

LINEARIZING NON-LINEAR INVERSE PROBLEMS AND AN APPLICATION TO INVERSE BACKSCATTERING

PLAMEN STEFANOV AND GUNTHER UHLMANN

ABSTRACT. We propose an abstract approach to prove local uniqueness and conditional Hölder stability to non-linear inverse problems by linearization. The main condition is that, in addition to the injectivity of the linearization A , we need a stability estimate for A as well. That condition is satisfied in particular, if A^*A is an elliptic pseudo-differential operator. We apply this scheme to show uniqueness and Hölder stability for the inverse backscattering problem for the acoustic equation near a constant sound speed.

1. INTRODUCTION

Non-linear inverse problems are often linearized, and reduced to the problem of the injectivity of their linearization. If that linearization is an injective map with a closed range, then this implies local uniqueness and Lipschitz stability, see Theorem 1 below. Most inverse problems in Mathematical Physics however are ill-posed and do not fall in this category. The mere fact that the linearization is injective, if it is, is not enough for local injectivity of the original problem when the closed range condition fails. The closed range condition is also equivalent to a linear stability estimate, see (11) below. The main purpose of this work is to propose a systematic approach to treat inverse problems for which the closed range condition may still hold if we replace the original spaces with new ones satisfying interpolation estimates, see (11). A typical example is to replace C^k or H^s by $C^{k'}$, respectively $H^{s'}$, with different k' , s' . A sufficient condition for the linear stability estimate, besides the injectivity of the linearization A , is A^*A to be an elliptic pseudo-differential operator (Ψ DO). The later is a standard consequence of the theory of elliptic Ψ DOs and elliptic Ψ DOs. Our main result in this direction is Theorem 2 that states, roughly speaking, that linearization plus an appropriate stability estimate of the linearized map imply local injectivity and a conditional Hölder stability for the non-linear map.

We recently used this approach for studying boundary rigidity/lens rigidity questions for compact Riemannian manifolds with boundary [17, 18, 20]. Some of these ideas have been used before, on a case by case basis, for proving local uniqueness with or without a Hölder stability, see for example [6, 10, 11, 15, 16]. Our goal is to systematize this approach, and in particular to understand what makes (at least a large enough class of) ill-posed inverse problems ill posed, and in what cases one can get local uniqueness by linearization. In particular, we show why many severely ill-posed inverse problems cannot be treated this way — the linearization is unstable in any pair of Sobolev spaces. The most famous example of this type is Calderón's problem [4, 22, 25].

We give an application to the inverse backscattering problem for the acoustic equation. In this problem, we are trying to determine the sound speed or, equivalently, the index of refraction, of a medium by measuring the scattering amplitude with the direction of incidence of a plane wave opposite to the direction of the reflected wave. In other words, we are measuring the echos produced

First author partly supported by NSF Grant DMS-0800428.

Second author partly supported by NSF and a Walker Family Endowed Professorship.

by plane waves. This problem arises for instance in ultrasound tomography. We prove injectivity near constant sound speeds and we give a Hölder type conditional stability estimate, see Theorem 4. Conditional Lipschitz stability estimates in different norms were given in [26] based on the local uniqueness proof in [16].

In section 2 we prove the main result — Theorem 2. In section 3, we give sufficient conditions for the stability of the linearized inverse problem required by Theorem 2. Those conditions are based on standard elliptic Ψ DO theory and Fredholm theory. Finally, in section 4, we give the application to the inverse backscattering problem.

2. AN ABSTRACT INVERSE PROBLEM

We describe our abstract inverse problem now. Let $\mathcal{A} : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be a continuous non-linear map between two Banach spaces. Consider the “inverse problem”:

$$(1) \quad \text{Given } h \in \text{Ran}(\mathcal{A}), \text{ find } f \text{ so that } \mathcal{A}(f) = h.$$

Here, we study only the (local) uniqueness and stability questions for (1).

Definition 1. *We say that there is a weak local uniqueness for (1) near $f_0 \in \mathcal{B}_1$, if there exists a neighborhood of f_0 so that for any f in that neighborhood that solves*

$$(2) \quad \mathcal{A}(f) = \mathcal{A}(f_0),$$

we have $f = f_0$.

Definition 2. *We say that there is a strong local uniqueness for (1) near $f_0 \in \mathcal{B}_1$, if there exists a neighborhood of f_0 so that for any f_1, f_2 in that neighborhood with*

$$(3) \quad \mathcal{A}(f_1) = \mathcal{A}(f_2),$$

we have $f_1 = f_2$.

Besides uniqueness, we are often interested in stability estimates. By a classical argument, if $\mathcal{A} : U \rightarrow V$ is injective, where U and V are open, and $K \subset U$ is compact, then \mathcal{A}^{-1} , restricted to $\mathcal{A}(K)$, is continuous. The compactness assumption is one of the ways ill-posedness of many (but not all) inverse problems manifests itself. The continuity of \mathcal{A}^{-1} is viewed as stability, and such kind of results are called conditional stability. One of the central problems in Inverse Problems is to estimate the modulus of continuity of \mathcal{A}^{-1} (restricted appropriately). In other words, we want to find a function $\phi(t)$, $t \rightarrow 0$, so that $\phi \rightarrow 0$, as $t \rightarrow 0$, and

$$\|f_1 - f_2\|_{\mathcal{B}_1} \leq \phi(\|\mathcal{A}(f_1) - \mathcal{A}(f_2)\|_{\mathcal{B}_2})$$

for all f_1, f_2 in some neighborhood of a fixed f_0 , and possibly also, restricted to a compact subset K of \mathcal{B}_1 (hence the stability is conditional). The estimate is of Hölder type, if one can choose $\phi(t) = Ct^\alpha$, $0 < \alpha \leq 1$, and of Lipschitz type, if $\phi(t) = Ct$. We view in this paper inverse problems that allow Hölder estimates as “stable” (Lipschitz stability occurs rarely); while any other are viewed as “unstable”. Recall that for the Electric Impedance Tomography problem, see [4, 22, 25], $\phi(t) = C(\log(1/t))^{-\mu}$, $\mu > 0$, see [2], that tends to 0, as $t \rightarrow 0$, much slower than t^α .

The choice of the compact set K also matters. If it is “too small”, the conditional stability is trivial and not interesting. For example, if K is finite dimensional, and \mathcal{A} is smooth enough, one always has a Lipschitz stability estimate under the assumption that \mathcal{A} has an injective differential, see below. If \mathcal{B}_1 is $H^s(M)$ or $C^k(M)$ with a compact set (or manifold) M , we are interested whether one can choose K to be $H^{s_1}(M)$ or $C^{k_1}(M)$ with $s_1 > s$, or $k_1 > k$. A subspace K consisting of real analytic functions only, for example, will be considered “too small”.

2.1. **“Inverse Problems” in \mathbf{R}^n .** We start with the trivial case when $\mathcal{B}_1 = \mathbf{R}^n$, $\mathcal{B}_2 = \mathbf{R}^m$. Then the problem is reduced to solving (1) with $\mathcal{A} : \mathbf{R}^n \rightarrow \mathbf{R}^m$. Assume that $\mathcal{A} \in C^2$ ($\mathcal{A} \in C^1$ actually suffices, see Theorem 1 below). Let A_{x_0} be the differential of \mathcal{A} at $x = x_0$. Then

$$(4) \quad \mathcal{A}(x) = \mathcal{A}(x_0) + A_{x_0}(x - x_0) + R_{x_0}(x) \quad \text{with } |R_{x_0}(x)| \leq C_{x_0}|x - x_0|^2,$$

for x near x_0 . Assume now that A_{x_0} is injective (which can happen only when $m \geq n$). This immediately implies the estimate

$$(5) \quad |h| \leq C_0 |A_{x_0} h|, \quad \forall h \in \mathbf{R}^n.$$

The easiest way to prove (5) is to note that one can choose $1/C_0$ to be the minimum of $|Ah|$ for h on the (compact) unit sphere. Then the local uniqueness question about the number of possible solutions of $\mathcal{A}(x) = \mathcal{A}(x_0)$ near $x = x_0$ can be answered as follows. We have

$$\begin{aligned} |x - x_0|/C_0 \leq |A_{x_0}(x - x_0)| &\leq |\mathcal{A}(x) - \mathcal{A}(x_0)| + |R_{x_0}(x)| \\ &\leq |\mathcal{A}(x) - \mathcal{A}(x_0)| + C_{x_0}|x - x_0|^2. \end{aligned}$$

Therefore, if $|x - x_0| \leq (2C_0C_{x_0})^{-1}$, one gets

$$(6) \quad |x - x_0| \leq 2C_0 |\mathcal{A}(x) - \mathcal{A}(x_0)|.$$

We therefore have local uniqueness and a Lipschitz stability estimate. Moreover, one can choose C_{x_0} to be locally independent of x_0 , if $\mathcal{A} \in C^2$, and the same applies to C_0 . Thus one gets that there exists a neighborhood U of x_0 in \mathbf{R}^n , so that for any $x_1 \in U$, $x_2 \in U$, the equality $\mathcal{A}(x_1) = \mathcal{A}(x_2)$ implies $x_1 = x_2$. Moreover, similarly to (6), one has

$$(7) \quad |x_1 - x_2| \leq 2C_0 |\mathcal{A}(x_1) - \mathcal{A}(x_2)|, \quad x_1 \in U, x_2 \in U.$$

Note that \mathcal{B}_2 can be infinite dimensional here, and the arguments do not change. In particular, if $\mathcal{A} \in C^{1,\mu}$, $\mu > 0$, and A_{x_0} is injective, then one always has local uniqueness and the stability estimate (7) of Lipschitz type. If \mathcal{B}_1 is infinite dimensional, then $\mathcal{A}|_K$ has that property for any finite dimensional K .

2.2. **A well posed inverse problem. The local injectivity theorem in Banach spaces.** We return to the case where $\mathcal{B}_{1,2}$ are finitely dimensional Banach spaces. Consider now a truly infinitely dimensional version of our abstract inverse problem under assumptions that make it well-posed, and therefore, not typical. The main assumption is that $A_{x_0} : \mathcal{B}_1 \rightarrow A_{x_0}(\mathcal{B}_1)$ is an isomorphism, i.e., that the differential A_{x_0} is not only injective but also has a closed range. Then we have the following theorem that is closely related to the inverse function theorem and the implicit function theorems in Banach spaces (A_{x_0} is assumed to be an isomorphism from \mathcal{B}_1 to \mathcal{B}_2 then).

Theorem 1 (Local Injectivity Theorem, [1]). *Let $\mathcal{A} : U \subset \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be C^1 , with $U \ni f_0$ open, let A_{f_0} be injective and let it have a closed range. Then there exists a neighborhood $V \subset U$ of f_0 , on which \mathcal{A} is injective. Moreover, the inverse $\mathcal{A}^{-1} : \mathcal{A}(V) \rightarrow U$ is Lipschitz continuous.*

This theorem shows that we have the same conclusions as in the finitely dimensional case, with a Lipschitz stability of non-conditional type.

We refer to [1] for the definition of the differential A_{f_0} of \mathcal{A} at a fixed f_0 and for a definition of (a Fréchet) differentiable map. In particular, if the Gâteaux derivative (the directional derivative in all directions) exists in some open set and it is continuous there, then \mathcal{A} is Fréchet differentiable as well and the two derivatives coincide. Then \mathcal{A} is said to be C^1 .

2.3. An abstract ill-posed inverse problem. We turn our attention now to an abstract inverse problem that may not be well posed. In order to study the inverse problem (1) by linearization, we assume that the differential A_{f_0} of \mathcal{A} exists at $f = f_0$ with some f_0 . Actually, we will assume something slightly stronger — a quadratic estimate on the remainder, that roughly speaking means that \mathcal{A} is $C^{1,1}$. Namely, we assume that

$$(8) \quad \mathcal{A}(f) = \mathcal{A}(f_0) + A_{f_0}(f - f_0) + R_{f_0}(f), \quad \text{with } \|R_{f_0}(f)\|_{\mathcal{B}_2} \leq C_{f_0} \|f - f_0\|_{\mathcal{B}_1}^2,$$

for f in some neighborhood in \mathcal{B}_1 of some $f_0 \in \mathcal{B}_1$.

It may happen that we have the weaker estimate

$$(9) \quad \|R_{f_0}(f)\|_{\tilde{\mathcal{B}}_2} \leq C_{f_0} \|f - f_0\|_{\tilde{\mathcal{B}}_1}^2$$

for f in some neighborhood in $\tilde{\mathcal{B}}_1$ of some $f_0 \in \tilde{\mathcal{B}}_1$, where $\tilde{\mathcal{B}}_1 \subset \mathcal{B}_1$, $\tilde{\mathcal{B}}_2 \supset \mathcal{B}_2$. Here and below, $\tilde{\mathcal{B}}_1 \subset \mathcal{B}_1$, $\tilde{\mathcal{B}}_2 \supset \mathcal{B}_2$ mean also that

$$(10) \quad \|\cdot\|_{\mathcal{B}_1} \leq C \|\cdot\|_{\tilde{\mathcal{B}}_1}, \quad \|\cdot\|_{\tilde{\mathcal{B}}_2} \leq C \|\cdot\|_{\mathcal{B}_2}.$$

In such case, we can simply replace \mathcal{B}_2 by $\tilde{\mathcal{B}}_2$; and \mathcal{B}_1 by $\tilde{\mathcal{B}}_1$. In the boundary rigidity problem [18], actually $\tilde{\mathcal{B}}_1 = C^1(M) \supset \mathcal{B}_1 = C^2(M)$, i.e., (8) is even stronger. Keeping $\tilde{\mathcal{B}}_1$, $\tilde{\mathcal{B}}_2$ and assuming appropriate interpolation estimates would improve a bit the exponent $\mu_1\mu_2$ in the Hölder exponent in Theorem 2 below but for the sake of simplicity, we will not pursue this. In other words, we have some freedom how to choose our spaces $\mathcal{B}_{1,2}$, and we choose them in a way that makes (8) true. This choice might not be the optimal for the analysis of A_{f_0} however.

Assume now that A_{f_0} is injective. This does not imply automatically a stability estimate of the type (5), of course. Assume in addition, that such an estimate holds in some norms, i.e.,

$$(11) \quad \|h\|_{\mathcal{B}'_1} \leq C \|A_{f_0}h\|_{\mathcal{B}'_2}, \quad \forall h \in \mathcal{B}_1,$$

with some Banach spaces $\mathcal{B}'_2 \subset \mathcal{B}_2$, $\mathcal{B}'_1 \supset \mathcal{B}_1$. In applications, we want not only such \mathcal{B}'_1 , \mathcal{B}'_2 to exist (they always do, one can set $\mathcal{B}'_1 = \mathcal{B}_1$, $\|A_{f_0}h\|_{\mathcal{B}'_2} = \|h\|_{\mathcal{B}_1}$ on $\text{Ran } A_{f_0}$) but we also want them to be “reasonable” spaces independent of f_0 , typically some H^s or C^k spaces. Then, if $\mathcal{B}_1 = \mathcal{B}'_1$, $\mathcal{B}_2 = \mathcal{B}'_2$ one can apply Theorem 1 to prove local injectivity and a Lipschitz stability estimate of the type (7). The inverse problem is well-posed then.

If the pairs of spaces in (8) and (11) cannot be chosen to be the same, and often that is the case, one can still prove local uniqueness and a stability estimate but of a Hölder and conditional type, if certain interpolation estimates are satisfied, as we show below. Then the inverse problem is ill-posed but we think of it as mildly ill-posed.

Before formulating this as a formal statement, we will give an example where (11) does not hold (in any Sobolev spaces), and there is no local uniqueness.

Example 1. Let l^2 be the Hilbert space of sequences $x = \{x_k\}_{k=1}^\infty$ with norm $\|x\|^2 = \sum |x_k|^2$. The Sobolev spaces h^s are defined through the norms $\|x\|_{h^s}^2 = \sum k^{2s} |x_k|^2$. We think of the components of x as the Fourier coefficients of a 2π -periodic function, which explains why we call h^s a Sobolev space. Define the non-linear map \mathcal{A} by

$$(12) \quad \mathcal{A}(x) = Ex - (x, a)x,$$

where

$$a = \{1/k\}, \quad E = \text{diag}\{e^{-k}\}.$$

Clearly, $\mathcal{A}(0) = 0$, the linearization of \mathcal{A} near $x = 0$ is $A_0 = E$, and the latter is an injective map. Moreover, there is an inverse E^{-1} with a dense domain, but E^{-1} is unbounded as an

operator from any Sobolev space to any other one, thus (11) does not hold in such spaces. Since $[\mathcal{A}(x)]_k = e^{-k}x_k - x_k \sum x_m/m$, we get that

$$x^{(k)} = (0, \dots, ke^{-k}, 0, \dots),$$

where the non-zero entry is the k -th one, is a solution to $\mathcal{A}(x) = 0$. Therefore, in any neighborhood of 0 in l^2 there are infinitely many solutions of $\mathcal{A}(x) = 0$ despite the fact that the linearization A_0 is injective. Moreover, for any s , in any neighborhood of 0 in h^s , there are still infinitely many solutions. Therefore, there is no local uniqueness in this case, neither weak nor strong, in any Sobolev space. One can still get local uniqueness for $\|x\|_* \ll 1$ by choosing an h^∞ type of norm $\|\cdot\|_*$ with an exponential weight, namely $\|x\|_* = \|E^{-1}x\|$. This however translates into a neighborhood of the origin of 2π -periodic functions that involves certain real analytic functions only. We think of such topology as “unreasonably restrictive”.

Based on that example, one can consider the following map in $L^2(\mathbf{R}^n)$

$$\mathcal{A}(f) = \phi * f - (f, a)f.$$

Choose $\hat{\phi}(\xi) = \sum \chi_{\{k-1 \leq |\xi| \leq k\}}(\xi) e^{-k|\xi|}$, where χ_K stands for the characteristic function of K . Also, fix $a \in L^2$ with $|\hat{a}(\xi)| \geq (1+|\xi|)^{-m}$ for some m . Then $f = 0$ solves $\mathcal{A}(f) = 0$, and \mathcal{A} has an injective differential at $f = 0$. Also, if \hat{f} is supported in $\{k-1 \leq |\xi| \leq k\}$, and \hat{f} solves $(\hat{f}, \hat{a}) = (2\pi)^n e^{-k|\xi|}$, such an f would also solve $\mathcal{A}(f) = 0$. This gives us a sequence of such solutions, converging exponentially fast to 0 in any H^s .

Our main abstract theorem for linearizing inverse problems with a “stable” linearization is the following.

Theorem 2. *Let \mathcal{A} be as above, and assume (8), (11) with $\mathcal{B}_1 \subset \mathcal{B}'_1$, $\mathcal{B}'_2 \subset \mathcal{B}_2$ as above. Assume also that there exist Banach spaces $\mathcal{B}''_2 \subset \mathcal{B}'_2$, $\mathcal{B}''_1 \subset \mathcal{B}_1$ so that A_{f_0} continuously maps \mathcal{B}''_1 into \mathcal{B}''_2 and the following interpolation estimates hold*

$$(13) \quad \|u\|_{\mathcal{B}'_2} \leq C \|u\|_{\mathcal{B}''_2}^{\mu_2} \|u\|_{\mathcal{B}''_1}^{1-\mu_2}, \quad \|h\|_{\mathcal{B}_1} \leq C \|h\|_{\mathcal{B}''_1}^{\mu_1} \|h\|_{\mathcal{B}'_1}^{1-\mu_1} \quad \mu_1, \mu_2 \in (0, 1], \quad \mu_1 \mu_2 > 1/2.$$

(a) *For any $L > 0$ there exists $\epsilon > 0$, so that for any f with*

$$(14) \quad \|f - f_0\|_{\mathcal{B}_1} \leq \epsilon, \quad \|f\|_{\mathcal{B}''_1} \leq L,$$

one has the conditional stability estimate

$$(15) \quad \|f - f_0\|_{\mathcal{B}_1} \leq C(L) \|\mathcal{A}(f) - \mathcal{A}(f_0)\|_{\mathcal{B}''_2}^{\mu_1 \mu_2}, \quad C(L) = CL^{2-\mu_1-\mu_2}.$$

In particular, there is a weak local uniqueness near f_0 , i.e., if $\mathcal{A}(f) = \mathcal{A}(f_0)$, then $f = f_0$.

(b) *Assume in addition that there is a Banach space $\mathcal{K} \subset \mathcal{B}''_1$ so that (11) holds for f_0 replaced with f close enough to f_0 in \mathcal{K} , and $A_f : \mathcal{B}''_1 \rightarrow \mathcal{B}''_2$ is uniformly bounded for such f . Then there exists $\epsilon > 0$, so that for any f_1, f_2 with*

$$(16) \quad \|f_1 - f_0\|_{\mathcal{K}} \leq \epsilon, \quad \|f_2 - f_0\|_{\mathcal{K}} \leq \epsilon,$$

one has the conditional stability estimate

$$(17) \quad \|f_1 - f_2\|_{\mathcal{B}_1} \leq C \|\mathcal{A}(f_1) - \mathcal{A}(f_2)\|_{\mathcal{B}''_2}^{\mu_1 \mu_2}.$$

In particular, there is a strong local uniqueness near f_0 , i.e., if $\mathcal{A}(f_1) = \mathcal{A}(f_2)$, then $f_1 = f_2$.

A theorem of a similar type has been proven by Pagani [10, 11].

Remark 1. Typical choices of the Banach spaces above are C^k or H^s spaces. Interpolation estimates for those norms are well known. In \mathbf{R}^n , for example, one has

$$(18) \quad \|u\|_{H^s} \leq C \|u\|_{H^{s_1}}^{\alpha_1} \|u\|_{H^{s_2}}^{\alpha_2}, \quad s = s_1\alpha_1 + s_2\alpha_2, \quad \alpha_1 + \alpha_2 = 1, \quad \alpha_1 \geq 0, \alpha_2 \geq 0.$$

This can be easily verified by using the Fourier transform. Similar estimates hold for Sobolev spaces and the C^k spaces in domains/manifolds, see [24]. For example, if M is a compact manifold with boundary, then

$$(19) \quad \|u\|_{C^k(M)} \leq C \|u\|_{C^{k_1}(M)}^{\alpha_1} \|u\|_{C^{k_2}(M)}^{\alpha_2}, \quad k = k_1\alpha_1 + k_2\alpha_2, \quad \alpha_1 + \alpha_2 = 1, \quad \alpha_1 \geq 0, \alpha_2 \geq 0.$$

Interpolation estimates that include an H^s and a C^k norm can be obtained by using Sobolev embedding theorems.

Remark 2. Assume, for example, that all spaces above are Sobolev spaces. Then one can make the exponent $\mu_1\mu_2$ in (15), (17) arbitrary close to 1, thus making the stability estimate “almost Lipschitz”. The price to pay for that is to assume that f is a priori bounded in H^s with $s \gg 1$, see (18). Alternatively, one may try to choose \mathcal{B}'_1 , \mathcal{K} close to \mathcal{B}_1 , which is the natural limit but that will decrease μ_1 . Since $\mu_1 > 1/(2\mu_2) \geq 1/2$, this puts a limit on how close \mathcal{B}'_1 , \mathcal{K} can be to \mathcal{B}_1 . If $\mathcal{B}_1 = H^s$, $\mathcal{B}'_1 = H^{s'}$, for example, $s' < s$, then we will have at least $\mathcal{B}'_1 = H^{s''}$ with s'' so $(s' + s'')/2 > s$. Therefore, we cannot take $s'' > s$ arbitrary close to s unless $s' = s$. The choice of s'' in this case is further restricted by μ_2 .

Remark 3. The second inequality in (14) is a typical compactness condition, when we work with Sobolev spaces or C^k spaces in a bounded domain or on a compact manifold. It may seem strange that there is no such explicit condition in (b). It is actually there, since (16) implies such a condition with \mathcal{B}'_2 replaces by \mathcal{K} . Actually, using interpolation estimates, one could write (14) in the form (16), and vice -versa, with a slight change of the Banach spaces (i.e., small change of k in C^k , etc.). We formulated (16) in a form different than (14) in order to avoid introducing yet another Banach space. What makes (15) and (17) conditional estimates is that in the conditions (14), respectively (16), there is at least one norm stronger than the norm in the l.h.s. of (15), respectively (17). Note also that (15), (17) can be formulated with different choices of the norms.

Proof of Theorem 2. We start with (a). By (8) and (10),

$$\|A_{f_0}(f - f_0)\|_{\mathcal{B}_2} \leq \|\mathcal{A}(f) - \mathcal{A}(f_0)\|_{\mathcal{B}_2} + C\|f - f_0\|_{\mathcal{B}'_1}^2$$

By (11) and the Hölder inequality $(a + b)^\mu \leq a^\mu + b^\mu$ for $a \geq 0, b \geq 0$, we get

$$\|A_{f_0}(f - f_0)\|_{\mathcal{B}'_2} \leq CL^{1-\mu_2} \left(\|\mathcal{A}(f) - \mathcal{A}(f_0)\|_{\mathcal{B}_2}^{\mu_2} + \|f - f_0\|_{\mathcal{B}'_1}^{2\mu_2} \right).$$

The constant $CL^{1-\mu_2}$ comes from estimating the term $\|A_{f_0}(f - f_0)\|_{\mathcal{B}'_2}^{1-\mu_2}$ that is bounded by the assumption on A_{f_0} and by the second inequality in (14). By (11), and the second interpolation inequality in (13),

$$\|f - f_0\|_{\mathcal{B}_1} \leq CL^{2-\mu_1-\mu_2} \left(\|\mathcal{A}(f) - \mathcal{A}(f_0)\|_{\mathcal{B}_2}^{\mu_1\mu_2} + \|f - f_0\|_{\mathcal{B}'_1}^{2\mu_1\mu_2} \right).$$

Since $2\mu_1\mu_2 > 1$, for $\epsilon \ll 1$ we get (15). Note that the condition on ϵ has the form $\epsilon \leq (CL^{2-\mu_1-\mu_2})^{-1}$, where C depends on \mathcal{A} only. This proves (a).

To prove (b) we note that the same proof can be applied under the assumptions of (b). Indeed, we start with

$$\|A_{f_1}(f_2 - f_1)\|_{\mathcal{B}_2} \leq \|\mathcal{A}(f_2) - \mathcal{A}(f_1)\|_{\mathcal{B}_2} + C\|f_2 - f_1\|_{\mathcal{B}'_1}^2.$$

Then we proceed as above under the condition $\|f_1 - f_0\|_{\mathcal{K}} \leq \epsilon_1 \ll 1$ that would guarantee (11) with f_0 replaced by f_1 . Then (17) holds if $\|f_2 - f_1\|_{\mathcal{B}_1} \leq \epsilon_2 \ll 1$. Those two conditions certainly hold if ϵ in (16) is small enough. \square

Remark 4. The variety of norms in the theorem above is typical. For example, consider the boundary rigidity problem, where one has to recover a Riemannian metric g on a compact manifold M , up to a group of isometries, from the distance function $\rho(x, y)$ restricted to $\partial M \times \partial M$, see e.g., [17, 21]. Linearizing, we get a tensor tomography problem: recover a symmetric 2-tensor field f from its integrals

$$(20) \quad If(\gamma) = \int f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t) dt$$

along all maximal geodesics in M . Since this transform vanishes on potential tensors (symmetric covariant derivatives of vector fields v vanishing on ∂M), we restrict I on solenoidal (i.e., divergence free) tensors that form an orthogonal complement to potential ones.

The role of A_{g_0} then is played by I related to g_0 . We apply the adjoint operator I^* taken in L^2 spaces with respect to suitable measures. The role of A_{g_0} now is played by I^*I , and we made the following choices, see [21]:

$$(21) \quad \mathcal{B}_1 = C^2(M), \quad \mathcal{B}'_1 = L^2(M), \quad \mathcal{B}''_1 = H^s(M), \quad s \gg 1, \quad \mathcal{K} = C^k(M), \quad k \gg s,$$

$$(22) \quad \mathcal{B}_2 = L^\infty(M_1), \quad \mathcal{B}'_2 = \tilde{H}^2(M_1), \quad \mathcal{B}''_2 = H^{s-1}(M_1),$$

where $M_1 \ni M$, and $\tilde{H}^2(M_1)$ is an appropriate Hilbert space such that $H^2(M_1) \subset \tilde{H}^2(M_1) \subset H^1(M_1)$. Here and below, $A \Subset B$ means that the closure of A is contained in the interior of B ; and $B \ni A$ means that $A \Subset B$. There are other complications there, coming from the fact that we have to work with equivalence classes of isometric metrics. Then I^*I acting on symmetric tensors in $H^s(M)$, extended as zero to M_1 , is not a bounded operator with values in $H^{s-1}(M_1)$ even though I^*I is a Ψ DO of order -1 . The reason is that $f \in H^s(M)$, extended as zero, belongs to $H^s(M_1)$ only if the first s normal derivatives of f (assuming that s is integer) vanish on ∂M . This is the reason for introducing the space \tilde{H}^2 . A stable recovery of the derivatives of the metric on ∂M from the boundary distance function is used in [21] to deal with this problem. As a result, we prove a Hölder type of stability estimate for the boundary rigidity problem, using ideas similar to those in the proof of Theorem 2.

3. STABILITY OF LINEAR INVERSE PROBLEMS

Most of the material in this section is based on the theory of elliptic Ψ DOs and on the theory of Fredholm operators, see e.g., [23]. We give proofs for the sake of completeness of the exposition.

Let $A : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be a bounded linear map between two Banach spaces. Assume in addition, that A is injective. We are interested whether one has the stability estimate

$$(23) \quad \|f\|_{\mathcal{B}_1} \leq C\|Af\|_{\mathcal{B}_2}$$

with some constant $C > 0$. It is well known, that if A is smoothing, and \mathcal{B}_1 and \mathcal{B}_2 are among C^k or H^s with finite k , or s , then there cannot be such a stability estimate. We will quickly review those arguments below.

First, note that (23) is not equivalent to invertibility, because (the closure of) the range $\text{Ran } A$ of A can be much smaller than \mathcal{B}_2 . On the other hand, one can replace \mathcal{B}_2 by $\mathcal{B}'_2 = \overline{\text{Ran } A}$, and then (23) is equivalent to the invertibility of $A : \mathcal{B}_1 \rightarrow \mathcal{B}'_2$, as the next proposition shows.

Proposition 1. *(23) holds if and only if A is injective and $\text{Ran } A$ is closed.*

Proof. Assume that $\text{Ran } A$ is closed and A is injective. Then $A : \mathcal{B}_1 \rightarrow \mathcal{B}'_2$ has range that coincides with \mathcal{B}'_2 . Since A is injective, then A^{-1} exists with domain the whole \mathcal{B}'_2 . By the open mapping theorem, A^{-1} is bounded, hence (23) holds.

Assume (23). Then the injectivity and the closedness of $\text{Ran } A$ follow easily. \square

Proving that $\text{Ran } A$ is closed is not always straightforward. Proving that it is not (when it is not), is usually easier. As an example, suppose that $\mathcal{B}_1 = \mathcal{B}_2 = L^2(\Omega)$, where Ω is a domain in \mathbf{R}^n with smooth boundary, or $\Omega = \mathbf{R}^n$. If there exists $s > 0$ so that $Af \in H^s(\Omega)$ for any $f \in L^2(\Omega)$, then (23) cannot be true. Indeed, if we know that A has a trivial cokernel (for example, because it is injective and self-adjoint, as the operator N_w below), then $\text{Ran } A$ cannot be closed. In the general case, assume that $\text{Ran } A$ is closed. Then the latter is a Banach space as well, and A is invertible on it. But A is compact, so we get that the identity on $L^2(\Omega)$ is compact, that is not true.

On the other hand, (23) may still hold with $\mathcal{B}_2 = H^s(\Omega)$, if $\text{Ran } A$ is closed in $H^s(\Omega)$. Take for example $A = (1 - \Delta)^{-1}$ on $L^2(\mathbf{R}^n)$, and $s = 2$. Then one can ask the more general question: is there a stability estimate of the type

$$(24) \quad \|f\|_{L^2} \leq C \|Af\|_{H^s}$$

for some $s > 0$? If so, it will be a weaker substitute for (23), if the latter does not hold but it will still allow us to apply Theorem 2. For (24) to be true, \mathcal{A} must map L^2 into H^s , therefore, this is the same question as before, with $\mathcal{B}_2 = H^s$.

Example 2 (the weighted X-ray transform). In \mathbf{R}^n , let I_w be the weighted X-ray transform

$$(25) \quad I_w f(x, \theta) = \int_{\mathbf{R}} w(x + t\theta, \theta) f(x + t\theta) dt, \quad x \in \mathbf{R}^n, \theta \in S^{n-1},$$

where w is a continuous function¹. Let us restrict I_w to functions supported in $\bar{\Omega}$, where $\Omega \subset \mathbf{R}^n$ is a strictly convex domain with a smooth boundary. One can assume then that $(x, \theta) \in \partial_- S\Omega$, which means that $x \in \partial\Omega$, and $\theta \in S^{n-1}$ points into Ω . We equip $\partial_- S\Omega$ with the measure $d\Sigma = |\theta \cdot \nu(x)| dS_x d\theta$, where dS_x is the measure on $\partial\Omega$ induced by the Euclidean measure in \mathbf{R}^n , and $d\theta$ is the measure on S^{n-1} . Then one can show that $I_w : L^2(\Omega) \rightarrow L^2(\partial_- S\Omega, d\Sigma)$ [14], see also the remark following (28) below. Let

$$(26) \quad N_w = I_w^* I_w : L^2(\Omega) \rightarrow L^2(\Omega)$$

be the ‘‘normal operator’’. We can think of the lines through Ω as a manifold, and $\partial_- S\Omega$ then is a global chart for them (singular at the lines tangent to $\partial\Omega$). The measure $d\Sigma$ remains invariant if we replace $\partial\Omega$ by another strictly convex surface encompassing Ω . Thus one can expect that N_w will not depend on $\partial\Omega$. Indeed, one can easily get that (see [17])

$$(27) \quad N_w f(x) = c_n \int \frac{W(x, y) f(y)}{|x - y|^{n-1}} dy$$

with

$$(28) \quad W(x, y) = \bar{w}\left(x, -\frac{x-y}{|x-y|}\right) w\left(y, -\frac{x-y}{|x-y|}\right) + \bar{w}\left(x, \frac{x-y}{|x-y|}\right) w\left(y, \frac{x-y}{|x-y|}\right).$$

Now we think of N_w as the operator defined by the formula above, with $w(x, \theta)$ extended for all x . It is easy to see that N_w maps $L^2_{\text{comp}}(\mathbf{R}^n)$ into $L^2_{\text{loc}}(\mathbf{R}^n)$. In particular, N_w is bounded on $L^2(\Omega)$. That shows that $I_w : L^2(\Omega) \rightarrow L^2(\partial_- S\Omega, d\Sigma)$ is bounded.

¹In the published version, the weight in (25) wrongly appears as $w(x, \theta)$.

Let $\Omega_1 \ni \Omega$ be another domain with the properties of Ω . Then $N_w : L^2(\Omega) \rightarrow L^2(\Omega_1)$ is also bounded, and I_w is injective if and only if $N_w : L^2(\Omega) \rightarrow L^2(\Omega_1)$ is injective, where one can replace Ω_1 by Ω as well, see Lemma 1 below.

If $w \in C^\infty(\bar{\Omega})$, then N_w is a Ψ DO of order -1 , as we will see below. Thus N_w maps $L^2(\Omega)$ into $H^1(\Omega)$ and a stability estimate for N_w of the kind (24), with $\mathcal{B}_1 = \mathcal{B}_2 = L^2(\Omega)$, or $\mathcal{B}_2 = L^2(\Omega_1)$, is not possible. If $w \in C^1(\bar{\Omega})$ only, then one can still show that N_w maps $L^2(\Omega)$ into $H^1(\Omega)$ using the theory of operators with singular kernels, see [8] for details. We then change \mathcal{B}_2 and replace it with $H^1(M_1)$. The question we ask now is the following: assuming that I_w (and therefore N_w) is injective for some class of w 's, what conditions on w would guarantee the stability estimate

$$(29) \quad \|f\|_{L^2(\Omega)} \leq C \|N_w f\|_{H^1(\Omega_1)}, \quad \forall f \in L^2(\Omega).$$

We will show below that a sufficient condition for that is that N_w is elliptic, that can be easily formulated in terms of w .

We return to the general case and the estimate (23). In the example above, if N_w is elliptic, then it is also a Fredholm operator. Recall that $F : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is called Fredholm, if there exist bounded operators $A : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ and $B : \mathcal{B}_2 \rightarrow \mathcal{B}_1$ so that $AF - \text{Id}$ and $FB - \text{Id}$ are compact. Then $\text{Ran } F$ is necessarily closed, and F has a finitely dimensional kernel $\text{Ker } F$ and a finitely dimensional cokernel $\text{Coker } A = \mathcal{B}_2 / \text{Ran } A$. Equivalently, F is Fredholm if its kernel and cokernel are finitely dimensional.

In case the inverse problem is over-determined, we cannot expect that A would have a finite cokernel, as in the example above. For this reason, if \mathcal{B}_2 is a Hilbert space, we will study whether $N = A^*A$ is a Fredholm operator, and the consequences of that. In the applied literature, applying A^* is viewed a ‘‘back-projection’’ that, in inverse boundary value problems, returns our data from the boundary back to the domain. Another reason to study A^*A instead of A is that, as in the example above, A might be a Fourier Integral Operator (FIO) but A^*A is a Ψ DO, thus easier to study. The latter is not always true for any FIO, so some conditions are needed. In case of integrals along geodesics, see (20), this is guaranteed by the assumption of no conjugate points, see sections 3 and 5 in [17], where the injectivity of the exponential map is used to guarantee that the Schwartz kernel of A^*A has singularities on the diagonal only. If the metric is Euclidean, there are no conjugate points, of course. Again the example above shows that we may need to restrict N to a subspace \mathcal{L} of \mathcal{H} in order to avoid working with Ψ DOs on manifolds with boundary. Then $F|_{\mathcal{L}}$ may not have finite cokernel anymore. For this reason, we will only study operators F that are *upper semi-Fredholm*, i.e., $\text{Ker } F$ is finite dimensional, and $\text{Ran } F$ is closed. Equivalently, F is upper semi-Fredholm, if there exists a bounded left inverse A modulo compact operators ($AF - \text{Id}$ is compact). Note that if $N = A^*A$ is Fredholm, then A itself is upper semi-Fredholm.

We are not losing generality when replacing A by $N = A^*A$ as the next lemma shows. Note that also stability estimates for N of the type (29) and (31) below imply stability estimates for A of the type (23), (24) as well.

Lemma 1. *Let $A : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be bounded, where $\mathcal{B}_{1,2}$ are Hilbert spaces, and let $\mathcal{L} \subset \mathcal{B}_1$ be a (closed) subspace. The following statements are equivalent.*

- (a) $A : \mathcal{L} \rightarrow \mathcal{B}_2$ is injective;
- (b) $A^*A : \mathcal{L} \rightarrow \mathcal{B}_1$ is injective.
- (c) $\Pi_{\mathcal{L}} A^*A : \mathcal{L} \rightarrow \mathcal{L}$ is injective, where $\Pi_{\mathcal{L}}$ is the orthogonal projection to \mathcal{L} .

Proof. Let $f \in \mathcal{L}$. Then

$$(30) \quad (A^*A f, f)_{\mathcal{B}_1} = \|A f\|_{\mathcal{B}_2}^2.$$

This proves the implication (a) \Rightarrow (b). If $Af = 0$ with $f \in \mathcal{L}$, then $A^*Af = 0$ as well, therefore (b) \Rightarrow (a). The equivalence of (c) and (a) follows in the same way. Indeed, assume (a) and let $\Pi_{\mathcal{L}}A^*Af = 0$ with some $f \in \mathcal{L}$. Then by (30), $Af = 0$, hence $f = 0$. Assume now (c) and let $Af = 0$ with $f \in \mathcal{L}$. Apply $\Pi_{\mathcal{L}}A^*$ to get $\Pi_{\mathcal{L}}A^*Af = 0$, hence $f = 0$. \square

Proposition 2. *Let $N : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be upper semi-Fredholm. Assume in addition, that N is injective on some (closed) subspace $\mathcal{L} \subset \mathcal{H}$. Then:*

(a) *There exists $C > 0$, so that*

$$(31) \quad \|f\|_{\mathcal{B}_2} \leq C\|Nf\|_{\mathcal{B}_1}, \quad \forall f \in \mathcal{L}.$$

(b) *Estimate (31) remains true for any other bounded operator close enough to N with a uniform $C > 0$.*

Clearly, one can always take $\mathcal{L} = (\text{Ker } N)^\perp$ if \mathcal{B}_1 is a Hilbert space. The way we formulated the proposition above however is intended to underline the fact that if we can prove injectivity on some explicitly given subspace that is of interest to us ($L^2(\Omega)$ in the example above), then we automatically have stability.

Proof of Proposition 2. The operator $N : \mathcal{L} \rightarrow N(\mathcal{L})$ is bounded, injective, and has a closed range. By Proposition 1, (31) holds. This proves (a).

To prove (b), it is enough to write

$$(32) \quad \|f\|_{\mathcal{B}_1}/C \leq \|Nf\|_{\mathcal{B}_2} \leq \|(N - \tilde{N})f\|_{\mathcal{B}_2} + \|\tilde{N}f\|_{\mathcal{B}_2}.$$

\square

Remark 5. In the tensor tomography problems, see (20), one has to use a finer version of Proposition 2(b). Then $N = N_g$ depends continuously on the metric g , and $N_g : H^1(M) \rightarrow \tilde{H}^2(M_1)$, where $M_1 \ni M$ and tensors in M are always extended as 0 outside M . The space $\tilde{H}^2(M_1)$ has the property that $H^2(M_1) \subset \tilde{H}^2(M_1) \subset H^1(M_1)$. On the other hand, $\tilde{H}^2(M_1)$ is “too large”, and $\text{Ran } N_g$ is not closed there. It turns out, that one can construct a left parametrix Q_g so that $Q_g N_g = \text{Id} + K_g$, where K_g is a smoothing operator. However, $Q_g : \tilde{H}^2(M_1) \rightarrow H^1(M)$ is not bounded; we can show only that $Q_g : \tilde{H}^2(M_1) \rightarrow L^2(M)$ is bounded. On the other hand, $Q_g N_g$ is bounded on $H^1(M)$, of course. This requires some modification of the arguments above, and the resulting estimate is

$$\|f^s\|_{L^2(M)} \leq C\|N_g f\|_{\tilde{H}^2(M)}, \quad \forall f \in H^1(M)$$

with $C > 0$ locally uniform in g . Here f^s is the orthogonal projection of f onto a certain space, called solenoidal tensors, where N_g is injective by assumption. Note that (32) does not apply with $\mathcal{B}_1 = L^2(M)$ and $\mathcal{B}_2 = \tilde{H}^2(M_1)$ because N_g is not bounded as an operator between those two spaces. On the other hand, one cannot replace \mathcal{B}_1 by $H^1(M)$ because $N_g(H^1(M))$ is not closed in $\tilde{H}^2(M)$. For more details, and for the needed modifications in the arguments, we refer to [17, 18].

The following proposition is well known, see e.g., [23]. We define Sobolev spaces on a compact manifold by fixing a finite atlas. We assume in the proposition below that N is of order $-m \leq 0$ in view of applications to tomography problems. The order $-m$ does not need to be non-positive, and one can study N as an operator mapping H^s into H^{s+m} for s not necessarily 0 by the usual technique — applying powers of $1 - \Delta$ to the left and right.

Proposition 3. *Let $N \in \Psi^{-m}(M)$, $m \geq 0$ be an elliptic Ψ DO on a compact manifold M .*

(a) *Then for any $l \geq 0$, one has*

$$(33) \quad \|f\|_{L^2(M)} \leq C_l \left(\|Nf\|_{H^m(M)} + \|f\|_{H^{-l}(M)} \right).$$

(b) Assume in addition, that N is injective on some (closed) subspace $\mathcal{L} \subset L^2(M)$. Then there exists $C > 0$, so that

$$(34) \quad \|f\|_{L^2(M)} \leq C\|Nf\|_{H^m(M)}, \quad \forall f \in \mathcal{L}.$$

Proof. Part (a) follows directly from the existence of a parametrix. Indeed, there is $Q \in \Psi^m$ so that $QAf = f + Kf$, where K has a smooth kernel. Note that for this proof it is only enough to have a parametrix Q of finite order l , i.e., to have K that maps L^2 into H^l . Then (33) follows directly.

Note that the inclusion $L^2 \rightarrow H^{-l}$ is a compact map. Part (b) follows from Proposition 2 above.

An alternative way to prove (b) is to notice that it follows directly from (a), by [23, Proposition 5.3.1]. \square

Recall that $\Psi^{-m}(M)$ is a Fréchet space with semi-norms $p_k(A)$, $k = 1, 2, \dots$ given by the semi-norms (they are actually norms) of its amplitude in the finitely many charts (U_j, χ_j) of M . The norms p_k are given by

$$p_k(a) = \max_j \sup_{x \in \bar{U}_j, \xi \in \mathbf{R}^n} \max_{|\alpha|+|\beta|+|\gamma| \leq k} (1 + |\xi|)^{m+|\gamma|} |\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a_j(x, y, \xi)|,$$

where a_j is the full symbol of A in the j -th coordinate chart.

Proposition 4. *There exists $k \gg 1$, depending only on $\dim M$ with the following property. Let $A = A_0 \in \Psi^{-m}$ satisfy the assumptions of Proposition 3(b). Then (31) holds for any $A \in \Psi^{-m}$ close enough to A_0 in the norm p_k with a constant C independent of A .*

Proof. Note that Ψ^{-m} equipped with the norm p_k , $k \geq 2n + m + 1$, is a subspace of the space of linear bounded operators mapping $L^2(M)$ into $H^m(M)$, see the appendix in [16]. Now the proposition follows from Proposition 2(b). \square

Remark 6. Proposition 3 can be applied to compact manifolds M with boundary; then we require the symbol to be defined and to satisfy the symbol estimates for x in the closed M . One can then extend it as an elliptic symbol to some extension of M without boundary, and treat $L^2(M)$ as \mathcal{L} . The same remark applies to Proposition 4 because one can choose the extension of the symbol in a way that is continuous in any fixed p_k norm.

Finally, we formulate a direct corollary of these statements.

Corollary 1. *Let $A : L^2(M) \rightarrow \mathcal{H}$, be a bounded linear operator, where M is a compact manifold without boundary, and \mathcal{H} is a Hilbert space. Assume that $N = A^*A$ is an elliptic Ψ DO of order $-m \leq 0$. Assume also that A is injective on a (closed) subspace \mathcal{L} of $L^2(M)$.*

(a) *Then $N : \mathcal{L} \rightarrow \mathcal{L}$ is onto, and*

$$(35) \quad \|f\|_{L^2(M)} \leq C\|Nf\|_{H^m(M)}, \quad \forall f \in \mathcal{L}.$$

(b) *Estimate (35) remains true with an uniform $C > 0$ if N is replaced by any other operator close enough to N in the operator norm $L^2(M) \rightarrow \mathcal{H}$, and in particular for all Ψ DOs in a small enough neighborhood of A in the norm p_k , $k \gg 1$.*

Proof. The only statement left to prove is that $N : \mathcal{L} \rightarrow \mathcal{L}$ is onto. This follows from the fact that N is injective and self-adjoint, hence it has a trivial cokernel. Next, since N is Fredholm, its range is closed. \square

In particular, by Lemma 1, Corollary 1(b) implies that A remains injective under a small perturbation (which also follows from the fact that $A|_{\mathcal{L}}$ is upper semi-Fredholm). Also, (35) implies an estimate of the kind $\|f\| \leq C\|Af\|_*$ with a suitable norm $\|\cdot\|_*$.

Note that Corollary 1 can be applied to manifolds with boundary as in the example below.

Example 3. We return to Example 2. Let $\Omega_1 \ni \Omega$ as before. Assume that $N_w = I_w^* I_w$ is an elliptic Ψ DO (of order -1) in Ω_1 . To apply Corollary 1 directly, one can embed Ω_1 into a compact manifold and extend N_w as an elliptic operator there in a way independent of w . Alternatively, one can work with the parametrix in Ω_1 and eventually restrict to $\bar{\Omega}$. By (28), the principal symbol of N_2 is given by (see [17, 18, 8] for a derivation in even more general cases)

$$\sigma_p(N_w)(x, \xi) = 2\pi \int_{S^{n-1}} |w(x, \theta)|^2 \delta(\xi \cdot \theta) d\theta,$$

where δ is the Dirac delta function, and clearly, $\sigma_p(N_w)$ is homogeneous in ξ of order -1 . The singularity at $\xi = 0$ can be cut-off. Since w is at least continuous, we see that the ellipticity assumption is equivalent to the following:

$$(36) \quad \forall (x, \zeta) \in \Omega \times S^{n-1}, \exists \theta \in S^{n-1}, \theta \perp \zeta; \text{ so that } w(x, \theta) \neq 0.$$

In invariant terms, $(x, \xi) \in S^* \Omega$, while $(x, \theta) \in S\Omega$. Another way to express the same is to say that

$$(37) \quad N^*(\text{supp}^0 w) \supset T^* \Omega,$$

where $\text{supp}^0 w$ is the open set where $w \neq 0$. Under this condition, using the continuity of w in $S\bar{\Omega} \cong \bar{\Omega} \times S^{n-1}$, one can get the same with Ω replaced by Ω_1 . By Proposition 3, we then get

$$(38) \quad \|f\|_{L^2(\Omega)} \leq C \|N_w f\|_{H^1(\Omega_1)},$$

under the assumption that I_w is injective on $L^2(\Omega)$. Since N_w is elliptic, it is enough to assume only that I_w is injective on $C_0^\infty(\Omega)$. Without that assumption, one has (33); and (38) on $(\text{Ker } I_w)^\perp$.

To analyze the continuous dependence on w , let us note first that given $k \geq 0$, one can construct a bounded extension operator extending functions in $C^k(\bar{\Omega})$ to $C^k(\bar{\Omega}_1)$. With the aid of this operator, one can map a neighborhood of some $w_0 \in C^k(\bar{\Omega})$ to some neighborhood of an extended w_0 in $C^k(\bar{\Omega}_1)$. Then we can apply Corollary 1 to get the following.

Theorem 3. *Let $w \in C^1(\bar{\Omega} \times S^{n-1})$ satisfy (36). Assume that I_w is injective on $C_0^\infty(\Omega)$. Then $N_w : L^2(\Omega) \rightarrow L^2(\Omega)$ is onto, and there exists a constant $C > 0$ so that*

$$(39) \quad \|f\|_{L^2(\Omega)} \leq C \|N_w f\|_{L^2(\Omega_1)}, \quad \forall f \in L^2(\Omega).$$

Moreover, (39) remains true under small $C^1(\bar{\Omega})$ perturbations of w .

The transform I_w is injective for a dense open set of weights satisfying (23), including real analytic ones, see [8]; but not for all positive smooth weights, see [3].

Without the injectivity assumption, (39) holds on $(\text{Ker } I_w|_{C_0^\infty(\Omega)})^\perp$. Theorem 3 is a weaker version of the results in [8] and is presented here just to illustrate the method. Note that $w \in C^k$ with $k = 1$ is not enough to guarantee that operator convergence in the $p_k(N)$ norm implies convergence in the norm $\|N\|_{L^2(\Omega) \rightarrow H^1(\Omega_1)}$, viewing $N = N_w$ as a Ψ DO. On the other hand, N_w is also an operator with singular kernel, see (28), and $w \in C^1$ is enough for $L^2 \rightarrow H^1$ continuity. We refer to [8] for details.

We show finally that for an injective Ψ DO, ellipticity is a necessary and sufficient condition for the estimate (35) in Corollary 1 to hold.

Proposition 5.

(a) Let N be a Ψ DO of order $-m \leq 0$ on a compact manifold M . Then (34) holds for any $f \in L^2(M)$ if and only if N is injective and elliptic.

(b) Let M be a compact manifold with boundary and let M_1 be another compact manifold so that $M_1 \ni M$. Assume that N is a Ψ DO of order $-m \leq 0$ defined on some neighborhood of M . Then

$$(40) \quad \|f\|_{L^2(M)} \leq C \|Nf\|_{H^m(M_1)}, \quad \forall f \in L^2(M)$$

if and only if N is injective on $L^2(M)$ and elliptic on $\pi^{-1}(M)$; more precisely,

$$\inf_{x \in M} \liminf_{|\xi| \rightarrow \infty} |\xi|^m |p_0(x, \xi)| > 1/C,$$

where p_0 is the principal symbol of N .

Proof. To prove (a), we will work locally near a fixed point on M . In fixed local coordinates near x_0 , let $p(x, \xi)$ be the symbol of N , multiplied by a standard cut-off supported near some x_0 . We will study $p(x, D)$ in \mathbf{R}^n first. Fix $\xi_0 \neq 0$ and apply N to the normalized ‘‘coherent state’’

$$f_\lambda(x) = (\lambda/\pi)^{n/4} e^{i\lambda x \cdot \xi_0 - \lambda|x-x_0|^2/2}.$$

Then

$$\hat{f}_\lambda(\lambda\xi) = (\lambda/\pi)^{n/4} (2\pi/\lambda)^{n/2} e^{-i\lambda(\xi-\xi_0) \cdot x_0 - \lambda|\xi-\xi_0|^2/2}.$$

Then we get

$$[p(x, D)f_\lambda](x) = e^{i\lambda x_0 \cdot \xi_0} (\lambda/\pi)^{n/4} (2\pi\lambda)^{-n/2} \int e^{i\lambda(x-x_0) \cdot \xi - \lambda|\xi-\xi_0|^2/2} p(x, \lambda\xi) d\lambda\xi.$$

For any $C > 0$, the contribution to the integral above coming from integrating over $|\xi - \xi_0| > 1/C$ is $O(\lambda^{-N})$, $\forall N > 0$. To estimate the effect of replacing $p(x, \lambda\xi)$ by $p(x, \lambda\xi_0)$ in the integral above, notice first that $p(x, \lambda\xi) - p(x, \lambda\xi_0) = \zeta(x, \xi, \xi_0, \lambda) \cdot (\xi - \xi_0)$ with $\zeta = O(1)$, therefore

$$(41) \quad \int_{|\xi-\xi_0| < C} |\xi - \xi_0| e^{-\lambda|\xi-\xi_0|^2/2} d\xi = O(\lambda^{-n})$$

(pass to polar coordinates $r\omega = \xi - \xi_0$). Thus we get

$$\begin{aligned} [p(x, D)f_\lambda](x) &= e^{i\lambda x_0 \cdot \xi_0} (\lambda/2\pi)^{n/2} (\lambda/\pi)^{n/4} \left[\int e^{i\lambda(x-x_0) \cdot \xi - \lambda|\xi-\xi_0|^2/2} p(x, \lambda\xi_0) d\xi + O(\lambda^{-n}) \right] \\ &= e^{i\lambda x_0 \cdot \xi_0} (\lambda/2\pi)^{n/2} (\lambda/\pi)^{n/4} \left[p(x, \lambda\xi_0) (2\pi/\lambda)^{n/2} e^{i\lambda(x-x_0) \cdot \xi_0 - \lambda|x-x_0|^2/2} + O(\lambda^{-n}) \right] \\ &= p(x, \lambda\xi_0) f_\lambda(x) + O(\lambda^{-n/4}). \end{aligned}$$

Assume now that (34) holds. It is enough to consider the case $m = 0$. Then, choosing $f = f_\lambda$, we get

$$(42) \quad 1/C \leq \|p(\cdot, \lambda\xi_0) f_\lambda(\cdot)\| + O(\lambda^{-n/4}).$$

The function $\phi_\lambda(x - x_0) = |f_\lambda(x)|^2 = (\lambda/\pi)^{n/2} e^{-\lambda|x-x_0|^2}$ is normalized so that it has L^1 norm equal to 1. Therefore,

$$(43) \quad \|p(\cdot, \lambda\xi_0) f_\lambda(\cdot)\|^2 = \int \phi_\lambda(x - x_0) |p(x, \lambda\xi_0)|^2 dx = |p(x_0, \lambda\xi_0)|^2 + O(\lambda^{-n})$$

by the argument already used above, see (41). Relations (42) and (43) show that p is elliptic at (x_0, ξ_0) . More precisely, we get

$$(44) \quad |p(x, \xi)| \geq 1/C - C_1 |\xi|^{-n/4}, \quad \forall (x, \xi) \in T^*M,$$

where $C > 0$ is the constant C in (34), and $C_1 > 0$ depends on p only.

To relate N and $p(x, D)$, note that in a fixed coordinates system near some x_0 , $R = N - p(x, D)$ is smoothing when acting on distributions supported near x_0 . Consider then $(N - p(x, D))\chi f_\lambda$, where χ is a standard cut-off near x_0 . Then

$$p(x, D)f_\lambda = p(x, D)(1 - \chi)f_\lambda + p(x, D)\chi f_\lambda = p(x, D)(1 - \chi)f_\lambda + (N - R)\chi f_\lambda.$$

Now, $(1 - \chi)f_\lambda = O(e^{-\lambda/C})$, $N\chi f_\lambda = O(\lambda^{-\infty})$ as can be seen by integrating by parts and using the smoothness of the kernel of R . Therefore, $N\chi f_\lambda = p(x, D)f_\lambda + O(\lambda^{-\infty})$, and this completes the proof of (a).

To prove (b), we proceed in the same way in a neighborhood of any $x_0 \in M^{\text{int}}$ but we apply $p(x, D)$ to χf_λ instead to f_λ , where χ is a standard cut-off near x_0 supported in M . As x_0 gets closer to ∂M , the derivatives of χ will have larger norms. This will affect the constant C_1 in (44) that may not be uniform in x . The conclusion (b) is still true, however. Note that if N has a polyhomogeneous symbol, then the sub-principal symbol must be uniformly bounded for $x \in M$, because we assume that N is a Ψ DO in the larger M_1 . Then C_1 is actually uniformly bounded for $x \in M$. \square

3.1. Non-sharp linear stability estimates. In view of the analysis in Section 2, we are also interested whether an estimate of the type (11), weaker than (35) can still be proven when N is not an elliptic Ψ DO. In other words, what conditions would imply (35) in different H^s spaces? Ellipticity of N is not necessary anymore, and for certain classes of hypoelliptic operators, one can still have (35) with a loss of one derivative, for example. We are not going to review those classes of operators. Without a proof, we will only mention that if the full symbol of N vanishes on some open conic set, (35) cannot hold in any Sobolev spaces. This is well known and used in tomography to determine subsets of lines/curves on which the X-ray transform can or cannot determine the function we integrate in a stable way, see e.g., [12]. In particular, in Example 2, if $T^* \setminus N^*(\text{supp } w)$ contains an open conic set, see (37), then no $H^{s_1} \rightarrow H^{s_2}$ stability estimate of the type (38) is possible.

3.2. Injectivity of the differential A_{f_0} . In this section so far we were trying to find sufficient conditions that would guarantee a stability estimate under the assumption that the differential A_{f_0} is known to be continuous. It is not the scope of this work to study in detail the typical approaches to prove that the differential is continuous. We will only briefly mention some of them: smallness of the coefficients is often used when the constant coefficient case is easy to study directly. Another technique is the use of the Analytic Fredholm Theorem [13] that, when can be applied, gives injectivity of A_f for generic f 's. When the unknown coefficient is in the principal symbol, as in the boundary rigidity question and related hyperbolic inverse problems [19], the calculus of the analytic Ψ DOs can be applied, see e.g., [18].

4. AN EXAMPLE: THE INVERSE BACK-SCATTERING PROBLEM FOR THE ACOUSTIC WAVE EQUATION

Consider the acoustic wave equation

$$(45) \quad (\partial_t^2 - c^2(x)\Delta)u = 0, \quad t \in \mathbf{R}, x \in \mathbf{R}^3$$

with a variable speed $c(x) > 0$ that equals 1 for large x , i.e.,

$$(46) \quad c(x) = 1 \quad \text{for } |x| > \rho$$

with some $\rho > 0$. Let $S(s, \omega, \theta)$, where $(s, \omega, \theta) \in \mathbf{R} \times S^2 \times S^2$, be the scattering kernel associated with c , see below. In [16] we showed that if c is close enough to 1 in the C^9 norm, then the back-scattering kernel $S(s, -\theta, \theta)$ determines c uniquely.

We recall some facts about the time-dependent scattering theory for (45), see [16]. For simplicity, we work in space dimension $n = 3$. Given $\theta \in S^2$, we denote by $u(t, x, \theta)$ the *distorted plane wave* defined as the solution of (45) satisfying

$$(47) \quad u|_{t \ll 0} = \delta(t - x \cdot \theta).$$

Then we set $u_{\text{sc}} = u - \delta(t - x \cdot \theta)$. In the Lax-Phillips scattering theory [9], see also [5], the *asymptotic wave profile* u^\sharp of u is defined by

$$u_{\text{sc}}^\sharp(s, \omega, \theta) = \lim_{t \rightarrow \infty} (t + s) \partial_t u_{\text{sc}}(t, (t + s)\omega, \theta),$$

where $s \in \mathbf{R}$, $\omega \in S^2$. The *scattering kernel* $S(s, \omega, \theta)$ is given by

$$S(s, \omega, \theta) = -\frac{1}{2\pi} u_{\text{sc}}^\sharp(s, \omega, \theta).$$

Then $S(s' - s, \omega', \omega)$ is the Schwartz kernel of $\mathcal{R}(\mathcal{S} - \text{Id})\mathcal{R}^{-1}$, where \mathcal{S} is the scattering operator, and \mathcal{R} is the Lax-Phillips translation representation.

One can formulate this problem with stationary data in mind. The scattering amplitude a , that is also defined through the stationary scattering theory, is essentially just a Fourier transform (a stable and invertible operation) of S in the s variable, more precisely,

$$(48) \quad \frac{i\lambda}{2\pi} a(\lambda, \omega, \theta) = \int e^{-is\lambda} S(s, \omega, \theta) ds,$$

see [16].

Set $\omega = -\theta$. Then

$$(49) \quad \text{supp } S(\cdot, -\theta, \theta) \subset (-\infty, 2\rho].$$

Let P_θ be the plane $x \cdot \theta = -\rho$, and let “dist” be the distance function in the metric $c^{-2} dx^2$. Fix T so that

$$(50) \quad T > 2 \max\{\text{dist}(P_\theta, x); x \in B(0, \rho), \theta \in S^2\} - 2\rho,$$

where $B(0, \rho)$ is the ball with center 0 and radius ρ , see also (61). We use the information contained in S for $s \geq -T$ only. We will denote by h the Heaviside function (the characteristic function of \mathbf{R}_+). Our data now is

$$(51) \quad S_T(s, -\theta, \theta) := h(s + T)S(s, -\theta, \theta).$$

Instead of working with S_T , we will work with its stationary analog a_T defined by

$$(52) \quad a_T(\lambda, -\theta, \theta) := \frac{2\pi}{i\lambda^3} \int e^{-is\lambda} S_T(s, \omega, \theta) ds, \quad \lambda > 0.$$

The extra factor λ^{-2} there, compared to (48), is there for convenience. Clearly, $\lambda^2 a_T = \hat{S}_T * \hat{\chi}_T$, where the hat indicates Fourier transform w.r.t. s . Getting a stability estimate in terms of a_T instead of a is even stronger because it uses less information. Since a_T is analytic in λ , we can throw away any finite interval without losing information. Our next theorem shows that we are not losing stability, either. We fix $\lambda_0 \geq 0$ and restrict λ to the interval $\lambda \geq \lambda_0$.

Theorem 4. *There exist $\varepsilon > 0$, $k \geq 2$, $\mu \in (0, 1)$ with the following property. For any two \tilde{c} , c satisfying (46) and*

$$(53) \quad \|c - 1\|_{C^k} + \|\tilde{c} - 1\|_{C^k} \leq \varepsilon,$$

and for any $\lambda_0 \geq 0$, we have

$$(54) \quad \|\tilde{c} - c\|_{L^\infty} \leq C \left(\sup_{\lambda > \lambda_0, \theta \in S^2} (1 + \lambda)^{-2} |(\tilde{a}_T - a_T)(\lambda, -\theta, \theta)| \right)^\mu,$$

where T is fixed so that it satisfies (50) w.r.t. both \tilde{c} and c .

Proof. We start with standard geometric optics arguments, see also [16, Prop. 3.1]. Let $h_j(t) = t^j/j!$ for $t \geq 0$, and $h(t) = 0$ for $t < 0$; then $h'_j = h_{j-1}$, $j \geq 0$, with the convention $h_0 = h$, $h_{-1} = \delta$.

Proposition 6. *If $c \in C^\infty$, and $\varepsilon \ll 1$, then $\forall N$,*

$$(55) \quad u = \sum_{j=-1}^N \alpha_j(x, \theta) h_j(t - \phi(x, \theta)) + r_N(t, x, \theta)$$

for $|x| \leq \rho$, where ϕ solves the eikonal equation

$$(56) \quad c^2(x) |\nabla \phi|^2 = 1, \quad \phi|_{x \cdot \theta < -\rho} = x \cdot \theta;$$

α_j solve the corresponding transport equations; and $r_N \in C^N$.

Proof. The construction of α_j is standard, we will focus our attention on r_N . The latter solves

$$(\partial_t^2 - c^2(x)\Delta) r_N = (c^2 \Delta \alpha_N) h_N(t - \phi(x, \theta)), \quad r_N|_{t < -\rho} = 0.$$

By standard energy estimates, $r_N \in H_{\text{loc}}^{N+1}$, see also [16]. By Sobolev embedding theorems, we get $r_N \in C^{N-1}$. To prove that $r_N \in C^N$, we apply this to r_{N+1} ; then $\alpha_{N+1} h_{N+1}(t - \phi(x, \theta)) + r_{N+1} \in C^N$. \square

Note that $\rho + \phi(x, \theta) = \text{dist}(P_\theta, x)$. The expansion (55) is valid in $\overline{B(0, \rho)}$ if the eikonal equation has a smooth solution there. The latter is guaranteed by the following geometrical condition. Let $\gamma_{z, \theta}(t)$ be the geodesic in the metric $c^{-2} dx^2$ issued from z in the direction of θ . Then we want $z \in P_\theta$, and t to be coordinates in $\overline{B(0, \rho)}$. This is certainly true if c is close enough to a constant in C^2 that we assume. The leading coefficient α_{-1} solves a homogeneous transport equation, and in particular, $\alpha_{-1} > 0$. If $c = 1$, then $\alpha_{-1} = 1$, $\alpha_j = 0$ for $j \geq 0$, and $\phi = x \cdot \theta$.

Note that (55) remains true if $c \in C^k$, $k \gg 2$ but for some $N = N(k)$, and $N(k) \gg 1$ if $k \gg 2$. Moreover, $\{\alpha_j\}_{j=-1}^N$ depend continuously on $c \in C^k$. Therefore, under the assumptions of the theorem, for any fixed N ,

$$(57) \quad \|\alpha_{-1} - 1\|_{C^N} \leq C\varepsilon, \quad \|\alpha_j\|_{C^N} \leq C\varepsilon, \quad j = 1 \dots N, \quad \|r_N\|_{C^N} \leq C\varepsilon,$$

if $k \gg 1$. The C^N norm of α_j is taken in $\overline{B(0, \rho)} \times S^2$, while the norm of r_N is taken in for $(x, \theta) \in \overline{B(0, \rho)} \times S^2$, and t in any fixed finite interval (under the assumption $\varepsilon \ll 1$). Similarly,

$$(58) \quad \|\phi - x \cdot \theta\|_{C^N} \leq C\varepsilon,$$

see also [16].

Let c and \tilde{c} be two sound speeds, and denote by \tilde{S} , \tilde{u} the scattering kernel S and the solution u related to \tilde{c} . The following formula is proven in [16]:

$$(59) \quad (\tilde{S} - S)(s, \omega, \theta) = \frac{1}{8\pi^2} \partial_s^3 \iint (\tilde{c}^{-2} - c^{-2}) \tilde{u}(t, x, \theta) u(-s - t, x, -\omega) dt dx.$$

The integral above makes sense even though u and \tilde{u} are distributions.

By the finite speed of propagation for (45), see e.g. [7],

$$(60) \quad \text{supp } u(t, \cdot, \theta) \subset \{x; \phi(x, \theta) \leq t\}, \quad \text{supp } u(-s - t, \cdot, \theta) \subset \{x; \phi(x, \theta) \leq -s - t\}.$$

Therefore, for $s \geq -T$, on the support of the integrand in (60), we have

$$t \leq T - \phi(x, \theta), \quad -s - t \leq T - \phi(x, \theta),$$

see (60). This gives an upper bound for t and $-s - t$ for $|x| \leq \rho$. We want that upper bound to be so that the singular front of both waves involved in (48), with $\omega = -\theta$ there, would pass through the whole ball $B(0, \rho)$. Physically, this is equivalent to the requirement that we make our measurements for an interval of time so that the front of the wave $u(t, x, \theta)$ would have enough time to pass through the whole $B(0, \rho)$, and then return back a scattered signal. This explains the choice of T : we want T to be so that $T - \phi(x, \theta) > \phi(x, \theta)$, $\forall x \in B(0, \rho)$, i.e.,

$$(61) \quad T > 2 \max\{\phi(x, \theta); |x| \leq \rho, |\theta| = 1\}.$$

This is equivalent to (50). Therefore, we study S restricted to $-T \leq s \leq 2\rho$ (the upper bound is not a restriction actually, see (49)), and the length of that interval, assuming for a moment equality in (50), equals twice the length of the longest geodesic issued from some support plane $P_\theta = \{x \cdot \theta = -\rho\}$, and staying in $B(0, \rho)$.

For $x \in B(0, \rho)$, $s \geq -T$, and $t > T + \rho$, we have that $-s - t < -\rho$, and then $u(-s - t, x, \theta) = 0$, see (60). Similarly, for $x \in B(0, \rho)$, $s \geq -T$, and $-s - t > T + \rho$, we have that $t < -\rho$, and then $\tilde{u}(t, x, \theta) = 0$. This shows that in the integral (59), the solutions \tilde{u} and u can be replaced by the same solutions restricted to $t \leq T + \rho$, and this will not change the l.h.s. when $s > -T$. Let χ be a smooth function so that

$$(62) \quad \chi(t) = 1 \text{ for } -\rho \leq t \leq T + \rho, \quad \chi \in C_0^\infty(\mathbf{R}).$$

Set

$$(63) \quad u_T = \chi(t)u(t, x, \theta),$$

and we define \tilde{u}_T in a similar way. Note that the subscript T in S and u has a different meaning. The cut-off for $t < -\rho$ in (63) does not change u for $x \in B(0, \rho)$ because $u = 0$ there. Then

$$(64) \quad (\tilde{S}_T - S_T)(s, -\theta, \theta) = \frac{1}{8\pi^2} \partial_s^3 \iint (\tilde{c}^{-2} - c^{-2}) \tilde{u}_T(t, x, \theta) u_T(-s - t, x, \theta) dt dx.$$

Take Fourier transform of both sides to get

$$(65) \quad (\tilde{a}_T - a_T)(\lambda, -\theta, \theta) = -\frac{1}{4\pi} \int (\tilde{c}^{-2}(x) - c^{-2}(x)) \tilde{v}_T(x, \theta, \lambda) v_T(x, \theta, \lambda) dx,$$

where

$$(66) \quad v_T(x, \theta, \lambda) = \int e^{i\lambda t} u_T(t, x, \theta) dt.$$

Remark 7. Note that if we replace u_T by u on the r.h.s. above, we get the ‘‘distorted harmonic wave’’ solution $v = e^{i\lambda x \cdot \theta} + v_{\text{sc}}$, where v_{sc} is outgoing, i.e., it satisfies the Sommerfeld radiation conditions. In our case, v_T is a convolution of v with the Fourier transform of a certain characteristic function. That convolution does not affect the high λ asymptotic of v , so it is not surprising that below we get the expansion familiar from the stationary theory. The advantage that we have here is that we do not need to estimate the contribution of large t 's, and instead of using estimates of the cut-off resolvent for non-trapping systems, we can just use the geometric optics expansion (55) that

is more direct, easier to justify for c 's of finite smoothness, and easier to perturb near a fixed c . Furthermore, we get an even stronger result by discarding information ($t \gg 0$) that is not needed.

Linearizing (65), we get the following.

Proposition 7. *Let c, \tilde{c} be two speeds satisfying the assumptions of Theorem 4. Then*

$$(67) \quad (\tilde{a}_T - a_T)(\lambda, -\theta, \theta) = \int (\tilde{c}^{-2}(x) - c^{-2}(x)) v_T^2(x, \theta, \lambda) dx + R(\theta, \lambda)$$

where

$$(68) \quad \sup_{\theta \in S^2, \lambda > 0} (1 + \lambda)^{-2} |R(\theta, \lambda)| \leq C \|\tilde{c}^{-2} - c^{-2}\|_{L^\infty}^2.$$

Proof. Let w be defined as u but satisfying $w|_{t \leq 0} = h_2(t - x \cdot \theta)$, where, as above, $h_2(s) = s^2/2$, if $s \geq 0$, and $h_2(s) = 0$ otherwise. Then $h_2''' = \delta$, $u = \partial_t^3 w$, and $w \in H_{\text{loc}}^2$, $w_t \in H_{\text{loc}}^1$. The difference $\tilde{w} - w$ solves

$$(\partial_t^2 - c^2 \Delta)(\tilde{w} - w) = (\tilde{c}^2 - c^2) \Delta \tilde{w}, \quad (\tilde{w} - w)|_{t \leq 0} = 0.$$

By standard energy estimates, in any compact set,

$$\|\tilde{w}_t - w_t\|_{L_x^2} + \|\tilde{w} - w\|_{H_x^1} \leq C \|\tilde{c}^2 - c^2\|_{L^\infty}.$$

The constant C depends on \tilde{c} and c and is easy to see that it remains uniformly bounded, if c is bounded in C^1 that follows from the hypothesis (53). Then we get by (66),

$$(1 + \lambda)^{-2} \|\tilde{v}_T - v_T\|_{L^2} \leq C \|\tilde{c}^2 - c^2\|_{L^\infty}.$$

Compare (65) and (67) to get

$$\sup_{\lambda} |(1 + \lambda)^{-2} R(\theta, \lambda)| \leq C \|\tilde{c}^{-2} - c^{-2}\|_{L^\infty}^2, \quad \forall \theta,$$

where the L^2 norm is taken in any compact. □

We focus our attention now on the linear operator

$$(69) \quad Af(\lambda, \theta) = \int v_T^2(x, \theta, \lambda) f(x) dx, \quad f \in L^2(B(0, \rho)).$$

The geometric optics expansion (55) yields

$$(70) \quad v_T^2(x, \theta, \lambda) = e^{i2\lambda\phi(x, \theta)} (a_{-1}(x, \theta, \lambda) + a_0(x, \theta, \lambda) + \dots a_N(x, \theta, \lambda) + R_N(x, \theta, \lambda))^2, \quad \lambda \rightarrow \infty,$$

where

$$a_j = \int e^{i\lambda t} \alpha_j(x, \theta) \chi(t) h_j(t - \phi(x, \theta)) dt, \quad R_N = \int e^{i\lambda t} \chi(t) r_N(t, x, \theta) dt.$$

Set $\eta = 2\lambda\theta$, and extend ϕ by homogeneity (of order 1) w.r.t. to θ ; then $2\lambda\phi(x, \theta) = \phi(x, \eta)$. The leading term is

$$a_{-1}^2 = \chi^2(\phi(x, \theta)) \alpha_{-1}^2.$$

We have that $\phi(x, \theta)$ is $O(\varepsilon)$ close to $x \cdot \eta$, see (58). The natural choice for the space containing $\text{Ran } A$ is therefore $L^2(\mathbf{R}_\eta^n)$, that we will restrict to $|\eta| > 2\lambda_0$. In terms of the original variables, this space is isomorphic to $L^2((\lambda_0, \infty) \times S^2; \lambda^2 d\lambda d\theta)$. Clearly, if $c \in C^\infty$, (70) implies that the expression in the parentheses on the r.h.s. is a classical elliptic symbol. For $c \in C^k$, $k \gg 1$ the symbol satisfies the symbol estimates up to a finite order m only, with $m \gg 1$, when $k \gg 1$. To prove this, we first show that a_j is a symbol of order $-j$ (of finite regularity). The properties of r_N that we established do not prove that R_N satisfies the symbol estimate $|\lambda^j \partial_\lambda^j \partial_{x, \theta}^\alpha r_N| \leq C \lambda^{-N}$ but

it is clear that it satisfies $|\partial_\lambda^j \partial_{x,\theta}^\alpha r_N| \leq C\lambda^{-N}$. We can replace N by $N + m$ now and this argument is enough to show that A^*A is a Ψ DO with symbol of finite regularity m . This is summarized in the following.

Proposition 8. *Let $c \in C^k(\mathbf{R}^n)$ satisfy (46). Then*

$$v_T(x, \eta/|\eta|, |\eta|/2) = e^{i\lambda\phi(x,\eta)} a(x, \eta)$$

where ϕ and a belong to $C^N(\overline{B(0, \rho)} \times S^2)$ with $N = N(k) \rightarrow \infty$, as $k \rightarrow \infty$; $a(x, \eta)$ is a formal elliptic symbol of order 0 satisfying the symbol estimates with all its derivatives up to order N .

We now consider A as the operator given by (69) in the following spaces

$$(71) \quad A : \mathcal{B}'_1 := L^2(B(0, \rho)) \longrightarrow \mathcal{B}'_2 := L^2((\lambda_0, \infty) \times S^2; \lambda^2 d\lambda d\theta).$$

Then

$$(72) \quad A^*Af(x) = \int_{S^2} \int_{\lambda > \lambda_0} \int_{|x| \leq \rho} e^{-i\lambda 2(\phi(x,\theta) - \phi(y,\theta))} \bar{a}(x, \theta, \lambda) a(y, \theta, \lambda) f(y) dy \lambda^2 d\lambda d\theta.$$

By a standard argument, define $\xi = \xi(x, y, \eta)$ by $2\lambda\phi(x, \theta) - \phi(y, \theta) = (x - y) \cdot \xi$, where $\eta = 2\lambda\theta$ as above. The latter equation certainly has a solution for $|x| \leq \rho$ if c is close enough to a constant in C^k , $k \gg 1$, and that solution is close enough to $\xi = \eta$ in C^l with $l \gg 1$ when $k \gg 1$. Then we make a change of variables from η to ξ to get that A^*A is a Ψ DO of order 0 in a neighborhood of $B(0, \rho)$, and it is clearly elliptic.

To apply the results in section 3, and especially Proposition 3(b), we have to show that A^*A is injective. Set $M = B(0, \rho_1)$, where $\rho_1 > \rho$ is close enough to ρ so that the construction above remains valid on M . Then set $\mathcal{L} = L^2(B(0, \rho))$. Assume first that $c = 1$. Then

$$A^*Af(x) = \int_{S^2} \int_{\lambda > \lambda_0} \int_{|x| \leq \rho} e^{-i2\lambda(x-y) \cdot \theta} f(x) dy \lambda^2 d\lambda d\theta = \pi^3 h(|D| - 2\lambda_0) f,$$

where f is extended as zero outside $B(0, \rho)$. It is easy to see that $h(|D| - 2\lambda_0)$ is a Fredholm injective operator on $L^2(B(0, \rho))$, see [15] for any $\lambda_0 \geq 0$ (and it is the identity of $\lambda_0 = 0$). Therefore, the conditions of Proposition 3(b) are satisfied for $c = 1$ and we get

$$(73) \quad \|f\|_{L^2(B(0,\rho))}/C \leq \|A^*Af\|_{L^2(B(0,R))} \leq C\|f\|_{L^2(B(0,\rho))}$$

with some $R > \rho$. This show in particular that $A : \mathcal{B}'_1 \rightarrow \mathcal{B}'_2$ is bounded, depending continuously on $c \in C^k$, $k \gg 1$. That property is preserved if we increase ρ a bit. Therefore, $A^* : \mathcal{B}'_2 \rightarrow L^2(B(0, R))$ is bounded a well, and also depends continuously on $c \in C^k$, $k \gg 1$. Thus

$$(74) \quad \|f\|_{L^2(B(0,\rho))} \leq C\|Af\|_{\mathcal{B}'_2}.$$

This is the basic estimate that we need to apply Theorem 2. The spaces \mathcal{B}_1 and \mathcal{B}_2 are determined by the estimate in Proposition 7; set $\mathcal{B}_1 = L^\infty(B(0, \rho))$, and let \mathcal{B}_2 be the Banach space determined by the norm on the l.h.s. of (68). We need to show that the interpolation estimates (13) hold with a proper choice of $\mathcal{B}''_{1,2}$. Since

$$\|f\|_{\mathcal{B}_1} = \|f\|_{L^\infty(B(0,\rho))} \leq C_\varepsilon \|f\|_{H^{n/2+\varepsilon}},$$

$\forall \varepsilon > 0$, we use (18) to deduce that the second interpolation estimate in (13) holds with $\mathcal{B}''_1 = H^{s_1}(B(0, \rho))$, where $n/2 + \varepsilon = s_1(1 - \mu_1)$. Note that we can make μ_1 as close to 1 as we want by choosing s_1 large enough. Next, we need to choose \mathcal{B}''_2 so that the first interpolation estimate in (13) holds:

$$(75) \quad \|u\|_{\mathcal{B}'_2} \leq C \|u\|_{\mathcal{B}_2}^{\mu_2} \|u\|_{\mathcal{B}''_2}^{1-\mu_2}.$$

The following estimate follows from the Hölder inequality, and is actually a version of (18):

$$(76) \quad \int |u|^2 \lambda^2 d\lambda \leq \left(\int |u|^2 \lambda^{-4} d\lambda \right)^\alpha \left(\int |u|^2 \lambda^{2s} d\lambda \right)^{1-\alpha},$$

with $s = (1 + 2\alpha)/(1 - \alpha)$ for any $0 \leq \alpha < 1$. We therefore define \mathcal{B}_2'' to be the Hilbert space with a norm

$$\|u\|_{\mathcal{B}_2''} = \left(\int |u|^2 \lambda^{2s_2} d\lambda \right)^{1/2},$$

with $s_2 > 1$, and then (75) holds with $\mu_2 = (s_2 - 1)/(s_2 + 2)$.

The condition $\mu_1 \mu_2 > 1/2$ is therefore equivalent to

$$\frac{s_2 - 1}{s_2 + 2} \left(1 - \frac{n}{2s_1} \right) > \frac{1}{2}.$$

This can always be achieved with $s_1 \gg 1$, $s_2 \gg 1$. The condition (16) is then guaranteed by (53), if $k \geq s_1$.

This completes the proof of Theorem 4. □

Acknowledgments. The authors thank G. Alessandrini for pointing out the references [10, 11].

REFERENCES

- [1] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, tensor analysis, and applications*, volume 75 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1988.
- [2] G. Alessandrini. Open issues of stability for the inverse conductivity problem. *J. Inverse Ill-Posed Probl.*, 15(5):451–460, 2007.
- [3] J. Boman. An example of nonuniqueness for a generalized Radon transform. *J. Anal. Math.*, 61:395–401, 1993.
- [4] A. P. Calderón. On an inverse boundary value problem. *Comput. Appl. Math.*, 25(2-3):133–138, 2006.
- [5] J. Cooper and W. Strauss. Scattering of waves by periodically moving bodies. *J. Funct. Anal.*, 47(2):180–229, 1982.
- [6] G. Eskin. Inverse scattering problem in anisotropic media. *Comm. Math. Phys.*, 199(2):471–491, 1998.
- [7] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.
- [8] B. Frigyik, P. Stefanov, and G. Uhlmann. The X-ray transform for a generic family of curves and weights. *J. Geom. Anal.*, 18(1):81–97, 2008.
- [9] P. D. Lax and R. S. Phillips. *Scattering theory*, volume 26 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, second edition, 1989. With appendices by Cathleen S. Morawetz and Georg Schmidt.
- [10] C. D. Pagani. Questions of stability for inverse problems. *Rend. Sem. Mat. Fis. Milano*, 52:599–608 (1985), 1982.
- [11] C. D. Pagani. Stability of a surface determined from measures of potential. *SIAM J. Math. Anal.*, 17(1):169–181, 1986.
- [12] E. T. Quinto. Singularities of the X-ray transform and limited data tomography in \mathbf{R}^2 and \mathbf{R}^3 . *SIAM J. Math. Anal.*, 24(5):1215–1225, 1993.
- [13] M. Reed and B. Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York, 1972.
- [14] V. A. Sharafutdinov. *Integral geometry of tensor fields*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.
- [15] P. Stefanov. Generic uniqueness for two inverse problems in potential scattering. *Comm. Partial Differential Equations*, 17(1-2):55–68, 1992.
- [16] P. Stefanov and G. Uhlmann. Inverse backscattering for the acoustic equation. *SIAM J. Math. Anal.*, 28(5):1191–1204, 1997.
- [17] P. Stefanov and G. Uhlmann. Stability estimates for the X-ray transform of tensor fields and boundary rigidity. *Duke Math. J.*, 123(3):445–467, 2004.
- [18] P. Stefanov and G. Uhlmann. Boundary rigidity and stability for generic simple metrics. *J. Amer. Math. Soc.*, 18(4):975–1003 (electronic), 2005.

- [19] P. Stefanov and G. Uhlmann. Stable determination of generic simple metrics from the hyperbolic Dirichlet-to-Neumann map. *Int. Math. Res. Not.*, 17(17):1047–1061, 2005.
- [20] P. Stefanov and G. Uhlmann. Local lens rigidity with incomplete data for a class of non-simple Riemannian manifolds. *submitted*, 2007.
- [21] P. Stefanov and G. Uhlmann. Integral geometry of tensor fields on a class of non-simple Riemannian manifolds. *Amer. J. Math.*, 130(1):239–268, 2008.
- [22] J. Sylvester and G. Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987.
- [23] M. E. Taylor. *Partial differential equations. I, II*, volumes 115, 116 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996. Basic theory.
- [24] H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1978.
- [25] G. Uhlmann. *Commentary on Calderón’s paper (29) ”On an inverse boundary value problem”*; *Selected Papers of A.P. Calderón*, edited by Alexandra Bellow, Carlos Kenig and Paul Malliavin, AMS (2008), 623–636.
- [26] J.-N. Wang. Stability estimate for an inverse acoustic backscattering problem. *Inverse Problems*, 14(1):197–207, 1998.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195