|^2 = 2 - (p\cdot\omega)^2 \in [1, 2].$$

We substitute in (4.15)

$$\theta_1 = \theta_1(\omega) = \theta_1\Big(\frac{\xi}{|\xi|}\Big), \qquad \theta_2 = \theta_2(\omega) = \theta_2\Big(\frac{\xi}{|\xi|}\Big)$$

with $\theta_j(\omega)$ as in (4.21). Next, in (4.20) we will set

$$\lambda = \lambda(\xi) = \frac{r}{2}\sqrt{2 - (p\cdot\omega)^2} = \frac{1}{2}\sqrt{2|\xi|^2 - (p\cdot\xi)^2}. \tag{4.22}$$

Notice that a priori $I_j = I(s,\theta_1,\theta_2)$, $F = F(\lambda,\theta_1,\theta_2)$. After the substitution (4.21) we get functions of (s,ω) and (λ,ω), respectively that we will denote by $I_j(s,\omega)$, $F(\lambda,\omega)$. Let us estimate the L^2-norm of $I_0 = I_0(s,\omega)$.

$$\begin{aligned}\int_{S^2}\int_{\mathbf{R}} |I_0(s,\omega)|^2\,ds\,d\omega &= \frac{1}{2\pi}\int_{S^2}\int_{\mathbf{R}} \lambda^2 |F(\lambda,\omega)|^2 d\lambda\,d\omega \\ &= \frac{1}{\pi}\int_{S^2}\int_{\mathbf{R}_+} |F(\lambda(r\omega),\omega)|^2 r^2 \left(\frac{1}{2}\sqrt{2-(p\cdot\omega)^2}\right)^3 dr\,d\omega \\ &= \frac{1}{8\pi}\int \left|F\Big(\lambda(\xi),\frac{\xi}{|\xi|}\Big)\right|^2 \left(2-\Big(\frac{p\cdot\xi}{|\xi|}\Big)^2\right)^{\frac{3}{2}} d\xi. \qquad (4.23)\end{aligned}$$

Let us denote $F(\lambda(\xi),\xi/|\xi|)$ simply by $F(\xi)$. Recall that F depends also on the parameter $p\in S^2$. We have

$$\frac{2^{-\frac{3}{2}}}{\sqrt{\pi}}\|F\|_{L^2(\mathbf{R}^3_\xi)} \le \|I_0\|_{L^2(\mathbf{R}\times S^2)} = \|I_0\|_{L^2([-2\rho,\tau]\times S^2)} \le \frac{2^{-\frac{3}{4}}}{\sqrt{\pi}}\|F\|_{L^2(\mathbf{R}^3_\xi)}. \qquad (4.24)$$

We are going next to estimate the norm of $I_j = I_j(s,\omega)$ in $L^2([-2\rho,\tau]\times S^2)$, $j=0,1,2$. We will show that $c_0\|m\| \le \|I_0\| = \|I_1+I_2\| \le c_1\varepsilon\|m\|$ with c_0, c_1 independent of m, p and ε, whence $m=0$.

To estimate $\|I_0\|$, it suffices by (4.24) to estimate the L^2-norm of F. Denote

$$\varphi(x,\xi) = \lambda(\xi)\Big(\phi_1\Big(x,\theta_1\Big(\frac{\xi}{|\xi|}\Big)\Big) + \phi_2\Big(x,\theta_2\Big(\frac{\xi}{|\xi|}\Big)\Big)\Big).$$

Thus (4.20) can be rewritten as

$$F(\xi) = \int_\Omega e^{i\varphi(x,\xi)}\alpha_1\alpha_2 \sum_{i,j=1}^3 \Big(\frac{\partial\phi_1}{\partial x_i}\frac{\partial\phi_2}{\partial x_j} + \delta_{ij} + d_{ij}\Big) m_{ij}\,dx \qquad (4.25)$$

with

$$\alpha_j = \alpha_j\Big(x,\theta_j\Big(\frac{\xi}{|\xi|}\Big)\Big), \qquad \phi_j = \phi_j\Big(x,\theta_j\Big(\frac{\xi}{|\xi|}\Big)\Big), \qquad j=1,2.$$

We introduce next the following class S^m_k of symbols. We say that $a = a(x,\xi) \in C^k(B_\rho\times \mathbf{R}^3\setminus\{0\})$ belongs to S^m_k iff there exists a constant $C\ge 0$, such that

$$\Big|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\Big| \le C|\xi|^{m-|\beta|} \qquad \text{for } x\in B_\rho,\quad \xi\in\mathbf{R}^3\setminus\{0\},\quad |\alpha|+|\beta|\le k. \qquad (4.26)$$

The optimal constant in (4.26) defines a norm in S^m_k. We say that $a = O(\varepsilon)$ in S^m_k iff $a\in S^m_k$ and the S^m_k-norm of a is $O(\varepsilon)$, in other words (4.26) holds with C replaced by $C\varepsilon$.

By Lemma 2.2 we have

$$\varphi(x,\xi) = x\cdot\xi + O(\varepsilon) \qquad \text{in } S^1_k. \qquad (4.27)$$

In (4.25) we have also

$$\begin{aligned}
\alpha_1\alpha_2 &= 1+O(\varepsilon) \qquad \text{in } S^0_{k-2},\\
\frac{\partial\phi_1}{\partial x_i} &= \frac{\xi_i+\left(-|\xi|p_i+\frac{p\cdot\xi}{|\xi|}\xi_i\right)}{\sqrt{2|\xi|^2-(p\cdot\xi)^2}}+O(\varepsilon) \qquad \text{in } S^0_{k-1},\\
\frac{\partial\phi_2}{\partial x_j} &= \frac{\xi_j-\left(-|\xi|p_j+\frac{p\cdot\xi}{|\xi|}\xi_j\right)}{\sqrt{2|\xi|^2-(p\cdot\xi)^2}}+O(\varepsilon) \qquad \text{in } S^0_{k-1},\\
d_{ij} &= O(\varepsilon) \qquad \text{in } S^0_k.
\end{aligned}$$

Proposition 4.1 *Let P denote the operator*

$$(Pf)(\xi)=\int_\Omega e^{i\varphi(x,\xi)}a(x,\xi)f(x)\,dx,$$

where $\varphi(x,\xi)$ is homogeneous of order 1 in ξ and for $x\in B_\rho$, $\xi\neq 0$ we have

$$\begin{aligned}
|\partial_x^\alpha\partial_\xi^\beta(\varphi(x,\xi)-x\cdot\xi)| &\le A\varepsilon|\xi|^{1-|\beta|}, \quad |\alpha|+|\beta|\le 9,\\
|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| &\le M|\xi|^{-|\beta|}, \quad |\alpha|+|\beta|\le 7
\end{aligned}$$

with some $A>0$, $M>0$. Then for $\varepsilon>0$ sufficiently small $P: L^2(\Omega)\to L^2(\mathbf{R}^3_\xi)$ is bounded and

$$\|Pf\|_{L^2(\mathbf{R}^3_\xi)}\le C_0M\|f\|_{L^2(\Omega)}$$

with $C_0=C_0(A)$. If, in addition, $a=1+O(\varepsilon)$ in S^0_7, then for $\varepsilon>0$ small enough

$$\frac{(2\pi)^3}{2}\|f\|_{L^2(\Omega)}\le\|Pf\|_{L^2(\mathbf{R}^3_\xi)}. \tag{4.28}$$

Proof. Proposition 4.1 was proven in [St-U]. For the sake of completeness below we will recall the proof. Consider P^*P. We have

$$(P^*Pf)(x)=\iint e^{-i(\varphi(x,\xi)-\varphi(y,\xi))}\overline{a(x,\xi)}a(y,\xi)f(y)\,dy\,d\xi. \tag{4.29}$$

The phase function above admits the representation

$$\varphi(x,\xi)-\varphi(y,\xi)=(x-y)\cdot\eta(x,y,\xi),$$

where

$$\eta(x,y,\xi)=\int_0^1(\nabla_x\varphi)(y+t(x-y),\xi)\,dt. \tag{4.30}$$

Here η is a homogeneous function of ξ of order 1. Let us extend the definition (4.26) of S^m_k to amplitudes $a(x,y,\xi)$ depending on y as well by replacing (4.26) by $|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma a(x,y,\xi)|\le$

$C|\xi|^{m-|\gamma|}$, $x\in B_\rho$, $y\in B_\rho$, $\xi\neq 0$, $|\alpha|+|\beta|+|\gamma|\le k$. Then $\eta=\xi+O(\varepsilon)$ in S^1_8. The equation $\eta=\eta(x,y,\xi)$ can be solved for ξ for ε small enough. The Jacobian $J:=|D\eta/D\xi|$ satisfies $J=1+O(\varepsilon)$ in S^0_7 and moreover, J is homogeneous in ξ. After the change of variables $\xi\to\eta$ in (4.29) we get

$$P^*Pf=\iint e^{-i(x-y)\cdot\eta}b(x,y,\eta)f(y)\tilde J(x,y,\eta)\,dy\,d\eta, \tag{4.31}$$

where $\tilde J(x,y,\eta)=J^{-1}(x,y,\xi)|_{\xi=\xi(x,y,\eta)}$, $b(x,y,\eta)=\overline{a(x,\xi)}a(y,\xi)|_{\xi=\xi(x,y,\eta)}$. Clearly, $b\tilde J\in S^0_7$ with norm $C(A)M^2$. We are in a position now to apply to (4.31) Theorem A.1 in [St-U], saying that $a(x,y,D)$ is bounded in L^2 with norm not exceeding CM, if $\int|\partial_x^\alpha\partial_y^\beta a(x,y,\xi)|dxdy\le M$, $|\alpha|+|\beta|\le 7$. This theorem is a straightforward generalization of a similar result for operators $a(x,D)$ (see Theorem 18.1.11$'$ in [H I]). More precisely, we apply the above mentioned theorem to the operator with amplitude $\chi(x)b(x,y,\eta)\tilde J(x,y,\eta)\chi(y)$, where $\chi\in C_0^\infty$, $\chi=1$ in Ω, $\chi=0$ outside B_ρ. This yields the first part of the proposition.

To prove the second assertion, notice that if $a=1+O(\varepsilon)$ in S^0_7, then $\tilde Jb=1+O(\varepsilon)$ in S^0_7 because we have the same for $\tilde J$. Therefore,

$$\|P^*P-(2\pi)^3Id\|_{\mathcal{L}(L^2(B_\rho))}\le C\varepsilon,$$

which yields immediately (4.28) for $\varepsilon>0$ small enough. □

By Proposition 4.1 and (4.25), F can be represented as $F=Pm$, where P is an operator as above (acting on matrix-valued functions). The amplitude a_{ij} is homogeneous in ξ of order 0, belongs to S^0_{k-2} and

$$a_{ij}=\left(2-\left(\frac{p\cdot\xi}{|\xi|}\right)^2\right)^{-1}\left[\left(1+\frac{p\cdot\xi}{|\xi|}\right)\frac{\xi_i}{|\xi|}-p_i\right]\left[\left(1-\frac{p\cdot\xi}{|\xi|}\right)\frac{\xi_j}{|\xi|}+p_j\right]+\delta_{ij}+O(\varepsilon)\qquad\text{in } S^0_{k-2}.$$

If $k\ge 9$, then by Proposition 4.1,

$$\begin{aligned} F &= \int_\Omega e^{i\varphi}\sum_{i,j=1}^3\left(\left[2-\left(\frac{p\cdot\xi}{|\xi|}\right)^2\right]^{-1}\left[\left(1+\frac{p\cdot\xi}{|\xi|}\right)\frac{\xi_i}{|\xi|}-p_i\right]\right.\\ &\quad\left.\times\left[\left(1-\frac{p\cdot\xi}{|\xi|}\right)\frac{\xi_j}{|\xi|}+p_j\right]+\delta_{ij}\right)m_{ij}\,dx+O(\varepsilon\|m\|)\qquad\text{in } L^2(\mathbf{R}^3_\xi).\end{aligned}$$

Using the fact that $\partial\varphi/\partial x_j=\xi_j+O(\varepsilon)$ in S^1_{k-1} and $m=0$ on the boundary (see (4.5)), we get

$$\begin{aligned}\frac{\xi_j}{|\xi|}\int_\Omega e^{i\varphi}m_{ij}\,dx &= \frac{1}{|\xi|}\int_\Omega e^{i\varphi}\frac{\partial\varphi}{\partial x_j}m_{ij}\,dx+O(\varepsilon\|m\|)\quad\text{in } L^2(\mathbf{R}^3_\xi)\\ &= -\frac{i}{|\xi|}\int_\Omega e^{i\varphi}\frac{\partial m_{ij}}{\partial x_j}\,dx+O(\varepsilon\|m\|)\quad\text{in } L^2(\mathbf{R}^3_\xi).\end{aligned}\tag{4.32}$$

Since by (4.5), $\sum_{i=1}^3 \partial m_{ij}/\partial x_i = 0$, $j=1,2,3$, we get

$$F = \int_\Omega e^{i\varphi}\Bigg(- \left[2 - \left(\frac{p\cdot\xi}{|\xi|}\right)^2\right]^{-1} \sum_{i,j=1}^3 m_{ij}p_ip_j + \operatorname{tr} m\Bigg)\,dx + O(\varepsilon\|m\|) \quad \text{in } L^2(\mathbf{R}^3_\xi). \tag{4.33}$$

Moreover, Proposition 4.1 allows us to conclude that the estimate on the remainder above is uniform in $p \in S^2$. By (4.24),

$$\|F_0\|_{L^2(\mathbf{R}^3)} \le C\|I_0\|_{L^2([-2\rho,\tau]\times S^2)} + O(\varepsilon\|m\|), \tag{4.34}$$

where F_0 denotes the integral term in (4.33).

Let us estimate now the norm of $I_1 = I_1(s,\omega)$.

$$\|I_1\|_{L^2([-2\rho,\tau]\times S^2)} \le C\varepsilon\| \int |m|\delta(s-\phi)\,dx\|_{L^2(\mathbf{R}\times S^2)},$$

where $\phi = \phi_1 + \phi_2$, $\phi_j = \phi_j(x,\theta_j(\omega))$, $j=1,2$ (see (4.21)). Since for any $f \in C^1(\mathbf{R})$ with $f=0$ outside $[-2\rho,\tau]$ we have $\|f\|_{L^2} \le C\|f'\|_{L^2}$, after approximating $|m| = (\sum_{ij}|m_{ij}|^2)^{1/2}$ with smooth functions, we get

$$\|I_1\|_{L^2([-2\rho,\tau]\times S^2)} \le C'\varepsilon\| \int |m|\delta'(s-\phi)\,dx\|_{L^2(\mathbf{R}\times S^2)}.$$

The integral above has a form similar to that of I_0 (see (4.16)) and therefore the analysis of I_0 yields

$$\|I_1\|_{L^2([-2\rho,\tau]\times S^2)} \le C''\varepsilon\|m\|. \tag{4.35}$$

And finally, for I_2 we have

$$\|I_2\|_{L^2([-2\rho,\tau]\times S^2)} \le C\varepsilon\|m\| \tag{4.36}$$

because (see (4.18)) I_2 is obtained from m by applying a Hilbert-Schmidt operator with kernel $\tilde{C}_{ij}$ having L^2-norm of the kind $O(\varepsilon)$, uniformly in the parameter $p \in S^2$.

Combining (4.15), (4.34) – (4.36) we obtain $F_0 = O(\varepsilon\|m\|)$ in L^2, in other words,

$$\int_\Omega e^{i\varphi(x,\xi)} \left(\sum_{i,j=1}^3 m_{ij}(x)p_ip_j - \left(2 - \left(\frac{p\cdot\xi}{|\xi|}\right)^2\right)\operatorname{tr} m(x)\right) dx = O(\varepsilon\|m\|) \quad \text{in } L^2(\mathbf{R}^3_\xi). \tag{4.37}$$

Recall that φ depends on $p \in S^2$ as well. As in the proof of Proposition 4.1 (we need here $k \ge 9$), let us apply the operator P^* to (4.37) to get

$$\iint_\Omega e^{i(x-y)\cdot\eta} \left(\sum_{i,j=1}^3 m_{ij}(y)p_ip_j - \left(\left(2 - \left(\frac{p\cdot\eta}{|\eta|}\right)^2\right)\operatorname{tr} m(y)\right) dy\,d\eta = O(\varepsilon\|m\|) \quad \text{in } L^2(\Omega_x). \tag{4.38}$$

Here, as in the proof of Proposition 4.1 we have made the change $\varphi(x,\xi)-\varphi(y,\eta)=(x-y)\cdot\eta(x,y,\xi)$, $\eta=\xi+O(\varepsilon)$ in S^1_{k-1} and used that fact that $\tilde J(x,y,\eta)=1+O(\varepsilon)$ in S^0_{k-2}. We can choose now successfully $p=e_1,e_2,e_3$ and sum up the corresponding equalities (4.38) to get

$$-4\iint_\Omega e^{i(x-y)\cdot\eta}\mathrm{tr}\, m(y)\,dy\,d\eta=O(\varepsilon\|m\|)\quad\text{in } L^2(\mathbf{R}^3_x).$$

In other words,

$$\|\mathrm{tr}\, m\|=O(\varepsilon\|m\|).$$

Going back to (4.38) we obtain

$$\sum_{i,j=1}^3 m_{ij}p_ip_j=O(\varepsilon\|m\|)\quad\text{in } L^2(\Omega),\ \forall p\in S^2.$$

Setting $p=e_1,e_2,e_3$, we get

$$\|m_{ii}\|=O(\varepsilon\|m\|),\quad i=1,2,3.$$

Setting $p=\frac{1}{\sqrt2}(e_i+e_j)$, $i\neq j$, we get

$$\|m_{ij}\|=O(\varepsilon\|m\|),\quad i\neq j.$$

Therefore, $\|m\|=O(\varepsilon\|m\|)$ which yields $m=0$ for ε sufficiently small and $k=9$ in (2.4). Going back to the notations at the beginning of this section, we see that $(\varphi_1\psi_1)^*g_1=(\varphi_2\psi_2)^*g_2$, therefore $(\psi_2^{-1}\varphi_2^{-1}\varphi_1\psi_1)^*g_1=g_2$. This completes the proof of Corollary 1.1.

5 The stability estimate

In this section we prove Theorem 1.1. First we need the following geometrical optics solution. For more details we refer to [CP].

Fix $(t^0,x^0)\in(0,\infty)\times\partial\Omega$ with t^0 sufficiently small and let $\chi\in C_0^\infty((0,\infty)\times\partial\Omega)$ be a cut-off function such that $\chi=1$ near (t^0,x^0). Then there exists a solution u of (1.1) that near (t^0,x^0) has the form

$$u=e^{i\lambda(t-\phi(x,\omega))}(A(x,\omega)+v(t,x,\omega,\lambda)),\tag{5.1}$$

where $\lambda>0$ is a large parameter, $\sum_{i,j=1}^3 g^{ij}(x^0)\omega_i\omega_j=1$, $\omega\cdot\nu(x^0)<0$ and

$$\|v(t,\cdot,\omega,\lambda)\|_{H^2}\le\frac{C}{\lambda}.\tag{5.2}$$

The phase function solves (in a neighborhood of x^0) the eikonal equation

$$\left\{\begin{array}{rcl}\sum_{i,j=1}^3 g^{ij}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j} &=& 1,\\ \phi|_{\partial\Omega} &=& x\cdot\omega,\\ \frac{\partial\phi}{\partial\nu}|_{\partial\Omega} &<& 0.\end{array}\right.\tag{5.3}$$

Recall that ν is the outer normal to $\partial\Omega$ and the third equation above implies that $\nabla\phi$ points into Ω. Since ω is not tangent to $\partial\Omega$ near x^0, (5.3) is non-characteristic and therefore well posed. For the amplitude A we have $A = \chi(t,x)$ for $x \in \partial\Omega$ and A solves the standard transport equations.

The construction of u is very similar to that of the solution v in Proposition 2.2 (see also [CP], [S-U]). First we construct a local solution as in Proposition 2.2. Then we extend g smoothly near $\partial\Omega$ such that $g = e$ outside a small neighborhood of $\partial\Omega$ and g satisfies (2.4) with $k = 9$. We propagate then the local solution backwards to $t = 0$, cut off the so obtained initial data so that it is zero in Ω and propagate forward.

In [S-U] it is shown that if two metrics have the same DN maps, they coincide at the boundary in suitable coordinates. We will adapt that proof to show a continuous dependence on the boundary. Let us define boundary normal coordinates near $\partial\Omega$ as follows. For x sufficiently close to the boundary, set $x_3 = \operatorname{dist}_g(x, \partial\Omega)$. If $x' := (x_1, x_2)$ are local coordinates on $\partial\Omega$, then in the new coordinates

$$\sum_{i,j=1}^{3} g^{ij}\xi_i\xi_j = \sum_{i,j=1}^{2} g^{ij}\xi_i\xi_j + \xi_3^2. \tag{5.4}$$

Suppose that we have two metrics g_1 and g_2 satisfying the assumptions of Theorem 1.1. Fix $x^0 \in \partial\Omega$ and let N_k be a local diffeomorphism mapping the original coordinates into its normal coordinates (x', x_3), corresponding to the metrics g_k, $k = 1, 2$. Set $h_k = N_k^* g_k$, $k = 1, 2$. Then h_k satisfies (5.4).

Proposition 5.1

$$\|h_1 - h_2\|_{L^\infty(\mathcal{O})} \le C\|\Lambda_{g_1} - \Lambda_{g_2}\|_*,$$

where $\mathcal{O}$ is a small neighborhood of x^0.

Proof. Let u_1, u_2 be the solution (5.1) associated with h_1, h_2 respectively defined in a neighborhood of (t^0, x^0) with some $t^0 > 0$. For (t,x) close to (t^0, x^0) we have

$$\begin{aligned}\Lambda_{h_k} u_k &= i\lambda e^{i\lambda(t - x\cdot\omega)} (\det h_k)^{\frac{1}{2}} \Big(\sum_{i,j=1}^{3} h_k^{ij} \nu_i \frac{\partial\phi_k}{\partial x_j} + O(\lambda^{-1}) \Big) \quad \text{in } H^{\frac{1}{2}}(\partial\Omega) \\ &= i\lambda e^{i\lambda(t - x\cdot\omega)} \Big((\det h_k)^{\frac{1}{2}} \frac{\partial\phi_k}{\partial x_3} + O(\lambda^{-1}) \Big) \quad \text{in } H^{\frac{1}{2}}(\partial\Omega),\end{aligned} \tag{5.5}$$

$k = 1, 2$. Let us choose $f \in C_0^\infty(\mathbf{R}_+ \times \partial\Omega)$ supported near (t^0, x^0), such that $\operatorname{supp} g \subset \{(t,x);\ \chi(t,x) = 1\}$ and consider

$$G(\lambda) = \frac{1}{i\lambda} \int_{\mathbf{R}_+ \times \partial\Omega} e^{-i\lambda(t - x\cdot\omega)} \left(\Lambda_{h_1} u_1 - \Lambda_{h_2} u_2 \right) f \, dt \, dS_x.$$

By (5.5),

$$\lim_{\lambda\to\infty} G(\lambda) = \int_{\mathbf{R}_+\times\partial\Omega} \sum_{i,j=1}^{3} \left((\det h_1)^{\frac{1}{2}} \frac{\partial\phi_1}{\partial x_3} - (\det h_2)^{\frac{1}{2}} \frac{\partial\phi_2}{\partial x_3} \right) f\, dt\, dS_x. \tag{5.6}$$

On the other hand,

$$|G(\lambda)| \le \frac{1}{\lambda} \|\Lambda_{h_1} - \Lambda_{h_2}\|_* \|u_1\|_{H^1} \|f\|_{L^2} + \frac{1}{\lambda} \|\Lambda_{h_2}\|_* \|u_1 - u_2\|_{H^1} \|f\|_{L^2},$$

where $\|\cdot\|_{H^1}$, $\|\cdot\|_{L^2}$ are the norms over $(\mathbf{R}_+\times\partial\Omega)\cap \operatorname{supp} f$. Since $\|u_1\|_{H^1} = O(\lambda)$, $\|u_1 - u_2\|_{H^1} = O(1)$ uniformly with respect to g_1, g_2 satisfying the assumptions of Theorem 1.1, we get

$$|G(\lambda)| \le C \left(\|\Lambda_{h_1} - \Lambda_{h_2}\|_* + O(\lambda^{-1}) \right) \|f\|_{L^2}. \tag{5.7}$$

Combining (5.6), (5.7), we get

$$\left| \int_{\mathbf{R}_+\times\partial\Omega} \left((\det h_1)^{\frac{1}{2}} \frac{\partial\phi_1}{\partial x_3} - (\det h_2)^{\frac{1}{2}} \frac{\partial\phi_2}{\partial x_3} \right) f\, dt\, dS_x \right| \le C \|\Lambda_{h_1} - \Lambda_{h_2}\|_* \|f\|_{L^2}. \tag{5.8}$$

The eikonal equation implies that on $\partial\Omega$

$$\frac{\partial\phi_k}{\partial x_3} = \left(1 - \sum_{i,j=1}^{2} h_k^{ij} \omega_i \omega_j \right)^{\frac{1}{2}}.$$

Picking suitable values of ω and bearing in mind that (5.8) holds for any $f \in L^2(\mathbf{R}_+\times\partial\Omega)$ supported near (t^0, x^0), we complete the proof of the proposition. $\square$

By Proposition 5.1, we have the same stability estimate at the boundary for g_1 and $(N_1^{-1}N_2)^* g_2$. Choosing a partition of unity, we get

Proposition 5.2

$$\|\tilde{g}_1 - \tilde{g}_2\|_{L^\infty(\partial\Omega)} \le C \|\Lambda_{g_1} - \Lambda_{g_2}\|_*,$$

where $\tilde{g}_1 = g_1$, $\tilde{g}_2 = \varphi^* g_2$ *and* $\|\varphi - Id\|_{C^{11}} \le C\varepsilon$.

We need here a modification of Proposition 3.1.

Proposition 5.3 *Suppose* $\psi : \bar{\Omega} \to \bar{\Omega}$ *solves the problem*

$$\begin{cases} (-\Delta_g + \varepsilon)\psi = 0 & \text{in } \Omega, \\ \psi|_{\partial\Omega} = Id. \end{cases} \tag{5.9}$$

Then if g *satisfies (2.4) with* $\varepsilon > 0$ *sufficiently small and* $k \ge 2$, ψ *is a diffeomorphism and*

$$\|\psi - Id\|_{C^{k+2,\mu}(\bar{\Omega})} \le C\varepsilon \tag{5.10}$$

with some $C > 0$. Moreover, for $\tilde{g} := \psi^* g$ we have

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} (\det \tilde{g})^{\frac{1}{2}} \tilde{g}^{i\alpha} = \varepsilon x_\alpha (\det \tilde{g})^{\frac{1}{2}} \quad \textit{in } \Omega, \quad \alpha = 1,2,3. \tag{5.11}$$

Proof. As before, denote $\Phi := \psi - Id$. Then

$$\begin{cases} (-\Delta_g + \varepsilon)\Phi_\alpha &= (\det g)^{-\frac{1}{2}} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (\det g)^{\frac{1}{2}} g^{i\alpha} - \varepsilon x_\alpha \quad \text{in } \Omega, \\ \Phi_\alpha|_{\partial\Omega} &= 0. \end{cases}$$

Applying standard elliptic estimates, we get (5.10). Next, since $(-\Delta_{\tilde{g}} + \varepsilon)\Phi = 0$, we get $(-\Delta_g + \varepsilon) Id = 0$, which implies (5.11). □

We prove next an analogue of Proposition 3.3.

Proposition 5.4 *Let g_1, g_2 satisfy the assumptions of Theorem 1.1. Let $\tilde{\tilde{g}}_k = \psi_k^* \tilde{g}_k$, where $\tilde{g}_k$, $k = 1,2$ are as in Proposition 5.2 and ψ_k, $k = 1,2$ solve (5.9). Then*

$$\|\tilde{\tilde{g}}_1 - \tilde{\tilde{g}}_2\|_{L^2(\partial\Omega)} \le C \left(\|\Lambda_{g_1} - \Lambda_{g_2}\|_* + \|\Lambda_{g_1} - \Lambda_{g_2}\|_{**} \right). \tag{5.12}$$

Proof. Let w_k, $k = 1,2$ solve (3.3) as in the proof of Proposition 3.3 with $\chi \in C_0^\infty(\mathbf{R}_+)$ such that $\int_0^\infty e^{-\sqrt{\varepsilon}t} \chi(t) dt = 1$. Define $\Psi_k(x, \lambda)$ by (3.5) and set $\psi_k(x) = \Psi_k(x, i\sqrt{\varepsilon})$, i.e.

$$\psi_k(x) = \int_0^\infty e^{-\sqrt{\varepsilon}t} w_k(t,x) \, dt,$$

$k = 1,2$. Then ψ_k solve (5.9). We have

$$\begin{aligned} \Big\| (\det \tilde{g}_1)^{\frac{1}{2}} \sum_{i,j=1}^{3} \tilde{g}_1^{ij} \nu_i \frac{\partial \psi_1}{\partial x_j} - (\det \tilde{g}_2)^{\frac{1}{2}} \sum_{i,j=1}^{3} \tilde{g}_2^{ij} \nu_i \frac{\partial \psi_2}{\partial x_j} \Big\|_{L^2(\partial\Omega)} & \\ \le \Big\| \int e^{-\sqrt{\varepsilon}t} (\Lambda_{g_1} - \Lambda_{g_2}) \chi(t) Id_x \, dt \Big\|_{L^2(\partial\Omega)} & \\ \le C \|\Lambda_{g_1} - \Lambda_{g_2}\|_{**}. & \end{aligned}$$

Using Proposition 5.2 and the fact that the tangential derivatives of $\tilde{g}_1$ and $\tilde{g}_2$ coincide, we get

$$\|\tilde{g}_1 - \tilde{g}_2\|_{L^2(\partial\Omega)} \le C \|\Lambda_{g_1} - \Lambda_{g_2}\|_{**}$$

which implies Proposition 5.4. □

We are ready now to begin with the proof of Theorem 1.1. Let g_1, g_2 satisfy the assumptions of Theorem 1.1. We define new metrics $\tilde{\tilde{g}}_1$ and $\tilde{\tilde{g}}_2$ as in Proposition 5.4 and in order to simplify the notations we denote them again by g_1, g_2. Then g_1, g_2 still satisfy the smallness assumption of Theorem 1.1. With m as in (4.4) we get by (5.11), (5.12),

$$\sum_{i,j=1}^{3} \frac{\partial m_{ij}}{\partial x_i} = O(\varepsilon\|m\|) \quad \text{in } L^2, \quad j=1,2,3, \qquad \text{and} \quad \|m|_{\partial\Omega}\|_{L^2(\partial\Omega)} \le C\delta, \tag{5.13}$$

where

$$\delta := \|\Lambda_{g_1} - \Lambda_{g_2}\|_* + \|\Lambda_{g_1} - \Lambda_{g_2}\|_{**}.$$

Instead of (4.3) we have

$$\begin{aligned} \int_0^T \int_{\partial\Omega} u_2(\Lambda_{g_1} - \Lambda_{g_2}) dS_x dt &= \int_0^T \int_\Omega \sum_{i,j=1}^{3} \left[(\det g_1)^{\frac12} g_1^{ij} - (\det g_2)^{\frac12} g_2^{ij}\right] \frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j}\, dx\, dt \\ &\quad - \int_0^T \int_\Omega \left[(\det g_1)^{\frac12} - (\det g_2)^{\frac12}\right] \frac{\partial u_1}{\partial t}\frac{\partial u_2}{\partial t}\, dx\, dt \end{aligned} \tag{5.14}$$

with u_1, u_2 as in (4.11). Here the left hand side is treated in distribution sense. With the notations of Section 4 (see (4.15) – (4.18)), (5.14) can be rewritten as

$$\int_0^T \int_{\partial\Omega} u_2(\Lambda_{g_1} - \Lambda_{g_2})u_1\, dS_x dt = I_0 - I_1 - I_2, \tag{5.15}$$

where $I_j = I_j(s, \theta_1, \theta_2)$, $j = 0, 1, 2$. Let us set $\theta_1 = \theta_1(\omega, p)$, $\theta_2 = \theta_2(\omega, p)$ as in (4.21) with $p \in S^2$ a parameter. Then I_j will depend on s, ω (and p) and we denote for simplicity the new function by $I_j(s, \omega)$ as before. Denote by $U_1(t, x, \omega)$ the solution to the first problem in (4.9) with δ replaced by h_1, thus in particular $\partial_t^2 U_1 = u_1$. Then we get from (5.15)

$$\partial_s^2 \int_0^T \int_{\partial\Omega} u_2(\Lambda_{g_1} - \Lambda_{g_2})U_1\, dS_x dt = I_0 - I_1 - I_2, \quad \forall (s, \omega). \tag{5.16}$$

We have for any s, ω

$$\left| \int_0^T \int_{\partial\Omega} u_2(\Lambda_{g_1} - \Lambda_{g_2})U_1\, dS_x dt \right| \le \|u_2|_{\partial\Omega}\|_{L^2([0,T]\times\partial\Omega)} \|\Lambda_{g_1} - \Lambda_{g_2}\|_* \|U_1|_{\partial\Omega}\|_{H^1([0,T]\times\partial\Omega)}. \tag{5.17}$$

It follows from Proposition 2.2 that $\|u_2|_{\partial\Omega}\|_{L^2([0,T]\times\partial\Omega)}$, $\|U_1|_{\partial\Omega}\|_{H^1([0,T]\times\partial\Omega)}$ are uniformly bounded for small ε. Let us take Fourier transform $\mathcal{F}_{s\to\lambda}$ of both sides of (5.16)

$$-\lambda^2 \mathcal{F}_{s\to\lambda} \int_0^T \int_{\partial\Omega} u_2(\Lambda_{g_1} - \Lambda_{g_2})U_1\, dS_x dt = \hat{I}_0 - \hat{I}_1 - \hat{I}_2.$$

By (5.17),

$$\left\| \mathcal{F}_{s\to\lambda} \int_0^T \int_{\partial\Omega} u_2(\Lambda_{g_1} - \Lambda_{g_2})U_1\, dS_x dt \right\|_{L^2(\mathbf{R}_\lambda \times S^2_\omega)} \le \|\Lambda_{g_1} - \Lambda_{g_2}\|_*.$$

Fix $R > 0$. Then

$$\|\hat{I}_0\|_{L^2([-R,R]\times S^2_\omega)} \leq C\|I_1 + I_2\|_{L^2([-2\rho,\tau]\times S^2)} + CR^2\|\Lambda_{g_1} - \Lambda_{g_2}\|_*.$$

By (4.35), (4.36),

$$\|\hat{I}_0\|_{L^2([-R,R]\times S^2_\omega)} \leq C\varepsilon\|m\| + CR^2\|\Lambda_{g_1} - \Lambda_{g_2}\|_*. \tag{5.18}$$

Reasoning as in Section 4 (see (4.19), (4.23), (4.24)), we get

$$\|\hat{I}_0\|_{L^2([-R,R]\times S^2_\omega)} = \int_{S^2}\int_{-R}^{R} \lambda^2 |F(\lambda,\omega)|^2 d\lambda\, d\omega \geq C\|F\|_{L^2(B_\rho)}, \tag{5.19}$$

where we denote as before $F(\xi) := F(\lambda(\xi), \xi/|\xi|)$. In this case m does not necessarily vanish on $\partial\Omega$, as in Section 4 and instead we have (5.13). Nevertheless, this is enough to show as in (4.32) that

$$\sum_{i=1}^{3} \frac{\xi_i}{|\xi|}\int_\Omega e^{i\varphi} m_{ij}\, dx = O(\varepsilon\|m\| + R^{\frac{1}{2}}\delta) \quad \text{in } L^2(B_R),\ j = 1,2,3.$$

So, (4.33) remains valid in our case and similarly to (4.34) one gets from (5.18), (5.19)

$$\|F_0\|_{L^2(B_R)} \leq C(\varepsilon\|m\| + R^2\delta)$$

with F_0 as in (4.33), (4.34). Similarly to (4.37),

$$\chi_R(\xi)\int_\Omega e^{i\varphi(x,\xi)}\left(\sum_{i,j=1}^{3} m_{ij}(x)p_ip_j - \left(2 - \left(\frac{p\cdot\xi}{|\xi|}\right)^2\right)\operatorname{tr} m(x)\right)dx = O(\varepsilon\|m\| + R^2\delta) \ \text{ in } L^2(\mathbf{R}^3_\xi), \tag{5.20}$$

where $\chi_R(\xi) = 1$ for $|\xi| \leq R$, $\chi_R(\xi) = 0$ otherwise. As in (4.38), let us apply P^* to (5.20) to get

$$\begin{aligned}\iint_\Omega e^{i(x-y)\cdot\eta}\chi_R(\xi(\eta,x,y))\left(\sum_{i,j=1}^{3} m_{ij}(y)p_ip_j - \left(\left(2 - \left(\frac{p\cdot\eta}{|\eta|}\right)^2\right)\operatorname{tr} m(y)\right)dy\, d\eta \\ = O(\varepsilon\|m\| + R^2\delta) \ \text{ in } L^2(\Omega_x).\end{aligned} \tag{5.21}$$

Therefore,

$$\begin{aligned}\iint_\Omega e^{i(x-y)\cdot\eta}\chi_R(\eta)\left(\sum_{i,j=1}^{3} m_{ij}(y)p_ip_j - \left(\left(2 - \left(\frac{p\cdot\eta}{|\eta|}\right)^2\right)\operatorname{tr} m(y)\right)dy\, d\eta \\ = O(\varepsilon\|m\| + R^2\delta) \ \text{ in } L^2(\Omega_x)\end{aligned}$$

(compare with (4.38)). This implies

$$\|\chi_\Omega\chi_R(D)m\| \leq C(\varepsilon\|m\| + R^2\delta). \tag{5.22}$$

In order to estimate $\hat{m}(\xi)$ for large ξ, consider

$$\int_\Omega \xi_j e^{ix\cdot\xi} m(x)\,dx = i\int_\Omega e^{ix\cdot\xi}\frac{\partial m(x)}{\partial x_j}dx - i\int_{\partial\Omega} e^{ix\cdot\xi}\nu_j(x)m(x)\,dS_x.$$

The first term in the right hand side above is $O(\varepsilon)$ as a function in $L^2(\mathbf{R}^3_\xi)$, because of (1.2). The second term belongs to $L^2_{-\alpha}$ with $\alpha > 1/2$ and

$$\left\| \int_{\partial\Omega} e^{ix\cdot\xi}\nu_j(x)m(x)\,dS_x \right\|_{L^2_{-\alpha}} \leq C\|m|_{\partial\Omega}\|_{L^2(\partial\Omega)} \leq C\delta.$$

Therefore,

$$\begin{aligned}
(1+R^2)^{1-\alpha}\int_{|\xi|>R}|\hat{m}(\xi)|^2 d\xi &\leq \int_{|\xi|>R}(1+|\xi|^2)^{1-\alpha}|\hat{m}(\xi)|^2 d\xi \\
&\leq C\varepsilon^2 + C\sum_{j=1}^3 \left\|(1+|\xi|^2)^{-\alpha/2}\int_{\partial\Omega} e^{ix\cdot\xi}\nu_j(x)m(x)\,dS_x\right\|^2_{L^2} \\
&\leq C(\varepsilon+\delta)^2.
\end{aligned}$$

Thus we get

$$\|\hat{m}\|_{L^2(\mathbf{R}^3\setminus B_R)} \leq CR^{\alpha-1}(\varepsilon+\delta), \quad \alpha > \frac{1}{2}. \tag{5.23}$$

Combining (5.22), (5.23), we get

$$\|m\| \leq C(\varepsilon\|m\| + R^2\delta + R^{\alpha-1}(\varepsilon+\delta)),$$

therefore

$$\|m\| \leq C(R^2\delta + R^{\alpha-1}).$$

Set $R = \delta^{-1/(3-\alpha)}$. Then we get

$$\|m\| \leq C\delta^{\frac{1-\alpha}{3-\alpha}}.$$

Note that $\sigma := (1-\alpha)/(3-\alpha) < 1/5$ and can be chosen as close to $1/5$ as we wish by choosing suitable $\alpha > 1/2$ close to $\alpha = 1/2$. This completes the proof of Theorem 1.1.

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