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QUASIMODES AND RESONANCES: SHARP LOWER BOUNDS

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Abstract

We prove that, asymptotically, any cluster of quasimodes close to each other approximates at least the same number of resonances, counting multiplicities. As a consequence, we get that the counting function of the number of resonances close to the real axis is bounded from below essentially by the counting function of the quasimodes.

1 Introduction

The purpose of this paper is to obtain sharp lower bounds of the number of resonances (scattering poles) close to the real axis. We consider a situation where one can construct real *quasimodes*, i.e., a sequence of approximate real "resonances" and corresponding approximate solutions supported in a fixed compact set. Our main result states, loosely speaking, that quasimodes are perturbed resonances near the real axis and that the number of resonances close to the real axis is at least equal to that of the quasimodes, counting multiplicities.

Quasimode constructions with polynomially small errors are known for a long time in various situations, see e.g. [C], [R], [L], [P1], [C-P] (see also [P2] for a construction with an exponentially small error). It has been an open problem however, for problems in unbounded domains, whether the mere fact that one can construct quasimodes implies existence of resonances close to them. In particular, it was not known whether an elliptic periodic trapped ray in obstacle scattering generated a sequence of resonances converging to the real axis, although a construction of quasimodes in this case was available. The first result in this direction appeared in [St-V2, Lemma 1] in the study of resonances caused by Rayleigh surface waves in linear elasticity. It was shown, for general compactly supported perturbations in odd dimensional spaces, that existence of real quasimodes

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with polynomially small error implied existence of resonances converging to the real axis at the same rate. An important role in the proof of that lemma played an *a priori* exponential estimate on the cut-off resolvent, established by Zworski (see the remarks before Lemma 1 in section 3). The method in [St-V2] however, was not sensitive enough to obtain information on the density of those resonances, it could only prove their existence. An asymptotic formula for the Rayleigh resonances for the specific problem studied in [St-V1], [St-V2] for convex obstacle was obtained in [Sj-V].

A major step ahead was made by Tang and Zworski [T-Z], who considered any space dimension and non necessarily compactly supported perturbations. They observed that one can localize not only near the real axis as done in [St-V2, Lemma 1], but in fact, one can localize near a quasimode to obtain that if the quasimode is large enough, then there is always a resonance close to it. This confirmed the expectation that quasimodes are perturbed resonances. The results in [T-Z] also imply lower bounds on the number of resonances near the real axis. For any known construction we get at least linear bound. If quasimodes are "well distributed" in some sense, one could also obtain finer bounds. However, if quasimodes are distributed in a "unregular" way, more precisely, if there can be multiple quasimodes or clusters of quasimodes too close to each other, then the results in [T-Z] could only prove that anyone of those multiple quasimodes or clusters produces one resonance only. This restricts the possibility of obtaining sharp lower bounds in those situations.

In the present paper we show that such clusters of quasimodes produce (asymptotically) at least the same number of resonances, see Theorem 1 and Corollary 1. To prove this, we develop further the ideas in [St-V2], [T-Z]. We then use these ideas to compare the counting function of quasimodes and resonances, respectively. To this end, using known upper polynomial bounds on the number of resonances (see (2), (3)), we get similar bounds on the number of quasimodes; then we group quasimodes in such disjoint clusters, see the proof of Theorem 2. Next, we estimate their lengths from above and the distance between them from below and then apply the local result of Corollary 1, obtaining at least as many resonances in some neighborhood of any of those clusters and proving that those neighborhoods still do not intersect. This implies a lower bound on the number of resonances asymptotically equal to the number of quasimodes.

The results we prove and especially Theorem 2, reduce the problem of obtaining lower bounds of the number of resonances to that of estimating the number of quasimodes. It allows us to obtain the optimal lower bound for any construction of almost orthogonal quasimodes as long as we can control the density of quasimodes. We would like to make the obvious remark that the lower bound we obtain is connected with the specific quasimodes we start with, one may have other resonances close to the real axis having different nature. As a possible application of our results we consider the classical obstacle problem with an elliptic periodic broken ray (see section 4). In this case we obtain the lower bound cr^n suggested by the quasimode construction in [P1].

This paper is organized as follows. In Section 2 we state the main results. The proofs are in Section 3. In Section 4 we present an application to the case of an elliptic broken ray in obstacle scattering.

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2 Introduction and Statement of the Main Results

We consider first the "black box scattering" framework as developed in [Sj-Z1], [Sj-Z2], [Sj] (see also [T-Z]). We refer to those works for details. For the sake of simplicity and to avoid repeating the assumptions for noncompactly supported perturbations, we restrict our exposition to the case of compactly supported perturbations of the Laplacian. Our main results however, hold under the general assumptions in [T-Z] provided that the polynomial estimates (2), (3) hold either for the number of the resonances or for the number of quasimodes.

Let $\mathcal{H}$ be a complex Hilbert space with

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbf{R}^n \setminus B(0, R_0)),$$

where $R_0 > 0$ is fixed and $B(0, R_0)$ is the ball with center 0 and radius R_0. For each $h \in (0, h_0]$ we have a unbounded self-adjoint operator

$$P(h) : \mathcal{H} \longrightarrow \mathcal{H}$$

with domain $D(P(h))$ independent of h whose projection onto $L^2(\mathbf{R}^n \setminus B(0, R_0))$ coincides with $H^2(\mathbf{R}^n \setminus B(0, R_0))$. It is also required that

$$\mathbf{1}_{B(0,R_0)}(P(h) + i)^{-1} : \mathcal{H} \to \mathcal{H}$$

is compact, where $\mathbf{1}_{B(0,R_0)}$ denotes the orthogonal projector onto $\mathcal{H}_{R_0}$ and we define similarly $\mathbf{1}_{\mathbf{R}^n \setminus B(0,R_0)}$. Next, we assume that

$$\mathbf{1}_{\mathbf{R}^n \setminus B(0,R_0)} P(h)u = -h^2 \Delta(u|_{\mathbf{R}^n \setminus B(0,R_0)}).$$

Having $P(h)$, one constructs a self-adjoint operator $P^\sharp(h)$ on $\mathcal{H}^\sharp = \mathcal{H}_{R_0} \oplus L^2(M \setminus B(0, R_0))$, where $M = (\mathbf{R} \setminus R\mathbf{Z})^n$ for some $R \gg R_0$. Denoting by $N(P^\sharp(h), [-\lambda, \lambda])$ the number of eigenvalues (counting multiplicities) in $[-\lambda, \lambda]$, one also assumes that

$$N(P^\sharp(h), [-\lambda, \lambda]) = O((\lambda/h^2)^{n^\sharp/2}), \quad \forall \lambda > 1 \tag{1}$$

with some $n^\sharp \geq n$.

Then one defines resonances $\operatorname{Res} P(h)$ of $P(h)$ by the method of complex scaling [Sj-Z1], [Sj]. They are also the poles of the meromorphic continuation of the cut-off resolvent $\chi(P(h) - z)^{-1}\chi$ from $\operatorname{Im} z < 0$ into a conic neighborhood of the real line in the upper half-plane, where $\chi \in$

C_0^∞ is a cut-off function with $\chi = 1$ near $B(0, R_0)$. The poles of that cut-off resolvent and their multiplicities do not depend on the particular choice of χ. Notice that here we accept the convention that scattering poles are in the upper half-plane. Our interest is in applications to classical situations, where P is as before, but independent of h. Then we only need to verify the assumptions above for $h = 1$ and set $P(h) = h^2 P$. In this case we define resonances of P, denoted by $\mathrm{Res} P$, as the poles of the meromorphic continuation of the cut-off resolvent $\chi(P - \lambda^2)^{-1}\chi$ from the lower half-plane into a conic neighborhood of the real line in the upper half-plane. The relationship between semi-classical resonances z of $P(h) = h^2 P$ and the classical ones λ of P is $\lambda^2 = h^{-2} z$. Since P is self-adjoint, resonances of P form a set symmetric about the imaginary axis. We will be interested in those resonances that lie in $\mathrm{Re}\,\lambda > 0$. We always include quasimodes and resonances with their multiplicities. By definition, the multiplicity of a resonance z_0 of $P(h)$ or a resonance λ_0 of P is the rank of the operator

$$\frac{1}{2\pi i}\oint_{|z-z_0|\ll 1} \chi(P(h) - z)^{-1}\chi\, dz, \quad \text{or} \quad \frac{1}{2\pi i}\oint_{|\lambda-\lambda_0|\ll 1} \chi(P - \lambda^2)^{-1}\chi\, \lambda d\lambda,$$

respectively.

We assume finally that we have a polynomial estimate on the number of resonances in a small neighborhood of the real axis:

$$\#\{z \in \mathrm{Res} P(h);\ 0 < a \le |z| \le b;\ 0 \le \mathrm{Im}\, z \le h^N\} \quad \le \quad C_{a,b} h^{-n^\sharp}, \tag{2}$$

$$\#\{\lambda \in \mathrm{Res} P;\ 1 \le |\lambda| \le r;\ 0 \le \mathrm{Im}\,\lambda \le |\lambda|^{-N}\} \quad \le \quad C r^{n^\sharp}, \quad r > 1, \tag{3}$$

for some $N > 0$. If the power $n^\sharp$ is different from that in (1), then we denote by $n^\sharp$ the largest number of the two. Estimates of this type were proven in [M], [Z1], [Sj-Z1], [Sj-Z2] [V], [Sj]. We notice that for the proofs of the results below we need this estimate either for the counting function of the resonances or for that of the quasimodes.

We are now ready to state our main results.

Theorem 1 *Let $P(h)$ be an operator satisfying the hypotheses above and let $0 < a_0 \le a(h) \le b(h) \le b_0 < \infty$ be two functions. Assume that there exists a sequence $\{h_l\}_{l=1}^\infty \subset (0, h_0]$ with $h_l \to 0$ as $l \to \infty$ having the following property: For any $h \in \{h_l\}_{l=1}^\infty$, there exist an integer $m(h) \ge 1$, finite sets $\{E_j(h)\}_{j=1}^{m(h)} \subset [a(h), b(h)]$ and $\{u_j(h)\}_{j=1}^{m(h)} \subset \mathcal{H}$, such that*

$$\begin{cases} \|(P(h) - E_j(h))u_j(h)\|_{\mathcal{H}} \le R(h), \quad j = 1, 2, \ldots, m(h), \\ \left|(u_i(h), u_j(h))_{\mathcal{H}} - \delta_{ij}\right| \le R(h), \quad i, j = 1, 2, \ldots, m(h), \\ \mathrm{supp}\,(\mathbf{1}_{\mathbf{R}^n \setminus B(0,R_0)} u_j(h)) \subset K \subset\subset \mathbf{R}^n, \quad j = 1, 2, \ldots, m(h) \end{cases}$$

with some function $R(h) = O(h^\infty)$. Then, for any positive function $S(h)$ satisfying $S(h) \ge h^{-n^\sharp - 1} R(h)$ and $De^{-D/h} \le S(h) = O(h^\infty)$ for some constant $D > 0$, and for any integer $k \ge 1$, there exists $h(S, k) > 0$ such that for all $h < h(S, k)$, $h \in \{h_l\}_{l=1}^\infty$, the operator $P(h)$ has at least $m(h)$ resonances (counting multiplicities) in the set

$$z \in [a(h) - 6h^k, b(h) + 6h^k] + i[0, 2S(h)h^{-n^\sharp - 1}]. \tag{4}$$

Remark 1. The constant 6 appearing above is not significant and can be replaced by one. We keep it however in order to conform with the notation in [T-Z].

Remark 2[1]**.** It is enough above to assume that $\left|(u_i(h), u_j(h))_{\mathcal{H}} - \delta_{ij}\right| \leq \alpha/m(h)$, $i, j = 1, 2, \dots, m(h)$, with $\alpha < 1$.

Remark 3. In the formulation of Theorem 1 we assume that h belongs to a sequence $\{h_l\}$ having in mind applications to the classical setting. However, we would like to note that Theorem 1 remains true if we assume that h belongs to an interval $(0, h_1)$. To see that it is enough to observe that Lemma 2 in Section 3 holds for h in an interval as well.

In the classical setting Theorem 1 implies the following.

Corollary 1 *Let P be an operator (independent of h) satisfying the assumptions above with $h = 1$. Assume that there exists a sequence $m_l \geq 1$, $l = 1, 2, \dots$ with the following property: for any $l = 1, 2, \dots$ we have m_l numbers $\lambda_{l,j} > 0$, $j = 1, \dots, m_l$ such that $a_l \leq \lambda_{l,j} \leq b_l$, $j = 1, \dots, m_l$ with $b_l \geq a_l \nearrow \infty$, as $l \to \infty$, $b_l/a_l \leq C$ and there exist $u_{l,j} \in \mathcal{H}$, $j = 1, \dots, m_l$ satisfying*

$$\begin{cases} \|(P - \lambda_{l,j}^2)u_{l,j}\|_{\mathcal{H}} \leq R(\lambda_{l,j}), \\ \left|(u_{l,i}, u_{l,j})_{\mathcal{H}} - \delta_{ij}\right| \leq R(\lambda_{l,j}), \\ \text{supp}\,(\mathbf{1}_{\mathbf{R}^n \setminus B_{R_0}} u_{l,j}) \subset K \subset\subset \mathbf{R}^n \end{cases}$$

for any $i, j = 1, \dots, m_l$ with some function $R(\lambda) = O(\lambda^{-\infty})$. Then, for any function $S(\lambda) \geq 2\lambda^{2n^\sharp+3}R(\lambda)$ satisfying $De^{-\lambda D} \leq S(\lambda) = O(\lambda^{-\infty})$ with some $D > 0$ and for any integer $k \geq 1$ there exists $\lambda(S, k) > 0$, such that for any $a_l > \lambda(S, k)$ the operator P has at least m_l resonances (counting multiplicities) in the set

$$\lambda \in [a_l - a_l^{-k}, b_l + a_l^{-k}] + i[0, S(a_l)].$$

The corollary above enables us to obtain sharp lower bounds on the number of the scattering poles close to the real axis.

Theorem 2 *Let P be an operator (independent of h) satisfying the assumptions above with $h = 1$. Assume that there exist infinitely many real quasimodes of P, i.e., a sequence $\{\lambda_j, u_j\}_{j=1}^{\infty}$, where $0 < \lambda_j \nearrow \infty$, $u_j \in \mathcal{H}$ and*

$$\begin{cases} \|(P - \lambda_j^2)u_j\|_{\mathcal{H}} \leq R(\lambda_j), \\ |(u_i, u_j)_{\mathcal{H}} - \delta_{ij}| \leq R(\lambda_j), \\ \text{supp}\,(\mathbf{1}_{\mathbf{R}^n \setminus B_{R_0}} u_j) \subset K \subset\subset \mathbf{R}^n, \end{cases}$$

$i, j = 1, 2, \dots$ with a decreasing function $R(\lambda) = O(\lambda^{-\infty})$. Fix a positive function $S(\lambda) = O(\lambda^{-\infty})$ such that $S(\lambda) \geq 4\lambda^{2n^\sharp+3}R(\lambda)$ and $-\gamma S \leq S' \leq 0$ with some $\gamma > 0$. Denote

$$\begin{aligned} N_{\text{quasi}}(r) &= \#\{\lambda_j;\ \lambda_j \leq r\}, \\ N_{\text{res}}(r) &= \#\{\lambda \in \text{Res}\,P;\ 1 < \text{Re}\,\lambda \leq r,\ 0 \leq \text{Im}\,\lambda \leq S(\text{Re}\,\lambda)\}. \end{aligned}$$

[1]due to M. Zworski

Then for any $k \geq 1$ there exists a constant C_k such that

$$N_{\text{res}}(r) \geq N_{\text{quasi}}(r - r^{-k}) - C_k, \quad \forall r \geq 1. \tag{5}$$

An immediate application of Theorem 2 yields the following.

Corollary 2 *Under the assumptions of Theorem 2, assume that*

$$N_{\text{quasi}}(r) \geq p(r) + q(r), \quad r \geq 1$$

with some C^1 function $p(r) \to \infty$ as $r \to \infty$ with polynomially bounded derivative and a remainder term $q(r)$. Then for k large enough we have

$$N_{\text{res}}(r) \geq p(r) + q(r - r^{-k}) - C_k, \quad \forall r > 1.$$

If $N_{\text{quasi}}(r)$ has asymptotic expansion or more generally if it can be bounded below by some asymptotic expansion like

$$N_{\text{quasi}}(r) \geq \sum_{m=0}^{N} \alpha_m r^{n^\sharp - m} + o(r^{n^\sharp - N}),$$

as $r \to \infty$ with $0 \leq N < n^\sharp$, then

$$N_{\text{res}}(r) \geq \sum_{m=0}^{N} \alpha_m r^{n^\sharp - m} + o_1(r^{n^\sharp - N}), \quad \forall r > 1$$

with the same coefficients α_m and a possibly different remainder term.

Remark 4. The constant C_k appearing in the estimates above can be expected because the quasimode construction is asymptotic and adding or removing a finite number of quasimodes does not make a difference. However, if we want to estimate the number $N_{\text{res}}(r_2) - N_{\text{res}}(r_1)$ of resonances with real parts between r_1 and r_2 with $1 \ll r_1 \ll r_2$, then the proof of Theorem 2 implies that $N_{\text{res}}(r_2) - N_{\text{res}}(r_1) \geq N_{\text{quasi}}(r_2 - r_2^{-k}) - N_{\text{quasi}}(r_1 + r_1^{-k})$ for r_1 large enough (see also next remark).

Remark 5. Following a remark in [T-Z] due to Shu Nakamura, one can replace h^k in Lemma 2 in Section 3 by a function $\omega(h) = O(h^\infty)$ such that

$$\frac{\omega^2(h)}{S(h) h^{-n^\sharp - 1}} \to \infty, \quad \text{as } h \to 0.$$

Then the results in [T-Z] indicate that for h small enough there is a resonance near any quasimode at a distance not greater than $\omega(h)$. We can also replace $6h^k$ in the statement of Theorem 1 by $\omega(h)$ and claim the same, including the multiplicities. We can do the same thing in Corollary 1. This implies the following property: Under the assumptions of Theorem 2, denote by $\{\lambda_j\}_{j=1}^\infty$ and $\{r_k\}_{k=1}^\infty$ the quasimodes and resonances, respectively, ordered (by their real parts) and counted with their multiplicities. Then there exists a subsequence of resonances $\{r_{k_j}\}_{j=1}^\infty$, such that

$$|\text{Re}(r_{k_j} - \lambda_j)| \leq \omega(\lambda_j) = O(\lambda_j^{-\infty}), \quad |\text{Im}(r_{k_j} - \lambda_j)| \leq S(\lambda_j) = O(\lambda_j^{-\infty}), \quad j \gg 1,$$

where $S(\lambda)$ is as in Corollary 1 and $\omega(\lambda)$ can be chosen to be $\omega(\lambda) = \lambda^{n^\sharp} S^{1/2}(\lambda)$.

3 Proof of the main results

Before proceeding with the proof of Theorem 1, we will consider first a simpler case — when resonances are replaced by eigenvalues and we have quasimodes. This example is not necessary for the proof below, but it illustrates that in this case our results look natural and admit a simple proof because we can use the spectral theorem. Consider a situation similar to that in Corollary 1. Let P be a self-adjoint operator (independent of h) with discrete spectrum in a Hilbert space $\mathcal{H}$. Suppose that the counting function for the eigenvalues of P admits the bound $C_0 r^{n^\sharp}$. Assume that in the interval $[a,b]$, $1 \ll a < b$, $b-a < 1$, we have m quasimodes $\mu_j = \lambda_j^2$, $j = 1,\dots,m$ similar to those in Corollary 1, more precisely there exist $\mu_j \in [a,b]$, $u_j \in \mathcal{H}$, $j = 1,\dots,m$ such that $\|(P-\mu_j)u_j\| \le R(\mu_j)$, $|(u_i,u_j) - \delta_{ij}| \le R(\mu_j)$ with $R(\mu) = O(\mu^{-\infty})$. We also assume that $m \le 2C_0 a^{n^\sharp}$ which in fact follows from the assumptions already made similarly to the proofs below. Then u_j admit an orthogonal decomposition $u_j = u_j' + u_j''$, with $u_j' := \Pi_{[a-\delta,b+\delta]} u_j$, where $\Pi_{[a-\delta,b+\delta]}$ is the spectral projector of the interval $[a-\delta, b+\delta]$, $\delta > 0$. Clearly,

$$\begin{aligned} \delta^2 \|u_j''\|^2 &\le \|(P-\mu_j)u_j''\|^2 \le \|(P-\mu_j)u_j'\|^2 + \|(P-\mu_j)u_j''\|^2 \\ &= \|(P-\mu_j)u_j\|^2 \le R^2(\mu_j). \end{aligned}$$

So, if we choose $\delta = a^{n^\sharp+1} R(a)$, we get $\|u_j''\| \le a^{-n^\sharp-1}$ and

$$|(u_i', u_j') - \delta_{ij}| \le R(a) + 2(1+R(a))^{1/2} a^{-n^\sharp-1} + a^{-2n^\sharp-2} = O(a^{-n^\sharp-1}).$$

Since m is $O(a^{n^\sharp})$, we get for a large enough (see Lemma 4) that $u_1', \dots, u_m'$ are linearly independent. Therefore, $\Pi_{[a-\delta,b+\delta]}\mathcal{H}$ is at least m-dimensional, which proves that the number of eigenvalues in $[a - a^{n^\sharp+1}R(a), b + a^{n^\sharp+1}R(a)]$, counting the multiplicities, is at least m. This corresponds well to the result in Corollary 1. Those arguments show also that $u_j = u_j' + O(a^{-n^\sharp-1})$ with $u_j' \in \Pi_{[a-\delta,b+\delta]}\mathcal{H}$. By choosing $\delta = R^{1/2}(a)$ above, we can make the remainder $O(a^{-\infty})$. This shows that u_j approximate certain linear combinations of eigenfunctions with eigenvalues in some small neighborhood of $[a,b]$. It is interesting to note that in case of a multiple quasimode or cluster of quasimodes, this does not necessarily imply that each u_j is close to a single eigenfunction. A typical quasimode construction is a set of functions asymptotically concentrated near some periodic ray(s). Although we get that u_j', $j = 1,\dots,m$ are also concentrated there, this is not necessarily true for some linearly independent system of m eigenfunctions, because among the eigenfunctions spanning $\{u_j'\}$ we may have for example functions asymptotically concentrated both near the ray(s) under consideration and near other rays not involved in that quasimode construction. A similar remark applies to the resonance case, see Remark 6 below.

We start the proofs with recalling two lemmas from [T-Z]. The first lemma states an *a priori* exponential estimate of the cut-off resolvent outside small neighborhoods of the resonances. An estimate of this type was first proved by M. Zworski [Z2] for scattering by obstacles using methods developed by Melrose [M] for obtaining upper bounds on the number of scattering poles. A similar estimate (kindly suggested to the authors by M. Zworski) was next proved in [St-V1], [St-V2] for more general cases and was used for proving existence of infinitely many poles near the real axis caused by the Rayleigh surface waves. The lemma below, belonging to Tang and Zworski, extends this estimate to the semiclassical framework of "black-box scattering" and is based on techniques developed by Sjöstrand [Sj].

Lemma 1 ([T-Z]) *There exists $\theta \in (0, \pi)$, such that for any simply connected compact set $\tilde{\Omega} \subset \{z \in \mathbf{C};\ \max(-\pi, 2\theta - 2\pi) < \arg z < 2\theta\}$ and positive function $g(h) \ll 1$ defined on $0 < h < h_0$, there exist constants $A = A(\tilde{\Omega}) > 0$ and h_1 with $0 < h_1 < h_0$ such that*

$$\|\chi(P(h) - z)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \le Ae^{Ah^{-n^\sharp}\ln(1/g(h))}, \quad \forall z \in \tilde{\Omega} \setminus \bigcup_{z_j \in \mathrm{Res}\, P(h) \cap \tilde{\Omega}} D(z_j, g(h)),$$

where $D(z_j, g(h)) := \{z \in \mathbf{C};\ |z - z_j| \le g(h)\}$.

The number θ is actually connected with the size of a conic neighborhood of $\mathbf{R}^n$, where one can extend holomorphically the coefficients of $P(h)$ outside $B(0, R_0)$ in case of non-compact perturbations (see [Sj], [T-Z]).

The next lemma allows us to estimate the growth of the cut-off resolvent under the assumption that it is holomorphic (there are no resonances) in some region near the real axis. As mentioned in the Introduction, a lemma of this type appeared first in [St-V2] and was used to prove existence of infinite number of scattering poles for the elasticity system with Neumann boundary conditions in the following way. That lemma implied an a priori estimates of the resolvent on the real axis contradicting the existence of quasimodes. In order to prove the lemma, under the assumption that there are no resonances near the real axis, we applied the maximum principle for unbounded domains (the Phragmén-Lindelöf principle) in a neighborhood of the real axis bounded by two curves approaching the real axis polynomially fast. To estimate the resolvent on those curves, the following observations were crucial: in the lower half-plane one has standard bounds of the resolvent; while in the upper half-plane one can use the a priori exponential estimate on the cut-off resolvent similar to that in the lemma above. Multiplying by a suitably chosen holomorphic function that would compensate for the exponential growth on the upper curve, one is in a position therefore to apply the maximum principle. Tang and Zworski [T-Z] observed that one can actually localize those arguments in a rectangle near a fixed quasimode by multiplying by a suitable holomorphic "cut-off" function that is exponentially small at the left and right sides of that rectangle neighborhood and uniformly bounded from below in a smaller set.

Lemma 2 ([T-Z]) *Let $\{h_l\}_{l=1}^{\infty} \subset \mathbf{R}_+$ be a sequence such that $h_l \to 0$, as $l \to \infty$. Suppose that $F(z, h)$, $h \in \{h_l\}_{l=1}^{\infty}$ is a holomorphic function of z defined in a neighborhood of*

$$\Omega(h) = [E(h) - 5h^k, E(h) + 5h^k] + i[-S(h), S(h)h^{-n^\sharp - 1}]$$

with $E(h) \in \mathbf{R}$, where $S(h)$ is as in Theorem 1. If $F(z, h)$ satisfies

$$\begin{aligned} |F(z,h)| &\le Ae^{Ah^{-n^\sharp}\ln(1/hS(h))} \quad \text{on } \Omega(h), \\ |F(z,h)| &\le 1/|\mathrm{Im}\, z| \quad \text{on } \Omega(h) \cap \{\mathrm{Im}\, z < 0\}, \end{aligned}$$

then there exists $h_1 = h_1(S, A, k) > 0$, $B = B(S, A, k) > 0$ *such that*

$$|F(z,h)| \le B/S(h), \quad \forall z \in [E(h) - h^k, E(h) + h^k]$$

for $h \le h_1$, $h \in \{h_l\}_{l=1}^{\infty}$.

An inspection of the proof of the lemma in [T-Z] shows that B and h_1 are independent of the choice of $E(h)$ and $F(z,h)$ as long as the constant A appearing in the exponential estimate above remains uniform. To see this, it is enough to note that the proof is based on application of the maximum principle in $\Omega(h)$ to the product of F and an auxiliary function depending on S and k only. Then h_1 and B depend on the properties of that function and on the exponential bound above used to estimate the maximum of F on $\partial\Omega(h)$. We will apply Lemma 1 to $\chi(P(h) - z)^{-1}\chi$ using Lemma 2. Then the uniformity of A will be fulfilled, if $[E(h) - 5h^k, E(h) + 5h^k] \subset (a_0, b_0)$ with $0 < a_0 < b_0$ which will be always true.

We refer to [T-Z] for proof of Lemma 1 and Lemma 2.

Fix a cut-off function $\chi \in C_0^\infty(\mathbf{R}^n)$ with $\chi = 1$ near K. In next lemma we will not indicate the dependence on h.

Lemma 3 *Let* $\tilde{\chi}$ *be another cut-off function with* $\tilde{\chi} = 1$ *near* K *an let* z_0 *be a pole of* $\chi(P-z)^{-1}\tilde{\chi}$, *i.e.,*

$$\chi(P - z)^{-1}\tilde{\chi} = A_0(z) + \sum_{j=1}^{N}(z - z_0)^{-j}A_j \tag{6}$$

with $A_0(z)$ *holomorphic near* $z = z_0$, $A_N \neq 0$ *and* $N \ge 1$. *Let* χ_j, $j = 1, \dots, N-1$ *be* C_0^∞-*functions such that* $\chi_1 = 1$ *near* K, $\operatorname{supp}\chi_j \subset \{\chi_{j+1} = 1\}$, $j = 1, \dots, N-2$ *and* $\operatorname{supp}\chi_{N-1} \subset \{\tilde{\chi} = 1\}$. *Then*

$$A_j\chi_1 = A_1(P - z_0)\chi_{j-1}(P - z_0)\chi_{j-2}\dots\chi_2(P - z_0)\chi_1, \quad j = 2, \dots, N.$$

Proof. Let us multiply (6) by $(P - z)$ on the right. We get

$$\begin{aligned}
&\chi\tilde{\chi} + \chi(P - z)^{-1}[\tilde{\chi}, P] \\
&\quad = A_0(z)(P - z) + \sum_{j=1}^{N}(z - z_0)^{-j}A_j(P - z_0 - z + z_0) \\
&\quad = A_0(z)(P - z) + \sum_{j=1}^{N}\left((z - z_0)^{-j}A_j(P - z_0) - (z - z_0)^{-j+1}A_j\right) \\
&\quad = A_0(z)(P - z) - A_1 + \sum_{j=1}^{N}(z - z_0)^{-j}\left(A_j(P - z_0) - A_{j+1}\right)
\end{aligned}$$

with the convention $A_{N+1} = 0$. Multiply by χ_l on the right and equate the singular powers of $z - z_0$ to get

$$A_j(P - z_0)\chi_l = A_{j+1}\chi_l, \quad j, l = 1, \dots, N-1.$$

Therefore, for $j = 2, \dots, N$,

$$\begin{aligned}
A_j\chi_1 &= A_{j-1}(P-z_0)\chi_1 = A_{j-1}\chi_2(P-z_0)\chi_1 \\
&= A_{j-2}(P-z_0)\chi_2(P-z_0)\chi_1 = A_{j-2}\chi_3(P-z_0)\chi_2(P-z_0)\chi_1 \\
&= A_{j-3}(P-z_0)\chi_3(P-z_0)\chi_2(P-z_0)\chi_1 = A_{j-3}\chi_4(P-z_0)\chi_3(P-z_0)\chi_2(P-z_0)\chi_1 \\
&= \dots \\
&= A_1(P-z_0)\chi_{j-1}(P-z_0)\chi_{j-2}\dots\chi_2(P-z_0)\chi_1.
\end{aligned}$$

This proves the lemma. □

Assume that $\tilde{\chi} = 1$ on supp χ in Lemma 3. Then we can choose $\chi_1 = \chi$. Multiply (6) by χ on the right to get

$$\chi(P(h) - z)^{-1}\chi = A_0(z,h)\chi + \sum_{j=1}^{N}(z - z_0(h))^{-j}A_1(h)Q_j(h), \tag{7}$$

where $Q_j(h)$, $j \geq 2$ are unbounded but $A_1(h)Q_j(h)$ are bounded operators. Notice that $A_1(h)$ is a finite rank operator, and by definition, Rank$(A_1(h))$ is the multiplicity of $z_0(h)$ and this rank is independent of the choice of the cut-off functions χ, $\tilde{\chi}$ in (6). The above lemma says that the range of the singular part of the cut-off resolvent is the same as the range of the residue A_1.

Proof of Theorem 1. We are going to assume that

$$m(h) < Ch^{-n^\sharp}, \quad h \in \{h_l\}_{l=1}^{\infty}. \tag{8}$$

At the end of the proof we will show that, in fact, this is always true.

Assume from now on that $h \in \{h_l\}_{l=1}^{\infty}$. Fix $0 \leq \chi \leq 1$ as in Lemma 3. Let $z_1(h), z_2(h), \dots, z_{M(h)}(h)$ be all distinct poles of $\chi(P(h) - z)^{-1}\chi$ in (4) and denote by $A_1^{(j)}(h)$, $j = 1, \dots, M(h)$ the corresponding residua. The sum of the ranks of all $A_1^{(j)}(h)$ is equal to the total number of resonances in (4). Denote by $\Pi(h)$ the orthogonal projector in $\mathcal{H}$ onto $\cup_{j=1}^{M(h)} A_1^{(j)}(h)\mathcal{H}$ and set $\Pi'(h) = \mathrm{Id} - \Pi(h)$. Then Rank $\Pi(h)$ does not exceed the total number of resonances in (4) counting multiplicities. Our goal is to prove that the latter is at least $m(h)$ and this will be achieved if we prove that Rank $\Pi(h) \geq m(h)$.

By (7), $\Pi'(h)\chi(P(h) - z)^{-1}\chi$ is holomorphic in (4). Then $\Pi'(h)\chi(P(h) - z)^{-1}\chi$ is holomorphic in

$$z \in \Omega_6(h) := [a(h) - 6h^k, b(h) + 6h^k] + i[-S(h), 2S(h)h^{-n^\sharp - 1}].$$

Set $E_s(h) = a(h) + s(b(h) - a(h))$, $s \in [0, 1]$. Then $\Pi'(h)\chi(P(h) - z)^{-1}\chi$ is holomorphic in a neighborhood of

$$\Omega_{5,s}(h) := [E_s(h) - 5h^k, E_s(h) + 5h^k] + i[-S(h), S(h)h^{-n^\sharp-1}] \quad \text{for any } s \in [0, 1]. \tag{9}$$

We choose $g(h) := hS(h)$ in Lemma 1. For $h \ll 1$, $\Omega_6(h)$ is included in a fixed compact set satisfying the requirements of Lemma 1, thus

$$\|\chi(P(h) - z)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \leq Ae^{Ah^{-n^\sharp}\ln(1/hS(h))}, \quad z \in \Omega_6(h) \setminus \bigcup_{z_j \in \operatorname{Res} P(h)} D(z_j, hS(h)). \tag{10}$$

This implies

$$\|\Pi'(h)\chi(P(h)-z)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \leq Ae^{Ah^{-n^\sharp}\ln(1/hS(h))}, \quad z \in \Omega_6(h)\setminus \bigcup_{z_j \in \operatorname{Res} P(h)} D(z_j, hS(h)). \tag{11}$$

We would like to prove this estimate in the whole $\Omega_{5,s}(h)$ using the fact that $\Pi'(h)\chi(P(h)-z)^{-1}\chi$ is actually holomorphic in the larger domain $\Omega_6(h)$. Notice that (11) holds in $\Omega_{5,s}(h)$ with the exclusion of the disks $D(z_j, hS(h))$. Some of those z_j's can lie outside $\Omega_{5,s}(h)$ and even outside $\Omega_6(h)$ and some of the disks can overlap. We claim that if some connected union of such disks has common points with $\Omega_{5,s}(h)$, then it lies entirely in $\Omega_6(h)$. This follows easily from the following. The distance between any point in $\partial\Omega_{5,s}(h)$ and the exterior of $\Omega_6(h)$ in $\operatorname{Im} z > 0$ is at least $h^{-n^\sharp-1}S(h)$ for h sufficiently small. On the other hand, because of (8), the diameter of each maximal connected set having common points with $\Omega_{5,s}(h)$, which is a union of such disks centered in $\Omega_6(h)$, does not exceed $m(h)hS(h) \leq Ch^{-n^\sharp+1}S(h)$. There can be also disks centered outside $\Omega_6(h)$ and intersecting $\Omega_6(h)$, not included in the those considerations but they do not have common points with those unions of disks because $Ch^{-n^\sharp+1}S(h) \ll h^{-n^\sharp-1}S(h)$. This proves the claim. The lemma does not guarantee that estimate (11) is fulfilled in the interior of any such union of disks, but since the latter lies in $\Omega_6(h)$, it is fulfilled on the boundary. Applying the maximum principle in each such set, we get that the estimate holds inside as well (see Figure 1). Therefore,

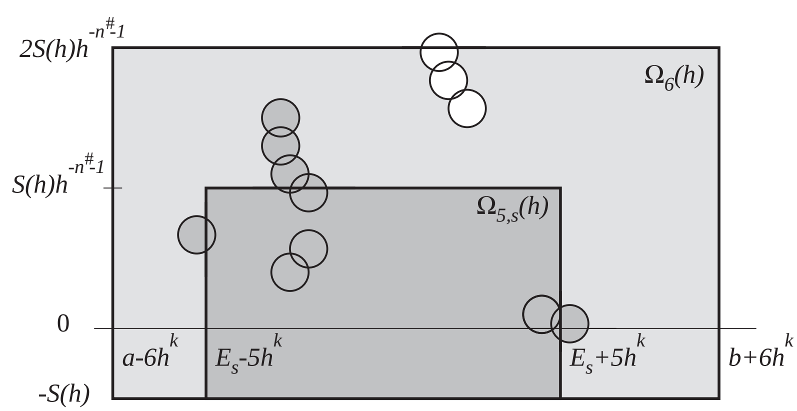

Figure 1

$$\|\Pi'(h)\chi(P(h)-z)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \leq Ae^{Ah^{-n^\sharp}\ln(1/hS(h))}, \quad \forall z \in \Omega_{5,s}(h), \quad \forall s \in [0,1]. \tag{12}$$

Here A is uniform in s.

Since $P(h)$ is selfadjoint, for $\operatorname{Im} z < 0$ we have the standard estimate

$$\|\Pi'(h)\chi(P(h)-z)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \leq \|\chi(P(h)-z)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \leq 1/|\operatorname{Im} z|.$$

By this and (12) we conclude from Lemma 2 that

$$\|\Pi'(h)\chi(P(h)-z)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \leq \frac{B}{S(h)} \quad \text{for } z \in [E_s(h)-h^k, E_s(h)+h^k] \tag{13}$$

for $0 < h < h_1(S,A,k)$ and $B = B(S,A,k)$. Here h_1, B are independent of s, thus we get the estimate above for $z \in [a(h)-h^k, b(h)+h^k]$.

We have $u_j(h) = \chi u_j(h) = \chi(P(h)-z)^{-1}\chi(P(h)-z)u_j(h)$ for $\operatorname{Im} z < 0$ and therefore for $z \notin \operatorname{Res} P(h)$. Let us multiply this by $\Pi'(h)$ to get $\Pi'(h)u_j(h) = \Pi'(h)\chi(P(h)-z)^{-1}\chi(P(h)-z)u_j(h)$. Since $\Pi'(h)\chi(P(h)-z)^{-1}\chi$ is holomorphic in (4), we can set $z = E_j(h)$ above. Hence, for $j = 1,\dots,m(h)$ with h as above, we have

$$\begin{aligned} \|\Pi'(h)u_j(h)\|_{\mathcal{H}} &= \|\Pi'(h)\chi(P(h)-E_j(h))^{-1}\chi(P(h)-E_j(h))u_j(h)\|_{\mathcal{H}} \\ &\leq \|\Pi'(h)\chi(P(h)-E_j(h))^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}}\|(P(h)-E_j(h))u_j(h)\|_{\mathcal{H}} \\ &\leq B\frac{R(h)}{S(h)} \leq Bh^{n^\sharp+1}. \end{aligned} \tag{14}$$

for h small enough. Since $u_j(h)$ form an orthonormal system up to an error $R(h)$, we get

$$\left|(\Pi(h)u_i(h), \Pi(h)u_j(h))_{\mathcal{H}} - \delta_{ij}\right| \leq 2\,(1+R(h))^{\frac{1}{2}}\, Bh^{n^\sharp+1} + B^2h^{2n^\sharp+2} + R(h).$$

Lemma 4 *Let $f_1, f_2, \dots, f_N$ be N vectors in the Hilbert space $\mathcal{H}$ with*

$$\left|(f_i, f_j)_{\mathcal{H}} - \delta_{ij}\right| \leq \varepsilon, \quad i,j = 1,\dots,N.$$

If $\varepsilon < 1/N$, then $f_1, f_2, \dots, f_N$ are linearly independent.

Proof. Assume that those vectors are linearly dependent. Then $a_1 f_1 + \ldots + a_N f_N = 0$ with $(a_1,\dots,a_N) \neq 0$. Without loss of generality we may assume that $0 \neq |a_N| \geq |a_j|$, $j = 1,\dots,N-1$. Divide by a_N to get

$$f_N = c_1 f_1 + \ldots + c_{N-1} f_{N-1}, \quad |c_j| \leq 1, \quad j = 1,\dots,N-1.$$

Multiply this by f_N to get

$$1-\varepsilon \le \|f_N\|^2_{\mathcal{H}} = c_1(f_1, f_N)_{\mathcal{H}} + \ldots + c_N(f_{N-1}, f_N)_{\mathcal{H}} \le (N-1)\varepsilon,$$

thus $1 \le N\varepsilon$, which proves the lemma. □

From Lemma 4 we therefore get that if

$$2\,(1+R(h))^{\frac{1}{2}}\, Bh^{n^\sharp+1} + B^2 h^{2n^\sharp+2} + R(h) < 1/m(h), \tag{15}$$

then $\Pi(h)u_j(h)$, $j=1,\ldots,m(h)$ are linearly independent. Condition (15) is fulfilled for small h because of (8). Therefore we get $\mathrm{Rank}(\Pi(h)) \ge m(h)$. This proves the theorem under the assumption (8).

We will show now that the assumption (8) made at the beginning of the proof is not restrictive. Assume that $m(h_{l_j})/h_{l_j}^{-n^\sharp} \to \infty$, $j=1,2,\ldots$ for some subsequence $\{h_{l_j}\}$ of $\{h_l\}_{l=1}^\infty$. We can remove some quasimodes to make sure that $m(h) \le Ch^{-n^\sharp-1}$ and keep the limit above. Then we get as above that the number of resonances of $P(h)$ in $[a_0, b_0]$ would not be $O(h^{-n^\sharp})$, which contradicts (2).

This completes the proof of Theorem 1. □

Remark 6. By (14) we get that $u_j(h) = \Pi(h)u_j(h) + O(h^{n^\sharp+1})$ and by choosing $S(h)$ so that $R(h)/S(h) = O(h^\infty)$ we can achieve that the remainder is actually $O(h^\infty)$. This property is similar to what we know about the case of eigenfunctions (see the discussion at the beginning of this section). Namely, u_j approximate functions in $\Pi(h)\mathcal{H}$, i.e., linear combinations of functions in $\cup_k A_1^{(k)}(h)\mathcal{H}$, where

$$A_1^{(k)}(h) = \frac{1}{2\pi i}\oint_{|z-z_k|\ll 1} \chi(P(h)-z)^{-1}\chi dz$$

and as before z_k are the resonances in $\Omega_6(h)$. But again, if $u_j(h)$ are concentrated around some set (*microsupport*) as $h\to 0$, this does not necessarily mean that we have functions in $A_1^{(k)}(h)\mathcal{H}$, such that each one corresponds to a single k rather to a combination of several k's with the same property. For example, the functions in each $A_1^{(k)}(h)\mathcal{H}$ may be asymptotically supported both near the microsupport of this quasimode construction and the microsupport of some other $\tilde{u}_j(h)$'s that happen to have quasimodes close to those we consider.

Proof of Corollary 1. Set $h_l := a_l^{-1}$, $P(h) := h^2P = a_l^{-2}P$, $h \in \{h_l\}_{l=1}^\infty$. In this proof, denote the function R from Theorem 1 by R_T. Then $P(h)$ satisfies the assumptions of Theorem 1 with $a(h)=1$, $b(h) = b_l^2/a_l^2$, $E_j(h) = a_l^{-2}\lambda_{l,j}^2$, $m(h) = m_l$, $R_T(h) = R(a_l) = R(h^{-1})$, $u_j(h) = u_{l,j}$. We obtain therefore existence of $l(S_T, k) > 0$, such that for $l > l(S_T, k)$ the operator $\chi(a_l^{-2}P - z)^{-1}\chi$ has at least m_l poles in the set $z \in [1-6a_l^{-k}, b_l^2/a_l^2 + 6a_l^{-k}] + i[0, 2S_T(a_l^{-1})a_l^{n^\sharp+1}]$, where S_T is any function satisfying the assumptions for S in Theorem 1. By estimating the square root of that set, we conclude that $\chi(a_l^{-2}P-\mu^2)^{-1}\chi$ has at least m_l poles in $\mu \in [1-4a_l^{-k}, a_l^{-1}b_l(1+4a_l^{-k})] + i[0, 2S_T(a_l^{-1})a_l^{n^\sharp+1}]$ for l large enough. Setting $\lambda = a_l\mu$, we get at least m_l resonances of P in $[a_l - 4a_l^{-k+1}, b_l + 4(b_l/a_l)a_l^{-k+1}] + i[0, 2S_T(a_l^{-1})a_l^{n^\sharp+2}]$. Using the fact that $4/a_l < 1$, $4b_l/a_l^2 < 1$ for large l, we get the conclusion of the corollary with k replaced by $k-2$ and $S(\lambda) = 2S_T(\lambda^{-1})\lambda^{n^\sharp+2}$. To complete the proof it remains to show that if $S(\lambda)$ satisfies the assumptions of Corollary 1, then $S_T(h)$ determined by $S(\lambda) = 2S_T(\lambda^{-1})\lambda^{n^\sharp+2}$, $\lambda = h^{-1}$, satisfies the assumptions of Theorem 1. Indeed, we have $S_T(h) = \frac{1}{2}S(h^{-1})h^{n^\sharp+2}$ thus $S_T(h) \geq h^{-n^\sharp-1}R(h^{-1}) = h^{-n^\sharp-1}R_T(h)$ which is one of the hypotheses on S_T in Theorem 1. The other condition about the exponential bound from below follows from the similar condition in Corollary 1. □

Proof of Theorem 2. We will assume first that

$$N_{\text{quasi}}(r) \leq C_0(r^{n^\sharp}+1) \tag{16}$$

with some $C_0 > 0$. Fix $k \geq n^\sharp + 1$. For any index j denote $I_j := \{\lambda_j\} + 2(-\lambda_j^{-k}, \lambda_j^{-k}] = (\lambda_j - 2\lambda_j^{-k}, \lambda_j + 2\lambda_j^{-k}]$. Here and below we use the notation $A + B := \{a+b;\ a \in A,\ b \in B\}$ for any two number sets A and B, and $\{\lambda_j\}_{j=1}^\infty$ is the set of quasimodes.

Consider

$$\bigcup_{j\geq 1}\left(\{\lambda_j\} + 2(-\lambda_j^{-k}, \lambda_j^{-k}]\right). \tag{17}$$

We claim that the set above consists of infinitely many disjoint intervals. Indeed, fix $a \gg 1$. If we assume that the whole interval $[a, a+1]$ is covered by (17), then we get that $[a, a+1]$ should contain at least $\frac{1}{5}a^k$ quasimodes. Since $k \geq n^\sharp + 1$, we get a contradiction with (16).

Therefore, (17) consists of a sequence of disjoint intervals $(A_l, B_l]$, $l = 1, 2, \dots$. Moreover, the argument above proves that $B_l - A_l \leq 5C_0B_l^{-k+n^\sharp}$, $l \gg 1$. Each $(A_l, B_l]$ contains $m_l \geq 1$ quasimodes $\lambda_{l,j}$, $j = 1, \dots, m_l$, and $\bigcup_{j=1}^{m_l}\left(\{\lambda_{l,j}\} + 2(-\lambda_{l,j}^{-k}, \lambda_{l,j}^{-k}]\right) = (A_l, B_l]$. Denote by $(a_l, b_l] \subset (A_l, B_l]$ the minimal interval containing $\bigcup_{j=1}^{m_l}\left(\{\lambda_{l,j}\} + (-\lambda_{l,j}^{-k}, \lambda_{l,j}^{-k}]\right)$. Then $B_l - b_l > b_l^{-k}$, $A_l - a_l > \frac{1}{2}a_l^{-k}$ and therefore,

$$0 < b_l - a_l \leq 5C_0b_l^{-k+n^\sharp}, \quad b_l^{-k} < a_{l+1} - b_l, \quad \forall l \gg 1 \tag{18}$$

(See Figure 2.) Fix a function S satisfying the assumptions of Theorem 2 and apply Corollary 1 to $[a_l, b_l]$ with S there replaced by $S/2$. Clearly, $S/2$ satisfies the assumptions of Corollary 1. We get that the number of resonances in

$$(a_l, b_l] + \frac{1}{4}[-a_l^{-k}, a_l^{-k}] + i[0, \frac{1}{2}S(a_l)] \tag{19}$$

is at least equal to the number m_l of quasimodes in $(a_l, b_l]$ for l large enough (it is easy to see that we can put the constant $1/4$ there by increasing k). By (18), those rectangles do not overlap for

$l \gg 1$. We claim that this implies that

$$N_{\rm quasi}(b_l) - N_{\rm quasi}(a_l) = m_l \le N_{\rm res}(b_l + \frac{1}{4}a_l^{-k}) - N_{\rm res}(a_l - \frac{1}{4}a_l^{-k}). \tag{20}$$

To prove the latter, in view of (19) and the definition of $N_{\rm res}$, it is enough to show that

$$(a_l, b_l] + \frac{1}{4}[-a_l^{-k}, a_l^{-k}] + i[0, \frac{1}{2}S(a_l)] \subset \{\lambda;\ a_l - \frac{1}{4}a_l^{-k} \le \operatorname{Re}\lambda \le b_l + \frac{1}{4}a_l^{-k},\ 0 \le \operatorname{Im}\lambda \le S(\operatorname{Re}\lambda)\}.$$

Let us first observe $-\gamma S \le S' \le 0$ implies easily that $S(a) \le 2S(a + 1/(2\gamma))$, $\forall a$. Therefore, $S(a_l) \le 2S(\operatorname{Re}\lambda)$ for $\operatorname{Re}\lambda$ as above provided that $a_l \gg 1$. This implies the inclusion above which in turn proves our claim.

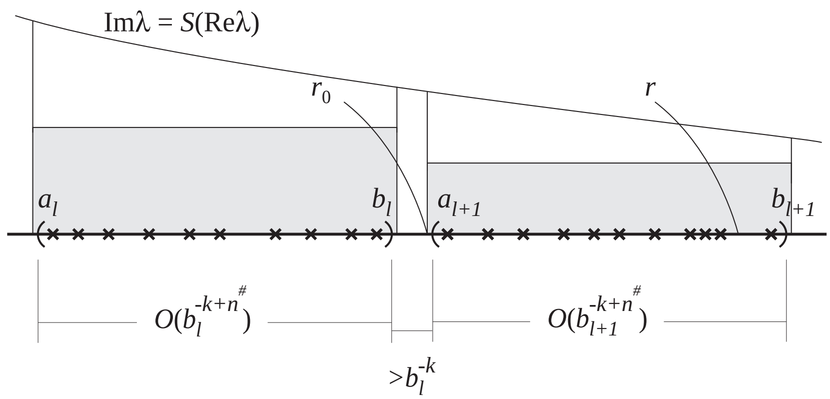

Figure 2

Fix a real r_0 in the gap between two rectangles (19), i.e.,

$$b_l + \frac{1}{4}a_l^{-k} \le r_0 \le a_{l+1} - \frac{1}{4}a_{l+1}^{-k}, \quad \text{for some } l \gg 1. \tag{21}$$

Then, summing up inequalities (20) with l there replaced by $l, l-1, \dots, l(k)$, we obtain

$$N_{\rm res}(r_0) - N_{\rm res}(a_{l(k)} - \frac{1}{4}a_{l(k)}^{-k}) \ge N_{\rm quasi}(r_0) - N_{\rm quasi}(a_{l(k)}), \tag{22}$$

where $l(k) > 0$ is that number for which all statements above hold with $l \ge l(k)$. Let $r > a_{l(k)}$ be any number. Denote by r_0 the closest number $r_0 \le r$ such that r_0 is of the type considered in (21) (see Figure 2). Then by (18), $0 \le r - r_0 \le 5C_0 r^{-k+n^\sharp}$ and by (22),

$$\begin{aligned} N_{\rm res}(r) &\ge N_{\rm res}(r_0) \ge N_{\rm quasi}(r_0) - \left(N_{\rm quasi}(a_{l(k)}) - N_{\rm res}(a_{l(k)} - \frac{1}{4}a_{l(k)}^{-k})\right) \\ &\ge N_{\rm quasi}(r - 5C_0 r^{-k+n^\sharp}) - C_k. \end{aligned}$$

Replacing k by $k+n^\sharp+1$, we see that we can replace $5C_0 r^{-k+n^\sharp}$ above by r^{-k} and thus complete the proof of (5).

It remains to show that (16) is always fulfilled. Assume the opposite, that for any $M > 0$ we have $N_{\text{quasi}}(r_j) > 2Mr_j^{n^\sharp}$ for a sequence $r_j \to \infty$.

For $m = 3, 4, \ldots$ do the following: first, remove all quasimodes from $(1, 2]$. Then, if $N_{\text{quasi}}(m) > Mm^{n^\sharp}$, remove some quasimodes from $(m-1, m]$ until we get $N_{\text{quasi}}(m) = Mm^{n^\sharp}$; if $N_{\text{quasi}}(m) \leq Mm^{n^\sharp}$, do nothing. Since by the previous step we have achieved that $N_{\text{quasi}}(m-1) \leq M(m-1)^{n^\sharp} < Mm^{n^\sharp}$, it is possible to arrange the equality $N_{\text{quasi}}(m) = Mm^{n^\sharp}$ by removing quasimodes from $(m-1, m]$ only. We have infinitely many m's for which we will have to remove quasimodes because assuming the opposite, we would get $N_{\text{quasi}}(m) \leq Mm^{n^\sharp}$, $\forall m = 2, 3, \ldots$, after removing a finite number of quasimodes, which contradicts our assumption. Thus, we have $N_{\text{quasi}}(m_j) = Mm_j^{n^\sharp}$ for a sequence $m_j \to \infty$ and $N_{\text{quasi}}(m) \leq Mm^{n^\sharp}$, $\forall m = 2, 3, \ldots$. The latter easily implies $N_{\text{quasi}}(r) \leq 2Mr^{n^\sharp}$ for all real r large enough, so (16) is fulfilled. Notice that $N_{\text{quasi}}(r)$ now is the counting function of the subset of the quasimodes obtained after the procedure above. Let us apply what we have already proved to this subset of quasimodes. By (5), $N_{\text{res}}(m_j + 1) \geq Mm_j^{n^\sharp} - C$, which contradicts (3) if we choose M large enough. $\square$

Proof of Corollary 2. Assume that $N_{\text{quasi}}(r) \geq p(r) + q(r)$ with $p(r)$ such that $|p'(r)| \leq Cr^N$, $\forall r \geq 1$ with some $N > 0$. Then $|p(r) - p(r - r^{-k})| \leq Cr^{-k+N} \to 0$, as $r \to \infty$, if $k > N$. Therefore, the term $p(r - r^{-k})$ in (5) can be replaced by $p(r)$ with changing the constant C_k.

4 An Application: Sharp lower bound on the number of the resonances generated by an elliptic broken ray.

In this section we apply Theorem 2 to the following classical problem. Let $\Omega \subset \mathbf{R}^n$ be a domain with a compact complement (*obstacle*) $\mathcal{O} = \mathbf{R}^n \setminus \Omega$ with smooth boundary. Let $P = -\Delta$ in $\mathcal{H} := L^2(\Omega)$ be the self-adjoint realization of the Laplacian with Dirichlet boundary conditions. Resonances of P can be defined by means of classical scattering theory as the poles of the meromorphic continuation of the cut-off resolvent. They are also the poles of the scattering matrix [L-P]. The (modified) Lax-Phillips Conjecture is that in case of trapped light rays there are infinitely many resonances in a strip around the real line. If the trapping is "strong" enough, one should have a sequence of resonances actually converging to the real axis.

A classical example of a trapped ray which is expected to produce many resonances near the real axis, is an elliptic broken ray. Quasimodes associated with such a ray were constructed in [P1] (see also [L]). In [St-V2] it was shown that there exists an infinite sequence of resonances converging rapidly to the real line as a consequence of the existence of the quasimodes. The results in [T-Z] provide at least a linear lower bound on the counting function of those resonances but the possibility of obtaining a sharp bound seems limited without additional arguments. Below we apply Theorem 2 to show that we have the optimal bound cr^n. Notice that in this case $n^\sharp = n$ in (3) (see [M]).

Next we sketch briefly some results from [P1]. Consider a broken periodic bicharacteristic in $T^*\Omega$ with vertices ρ_j, $j = 0, \dots, m$. Assume that

(H1) ρ_0 is an *elliptic* fixed point of the Poincaré map $\mathcal{P}$, i.e., all eigenvalues of $D\mathcal{P}(\rho_0)$ lie on the unit circle and are different from ± 1.

(H2) The Poincaré map $\mathcal{P}$ is 5-elementary, i.e., if $e^{i\alpha_j}$, $0 < |\alpha_j| < \pi$, $j = 1, \dots, n-1$ are the eigenvalues of $D\mathcal{P}(\rho_0)$, then $k_1\alpha_1 + \dots + k_{n-1}\alpha_{n-1} \neq 0$ for $k_1, \dots, k_{n-1}$ integers such that $1 \leq |k_1| + \dots + |k_{n-1}| \leq 5$.

Then one constructs the Birkhoff normal form of $\mathcal{P}$ near $\rho_0 = (0, 0)$ by

$$\mathcal{P}(\theta, I) = \left(\theta + \operatorname{grad} S(I) + O(|I|^N), I + O(|I|^{N+1})\right), \quad (\theta, I) \in T^{n-1} \times \mathbf{R}_+^{n-1},$$

where $S \in C^\infty$, $S(0) = 0$, $\operatorname{grad} S(0) = (\alpha_1, \dots, \alpha_{n-1})$. Finally, we require that

(H3) the Birkhoff form is non-degenerate, i.e., $\det D^2 S(0) \neq 0$.

Under those assumptions, Popov has constructed quasimodes of P with error function $O(\lambda^{-\infty})$ associated to that broken ray and has found an asymptotic formula for the counting function. He proved that

$$N_{\text{quasi}}(r) = \frac{\operatorname{meas}(G_E)}{n(2\pi)^n} r^n + O(r^{n-\gamma}),$$

where $\gamma > 0$ and G_E is a Cantor set with non-zero measure associated with the invariant tori of the Poincaré map. We refer to [P1] for the outline of the proof of this (see also [C-P, Sec. 4] for further details).

A direct application of Theorem 2 yields the following.

Theorem 3 *Let P be the Dirichlet Laplacian in $L^2(\Omega)$ and assume that there exists an elliptic periodic broken ray satisfying (H1), (H2) and (H3). Then there exists a positive function $S(\lambda) = O(\lambda^{-\infty})$, such that for the counting function*

$$N_{\text{res}}(r) = \#\{\lambda \in \operatorname{Res} P;\ 1 < \operatorname{Re}\lambda \leq r,\ 0 < \operatorname{Im}\lambda \leq S(\operatorname{Re}\lambda)\}$$

we have

$$N_{\text{res}}(r) \geq \frac{\operatorname{meas}(G_E)}{n(2\pi)^n} r^n - C r^{n-\gamma},$$

with some $\gamma > 0$, $C \geq 0$ and G_E as above.

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